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ON CONVERGENCE OF SUBSPACES GENERATED BY DILATIONS OF POLYNOMIALS. AN APPLICATION TO BEST LOCAL APPROXIMATION

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Abstract. We study the convergence of a net of subspaces generated by dilations of polynomials in a finite dimensional subspace. As a consequence, we extend the results given by Zó and Cuenya [*Advanced Courses of Mathematical Analysis II (Granada, 2004)*, 193–213, World Scientific, 2007] on a general approach to the problems of best vector-valued approximation on small regions from a finite dimensional subspace of polynomials.

1. Introduction

Suppose that $\{a_j\}$ is a data set. These data are values of a function and its derivatives at a point. If we want to approximate these data using a polynomial of degree at most l, which will be the best algorithm to use? A Taylor polynomial of degree l is probably the most natural procedure to use.

The problem of finding an optimal algorithm to approximate a finite number of data corresponding to a function is developed in the best local approximation theory.

In 1934, Walsh proved in [11] that the Taylor polynomial of degree l for an analytic function f can be obtained by taking the limit as $\epsilon \to 0$ of the best Chebyshev approximation to f from Π^l on the disk $|z| \leq \epsilon$. This paper was the first association between the best local approximation to a function f from Π^l in 0 and the Taylor polynomial for f at the origin. However, the concept of best local approximation has been introduced and developed more recently by Chui, Shisha, and Smith in [2]. Later, several authors [3, 4, 5, 6, 8, 9, 10, 12] have studied this problem.

We consider a family of function seminorms $\{\|\cdot\|_\epsilon\}_{\epsilon>0}$, acting on Lebesgue measurable functions $F : B \subset \mathbb{R}^n \to \mathbb{R}^k$, where B is the unit ball centered at the origin in $\mathbb{R}^n$. We will use the notation $F^\epsilon(x) = F(\epsilon x)$ and $\|F\|_\epsilon^* = \|F^\epsilon\|_\epsilon$. For $l \in \mathbb{N} \cup \{0\}$,

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we will denote by Π^l the class of algebraic polynomials in n variables of degree at most l, and Π^l_k the set $\{P = (p_1, \dots, p_k) : p_s \in \Pi^l\}$.

Let $\mathcal{A}$ be a subspace of Π^l_k and let $\{P_\epsilon\}_{\epsilon>0}$ be a net of best approximants to F from $\mathcal{A}$ with respect to $\|\cdot\|^*_\epsilon$, i.e.,

$$\|F - P_\epsilon\|^*_\epsilon \le \|F - P\|^*_\epsilon, \quad \text{for all } P \in \mathcal{A}. \tag{1.1}$$

If the net $\{P_\epsilon\}_{\epsilon>0}$ has a limit in $\mathcal{A}$ as $\epsilon \to 0$, this limit is called the *best local approximation to F from $\mathcal{A}$ in* 0. According to (1.1), we observe that P^ϵ_ϵ is a polynomial in

$$\mathcal{A}^\epsilon := \{P^\epsilon : P \in \mathcal{A}\} \subset \Pi^l_k \tag{1.2}$$

of best approximation to F^ϵ by elements of the class $\mathcal{A}^\epsilon$, with respect to the seminorm $\|\cdot\|_\epsilon$. We write it briefly as $P^\epsilon_\epsilon \in \mathcal{P}_{\mathcal{A}^\epsilon,\epsilon}(F^\epsilon)$. Note that $\mathcal{A}^\epsilon$ is a subspace generated by dilations of polynomials in $\mathcal{A}$.

From now on, we assume the following properties for the family of function seminorms $\|\cdot\|_\epsilon$, $0 \le \epsilon \le 1$.

(1) For $F = (f_1, \dots, f_k)$ and $G = (g_1, \dots, g_k)$, we have $\|F\|_\epsilon \le \|G\|_\epsilon$, for every $\epsilon > 0$, whenever $|f_s| \le |g_s|$, $s = 1, \dots, k$.
(2) If 1 is the function $F(x) = (1, \dots, 1)$, we have $\|1\|_\epsilon < \infty$, for all $\epsilon > 0$.
(3) For every $F \in C_k(B)$, we have $\|F\|_\epsilon \to \|F\|_0$, as $\epsilon \to 0$, where $C_k(B)$ is the set of continuous functions $F : B \subset \mathbb{R}^n \to \mathbb{R}^k$. Moreover, $\|\cdot\|_0$ is a norm on $C_k(B)$.

An important point to note here is that there exist positive constants $C = C(m,k)$ and $\epsilon(m,k)$ such that for every $0 < \epsilon \le \epsilon(m,k)$,

$$\frac{1}{C}\|P\|_0 \le \|P\|_\epsilon \le C\|P\|_0, \quad \text{for every } P \in \Pi^m_k \tag{1.3}$$

[13, Proposition 3.1].

In order to give an example of norms $\|\cdot\|_\epsilon$, $0 \le \epsilon \le 1$, with the properties (1)–(3), we recall a definition of convergence of measures given in [6]. See also [1] for the notion of weak convergence of measures in general.

Definition 1.1. Let μ_ϵ, $0 \le \epsilon \le 1$, be a family of probability measures on B. We say that the measures μ_ϵ converge weakly in the proper sense to the measure μ_0 if we have

$$\int_B f(x)\, d\mu_\epsilon(x) \to \int_B f(x)\, d\mu_0(x), \quad f \in C_1(B),$$

and $\mu_0(B') > 0$ for any ball $B' \subset B$.

The assumption on the measure μ_0 implies that

$$\|F\|_\epsilon = \|F\|_{L^p(\mu_\epsilon)} = \left(\int_B \|F\|^p d\mu_\epsilon\right)^{\frac{1}{p}}$$

is actually a norm on $C_k(B)$ for $\epsilon = 0$ and $1 \le p < \infty$, where $\|\cdot\|$ stands for any monotone norm on R^k. We use a monotone norm on R^k to ensure property (1) for the family of seminorms $\|\cdot\|_\epsilon$, $0 \le \epsilon \le 1$.

Let F be in $C_k(B)$; it is readily seen, by using the definition of weak convergence of measures, that there exists $\epsilon_0 = \epsilon_0(F) > 0$ such that if $\|F\|_\epsilon = \|F\|_{L^p(\mu_\epsilon)} = 0$, for some $0 < \epsilon \leq \epsilon_0$, then $F = 0$. Moreover we have that $\|F\|_\epsilon = \|F\|_{L^p(\mu_\epsilon)}$ converges as $\epsilon \to 0$ to the norm $\|F\|_0 = \|F\|_{L^p(\mu_0)}$ if $F \in C_k(B)$.

For more examples of nets of seminorms fulfilling conditions (1)–(3), we refer the reader to [13].

