ANNALES
POLONICI MATHEMATICI
XL (1983)

On convergence of successive approximations of some generalized contraction mappings

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Abstract. In this paper we consider the relation $x \in Fx$, where $F\colon X \to CN(X)$, X is a metric space and $CN(X)$ denotes the space of all non-empty and closed subsets of X. In Section 2 we formulate some existence theorems for the relation $x \in Fx$ by the method of successive approximations imposing certain general contraction conditions on the operator F. In particular, we generalize some results of [7], [9], [20], [21], [32], [33].

In Section 3 we use the results obtained to the case of the generalized metric space which was introduced in paper [41] (see also [27], [30]). This gives a possibility of an application of our theorems to certain functional and integro-functional equations (Theorems 3.1–3.2).

All our results have been obtained by using the idea of Ważewski's method [27], [28], [41] to general contraction mappings in orbitally complete metric spaces.

1. Notation, definitions and lemmas. Let $R_+ = \{g \in R\colon g \geqslant 0\}$, where R is the set of real numbers. Let $S(R_+)$ denote the set of all non-negative, non-increasing and bounded sequences $(g_n)_{n\in N} \in R_+^N$. In the sequel $S_0(R_+) \subset S(R_+)$ will denote the subclass of all sequences convergent to zero and for $(g_n)_{n\in N} \in S(R_+)$ we will write $g_n \searrow g$ if $\lim g_n = g$.

Let the mappings $a^i\colon R_+^5 \to R_+$, $i = 1, 2$, satisfy the following conditions:

(A) a^i is a mapping non-decreasing in each coordinate separately and monotonically continuous for $i = 1, 2$, i.e., the conditions $g_{k,n} \searrow g_k$, $k = 1, 2, \ldots, 5$, imply $\lim a^i(g_{1,n}, \ldots, g_{5,n}) = a^i(g_1, \ldots, g_5)$ for any $(g_1, \ldots, g_5) \in \Delta^5$, $\Delta = [0, h] \subset R_+$;

(B) equations

$$g = \dot{a}^i(g), \tag{1.1$_i$}$$

where $\dot{a}^i(g) = (a^i \circ \varphi_i)(g)$, $i = 1, 2$, $\varphi_1(g) = (g, g, 2g, g, 2g)$, $\varphi_2(g) = (g, 2g, g, 2g, g)$, have in Δ the unique solution $g = 0$.

For $c \in \Delta$ and $(q_n)_{n\in N_0} \in S_0(\Delta)$ ($S_0(\Delta)$ denotes the set of sequences (q_n) such that $q_n \searrow 0$ and $q_n \in \Delta$, $n = 0, 1, \ldots$) we define the sequence of successive approximations;

$$(A_n^i(c, q_0, \ldots, q_n))_{n\in N_0}, \quad N_0 = N \cup \{0\},$$

where

$$A_0^i(c, q_0) = c, \quad A_n^i(c, q_0, \ldots, q_n) = q_n + \hat{a}^i(A_{n-1}^i(c, q_0, \ldots, q_{n-1})),$$
$$n = 1, 2, \ldots, \ i = 1, 2.$$

Let us consider the equations

$$(1.2_i) \qquad g = \hat{a}^i(g) + q$$

on $[0, b]$, where q, $2b \in \Delta$, $i = 1, 2$. Let b^i be a solution of (1.2_i) on $[0, b]$, $i = 1, 2$. If for any solution g^i of equation (1.2_i) on the interval $[0, b]$ the inequality $g^i \leqslant b^i$ holds, then b^i is called a *maximal solution* of (1.2_i) on $[0, b]$ and is denoted by $m_i(q, b)$, $i = 1, 2$.

Lemmas 1.1–1.3 formulated below are a simple adaptation of the well-known lemmas included in papers [27], [28] (see also [30], p. 20–22). Thus we may omit their proofs.

LEMMA 1.1. *If assumptions* (A) *and* (B) *are satisfied,* $(q_n) \in S_0(\Delta)$ *and if there exists* $c \in \Delta$ *such that* $q_0 + \hat{a}^i(c) \leqslant c$, *then the sequence* $(A_n^i(c, q_0, \ldots \ldots, q_n))_{n \in N_0}$ *is convergent to* 0 *for* $i = 1, 2$.

LEMMA 1.2. *If assumptions* (A) *and* (B) *are satisfied and if there exist* q, $2b \in \Delta$ *such that* $q + \hat{a}^i(b) \leqslant b$, *then there exists a maximal solution* $m_i(q, b) \in \in [0, b]$ *of* (1.2_i), $i = 1, 2$. *Moreover, if there exists* $p \in [0, b]$ *such that* $p \leqslant \hat{a}^i(p) + q$, *then* $p \leqslant m_i(q, b)$ *for* $i = 1, 2$.

LEMMA 1.3. *Let assumptions* (A) *and* (B) *be satisfied on* $\Delta = R_+$ *and let for any* $q \in R_+$ *there exist a maximal solution* $m_i(q)$ *of the equution* $g = q + \hat{a}^i(g)$, $i = 1, 2$. *Then we have*:

1° *if for some* $b \in R_+$ *the inequality* $b \leqslant \hat{a}^i(b) + q$ *holds, then* $b \leqslant m_i(q)$ *for* $i = 1, 2$,

2° *if* $q' \leqslant q$, *then* $m_i(q') \leqslant m_i(q)$.

Let (X, d) be a metric space and let $N(X)$ be the family of all non-empty subsets of X. Let F_n: $X \to N(X)$ for $n = 1, 2, \ldots$ and w: $N \times X \to N$ be given functions.

Any sequence $(x_n)_{n \in N_0}$, $x_0 \in X$ and $x_n \in F_n^{w(n, x_{n-1})} x_{n-1}$, $n = 1, 2, \ldots$, is called an *orbit of the sequence of mappings* $(F_n)_{n \in N}$ with the start-point x_0, where $F^{n+1}(x) = F(F^n(x)) = \bigcup_{u \in F^n(x)} F(u)$ for each $x \in X$ and $n = 1, 2, \ldots$ The set of all orbits of a sequence $\mathscr{F} = (F_n)_{n \in N}$ starting from x_0 will be denoted by $O(\mathscr{F}, w, x_0)$.

A fundamental sequence $(x_n) \in O(\mathscr{F}, w, x_0)$ is called a *Cauchy orbit*. Obviously, $(x_n)_{n \in N} \in X^N$ is fundamental if and only if $d(x_n, x_{n+p}) \leqslant q_n$ for some $(q_n)_{n \in N} \in S_0(R_+)$ and $n \geqslant n_0$, $p \in N_0$. The set of all Cauchy orbits $(x_n) \in O(\mathscr{F}, w, x_0)$ is denoted by $CO(\mathscr{F}, w, x_0)$.

We say that (X, d) is an $(\mathscr{F}, w, x_0)$-*orbitally complete space* $[(\mathscr{F}, w)$-

orbitally complete space] if each $(x_n) \in CO(\mathcal{F}, w, x_0)$ is convergent to an element of X [(X, d) is an $(\mathcal{F}, w, x)$-orbitally complete space for each $x \in X$ separately].