We say that $F : B \subset \mathbb{R}^n \to \mathbb{R}^k$ has a Taylor polynomial of degree m at 0 if there exists $P \in \Pi_k^m$ such that

$$\|F - P\|_\epsilon^* = o(\epsilon^m), \quad \text{as } \epsilon \to 0.$$

It is well known that if a Taylor polynomial exists, it is unique [13, Proposition 3.3]; we denote it by $T_m = T_m(F)$. We write $F \in t^m$ if the function F has the Taylor polynomial of degree m at 0. Moreover, if $F \in t^m$ and $T_m(F) = \sum_{|\alpha|\leq m} C_\alpha x^\alpha$, then the Taylor polynomial of degree $l \leq m$ for F at 0 is given by $T_l(F) = \sum_{|\alpha|\leq l} C_\alpha x^\alpha$ [13, Proposition 3.5], where $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n$ with $\alpha_i \geq 0$ and $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n$. We set $\partial^\alpha F(0)$ for the vector $\alpha! C_\alpha$ with $\alpha! = \alpha_1!\alpha_2!\dots\alpha_n!$.

The problem of best local approximation with a family of function seminorms $\{\|\cdot\|_\epsilon\}_{\epsilon>0}$ satisfying (1)–(3) was considered in [13] for two types of approximation class $\mathcal{A}$ fulfilling $\Pi_k^m \subset \mathcal{A} \subset \Pi_k^l$ and

(c1) $\mathcal{A}^\epsilon = \mathcal{A}$, for each $\epsilon > 0$, or

(c2) if $P \in \mathcal{A}$ and $T_{m+1}(P) = 0$, then $P = 0$.

Firstly, the authors studied the asymptotic behavior of a normalized error function as $\epsilon \to 0$ [13, Theorems 4.2 and 4.5]. Secondly, they showed that there exists the best local approximation to F in 0 and is associated with a Taylor polynomial for F in 0 [13, Theorem 5.1]. In particular, if $\mathcal{A} = \Pi_k^m$ and $F \in t^m$, they proved that $P_\epsilon \to T_m(F)$ as $\epsilon \to 0$ [13, Theorem 3.1].

In this work we generalize the results found in [13], without the restrictions (c1) or (c2) given above. For this, it is essential to study the convergence of the net $\{\mathcal{A}^\epsilon\}$ as $\epsilon \to 0$.

This paper is organized as follows. In Section 2, we investigate the asymptotic behavior of $\{\mathcal{A}^\epsilon\}$. In Section 3, we study the asymptotic behavior of the error function $\epsilon^{-m-1}(F_\epsilon - P_\epsilon)^\epsilon$ for a suitable integer, and we show some results about the best local approximation in the origin which generalizes those of [13].

2. Asymptotic behavior of the net $\{\mathcal{A}^\epsilon\}$

In this section, we study the asymptotic behavior of the net $\{\mathcal{A}^\epsilon\}$ given in (1.2). We begin with the following definition.

Definition 2.1. Let $\mathcal{A} \subset \Pi_k^l$ be a subspace. We say that $P \in \lim_{\epsilon\to 0} \mathcal{A}^\epsilon$ if there exists a net $\{P_\epsilon\} \subset \mathcal{A}$ such that $\lim_{\epsilon\to 0} \|P - P_\epsilon^\epsilon\|_0 = 0$. We denote $\mathcal{B} = \lim_{\epsilon\to 0} \mathcal{A}^\epsilon$.

Remark 2.2. If $\mathcal{A} \subset \Pi_k^l$ is a subspace, then the sets $\mathcal{A}^\epsilon$ and $\mathcal{B}$ are also subspaces of Π_k^l. Furthermore, if $\mathcal{A}^\epsilon = \mathcal{A}$, for all $\epsilon > 0$, we have that $\mathcal{B} = \mathcal{A}$.

Next, we show a simple example of $\mathcal{A}^\epsilon = \mathcal{A}$.

Example 2.3. Set $n = 3$ and $\mathcal{A} = \operatorname{span}\{(x_1, x_1 + x_2 + x_3, x_1^2 + x_2^2)\}$. Then, clearly we obtain $\mathcal{A}^\epsilon = \operatorname{span}\{(\epsilon x_1, \epsilon(x_1 + x_2 + x_3), \epsilon^2(x_1^2 + x_2^2))\} = \mathcal{A}$.

Proposition 2.4. *Let $\mathcal{A}$ be a subspace of polynomials such that $\Pi_k^m \subset \mathcal{A}$ for some $m \in \mathbb{N} \cup \{0\}$ and $k \in \mathbb{N}$. Then $\Pi_k^m \subset \mathcal{A}^\epsilon$ for all $\epsilon > 0$. Moreover, $\Pi_k^m \subset \mathcal{B}$.*

Proof. Set $R_{\alpha,i}(x) = x^\alpha e_i$, $|\alpha| \le m$, $1 \le i \le k$, where $\{e_i\}_{i=1}^k$ is the canonical basis of $\mathbb{R}^k$. Then

$$\{R_{\alpha,i} : |\alpha| \le m, 1 \le i \le k\} \tag{2.1}$$

is a basis of the space Π_k^m. Since $\mathcal{A}^\epsilon$ is a subspace, we have $R_{\alpha,i} = \frac{1}{\epsilon^{|\alpha|}} R_{\alpha,i}^\epsilon \in \mathcal{A}^\epsilon$, and so $\Pi_k^m \subset \mathcal{A}^\epsilon$, for all $\epsilon > 0$. Finally, using the definition of $\mathcal{B}$, we obtain $\Pi_k^m \subset \mathcal{B}$. □

From now on, for any Lebesgue measurable function $F : B \subset \mathbb{R}^n \to \mathbb{R}^k$ we denote $T_{-1}(F) = 0$.

Proposition 2.5. *Let $\mathcal{A}$ be a subspace of Π_k^l and let $0 \le s + 1 \le l$ be an integer. If $P \in \mathcal{A}$ satisfies $T_s(P) = 0$ and $T_{s+1}(P) \neq 0$, then $T_{s+1}(P) \in \mathcal{B}$.*

Proof. For each $\epsilon > 0$ we define $Q_\epsilon = \frac{P}{\epsilon^{s+1}} \in \mathcal{A}$. Since $T_s(P) = 0$, it follows that $\|T_{s+1}(P) - Q_\epsilon^\epsilon\|_0 = \frac{\|(T_{s+1}(P) - P)^\epsilon\|_0}{\epsilon^{s+1}}$. So $\|T_{s+1}(P) - Q_\epsilon^\epsilon\|_0 = o(1)$ as $\epsilon \to 0$, and thus $T_{s+1}(P) \in \mathcal{B}$. □

The following sets will be needed throughout the paper. Let $\mathcal{A}$ be a non-zero subspace of Π_k^l. We define

$$A_{-1} := \mathcal{A} \quad \text{and} \quad A_j := \{P \in \mathcal{A} : T_j(P) = 0\}, \text{ for } 0 \le j \le l. \tag{2.2}$$

We note that

$$A_j \subset A_i, \quad \text{whenever } i < j.$$

Since $A_l \subset \{P \in \Pi_k^l : T_l(P) = 0\} = \{0\}$, we have

$$\{j : 0 \le j \le l \text{ and } A_j \neq \mathcal{A}\} \neq \emptyset \quad \text{and} \quad \{j : 0 \le j \le l \text{ and } A_j = \{0\}\} \neq \emptyset.$$

Set

$$s_0 = \min\{j : 0 \le j \le l \text{ and } A_j \neq \mathcal{A}\}$$

and

$$r_0 = \min\{j : 0 \le j \le l \text{ and } A_j = \{0\}\}.$$

It is easy to see that $0 \le s_0 \le r_0 \le l$, and

$$s_0, r_0 \in \{j : s_0 \le j \le r_0 \text{ and } A_j \subsetneq A_{j-1}\} =: J.$$

We can now formulate our main result which describes the limit set $\mathcal{B}$.

Theorem 2.6. *Let $\mathcal{A}$ be a non-zero subspace of Π_k^l. Then $\mathcal{B}$ is a subspace of $\Pi_k^{r_0}$ isomorphic to $\mathcal{A}$. Furthermore, under the above notation the following holds:*

(a) if $s_0 < r_0$ and $J \setminus \{r_0\} = \{s_0, \dots, s_N\}$ with $s_i < s_{i+1}$ for $N > 0$, then $\mathcal{B} = T_{r_0}(A_{s_N}) \oplus T_{s_N}(S_{s_N}) \oplus T_{s_{N-1}}(S_{s_{N-1}}) \oplus \cdots \oplus T_{s_0}(S_{s_0})$, where $A_{s_i} \oplus S_{s_i} = A_{s_{i-1}}$, $0 \le i \le N$;

(b) if $s_0 = r_0$, then $\mathcal{B} = T_{r_0}(\mathcal{A})$.

Proof. (a) Assume $s_0 < r_0$. Since every subspace of $A_{s_{i-1}}$, $0 \le i \le N$, has a complement, there exists a subspace $S_{s_i} \subset A_{s_{i-1}}$ such that

$$A_{s_i} \oplus S_{s_i} = A_{s_{i-1}}, \quad 0 \le i \le N. \tag{2.3}$$

In consequence,

$$\mathcal{A} = A_{s_N} \oplus S_{s_N} \oplus S_{s_{N-1}} \oplus \cdots \oplus S_{s_0}. \tag{2.4}$$

As $S_{s_i} \subset A_{s_{i-1}}$, $0 \le i \le N$, and $A_{r_0-1} = A_{s_N}$, we obtain

$$Q(x) = \begin{cases} \sum_{|\alpha| \ge s_i} \frac{\partial^\alpha Q(0)}{\alpha!} x^\alpha, & \text{if } Q \in S_{s_i},\ 0 \le i \le N; \\ \sum_{|\alpha| \ge s_{N+1}} \frac{\partial^\alpha Q(0)}{\alpha!} x^\alpha, & \text{if } Q \in A_{s_N}, \end{cases} \tag{2.5}$$

where $s_{N+1} = r_0$. Let $T_i : S_{s_i} \to \Pi_k^{s_i}$ be a linear operator defined by $T_i(P) = T_{s_i}(P)$, $0 \le i \le N$, and $T_{N+1} : \mathcal{A} \to \Pi_k^{s_{N+1}}$ be the linear operator given by $T_{N+1}(P) = T_{s_{N+1}}(P)$. We claim that

(i) T_i is an injective operator, $0 \le i \le N+1$.

(ii) $T_{s_{N+1}}(A_{s_N}) \cap \sum_{i=0}^N T_{s_i}(S_{s_i}) = \{0\}$.

(iii) If $N > 0$ then $T_{s_l}(S_{s_l}) \cap \Big(T_{s_{N+1}}(A_{s_N}) + \sum_{i=0, i\ne l}^N T_{s_i}(S_{s_i})\Big) = \{0\}$ whenever $l \ne i$.

Indeed, let $0 \le i \le N$. If $T_{s_i}(P) = T_{s_i}(Q)$ for some $P, Q \in S_{s_i}$, then $P - Q \in A_{s_i} \cap S_{s_i}$. So (2.3) implies that $P = Q$. On the other hand, if $T_{s_{N+1}}(P) = T_{s_{N+1}}(Q)$ with $P, Q \in \mathcal{A}$, then $P - Q \in A_{s_{N+1}} = \{0\}$, which proves (i). To prove (ii) we consider $Q_{N+1} \in A_{s_N}$ and $Q_i \in S_{s_i}$ such that $P = T_{s_{N+1}}(Q_{N+1}) = \sum_{i=0}^N T_{s_i}(Q_i)$. From (2.5) we see that

$$T_{s_{N+1}}(Q_{N+1})(x) = \sum_{|\alpha| = s_{N+1}} \frac{\partial^\alpha Q_N(0)}{\alpha!} x^\alpha \quad \text{and} \quad \sum_{i=0}^N T_{s_i}(Q_i) \in \Pi_k^{s_N}. \tag{2.6}$$

Therefore $P = 0$. Now, let $Q_{N+1} \in A_{s_N}$ and $Q_i \in S_{s_i}$ be such that

$$P = T_{s_l}(Q_l) = T_{s_{N+1}}(Q_{N+1}) + \sum_{i=0, i\ne l}^N T_{s_i}(Q_i). \tag{2.7}$$

From (2.5) it follows that

$$T_{s_i}(Q_i) = \sum_{|\alpha| = s_i} \frac{\partial^\alpha Q_i(0)}{\alpha!} x^\alpha, \quad 0 \le i \le N.$$

According to (2.6) and (2.7) we have $P = 0$, and (iii) is proved. Using (i)–(iii), we deduce that the subspace

$$T_{s_{N+1}}(A_{s_N}) + T_{s_N}(S_{s_N}) + T_{s_{N-1}}(S_{s_{N-1}}) + \cdots + T_{s_0}(S_{s_0})$$

is a direct sum isomorphic to $\mathcal{A}$. The proof concludes by proving

$$\mathcal{B} = T_{s_{N+1}}(A_{s_N}) \oplus T_{s_N}(S_{s_N}) \oplus T_{s_{N-1}}(S_{s_{N-1}}) \oplus \cdots \oplus T_{s_0}(S_{s_0}).$$

We observe that if $P \in S_{s_i} \setminus \{0\}$, then $T_{s_i}(P) \neq 0$ and $T_{s_i-1}(P) = 0$ by (2.3). So, Proposition 2.5 implies that $T_{s_i}(P) \in \mathcal{B}$. On the other hand, if $P \in A_{s_N} \setminus \{0\}$, we get $T_{s_N}(P) = 0$. Moreover, we have $T_{s_{N+1}}(P) \neq 0$. In fact, on the contrary, we see that $P \in A_{s_{N+1}} = \{0\}$. Proposition 2.5 now gives $T_{s_{N+1}}(P) \in \mathcal{B}$. Therefore,

$$T_{s_{N+1}}(A_{s_N}) \oplus T_{s_N}(S_{s_N}) \oplus T_{s_{N-1}}(S_{s_{N-1}}) \oplus \cdots \oplus T_{s_0}(S_{s_0}) \subset \mathcal{B}.$$