We say that (X, d) is an $(\mathcal{F}, w, x_0)$-*orbitally precompact space* [$(\mathcal{F}, w)$-*orbitally precompact space*] if each $(x_n) \in O(\mathcal{F}, w, x_0)$ has a Cauchy subsequence [(X, d) is an $(\mathcal{F}, w, x)$-orbitally precompact space for each $x \in X$ separately].

We say that (X, d) is an $(\mathcal{F}, w, x)$-*orbitally compact space* [$(\mathcal{F}, w)$-*orbitally compact space*] if each $(x_n) \in O(\mathcal{F}, w, x)$ has a convergent subsequence [(X, d) is an $(\mathcal{F}, w, x)$-orbitally compact space for each $x \in X$ separately].

Let $CN(X)$ be the family of all closed and non-empty subsets of X. Assume that a function $H: N(X) \times N(X) \to R_+$ has the properties:

(H_1) $H(A, A) = 0$ for each $A \in N(X)$,

(H_2) $H(A, B) = H(B, A)$ for each $A, B \in N(X)$,

(H_3) $H(A, B) \leqslant H(A, C) + H(C, B)$ for each $A, B, C \in N(X)$,

(H_4) for every pair of sequences $(A_n)_{n\in N}$, $(B_n)_{n\in N}$, A_n, $B_n \in CN(X)$, $n = 1, 2, \ldots$, every sequence $(x_n)_{n\in N}$ such that $x_n \in A_n$, $n \in N$, and every $(q_n)_{n\in N} \in S_0(R_+)$ there exists $(y_n)_{n\in N}$ such that $y_n \in B_n$ and $d(x_n, y_n) \leqslant H(A_n, B_n) + q_n$ for $n \in N$.

Obviously, the sequence $(A_n)_{n\in N}$, $A_n \in CN(X)$, $n \in N$, is convergent to A, $A \in CN(X)$, if and only if there exists $(q_n)_{n\in N} \in S_0(R_+)$ and a positive integer n_0 that $H(A_n, A) \leqslant q_n$ for $n \geqslant n_0$.

The mapping $T: X \to CN(X)$ will be called $(\mathcal{F}, w, x)$-*orbitally continuous* at $\dot{x} \in X$ [$(\mathcal{F}, w)$-*orbitally continuous* on X] if for every $(x_n) \in O(\mathcal{F}, w, x)$ and $\lim x_n = \dot{x}$ we have $\lim Tx_n = T\dot{x}$ [T is $(\mathcal{F}, w, x)$-orbitally continuous at each point $x \in X$ for every start-point $x \in X$ separately].

The above definitions of $(\mathcal{F}, w, x)$-orbital continuity of $T: X \to CN(X)$ are slight modifications of the well-known corresponding definitions considered e.g. in the papers of Ćirić.

Let the function $\delta: X \times CN(X) \to R_+$ be such that

(δ_1) $\delta(x, A) = 0 \Leftrightarrow x \in A$ for each $A \in CN(X)$ and $x \in X$,

(δ_2) $\delta(x, A) \leqslant d(x, y) + \delta(y, B) + H(A, B)$ for each x, $y \in X$ and each A, $B \in CN(X)$.

2. Some fixed-point theorems for multivalued mappings. Let (X, d) be a metric space, H and δ be as in Section 1. Assume that the mappings $F_n: X \to N(X)$, $n = 1, 2, \ldots$, have the properties:

(2.1) there exist a function $w: N \times X \to N$, a point $x_0 \in X$ and $2b \in \Delta$ such that

(a) $V_n x_{n-1} \in CN(B(x_0, b))$ for each $n = 1, 2, \ldots$ and $(x_n) \in O(\mathcal{F}, w, x_0)$, where $V_n = F_n^{w(n, x_{n-1})}$,

(b) (X, d) is an $(\mathcal{F}, w, x_0)$-orbitally complete space, (2.2) for every orbit $(x_n) \in O(\mathcal{F}, w, x_0)$ and for each n, $m \in N$, $n \neq m$, the inequality holds:

$$H(V_n x_{n-1}, V_m x_{m-1})$$
$$\leqslant a^{i(m,n)}\big(d(x_{n-1}, x_{m-1}), d(x_{n-1}, x_n), d(x_{m-1}, x_m), d(x_{n-1}, x_m), d(x_n, x_{m-1})\big),$$
$$i(m, n) = \begin{cases} 1 & \text{for } n < m, \\ 2 & \text{for } n > m. \end{cases}$$

a^i fulfils (A) and (B) on $\Delta = [0, h]$ for $i = 1, 2$ and there exist q_0, $q_1 \in \Delta$ such that

(a) $d(x_0, x_1) \leqslant q_0$ for some $x_1 \in V_1 x_0$,

(b) $q_0 + q_1 + \hat{a}^i(h) \leqslant h$ and $m_i(q_0) + q_1$, $h \in [0, b]$, where $m_i(q_0 + q_1, h)$ is a maximal solution of the equation $q_0 + q_1 + \hat{a}^i(g) = g$ in Δ, $i = 1, 2$,

(c) if $g \in R_+$ fulfils $g \leqslant q_0 + q_1 + \hat{a}^i(g)$, then $g \in \Delta$,

(2.3) for every convergent orbit $(x_n) \in O(\mathcal{F}, w, x_0)$ such that $\lim x_n = \bar{x}$ and for each k, $l \in N$, $k \neq l$, the inequality holds

$$H(V_k x_{k-1}, V_l x)$$
$$\leqslant a^3\big(d(x_{k-1}, x), \delta(x_{k-1}, V_k x_{k-1}), \delta(\bar{x}, V_l \bar{x}), \delta(x_{k-1}, V_l x_{k-1}), \delta(\bar{x}, V_k x_{k-1})\big),$$

where $a^3\colon \Delta^5 \to R_+$ is a non-decreasing and monotonically continuous mapping such that $g = 0$ is the unique solution of the equation

$$g = a^3(0, 0, g, g, 0) \quad \text{or} \quad g = a^3(0, g, 0, 0, g) \quad \text{on } \Delta.$$

THEOREM 2.1. *If assumptions* (2.1)–(2.3) *are fulfilled, then there exists* $x \in B(x_0, b)$ *such that* $x \in V_n x$ *for* $n = 1, 2, \ldots$

Proof. For $(q_n)_{n=2,3,\ldots} \in S_0(\Delta)$ such that $q_2 \leqslant q_1$ we define a sequence $(x_n)_{n \in N_0} \in O(\mathcal{F}, w, x_0)$ as follows:

(2.4) $\quad x_0, x_1$ – as in (2.2)(a), $x_n \in V_n x_{n-1}$ and

$$d(x_{n-1}, x_n) \leqslant H(V_{n-1} x_{n-2}, V_n x_{n-1}) + q_{n-1} - q_n, \qquad n = 2, 3, \ldots$$