On the other hand, if $P \in \mathcal{B}$, there exists $\{P_\epsilon\} \subset \mathcal{A}$ such that

$$\lim_{\epsilon \to 0} \|P - P_\epsilon^\epsilon\|_0 = 0. \tag{2.8}$$

Let $d_{N+1} = \dim(A_{s_N})$ and $d_i = \dim(S_{s_i})$, $0 \leq i \leq N$. We take $\{v_l\}_{l=1}^{d_{N+1}}$ and $\{w_{ir}\}_{r=1}^{d_i}$ bases of A_{s_N} and S_{s_i}, respectively. It is easy to check that for each $0 < \epsilon \leq 1$, $\{\epsilon^{-s_{N+1}} v_l\}_{l=1}^{d_{N+1}}$ is a basis of A_{s_N} and $\{\epsilon^{-s_i} w_{ir}\}_{r=1}^{d_i}$ is a basis of S_{s_i}, $0 \leq i \leq N$. According to (2.4), we have that there exist real numbers $D_{l,\epsilon}$ and $C_{i,r,\epsilon}$ such that

$$P_\epsilon = \sum_{l=1}^{d_{N+1}} \epsilon^{-s_{N+1}} D_{l,\epsilon} v_l + \sum_{i=0}^{N} \sum_{r=1}^{d_i} \epsilon^{-s_i} C_{i,r,\epsilon} w_{ir}.$$

From (2.5) it follows that

$$v_l(x) = \sum_{|\alpha| \geq s_{N+1}} \frac{\partial^\alpha v_l(0)}{\alpha!} x^\alpha \quad \text{and} \quad w_{ir}(x) = \sum_{|\alpha| \geq s_i} \frac{\partial^\alpha w_{ir}(0)}{\alpha!} x^\alpha. \tag{2.9}$$

So,

$$\begin{aligned} P_\epsilon^\epsilon(x) &= \sum_{l=1}^{d_{N+1}} D_{l,\epsilon} \epsilon^{-s_{N+1}} v_l^\epsilon(x) + \sum_{i=0}^{N} \sum_{r=1}^{d_i} C_{i,r,\epsilon} \epsilon^{-s_i} w_{ir}^\epsilon(x) \\ &= \sum_{l=1}^{d_{N+1}} D_{l,\epsilon} \sum_{|\alpha| = s_{N+1}} \frac{\partial^\alpha v_l(0)}{\alpha!} x^\alpha + \sum_{l=1}^{d_{N+1}} D_{l,\epsilon} \sum_{|\alpha| > s_{N+1}} \epsilon^{|\alpha| - s_{N+1}} \frac{\partial^\alpha v_l(0)}{\alpha!} x^\alpha \\ &\quad + \sum_{i=0}^{N} \sum_{r=1}^{d_i} C_{i,r,\epsilon} \sum_{|\alpha| = s_i} \frac{\partial^\alpha w_{ir}(0)}{\alpha!} x^\alpha + \sum_{i=0}^{N} \sum_{r=1}^{d_i} C_{i,r,\epsilon} \sum_{|\alpha| > s_i} \epsilon^{|\alpha| - s_i} \frac{\partial^\alpha w_{ir}(0)}{\alpha!} x^\alpha. \end{aligned}$$

Consequently

$$\begin{aligned} T_{s_j}(P_\epsilon^\epsilon)(x) = \sum_{i=0}^{j} \sum_{r=1}^{d_i} C_{i,r,\epsilon} \sum_{|\alpha| = s_i} \frac{\partial^\alpha w_{ir}(0)}{\alpha!} x^\alpha \\ + \sum_{i=0}^{j-1} \sum_{r=1}^{d_i} C_{i,r,\epsilon} \sum_{s_i < |\alpha| \leq s_j} \epsilon^{|\alpha| - s_i} \frac{\partial^\alpha w_{ir}(0)}{\alpha!} x^\alpha \end{aligned}$$

if $0 \le j \le N$, and

$$T_{s_{N+1}}(P_\epsilon^\epsilon)(x) = \sum_{l=1}^{d_{N+1}} D_{l,\epsilon} \sum_{|\alpha|=s_{N+1}} \frac{\partial^\alpha v_l(0)}{\alpha!} x^\alpha + \sum_{i=0}^{N} \sum_{r=1}^{d_i} C_{i,r,\epsilon} \sum_{|\alpha|=s_i} \frac{\partial^\alpha w_{ir}(0)}{\alpha!} x^\alpha$$
$$+ \sum_{i=0}^{N} \sum_{r=1}^{d_i} C_{i,r,\epsilon} \sum_{s_i<|\alpha|\le s_{N+1}} \epsilon^{|\alpha|-s_i} \frac{\partial^\alpha w_{ir}(0)}{\alpha!} x^\alpha.$$

From (2.9) it follows that

$$T_{s_{N+1}}(v_\ell)(x) = \sum_{|\alpha|=s_{N+1}} \frac{\partial^\alpha v_\ell(0)}{\alpha!} x^\alpha \quad \text{and} \quad T_{s_j}(w_{j.r})(x) = \sum_{|\alpha|=s_j} \frac{\partial^\alpha w_{j,r}(0)}{\alpha!} x^\alpha.$$

Thus, a straightforward computation yields

$$T_{s_0}(P_\epsilon^\epsilon)(x) = \sum_{r=1}^{d_0} C_{0,r,\epsilon} T_{s_0}(w_{0,r})(x), \tag{2.10}$$

$$\begin{aligned} T_{s_j}(P_\epsilon^\epsilon)(x) = T_{s_{j-1}}(P_\epsilon^\epsilon)(x) &+ \sum_{i=0}^{j-1} \sum_{r=1}^{d_i} C_{i,r,\epsilon} \sum_{s_{j-1}<|\alpha|\le s_j} \epsilon^{|\alpha|-s_i} \frac{\partial^\alpha w_{i,r}(0)}{\alpha!} x^\alpha \\ &+ \sum_{r=1}^{d_j} C_{j,r,\epsilon} T_{s_j}(w_{j,r})(x) \end{aligned} \tag{2.11}$$

if $1 \le j \le N$, and

$$\begin{aligned} T_{s_{N+1}}(P_\epsilon^\epsilon)(x) = T_{s_N}(P_\epsilon^\epsilon)(x) &+ \sum_{l=1}^{d_{N+1}} D_{l,\epsilon} T_{s_{N+1}}(v_\ell)(x) \\ &+ \sum_{i=0}^{N} \sum_{r=1}^{d_i} C_{i,r,\epsilon} \sum_{s_N<|\alpha|\le s_{N+1}} \epsilon^{|\alpha|-s_i} \frac{\partial^\alpha w_{i,r}(0)}{\alpha!} x^\alpha. \end{aligned} \tag{2.12}$$

From (2.8) and (2.10), we deduce that $T_{s_0}(P_\epsilon^\epsilon)(x) = \sum_{r=1}^{d_0} C_{0,r,\epsilon} T_{s_0}(w_{0,r})(x)$ is convergent as $\epsilon \to 0$. Since $\{T_{s_0}(w_{0,r})\}_{r=1}^{d_0}$ is a basis of $T_{s_0}(S_{s_0})$, there are real numbers $C_{0,r}$, $1 \le r \le d_0$, such that $C_{0,r,\epsilon} \to C_{0,r}$ as $\epsilon \to 0$. According to (2.8) and (2.11) it follows that $\sum_{r=1}^{d_1} C_{1,r,\epsilon} T_{s_1}(w_{1,r})(x)$ is convergent as $\epsilon \to 0$. Hence, there are real numbers $C_{1,r}$, $1 \le r \le d_1$, such that $C_{1,r,\epsilon} \to C_{1,r}$ as $\epsilon \to 0$, because $\{T_{s_1}(w_{1r})\}_{r=1}^{d_1}$ is a basis of $T_{s_1}(S_{s_1})$. Similarly, as $\{T_{s_{N+1}}(v_l)\}_{l=1}^{a}$ is a basis of $T_{s_{N+1}}(A_{s_N})$ and $\{T_{s_i}(w_{ir})\}_{r=1}^{d_i}$ is a basis of $T_{s_i}(S_{s_i})$, $0 \le i \le N$, (2.8) and (2.10)–(2.12) show that there are real numbers D_l and $C_{i,r}$ such that $D_{l,\epsilon,} \to D_l$ and $C_{i,r,\epsilon} \to C_{i,r}$ as $\epsilon \to 0$. In consequence,

$$P = \sum_{l=1}^{a} D_l T_{s_{N+1}}(v_l) + \sum_{i=0}^{N} \left(\sum_{r=1}^{d_i} C_{i,r} T_{s_i}(w_{ir}) \right),$$

and so $P \in T_{s_{N+1}}(A_{s_N}) \oplus T_{s_N}(S_{s_N}) \oplus T_{s_{N-1}}(S_{s_{N-1}}) \oplus \cdots \oplus T_{s_0}(S_{s_0})$.

(b) Now assume $s_0 = r_0$, i.e., $A_{s_0} = \{0\}$. Then $\mathcal{A}$ has the form (2.4) with $N = 0$, $A_{s_0} = \{0\}$ and $S_{s_0} = \mathcal{A}$. An analysis similar to the proof of (a) shows that T_{r_0} is an isomorphism and $\mathcal{B} = T_{s_0}(S_{s_0}) = T_{r_0}(\mathcal{A})$. □

The following corollary follows immediately from the proof of Theorem 2.6.

Corollary 2.7. *Let $\mathcal{A}$ be a non-zero subspace of Π_k^l. Then $\lim_{n\to\infty} \mathcal{A}^{\epsilon_n} = \mathcal{B}$ for any sequence $\{\epsilon_n\}$ of the net $\epsilon \downarrow 0$.*

Remark 2.8. $\mathcal{B}$ is isomorphic to $T_{r_0}(\mathcal{A})$.

Corollary 2.9. *Let $s \geq m+1$ and let $\mathcal{A} = \Pi_k^m \oplus A_{s-1}$ be such that $A_s = \{0\}$. Then $\mathcal{B} = \Pi_k^m \oplus T_s(A_{s-1})$ and the linear operator $T : \mathcal{A} \to \Pi_k^s$ given by $T(P) = T_s(P)$ defines an isomorphism between $\mathcal{A}$ and $\mathcal{B}$.*

Proof. We first claim that T is an injective operator. Indeed, if $T(P) = T(Q)$ for $P, Q \in \mathcal{A}$, then $T_s(P-Q) = 0$ and so $P - Q \in A_s$. Since $A_s = \{0\}$, we have $P = Q$.

Since $\mathcal{A}$ is isomorphic to $T(\mathcal{A})$, the proof concludes by proving $\mathcal{B} = \Pi_k^m \oplus T_s(A_{s-1}) = T_s(\mathcal{A})$.

Let A_j be the sets defined in (2.2). Since

$$\{0\} = A_s \subsetneq A_{s-1} = \cdots = A_m \subsetneq A_{m-1} \subsetneq \cdots \subsetneq A_0 \subsetneq \mathcal{A},$$

then $\mathcal{A} = A_{s-1} \oplus B_m \oplus B_{m-1} \oplus \cdots \oplus B_0$, where $A_i \oplus B_i = A_{i-1}$, $0 \leq i \leq m$. Therefore Π_k^m is isomorphic to $B_m \oplus \cdots \oplus B_0$. On the other hand, since $s_0 = 0$, $r_0 = s$ and $J \setminus \{r_0\} = \{0, 1, \ldots, m\}$, by Proposition 2.6 (a),

$$\mathcal{B} = T_s(A_{s-1}) \oplus T_m(B_m) \oplus \cdots \oplus T_0(B_0).$$

From the proof of Theorem 2.6, we obtain that $B_m \oplus \cdots \oplus B_0$ is isomorphic to $T_m(B_m) \oplus \cdots \oplus T_0(B_0)$, and consequently Π_k^m is isomorphic to $T_m(B_m) \oplus \cdots \oplus T_0(B_0) \subset \Pi_k^m$. Hence, $T_m(B_m) \oplus \cdots \oplus T_0(B_0) = \Pi_k^m$ and so $\mathcal{B} = T_s(A_{s-1}) \oplus \Pi_k^m = T_s(A_{s-1}) \oplus T_s(\Pi_k^m) = T_s(\mathcal{A})$. □

3. An application to best local approximation

Let $\{P_\epsilon\}$ be a net of best approximants to F from $\mathcal{A}$ with respect to $\|\cdot\|_\epsilon^*$, and let E_ϵ be the error function

$$E_\epsilon(F) = \frac{F^\epsilon - P_\epsilon^\epsilon}{\epsilon^{m+1}}.$$

If $F \in t^{m+1}$, then

$$F^\epsilon = T_{m+1}^\epsilon + \epsilon^{m+1} R_{m+1}^\epsilon,$$

where $R_{m+1} = \frac{F - T_{m+1}}{\epsilon^{m+1}}$, $\|R_{m+1}^\epsilon\|_\epsilon = o(1)$, and T_{m+1} is the Taylor polynomial of F of degree $m+1$ at 0. Moreover,

$$\lambda P_\epsilon^\epsilon \in \mathcal{P}_{\mathcal{A}^\epsilon,\epsilon}(\lambda F^\epsilon) \quad \text{and} \quad P^\epsilon + P_\epsilon^\epsilon \in \mathcal{P}_{\mathcal{A}^\epsilon,\epsilon}((P+F)^\epsilon), \quad \text{for } P \in \mathcal{A}.$$

The following proposition may be proved in much the same way as [13, Proposition 4.1]. However, we repeat the proof for completeness.

Proposition 3.1. *Let $\mathcal{A}$ be a non-zero subspace of Π_k^l with $l > m$, and let $\{P_\epsilon\}$ be a net of best approximants of F from $\mathcal{A}$ with respect to $\|\cdot\|_\epsilon^*$. If $F \in t^{m+1}$, $T_m \in \mathcal{A}$ and $\phi_{m+1} = T_{m+1} - T_m$, then*

$$E_\epsilon(F) = \phi_{m+1} + R_{m+1}^\epsilon - \mathcal{P}_{\mathcal{A}^\epsilon,\epsilon}(\phi_{m+1} + R_{m+1}^\epsilon),$$

where $\|R_{m+1}^\epsilon\|_\epsilon = o(1)$ as $\epsilon \to 0$.