At first we prove that for each $n = 1, 2, \ldots$ we have $x_n \in B(x_0, b)$. Obviously, $d(x_0, x_1) \leqslant q_0 \leqslant b$. For any $n \in N$ and $d(x_0, x_{n-1}) \leqslant b$ we have

$$\begin{aligned} d_n = d(x_0, x_n) &\leqslant d(x_0, x_1) + d(x_1, x_n) \\ &\leqslant q_0 + q_1 + H(V_1 x_0, V_n x_{n-1}) \\ &\leqslant q_0 + q_1 + a^1(d_{n-1}, d_{n-1}, d_{n-1} + d_n, d_n, 2d_n) \\ &\leqslant q_0 + q_1 + a^1(b, b, b + d_n, d_n, b + d_n). \end{aligned}$$

On account of (2.2)(b)–(c) we get $d_n \leqslant b$. Indeed, for $d_n > b$ and $d_n \leqslant q_0 + q_1 + \hat{a}^1(d_n)$ we would have by (2.2)(c) that $d_n \in \Delta$ and (2.2)(b) imply that $d_n \leqslant m_1(q_0 + q_1, h) \leqslant b$.

Now we prove that $(x_n) \in CO(\mathscr{F}, w, x_0)$. Obviously, for each $n = 1, 2, \ldots$ we have

$$d(x_0, x_n) \leqslant b = A_0^1(b, q_0),$$

where $A_0^1(b, q_0)$ is the first element of the sequence $\big(A_n^1(b, q_0, \ldots, q_n)\big)_{n \in N_0}$ (see Section 1).

Suppose that $d_p = d(x_p, x_{n+p}) \leqslant A_p^1(b, q_0, \ldots, q_p)$ for some $p \in N_0$. Then

$$\begin{aligned} d_{p+1} &= d(x_{p+1}, x_{n+p+1}) \\ &\leqslant q_{p+1} - q_{n+p+1} + H(V_{p+1} x_p, V_{n+p}) \\ &\leqslant q_{p+1} + a^1\big(d(x_p, x_{n+p}), d(x_p, x_{p+1}), d(x_{n+p}, x_{n+p+1}); \\ &\qquad\qquad d(x_p, x_{n+p+1}), d(x_{p+1}, x_{n+p})\big) \\ &\leqslant q_{p+1} + a^1\big(d(x_p, x_{n+p}), d(x_p, x_{1+p}), d(x_p, x_{n+p}) + d(x_p, x_{n+1+p}), \\ &\qquad\qquad d(x_p, x_{n+1+p}), d(x_p, x_{1+p}) + d(x_p, x_{n+p})\big) \\ &\leqslant q_{p+1} + \bar{a}^1\big(A_p^1(b, q_0, \ldots, q_p)\big) = A_{p+1}^1(b, q_0, \ldots, q_p). \end{aligned}$$

Analogously we get

$$d(x_p, x_{n+p}) \leqslant A_p^2(b, q_0, \ldots, q_{p-1})$$

and consequently $(x_n) \in CO(\mathscr{F}, w, x_0)$.

Now, (2.1)(b) implies that there exists $\dot{x} \in X$ such that $\lim x_n = \dot{x}$. From (2.1)(a) and from (δ_2) we get

$$\delta_n = \delta(\dot{x}, V_n \dot{x}) \leqslant d(\dot{x}, x_m) + H(V_m x_{m-1}, V_n \dot{x})$$

and from (2.3)

$$\begin{aligned} \delta_n &= \delta(\dot{x}, V_n \dot{x}) \\ &\leqslant d(\dot{x}, x_m) + a^3\big(d(x_{m-1}, \dot{x}), \delta(x_{m-1}, V_m x_{m-1}), \\ &\qquad\qquad \delta(\dot{x}, V_n \dot{x}), \delta(x_{m-1}, V_n \dot{x}), \delta(\dot{x}, V_m x_{m-1})\big). \end{aligned}$$

Thus

$$\delta_n \leqslant a^3(0, 0, \delta_n, \delta_n, 0)$$

and from the properties of a^3 we infer that $\delta_n = 0$. Therefore

$$\dot{x} \in V_n \dot{x} \quad \text{for } n = 1, 2, \ldots,$$

which completes the proof.

Remark 2.1. We may replace condition (2.1)(c) of Theorem 2.1 by an equivalent condition of the form (2.1)(c'): (X, d) is an $(\mathscr{F}, w, x_0)$-orbitally compact space.

Remark 2.2. Let $\psi_1(g) = (g, g, g, 2g, g)$ and $\psi_2(g) = (g, g, g, g, 2g)$ for $g \in R_+$ and let all assumptions (2.1)–(2.3) except (2.2)(c) be fulfilled for the functions $\hat{a}^1 = a \circ \psi_1$ and $\hat{a}^2 = a \circ \psi_2$. If for some orbit $(x_n) \in O(\mathscr{F}, w, x_0)$ of the form (2.4) the inequality $d(x_n, x_{n+1}) \leqslant b$ holds for $n = 1, 2, \ldots$, then there exists $\bar{x} \in B(x_0, b)$ such that $\bar{x} \in V_n \bar{x}$, $n = 1, 2, \ldots$

Remark 2.3. It is easy to see that if $\chi_i(g) = (g, g, g, g, g)$ for $i = 1, 2$, $g \in R_+$ and all assumptions (2.1)–(2.3) except (2.2)(c) are fulfilled for the function $\hat{a}^i = a \circ \chi_i$, $i = 1, 2, \ldots$, $g \in R_+$ and if $d(x_0, x_n) \leqslant b$ for each $n = 2, 3, \ldots$, then there exists $\bar{x} \in B(x_0, b)$ such that $\bar{x} \in V_n \bar{x}$ for $n = 1, 2, \ldots$

Now we give the global version of Theorem 2.1.

Let (X, d), H and δ be as above and let $F_n\colon X \to N(X)$, $n = 1, 2, \ldots$ Moreover, suppose that:

(2.5) there exists a function $w\colon N \times X \to N$ such that

(a) $V_n x_{n-1} \in CN(X)$ for $n = 1, 2, \ldots$, V_n as in Theorem 2.1, $(x_n) \in O(\mathscr{F}, w, x)$, $x \in X$,

(b) (X, d) is an $(\mathscr{F}, w)$-orbitally complete space,

(2.6) for every orbit $(x_n) \in O(\mathscr{F}, w, u)$, $u \in X$, and for each n, $m \in N$, $n \neq m$, the inequality of (2.2) holds and a^i fulfils (A) and (B) on $\Delta = R_+$ and for each $q \in R_+$ there exists a maximal solution $m_i(q) \in R_+$ of the equation $q + \hat{a}^i(b) = b$ for $i = 1, 2$,

(2.7) for every convergent orbit $(x_n) \in O(\mathscr{F}, w, x)$, $x \in X$, such that $\lim x_n = \bar{x}$ and for each $k, l \in N$, $k \neq l$, inequality (2.3) holds and $a^3\colon R_+^5 \to R_+$ has all the properties of (2.3) on $\Delta = R_+$.