Proof. Since $R_{m+1}^\epsilon = \frac{F^\epsilon - T_{m+1}^\epsilon}{\epsilon^{m+1}}$, then

$$\begin{aligned}\phi_{m+1} + R_{m+1}^\epsilon &= T_{m+1} - T_m + \frac{F^\epsilon - T_{m+1}^\epsilon}{\epsilon^{m+1}} = \frac{T_{m+1}^\epsilon - T_m^\epsilon}{\epsilon^{m+1}} + \frac{F^\epsilon - T_{m+1}^\epsilon}{\epsilon^{m+1}} \\ &= \frac{F^\epsilon - T_m^\epsilon}{\epsilon^{m+1}}.\end{aligned}$$

As $T_m \in \mathcal{A}$, we have

$$\begin{aligned}\phi_{m+1} + R_{m+1}^\epsilon - \mathcal{P}_{\mathcal{A}^\epsilon,\epsilon}(\phi_{m+1} + R_{m+1}^\epsilon) &= \frac{F^\epsilon - T_m^\epsilon}{\epsilon^{m+1}} - P_{\mathcal{A}^\epsilon,\epsilon}\left(\frac{F^\epsilon - T_m^\epsilon}{\epsilon^{m+1}}\right) \\ &= \frac{F^\epsilon - P_\epsilon^\epsilon}{\epsilon^{m+1}} = E_\epsilon(F).\end{aligned}$$

□

Next, we give a new result about the asymptotic behavior of the error without the conditions (c1) or (c2), which generalizes Theorems 4.2 and 4.5 of [13].

Theorem 3.2. *Let $\mathcal{A}$ be a non-zero subspace of Π_k^l with $l > m$. If $F \in t^{m+1}$, $T_m \in \mathcal{A}$ and $\phi_{m+1} = T_{m+1} - T_m$, then*

$$\|E_\epsilon(F)\|_\epsilon \to \inf_{P \in \mathcal{B}} \|\phi_{m+1} - P\|_0, \quad \text{as } \epsilon \to 0.$$

Proof. By Proposition 3.1,

$$E_\epsilon(F) = \phi_{m+1} + R_{m+1}^\epsilon - \mathcal{P}_{\mathcal{A}^\epsilon,\epsilon}(\phi_{m+1} + R_{m+1}^\epsilon), \tag{3.1}$$

where $\|R_{m+1}^\epsilon\|_\epsilon = o(1)$ as $\epsilon \to 0$. We first prove

$$\overline{\lim_{\epsilon\to 0}} \|E_\epsilon(F)\|_\epsilon \le \inf_{P \in B} \|\phi_{m+1} - P\|_0. \tag{3.2}$$

In fact, let $P \in \mathcal{B}$. By the definition of $\mathcal{B}$, there exists a net $\{Q_\epsilon\} \subset \mathcal{A}$ such that $\|P - Q_\epsilon^\epsilon\|_0 \to 0$, as $\epsilon \to 0$. In consequence, $\|P - Q_\epsilon^\epsilon\|_\epsilon = o(1)$, as $\epsilon \to 0$, by (1.3). Since $Q_\epsilon^\epsilon \in \mathcal{A}^\epsilon$ and $\|R_{m+1}^\epsilon\|_\epsilon = o(1)$, from (3.1) we obtain

$$\|E_\epsilon(F)\|_\epsilon \le \|\phi_{m+1} + R_{m+1}^\epsilon - Q_\epsilon^\epsilon\|_\epsilon \le \|\phi_{m+1} - Q_\epsilon^\epsilon\|_\epsilon + o(1), \quad \text{as } \epsilon \to 0. \tag{3.3}$$

By Property (3), $\|\phi_{m+1} - P\|_\epsilon \to \|\phi_{m+1} - P\|_0$, as $\epsilon \to 0$. Hence, using the triangle inequality we have

$$\begin{aligned}\big|\|\phi_{m+1} - Q_\epsilon^\epsilon\|_\epsilon - \|\phi_{m+1} - P\|_0\big| &\le \big|\|\phi_{m+1} - Q_\epsilon^\epsilon\|_\epsilon - \|\phi_{m+1} - P\|_\epsilon\big| \\ &\quad + \big|\|\phi_{m+1} - P\|_\epsilon - \|\phi_{m+1} - P\|_0\big| \\ &\le \|P - Q_\epsilon^\epsilon\|_\epsilon + \big|\|\phi_{m+1} - P\|_\epsilon - \|\phi_{m+1} - P\|_0\big| = o(1)\end{aligned}$$

as $\epsilon \to 0$. Now, according to (3.3) we get (3.2).

The proof finishes by observing that

$$\varliminf_{\epsilon\to 0}\|E_\epsilon(F)\|_\epsilon \geq \inf_{P\in\mathcal{B}}\|\phi_{m+1}-P\|_0. \tag{3.4}$$

Let $\epsilon \downarrow 0$ be a sequence such that $\lim_{\epsilon\to 0}\|E_\epsilon(F)\|_\epsilon = \varliminf_{\epsilon\to 0}\|E_\epsilon(F)\|_\epsilon$. We consider $P_\epsilon^\epsilon \in \mathcal{P}_{\mathcal{A}^\epsilon,\epsilon}(\phi_{m+1}+R_{m+1}^\epsilon)$. We claim that there exist constants $M, \epsilon_0 > 0$ such that

$$\|P_\epsilon^\epsilon\|_0 \leq M, \quad 0<\epsilon\leq\epsilon_0. \tag{3.5}$$

Indeed, as $0\in\mathcal{A}^\epsilon$ we get

$$\begin{aligned}\|P_\epsilon^\epsilon\|_\epsilon &\leq \|P_\epsilon^\epsilon-(\phi_{m+1}+R_{m+1}^\epsilon)\|_\epsilon+\|\phi_{m+1}+R_{m+1}^\epsilon\|_\epsilon\\ &\leq 2\|\phi_{m+1}+R_{m+1}^{\epsilon_n}\|_\epsilon\\ &\leq 2\|\phi_{m+1}\|_\epsilon+2\|R_{m+1}^\epsilon\|_\epsilon,\end{aligned} \tag{3.6}$$

for $0<\epsilon\leq 1$. By Proposition 3.1 and Property (3), we see that $2\|\phi_{m+1}\|_\epsilon + 2\|R_{m+1}^\epsilon\|_\epsilon \to 2\|\phi_{m+1}\|_0$, as $\epsilon\to 0$. So, from (1.3) and (3.6), we obtain (3.5).