From Theorem 2.1 and Lemma 1.3 we get

THEOREM 2.2. *If assumptions* (2.5)–(2.7) *are fulfilled, then there exists* $\bar{x} \in X$ *such that* $\bar{x} \in V_n \bar{x}$ *for* $n = 1, 2, \ldots$

EXAMPLE 2.1. Let (X, d) be a complete metric space and let F_n: $X \to CB(X)$, $n = 1, 2, \ldots$, where $CB(X)$ denotes the class of all non-empty, closed and bounded subsets of X. Let $\delta(x, A) = \inf\{d(x, y)\colon y \in A\}$ and H be a Hausdorff metric. If assumptions (2.5)–(2.7) are fulfilled, then the assertion of Theorem 2.2 is true.

Remark 2.4. If in Example 2.1 we take $a^i(g) = \alpha \cdot g$ for $i = 1, 2$, $g \in R_+$, $0 \leqslant \alpha < 1$, then we obtain the theorem of Nadler [32].

Remark 2.5. Let us consider the single-valued mapping $F\colon X \to X$, where (X, d) is a metric space. If $F_n = F$, $n = 1, 2, \ldots$, then condition (2.6) of Theorem 2.2 has the form

$$(2.6)\quad d(x_{n+1}, x_{m+1}) \leqslant a^{i(m,n)}\big(d(x_n, x_m), d(x_n, x_{n+1}), d(x_m, x_{m+1}), d(x_n, x_{m+1}), d(x_{n+1}, x_m)\big),$$

where

$$i(m, n) = \begin{cases} 1 & \text{for } n < m, \\ 2 & \text{for } n > m, \end{cases}$$

and a^i, $i = 1, 2$, is as in Theorem 2.2. If, additionally, (X, d) is an F-orbitally complete space and F is F-orbitally continuous, then there exists a fixed point of F (see also, for example, Massa [30]). If, furthermore, the inequality

$$\text{(W)} \qquad d(Fx, Fy) \leqslant a^i\big(d(x, y), d(x, Fx), d(y, Fy), d(x, Fy), d(y, Fx)\big)$$

holds for each $(x, y) \in X^2$, where $i = 1$ or $i = 2$ and a^i is as in Theorem 2.2, then the fixed point is unique.

In the case of $a(g) = a^i(g, 0, 0, 0, 0)$ we obtain a Ważewski-type condition (see [27] and [41]). It is easy to see that if condition (W) holds, then for each $x_0 \in X$ we have $F^n x_0 \in B(x_0, b)$ for $n = 1, 2, \ldots$, $b = \max\{b_1, b_2\}$, where b_i is a maximal solution on R_+ of the equation $q + \hat{a}^i(g) = g$.

Husain and Sehgal prove in [18] that if (X, d) is a complete metric space, a: $R_+^5 \to R_+$ is a continuous, non-decreasing function in each coordinate variable and satisfies the condition $a(t, t, t, t, t) < t$ for any $t > 0$, and if the mapping F: $X \to X$ satisfies the following conditions:

(a) $F^n x_0 \in B(x_0, r)$, $n = 1, 2, \ldots$ for some $x_0 \in X$, $r > 0$,

(b) $d(Fx, Fy) \leqslant a\big(d(x, y), d(x, Fx), d(y, Fy), d(x, Fy)\, d(y, Fx)\big)$ for all $x, y \in X$, then F has a unique fixed point in X.

At first, it is easy to see that the above assertion is true if we assume that the function a is monotonicaly continuous and (X, d) is an (F, x_0)-orbitally complete metric space (see Remark 2.3). On the other hand, in our case we may additionally write the following estimation: $d(\bar{x}, F^n x_0) \leqslant m_i(q_n)$ for $n \in N$, where $d(F^n x_0, F^{n+1} x_0) \leqslant q_n$ and $m_i(q_n)$ is the maximal solution of the equation $q_n + \hat{a}^i(g) = g$, $i = 1, 2$ (see [27] and [41]). Obviously, there exist Husain–Sehgal type functions for which condition (2.2)(c) is not fulfilled. Thus, in general, for Husain–Sehgal-type functions we cannot obtain the above estimates.

Remark 2.6. Let F: $X \to X$ and $d(Fx, Fy) \leqslant a\big(d(x, y)\big)$, where a: $P \to R_+$ is an upper semicontinuous function from the right on the closure of $P = \{d(x, y)\colon (x, y) \in X^2\}$ and satisfies $a(t) < t$ for all $t \in P \setminus \{0\}$, where (X, d) is an F-orbitally complete space. Then:

(a) $F^n x_0 \in B(x_0, r)$, $n = 1, 2, \ldots, r > 0$,

(b) F has a unique fixed point $\bar{x} \in X$ and $\lim F^n x = \bar{x}$ for each $x \in X$ (see Boyd and Wong [2]).

But, in general (see Remark 2.2), for Boyd–Wong type functions we cannot obtain the estimation: $d(\bar{x}, F^{n+1} x_0) \leqslant m(q_n)$ for $n \in N$, where $d(F^n x_0, F^{n+1} x_0) \leqslant q_n$ and $m(q_n)$ is the maximal solution of the equation $q_n + a(g) = g$.

Remark 2.7. Some theorems of Ćirić and Kubiaczyk result from Theorem 2.2:

(a) Ćirić [40] proves the theorem on the existence and uniqueness of the solution of the equation $x = Fx$ in an F-orbitally complete space, where $F\colon X \to X$ fulfils condition (b) of Remark 2.5 for $a(t_1, \ldots, t_5) = \max\{t_i\colon 1 \leqslant i \leqslant 5\}$, $0 \leqslant \lambda < 1$.

Let us consider the orbit (x_n), $x_{n+1} = Fx_n$, $x_0 \in X$, $n = 0, 1, \ldots$ Then we have

$$d(x_{n+1}, x_{m+1}) \leqslant \begin{cases} \lambda(1-\lambda)^{-1} \max\{d(x_n, x_m), d(x_n, x_{n+1}), d(x_n, x_{m+1})\} & \text{for } n < m, \\ \lambda(1-\lambda)^{-1} \max\{d(x_n, x_m), d(x_m, x_{m+1}), d(x_{n+1}, x_m)\} & \text{for } n > m. \end{cases}$$

Therefore we can take

$$a^1(g_1, g_2, g_3, g_4, g_5) = \lambda(1-\lambda)^{-1} \max\{g_1, g_2, g_4\} \quad \text{for } n < m,$$
$$a^2(g_1, g_2, g_3, g_4, g_5) = \lambda(1-\lambda)^{-1} \max\{g_1, g_3, g_5\} \quad \text{for } n > m.$$

The functions a^1 and a^2 fulfil all the assumptions of Theorem 2.2 and therefore the result of Ćirić follows from our Theorem 2.2.

(b) Kubiaczyk [24] consider mappings $F_1, F_2\colon X \to CB(X)$ for which the inequality holds:

(I) $H(F_1 x, F_2 y) \leqslant k\psi(\delta(x, F_1 x), \delta(y, F_2 y), d(x, y))$ for $(x, y) \in X^2$, $k < 1$, where $\psi\colon P \to [0, \infty)$, $P = Q^3$, $Q = \{d(x, y)\colon (x, y) \in X^2\}$, is right continuous and such that:

(α) $\psi(a, b, c) = \psi(b, a, c)$,

(β) $\psi(a, b, a) \leqslant \max\{a, b\}$ for all (a, b, c), (a, b, a), $(b, a, c) \in P$,

(γ) the orbit (x_n) defined by:

$$x_0 \in X, \quad x_1 \in F_1 x_0,$$
$$x_{2n} \in F_2 x_{2n-1} \quad \text{and} \quad d(x_{2n-1}, x_{2n}) \leqslant 1/\sqrt{k}\, H(F_1 x_{2n-2}, F_2 x_{2n-1}),$$
$$x_{2n+1} \in F_1 x_{2n} \quad \text{and} \quad d(x_{2n}, x_{2n+1}) \leqslant 1/\sqrt{k}\, H(F_2 x_{2n-1}, F_1 x_{2n})$$

contains a convergent subsequence.

Then (see [24]) $\lim x_n = \bar{x}$ and $\bar{x} \in F_i \bar{x}$, $i = 1, 2$.

For the orbit (x_n) we have

$$d(x_{n+1}, x_{m+1}) \leqslant \begin{cases} k(1-k)^{-1} \max\{d(x_n, x_m), d(x_n, x_{n+1})\} & \text{for } n < m, \\ k(1-k)^{-1} \max\{d(x_n, x_m), d(x_m, x_{m+1})\} & \text{for } n > m. \end{cases}$$

Therefore we can take

$$a^1(g_1, g_2, g_3, g_4, g_5) = k \cdot (1-k)^{-1} \max\{g_1, g_2\} \quad \text{for } n < m,$$
$$a^2(g_1, g_2, g_3, g_4, g_5) = k \cdot (1-k)^{-1} \max\{g_1, g_3\} \quad \text{for } n > m.$$

Thus, by Remark 2.6, this orbit is a Cauchy orbit. It is easy to see that the convergence of Cauchy orbits is equivalent to the existence of convergent subsequences of such orbits. Therefore the result of Kubiaczyk follows from Theorem 2.2.

Remark 2.8. Tasković [40] showed that Banach's result on contractions can be extended to f-contractions. $F\colon X \to X$ is an f-contraction mapping of a metric space (X, d) if for every $x, y \in X$ the inequality holds:

$$\text{(T)}\quad d(Fx, Fy) \leqslant f\big(\alpha_1(x, y)d(x, y), \alpha_2(x, y)d(x, Fx), \\ \alpha_3(x, y)d(y, Fy), \alpha_4(x, y)d(x, Fy), \alpha_5(x, y)d(y, Fx)\big),$$

where $f\colon R_+^5 \to R_+$ is increasing, semihomogeneous and such that the function $u\colon R_+ \to R_+$, $u(g) = f(g_1 \cdot g, \ldots, g_5 \cdot g)$ is continuous at the point $g = 1$ for each $(g_1, \ldots, g_5) \in R_+^5$, $\alpha_i\colon X^2 \to R_+$, $i = 1, \ldots, 5$. If condition (T) holds, then the Ćirić condition holds, too. Indeed, we have

$$f\big(\alpha_1(x, y)d(x, y), \alpha_2(x, y)d(x, Fx), \ldots, \alpha_5(x, y)d(y, Fx)\big) \\ \leqslant \lambda \max\{d(x, y), d(x, Fx), \ldots, d(y, Fx)\},$$

where $\lambda = \sup\{f(\alpha_1(x, y), \ldots, \alpha_5(x, y))\colon (x, y) \in X^2\} < 1$.

Now let (X, d), H and δ be as in Theorem 2.2. Let $F_n\colon X \to CN(X)$, $n = 1, 2, \ldots$ fulfil (2.5) and let, moreover,

(2.8) for every orbit $(x_n) \in O(\mathcal{F}, w\ , x)$, $x \in X$, and each $n, m \in N$, $n \neq m$ the inequality holds:

$$H(V_n x_{n-1}, V_m x_{m-1}) \\ \leqslant \alpha d(x_{n-1}, x_{m-1}) + \beta\big(d(x_{n-1}, x_n) + d(x_{m-1}, x_m)\big) + \gamma\big(d(x_{n-1}, x_m) + d(x_n, x_{m-1})\big),$$

where α, β, $\gamma \in R_+$ and $\alpha + 2\beta + 3\gamma < 1$,

(2.9) for every convergent orbit $(x_n) \in O(\mathcal{F}, w, x)$, $x \in X$, such that $\lim x_n = \bar{x}$ and for each $k, l \in N$, $k \neq l$, the inequality holds:

$$H(V_k x_{k-1}, V_l \bar{x}) \leqslant \beta' \delta(\bar{x}, V_l \bar{x}) + \gamma' \delta(x_{k-1}, V_l \bar{x}),$$

where β', $\gamma' \in R_+$ and $\beta' + \gamma' < 1$.

THEOREM 2.3. *If assumptions* (2.5) *and* (2.8)–(2.9) *are fulfilled, then there exists* $z \in X$ *such that* $z \in V_n z$ *for* $n = 1, 2, \ldots$ *Moreover, if in* (2.8)–(2.9), $w(n, x) = 1$ *for each* $(n, x) \in N \times X$, *then this assertion is also true with* $\alpha + 2\beta + 2\gamma < 1$.

Proof. Let $z_0 \in X$ and let z_1 be an arbitrary element of $V_1 z_0$. Now let $(q_n) \in S_0(R_+)$ and z_2 be such that $z_2 \in V_2 z_1$ and $d(z_1, z_2) \leqslant H(V_1 z_0, V_2 z_1) + q_1 - q_2$, $(1-\mu)d(z_1, z_2) \geqslant \nu q_2$, where $\mu = (\alpha + \beta + \mu)(1 - \beta - \gamma)^{-1}$ and $\nu = (1 - \beta - \gamma)^{-1}$.

Inductively, we assume that for $n = 2, 3, \ldots$, $z_n \in V_n z_{n-1}$ and $(1-\mu) \times d(z_{n-1}, z_n) \geqslant \nu q_n$, where μ and ν are as above.