In consequence, there exists a subsequence of $\{P_\epsilon^\epsilon\}$, which is denoted in the same way, and $P_0\in\Pi_k^l$ such that $P_\epsilon^\epsilon\to P$ uniformly on B, as $\epsilon\to 0$. Since $|\|\phi_{m+1}-P_\epsilon^\epsilon\|_\epsilon-\|\phi_{m+1}-P\|_0| \leq |\|\phi_{m+1}-P_\epsilon^\epsilon\|_\epsilon-\|\phi_{m+1}-P\|_\epsilon| + |\|\phi_{m+1}-P\|_\epsilon - \|\phi_{m+1}-P\|_0| \leq \|P-P_\epsilon^\epsilon\|_\epsilon + |\|\phi_{m+1}-P\|_\epsilon-\|\phi_{m+1}-P\|_0|$, using Property (3) we get

$$\|\phi_{m+1}-P\|_0 = \|\phi_{m+1}-P_\epsilon^\epsilon\|_\epsilon + o(1), \quad \text{as } \epsilon\to 0.$$

We observe that $P\in B$ by Corollary 2.7. Therefore, by Proposition 3.1,

$$\begin{aligned}\inf_{Q\in\mathcal{B}}\|\phi_{m+1}-Q\|_0 \leq \|\phi_{m+1}-P\|_0 &= \|\phi_{m+1}-P_\epsilon^\epsilon\|_\epsilon+o(1)\\ &\leq \|\phi_{m+1}+R_{m+1}^\epsilon-P_\epsilon^\epsilon\|_\epsilon+\|R_{m+1}^\epsilon\|_\epsilon\\ &= \|E_\epsilon(F)\|_\epsilon+\|R_{m+1}^\epsilon\|_\epsilon.\end{aligned}$$

So, $\inf_{Q\in\mathcal{B}}\|\phi_{m+1}-Q\|_0 \leq \lim_{\epsilon\to 0}\left(\|E_\epsilon(F)\|_\epsilon+\|R_{m+1}^\epsilon\|_\epsilon\right) = \varliminf_{\epsilon\to 0}\|E_\epsilon(F)\|_\epsilon$, and (3.4) is proved. □

The following result provides us with a useful and important property for a net of best approximants to F from $\mathcal{A}$.

Theorem 3.3. *Let $\mathcal{A}$ be a non-zero subspace of Π_k^l with $l>m$, and let $\{P_\epsilon\}$ be a net of best approximants of F from $\mathcal{A}$ with respect to $\|\cdot\|_\epsilon^*$. Assume $F\in t^{m+1}$, $T_m\in\mathcal{A}$ and $\phi_{m+1}=T_{m+1}-T_m$. If $\mathcal{C}$ is the cluster point set of the net $\left\{\frac{(P_\epsilon-T_m)^\epsilon}{\epsilon^{m+1}}\right\}$, as $\epsilon\to 0$, then $\mathcal{C}\neq\emptyset$. Moreover, each polynomial in $\mathcal{C}$ is a solution of the minimization problem*

$$\min_{P\in\mathcal{B}}\|\phi_{m+1}-P\|_0. \tag{3.7}$$

Proof. We observe that

$$\begin{aligned} E_\epsilon(F) = \frac{(F-P_\epsilon)^\epsilon}{\epsilon^{m+1}} &= \frac{(T_{m+1}-T_m)^\epsilon + (F-T_{m+1})^\epsilon - (P_\epsilon - T_m)^\epsilon}{\epsilon^{m+1}} \\ &= \frac{\phi_{m+1}^\epsilon - (P_\epsilon - T_m)^\epsilon}{\epsilon^{m+1}} + \frac{(F-T_{m+1})^\epsilon}{\epsilon^{m+1}} \\ &= \phi_{m+1} - \frac{(P_\epsilon - T_m)^\epsilon}{\epsilon^{m+1}} + \frac{(F-T_{m+1})^\epsilon}{\epsilon^{m+1}}. \end{aligned}$$

Then

$$\begin{aligned} &\left\| \phi_{m+1} - \frac{(P_\epsilon - T_m)^\epsilon}{\epsilon^{m+1}} \right\|_\epsilon - \frac{\|(F-T_{m+1})^\epsilon\|_\epsilon}{\epsilon^{m+1}} \\ &\qquad \le \|E_\epsilon(F)\|_\epsilon \le \left\| \phi_{m+1} - \frac{(P_\epsilon - T_m)^\epsilon}{\epsilon^{m+1}} \right\|_\epsilon + \frac{\|(F-T_{m+1})^\epsilon\|_\epsilon}{\epsilon^{m+1}}, \end{aligned}$$

and consequently,

$$\|E_\epsilon(F)\|_\epsilon = \left\| \phi_{m+1} - \frac{(P_\epsilon - T_m)^\epsilon}{\epsilon^{m+1}} \right\|_\epsilon + o(1), \quad \text{as } \epsilon \to 0,$$

since $F \in t^{m+1}$. By Theorem 3.2,

$$\inf_{P\in B} \|\phi_{m+1} - P\|_0 = \lim_{\epsilon\to 0} \left\| \phi_{m+1} - \frac{(P_\epsilon - T_m)^\epsilon}{\epsilon^{m+1}} \right\|_\epsilon . \tag{3.8}$$

According to (1.3), there exist constants $\epsilon_0, M > 0$ such that

$$\left\| \phi_{m+1} - \frac{(P_\epsilon - T_m)^\epsilon}{\epsilon^{m+1}} \right\|_0 \le M,$$

for all $0 < \epsilon \le \epsilon_0$. The equivalence of the norms in Π_k^l implies that the net $\left\{ \frac{(P_\epsilon - T_m)^\epsilon}{\epsilon^{m+1}} \right\}_{0<\epsilon\le\epsilon_0}$ is uniformly bounded on B. So, there exists a subsequence of $\left\{ \frac{(P_\epsilon - T_m)^\epsilon}{\epsilon^{m+1}} \right\}_{0<\epsilon\le\epsilon_0}$, which is denoted in the same way, and a polynomial P_0 such that

$$\frac{(P_\epsilon - T_m)^\epsilon}{\epsilon^{m+1}} \text{ converges to } P_0, \text{ uniformly on } B, \text{ as } \epsilon \to 0.$$

In consequence, $\mathcal{C} \neq \emptyset$.

On the other hand, if $P_0 \in \mathcal{C}$, there is a sequence $\epsilon \downarrow 0$ such that $\frac{(P_\epsilon - T_m)^\epsilon}{\epsilon^{m+1}} \to P_0$. Since $T_m \in \mathcal{A}$, we have $P_\epsilon - T_m \in \mathcal{A}$, and so $P_0 \in \mathcal{B}$ by Corollary 2.7. Finally, from Property (3) and (3.8) we conclude that

$$\inf_{P\in B} \|\phi_{m+1} - P\|_0 = \lim_{\epsilon\to 0} \left\| \phi_{m+1} - \frac{(P_\epsilon - T_m)^\epsilon}{\epsilon^{m+1}} \right\|_\epsilon = \|\phi_{m+1} - P_0\|_0 ,$$

i.e., P_0 is a solution of (3.7). □

The following theorem is an extension of [13, Theorem 5.1].