Then we have for each $n \in N$

$$d(z_n, z_{n+1}) \leq H(V_n z_{n-1}, V_{n+1} z_n) + q_n - q_{n+1}$$
$$\leq \alpha d(z_{n-1}, z_n) + \beta\big(d(z_{n-1}, z_n) + d(z_n, z_{n+1})\big) +$$
$$+ \gamma d(z_{n-1}, z_{n+1}) + q_n - q_{n+1}.$$

Therefore

$$d(z_n, z_{n+1}) \leq \mu d(z_{n-1}, z_n) + \nu(q_n - q_{n+1}).$$

Now, for each $n, p \in N$ we have

$$H(V_n z_{n-1}, V_{n+p} z_{n+p-1})$$
$$\leq \alpha d(z_{n-1}, z_{n+p-1}) + \beta d(z_{n-1}, z_n) + \beta \mu^p d(z_{n-1}, z_n) +$$
$$+ \gamma d(z_{n-1}, z_{n+p}) + \gamma d(z_n, z_{n+p-1}) + \nu(q_n - q_{n+p}).$$

Finally we obtain (for sufficiently large n, p)

$$H(V_n z_{n-1}, V_{n+p} z_{n+p-1})$$
$$\leq a^1\big(d(z_{n-1}, z_{n+p-1}), d(z_{n-1}, z_n), 0, d(z_{n-1}, z_{n+p}), d(z_n, z_{n+p-1})\big) + q_{n,p},$$

where

$$a^1(g_1, g_2, g_3, g_4, g_5) = \alpha g_1 + 2\beta g_2 + \gamma g_4 + \gamma g_5$$

for each $(g_1, \ldots, g_5) \in R_+^5$ and $q_{n,p} \searrow 0$ as $n, p \to \infty$.

Obviously, in the same way we show that

$$a^2(g_1, g_2, g_3, g_4, g_5) := \alpha g_1 + (2\beta + \gamma) g_3 + \gamma g_5$$

for each $(g_1, \ldots, g_5) \in R_+^5$.

All the assumptions of Theorem 2.2 for any orbit (z_n) as above are satisfied and Theorem 2.3 is true.

Remark 2.9. It is easy to see that conditions (2.8) and

$$(2.8') \quad H(V_n x_{n-1}, V_m x_{m-1})$$
$$\leq \alpha d(x_{n-1}, x_{m-1}) + \beta_1 d(x_{n-1}, x_n) + \beta_2 d(x_{m-1}, x_m) +$$
$$+ \gamma_1 d(x_{n-1}, x_m) + \gamma_2 d(x_n, x_{m-1}),$$

where $\alpha, \beta_1, \beta_2, \gamma_1, \gamma_2 \in R_+$ and $\alpha + \beta_1 + \beta_2 + \gamma_1 + \gamma_2 < 1$, are equivalent. This fact is well known for linear conditions (see e.g. Ghosh [14]) and follows from the symmetry of the metric d. Indeed, we have

$$d(x_n, x_m) \leq \alpha d(x_{n-1}, x_{m-1}) + \tfrac{1}{2}(\beta_1 + \beta_2)\big(d(x_{n-1}, x_n) + d(x_{m-1}, x_n)\big)$$
$$+ \tfrac{1}{2}(\gamma_1 + \gamma_2)\big(d(x_{n-1}, x_m) + d(x_n, x_{m-1})\big)$$

and we can take

$$\beta = \frac{(\beta_1 + \beta_2)}{2}, \quad \gamma = \frac{(\gamma_1 + \gamma_2)}{2}.$$

EXAMPLE 2.2. Let (X, d), H and δ be as in Example 2.1. Let $S, T: X \to CN(X)$ and let there exist $\alpha, \beta, \gamma \in R_+$ such that $\alpha+2\beta+2\gamma < 1$ and for each $x, y \in X$

$$H(Sx, Ty) \leqslant \alpha d(x, y)+\beta\big(\delta(x, Sx)+\delta(y, Ty)\big)+\gamma\big(\delta(x, Ty)+\delta(y, Tx)\big).$$

Then there exists a common fixed point of S and T in X. Indeed, if we take $w(n, x) = 1$, $F_{2k+1} = S$, $F_{2k+2} = T$ for $k = 0, 1, \ldots$, then the assertion follows from Theorem 2.3.

Remark 2.10. The theorem in Example 2.2 for $\gamma = 0$ and $S, T: X \to CB(X)$ was proved by Reich (see [35]).

Remark 2.11. If in Example 2.2 $S = T$ and $T: X \to X$, then we obtain the Kannan-type condition:

$$\text{(K)} \quad d(Tx, Ty) \leqslant \alpha d(x, y)+\beta\big(d(x, Tx)+d(y, Ty)\big)+\gamma\big(d(x, Ty)+d(y, Tx)\big),$$
$$(x, y)\in X^2, \ \alpha, \beta, \gamma \geqslant 0, \ \alpha+2\beta+2\gamma < 1.$$

Obviously, the assertion of the theorem in Example 2.2 is true if condition (K) holds (the fixed point of T is unique).

Remark 2.12. Non-linear contraction type conditions for single-valued mappings, which generalize the Banach contraction principle, have been considered by Browder [3], Rakotch [35], Ważewski [41], Kwapisz [27], [28], Boyd and Wong [2], Husain and Sehgal [18], Ćirić [7], [8], [9], Tasković [40] and others. Kannan [20] and Chatterja [6] considered condition (K) in particular cases. Bryant [4], Sehgal [18], Chi Song Wong [42], [43] and others impose (K)-type conditions on some iterate F^k rather than on mapping F. Sequences of generalized contractions are considered, for example, in [29] and [15].

Now let (X, d), H and δ be as in Theorem 2.3. Let $F_n: X \to N(X)$, $n = 1, 2, \ldots$, fulfil (2.5) and let, moreover,

(2.10) for every orbit $(x_n) \in O(\mathcal{F}, w, x)$, $x \in X$, and for each $n, m \in N$, $n \neq m$, the inequality holds:

$$H(V_n x_{n-1}, V_m x_{m-1})$$
$$\leqslant \alpha(x_{n-1}, x_{m-1})\, d(x_{n-1}, x_{m-1})+\beta(x_{n-1}, x_{m-1})\big(d(x_{n-1}, x_n)+d(x_{m-1}, x_m)\big)+$$
$$+\gamma(x_{n-1}, x_{m-1})\big(d(x_{n-1}, x_m)+d(x_n, x_{m-1})\big),$$

where $\alpha, \beta, \gamma: X^2 \to R_+$ and

$$\sup\{\alpha(x, y)+2\beta(x, y)+3\gamma(x, y): (x, y)\in X^2\} < 1,$$

(2.11) for every convergent orbit (x_n), $(x_n)\in O(\mathcal{F}, w, x)$, such that $\lim x_n = x$ and for $k, l \in N$, $k \neq l$, the inequality holds:

$$H(V_k x_{k-1}, V_l x) \leqslant \beta'(x_{k-1}, x)\,\delta(x, V_l x)+\gamma'(x_{k-1}, x)\,\delta(x_{k-1}, V_l x),$$

where $\beta', \gamma': X^2 \to R_+$ and $\sup\{\beta'(x, y)+\gamma'(x, y): (x, y)\in X^2\} < 1$.

THEOREM 2.4. *If all assumptions* (2.5) *and* (2.10)–(2.11) *are fulfilled, then there exists* $x \in X$ *such that* $x \in V_n x$ *for* $n = 1, 2, \ldots$

Proof. At first we notice that

$$\mu = \sup\{z_1(x, y): (x, y) \in X^2\} < 1,$$

where

$$z_1(x, y) = \big(\alpha(x, y) + \beta(x, y) + \gamma(x, y)\big)\big(1 - \beta(x, y) - \gamma(x, y)\big)^{-1}.$$

Therefore we can take

$$\nu = \sup\{(1 - \beta(x, y) - \gamma(x, y))^{-1}: (x, y) \in X^2\}$$

and thus the proof reduces to the proof of Theorem 2.3, where μ and ν are as above and, obviously, condition (2.11) imply the existence and uniqueness of the solution x.

Remark 2.12. From Theorem 2.4 we can obtain Theorem 1, proved by Kubiaczyk in [24]. For this purpose it is sufficient to take $w(n, x) = 1$, $F_{2k+1} = S$, $F_{2k+2} = T$ for each $k = 0, 1, \ldots$ in Theorem 2.4, assuming that (X, d) is complete and $S, T: X \to CB(X)$.

From Remark 2.11 and Theorem 2.4 we get

CONCLUSION 2.1(Ćirić [7]). If $F: X \to X$, (X, d) is an F-orbitally complete space and the inequality

$$d(Fx, Fy) \leqslant \alpha(x, y)d(x, y) + \beta_1(x, y)\big(d(x, Fx) + \beta_2(x, y)d(y, Fy)\big) + \\ + \gamma(x, y)\big(d(x, Fy) + d(y, Fx)\big)$$

holds for all $(x, y) \in X^2$, where

$$\sup\{\alpha(x, y) + \beta_1(x, y) + \beta_2(x, y) + 2\gamma(x, y): (x, y) \in X^2\} = \lambda < 1,$$

$\alpha, \beta_1, \beta_2, \gamma: X^2 \to R_+$, then

(i) there is a unique fixed-point $\bar{x}$ of F in X,

(ii) $\lim F^n x = \bar{x}$ for every $x \in X$,

(iii) $d(\bar{x}, F^n x) \leqslant \lambda(1-\lambda)^{-1} d(x, Fx)$.

Let (X, d), H and δ be as in Theorem 2.1. Suppose that $F_{2k+1} = S$, $F_{2k+2} = T$ for $k = 0, 1, \ldots$ and $S, T: X \to N(X)$ are such that (2.1) is fulfilled for a function w, where $w(2k-1, x) = w_1$ and $w(2k, x) = w_2$ for $k \in N$, $x \in X$. Assume that (2.2) is fulfilled and that

(2.12) S^{w_1}, T^{w_2} are $(\mathscr{F}, w, x_0)$-orbitally continuous on $B(x_0, b)$.

THEOREM 2.5. *If assumptions* (2.1)–(2.2) *and* (2.12) *are fulfilled, then there exists* $\bar{x} \in B(x_0, b)$ *such that* $\bar{x} \in S^{w_1}\bar{x}$ *and* $\bar{x} \in T^{w_2}\bar{x}$.

Proof. Obviously, by Theorem 2.1, there exists $\bar{x} \in B(x_0, b)$ such that $\lim x_n = \bar{x}$, Thus

$$\delta(\bar{x}, S^{w_1}\bar{x}) \leqslant d(\bar{x}, x_{2k+1}) + H(S^{w_1}x_{2k}, S^{w_1}\bar{x})$$

and from (2.12)

$$\delta(\bar{x}, S^{w1}\bar{x}) = 0.$$

On account of

$$\delta(\bar{x}, T^{w2}\bar{x}) \leqslant d(\bar{x}, x_{2k+1}) + H(S^{w1}x_{2k+1}, T^{w2}\bar{x})$$

we also have $\bar{x} \in T^{w2}\bar{x}$.

Now we are going to formulate two theorems which are simple generalizations of theorems of Kubiaczyk [24].

THEOREM 2.6. *Let* (X, d), H *and* δ *be as above and let* $S, T: X \to CN(X)$. *Moreover, suppose that:*

(2.14) *for each* $(x, A) \in X \times CN(X)$ *there exists* $\bar{x} \in A$ *such that* $d(x, \bar{x}) = \delta(x, A)$,

(2.15) *there exists* $\min \varphi_1(X)$ *or* $\min \varphi_2(X)$, *where* $\varphi_1(x) = \delta(x, Sx)$ *and* $\varphi_2(x) = \delta(x, Tx)$, $x \in X$,

(2.16) *there exists a function* $M: [CN(X)]^2 \to R_+$ *such that*

(a) *for each* $A, B \in CN(X)$, $H(A, B) \leqslant M(A, B)$,

(b) $d(x, y) \leqslant M(A, B)$ *for each* $A, B \in CN(X)$ *and* $x \in A$, $y \in B$,

(c) *for all distinct elements* x, y *of* X *the inequality holds:*

$$M(Sx, Ty) < a\big(d(x, y), \delta(x, Sx), \delta(y, Ty), \delta(x, Ty), \delta(y, Sx)\big),$$

where $a: R_+^5 \to R_+$ *is a non-decreasing function with* $a(g, g, g, 2g, 2g) \leqslant g$ *for each* $g \in R_+$.

Then

(1) *there exists a fixed point of* S *or* T *in* X,

(2) *if* $u \in Su$ *and* $v \in Tv$, *then* $u = v$.

THEOREM 2.7. *If all assumptions* (2.14)–(2.15) *and* (2.16) (a) (b) *are fulfilled and if*

(2.16′) (c) *for all distinct elements* x, y *of* X *the inequality holds*:

$$M(Sx, Ty) < \alpha(x, y)d(x, y) + \beta(x, y)\big(\delta(x, Sx) + \delta(y, Ty)\big) + \\ + \gamma(x, y)\big(\delta(x, Ty) + \delta(y, Sx)\big),$$

where α, β, γ: $X^2 \to R_+$ *and*

$$\sup\{\alpha(x, y) + 2\beta(x, y) + 2\gamma(x, y): (x, y) \in X^2\} \leqslant 1,$$

then the assertion of Theorem 2.6 *is true.*

Indeed, it is easy to see that

$$\sup\{\big(\alpha(x, y) + \beta(x, y) + \gamma(x, y)\big)\big(1 - \beta(x, y) - \gamma(x, y)\big)^{-1}: (x, y) \in X^2\} \leqslant 1$$

and the above assertion results from the proof of Theorem 2.4.

CONCLUSION 2.2. Let (X, d) be a compact metric space and S, T: $X \to CB(X)$. Let either S or T be a continuous mapping. Suppose that assumptions (2.16) (a) (b) and (2.16′) (c) of Theorem 2.7 are fulfilled. Then the assertion of Theorem 2.7 is true.