Theorem 3.4. *Let $\mathcal{A}$ be a non-zero subspace of Π_k^l with $l > m$, and let $\{P_\epsilon\}$ be a net of best approximants of F from $\mathcal{A}$ with respect to $\|\cdot\|_\epsilon^*$. Assume $m+1 = \min\{j : 0 \le j \le l$ and $A_j = \{0\}\}$, $F \in t^{m+1}$ with $T_m \in \mathcal{A}$, and set $\phi_{m+1} = T_{m+1} - T_m$. If the minimization problem* (3.7) *has a unique solution P_0, then $P_\epsilon \to T_m + P$, where $P \in \mathcal{A}$ is uniquely determined by the condition $T_{m+1}(P) = P_0 - T_m(P_0)$.*

Proof. Since (3.7) has a unique solution P_0, Theorem 3.3 implies that

$$\lim_{\epsilon\to 0} \frac{(P_\epsilon - T_m)^\epsilon}{\epsilon^{m+1}} = P_0.$$

In consequence, $\partial^\alpha(P_\epsilon - T_m)(0) \to 0$, $|\alpha| \le m$, and $\partial^\alpha(P_\epsilon - T_m)(0) \to \partial^\alpha P_0(0)$, $|\alpha| = m+1$, as $\epsilon \to 0$. Therefore

$$T_{m+1}(P_\epsilon - T_m)(x) \to \sum_{|\alpha|=m+1} \frac{\partial^\alpha P_0(0)}{\alpha!} x^\alpha =: R(x), \quad x \in B, \text{ as } \epsilon \to 0. \tag{3.9}$$

Let $T : \mathcal{A} \to \Pi_k^{m+1}$ be the linear operator defined by $T(P) = T_{m+1}(P)$. As $A_{m+1} = \{0\}$, an analysis similar to that in the proof of Corollary 2.9 shows that T is an injective operator. Since $T(\mathcal{A})$ is a closed subspace and $\{T_{m+1}(P_\epsilon - T_m)\} \subset T(\mathcal{A})$, (3.9) implies that there exists a unique $P \in \mathcal{A}$ such that $T_{m+1}(P) = R$. Hence $T_{m+1}(P_\epsilon - T_m - P) \to 0$ as $\epsilon \to 0$. As $A_{m+1} = \{0\}$ we see that $\|Q\| := \|T_{m+1}(Q)\|_0$ is a norm on $\mathcal{A}$, and so $P_\epsilon \to T_m + P$ as $\epsilon \to 0$. Finally, by Theorem 2.6, $\mathcal{B} \subset \Pi_k^{m+1}$, and consequently $P_0 - T_m(P_0) = T_{m+1}(P_0) - T_m(P_0) = R$. The proof is complete. □

Remark 3.5. If $\mathcal{A}$ satisfies the condition (c2), then $\mathcal{A} = \Pi_k^m \oplus A_m$ with $A_{m+1} = \{0\}$. By Corollary 2.9, $\mathcal{B} = \Pi_k^m \oplus T_{m+1}(A_m)$ and each element $P \in \mathcal{A}$ is uniquely determined by $T_{m+1}(P)$. So, we can rewrite the problem (3.7) in the following (equivalent) form:

$$\min_{Q+U\in\Pi_k^m\oplus A_m} \|\phi_{m+1} - (Q + T_{m+1}(U))\|_0. \tag{3.10}$$

The following result has been proved in [13, Theorem 5.1] and it is a consequence of Theorem 3.4.

Corollary 3.6. *Let $\Pi_k^m \subset \mathcal{A} \subset \Pi_k^l$ be a non-zero subspace that satisfies the condition (c2) and let $\{P_\epsilon\}$ be a net of best approximants of F from $\mathcal{A}$ with respect to $\|\cdot\|_\epsilon^*$. Assume $F \in t^{m+1}$. If the minimization problem* (3.10) *has a unique solution P_0, then $P_\epsilon \to T_m + P$, where $P \in \mathcal{A}$ is uniquely determined by the condition $T_{m+1}(P) = P_0 - T_m(P_0)$.*

In the following example we present a function $F \in \bigcap_{m=0}^\infty t^m$ such that $T_2(F) \notin \mathcal{A}$ and the net $\{T_i(P_\epsilon)\}$ does not converge for the same $i > m+1$.

Example 3.7. Set $B = [-1, 1]$, $\|G\|_\epsilon = \left(\int\limits_{-1}^{1} |G(x)|^2\, dx\right)^{\frac{1}{2}}$, $\mathcal{A} = \operatorname{span}\{1, x^2, x^3\}$, and $F(x) = x$. So

$$\|G\|_\epsilon^* = \left(\frac{1}{\epsilon}\int\limits_{-\epsilon}^{\epsilon} |G(x)|^2\, dx\right)^{\frac{1}{2}},$$

$A_0 = A_1 = \operatorname{span}\{x^2, x^3\}$, $A_2 = \operatorname{span}\{x^3\}$ and $A_3 = \{0\}$. Since $T_1(x^2) = 0$, we observe that the subspace $\mathcal{A}$ does not satisfy the condition (c2). Moreover, an straightforward computation shows that

$$\frac{\|F - T_0\|_\epsilon^*}{\epsilon^0} = \frac{\sqrt{6}}{3}\epsilon \qquad \text{and} \qquad \frac{\|F - T_s\|_\epsilon^*}{\epsilon^s} = 0, \quad s \in \mathbb{N},$$

where $T_0(x) = 0$ and $T_s(x) = x$. In consequence, $F \in t^m$ for all $m \in \mathbb{N} \cup \{0\}$, and $T_2(F) \notin \mathcal{A}$. Since $\int\limits_{-\epsilon}^{\epsilon} \left(x - \frac{7}{5\epsilon^2}x^3\right) x^i\, dx = 0$, $i = 0, 2, 3$, then $P_\epsilon(x) = \frac{7}{5\epsilon^2}x^3$ is the best approximant to F from $\mathcal{A}$ with respect to $\|\cdot\|_\epsilon^*$. Therefore $T_i(P_\epsilon)(x) \to 0$, for $i = 0, 1, 2$, but $T_3(P_\epsilon)(x)$ does not converge, as $\epsilon \to 0$. So, the best local approximation to F from $\mathcal{A}$ in 0 does not exist, and

$$\|E_\epsilon(F)\|_\epsilon = \frac{\|F - P_\epsilon\|_\epsilon^*}{\epsilon^3} = \frac{2\sqrt{6}}{15\epsilon^2} \to \infty, \quad \text{as } \epsilon \to 0.$$

We now give another example which shows that the condition $T_m \in \mathcal{A}$ is not necessary for the existence of the best local approximation.

Example 3.8. Set B, $\|\cdot\|_\epsilon^*$ and F as in Example 3.7, and we consider the subspace $\mathcal{A} = \operatorname{span}\{1, x^2\}$. It is clear that $A_0 = A_1 = \operatorname{span}\{x^2\}$, $A_2 = \{0\}$ and $\mathcal{B} = \mathcal{A}$. Moreover, we have $F \in t^2$, $T_1 \notin \mathcal{A}$, and $\mathcal{A}$ does not satisfy the condition (c2) since $T_1(x^2) = 0$. As $\int\limits_{-\epsilon}^{\epsilon} (x - 0)\, x^i\, dx = 0$, $i = 0, 2$, then $P_\epsilon(x) = 0$ is the best approximant to F from $\mathcal{A}$ with respect to $\|\cdot\|_\epsilon^*$. Therefore, the polynomial 0 is the best local approximation to F from $\mathcal{A}$ in 0.

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