CONCLUSION 2.3. (Kubiaczyk [24]). Let (X, d) and S, T be as in Conclusion 2.2. If assumption (2.16′) (c) is fulfilled for $M = \Delta$, where $\Delta(A, B) = \sup\{d(x, y): x \in A, y \in B\}$, then the assertion of Theorem 2.7 is true.

3. Some generalizations. Now let G be a non-empty set equipped with a binary operation $(x, y) \to x+y$ mapping $G \times G$ into G, such that $(G, +)$ is an Abelian group. Let $\leqslant$ be a partial ordering of G and suppose that the partially ordered set $(G, \leqslant)$ has the compatibility property:

(1) $g_1 \leqslant g_2$ implies $g_1+g \leqslant g_2+g$ for each g_1, g_2, $g \in G$.

Let $G_+ = \{g \in G: g \geqslant 0\}$ and assume that the partially ordered set $(G_+, \leqslant)$ has the additional property:

(2) $g_1+g_2 \leqslant g$ implies g_1, $g_2 \leqslant g$ for each g_1, g_2 $g \in G_+$.

Let $S(G_+)$ denote the set of all non-increasing sequences $(g_n) \in G_+^N$ with an operator "lim", $S(G_+) \in (g_n)_{n\in N} \to \lim g_n = g \in G_+$ such that for (g_n), $(g'_n) \in S(G_+)$ we have:

(1) $\lim(g_n+g'_n) = \lim g_n + \lim g'_n$;

(2) $g_n \leqslant g'_n$ for $n \in N$ implies $\lim g_n \leqslant \lim g'_n$;

(3) $(g_n) = (g)$ implies $\lim g_n = g$.

In the sequel $S_0(G_+) \subset S(G_+)$ will denote the subclass of all sequences convergent to zero.

We define a partial ordering $\leqslant$ on G^5 in the usual way: $\bar{g} \leqslant \bar{g}'$ or $\bar{g}' \geqslant \bar{g}$ if and only if $\bar{g}'-\bar{g} \in G_+^5$, $\bar{g} = (g_1, \ldots, g_5)$, $\bar{g}' = (g'_1, \ldots, g'_5)$. The new operator lim: $S(G_+^5) \to G_+^5$, where $\lim(g_{1,n}, \ldots, g_{5,n}) := (\lim g_{1,n}, \ldots, \lim g_{5,n})$, has the properties analogous to (1)–(3) (obviously, $(g_n) = (g_{1,n}, \ldots, g_{5,n}) \in S(G_+^5)$ if and only if $(g_{i,n}) \in S(G_+)$, $i = 1, \ldots, 5$).

Remark 3.1. It is easy to see that Lemmas 1.1–1.3 are true for the case, where a^i: $G_+^5 \to G_+$, $i = 1, 2$. In Lemmas 1.1–1.2 we must additionally assume that also $b \in \Delta$.

We say that (X, d) is a *generalized metric space*, if d: $X^2 \to G_+$ is a function such that

(a) $d(x, y) = 0 \Leftrightarrow x = y$ for each x, $y \in X$,

(b) $d(x, y) \leqslant d(x, z)+d(y, z)$ for each x, y, $z \in X$.

Let the operator lim: $X^N \to X$ fulfil the condition: a sequence $(x_n) \in X^N$ is convergent to $x \in X$ if there exists a sequence $(q_n) \in S_0(G_+)$ and a positive integer n_0 such that $d(x_n, x) \leqslant q_n$ for $n \geqslant n_0$; we then write, as usual,

$\lim x_n = x$. By A we denote the sequential closure of the set $A \subset X$, i.e. the set $\{x \in X$: there exists $(x_n) \in A^N$ such that $x = \lim x_n\}$. In general, the pair $(X, \lim)$ is not an L-space of Fréchet type (see for example Kuratowski [25], Kisyński [22]).

Remark 3.2. Mutatis mutandis we can write generalized versions of Theorems 2.1–2.7 in a generalized metric space; we will call them in the sequel Theorems 3.1–3.7, respectively. Proofs of Theorems 2.1–2.7 can be easily transformed to work in such a generalized setting. Let us state explicitly three of these theorems:

THEOREM 3.1. *Let (X, d) be a generalized metric space and let the order relation $\leqslant$ be linear on Δ. If assumptions* (2.1)–(2.3) *are fulfilled, then there exists $\bar{x} \in B(x_0, b)$ such that $\bar{x} \in V_n \bar{x}$ for $n = 1, 2, \ldots$*

THEOREM 3.2. *Let (X, d) be a generalized metric space. If assumptions* (2.5)–(2.7) *are fulfilled, then there exists $\bar{x} \in X$ such that $\bar{x} \in V_n \bar{x}$ for $n = 1, 2, \ldots$*

THEOREM 3.7. *Let (X, d) be a generalized metric space and suppose that assumptions* (2.14)–(2.16) (a) (b) *and* (2.16′) *are fulfilled. If, in addition, $(G, \leqslant)$ is a complete structure and the ordering is linear on $\varphi_1(X) \cup \varphi_2(X)$, then the assertion of Theorem* 2.7 *is true.*

EXAMPLE 3.1. Let (X, d) be a metric space and let $N(\varepsilon, A) = \{x \in X$: $d(x, c) < \varepsilon$ for some $c \in A$, $A \in CN(X)\}$. Then the function H: $CN(X) \times \times CN(X) \to R_+^\#$, given by

$$H(A, B) = \begin{cases} \inf\{\varepsilon > 0:\ A \subset N(\varepsilon, B) \text{ and } B \subset N(\varepsilon, A)\} & \text{if the infimum exists,} \\ \infty, & \text{otherwise,} \end{cases}$$

for $A, B \in CN(X)$, is a generalized Hausdorff metric, where $R^\#$ is the extended set of real numbers (see for example [17]). Let $\delta(x, A) = \inf\{d(x, y)$: $y \in A\}$ for $x \in X$, $A \in CN(X)$. Obviously, the functions H and δ are as in Theorem 3.2, where $G_+ = R_+^\# = \{g \in R^\#: g \geqslant 0\}$. If F_n: $X \to N(X)$, $n = 1, 2, \ldots$, $2b \in \Delta$ and assumptions (2.5)–(2.7) are fulfilled, then the assertion of Theorem 3.2 is true.

CONCLUSION 3.1 (Czerwik [10]). Let (X, d) be a complete metric space and let H and δ be as in Example 3.1. Let F: $X \to CN(X)$ have the property:

$$H(Fx, Fy) \leqslant a^i\big(d(x, y), \delta(x, Fx), \delta(y, Fy), \delta(x, Fy), \delta(y, Fx)\big), \quad i = 1, 2.$$

If a^1, a^2 are as in Theorem 3.2 with $G_+ = R_+^\#$, then there exists $\bar{x} \in X$ such that $\bar{x} \in F\bar{x}$. Indeed, it suffices to take $w(n, x) = 1$, $F_n = F$ for $n = 1, 2, \ldots$ and from Theorem 3.2 we obtain the assertion.

Let the function d: $X \times X \to (R_+^\#)^r$ and the space (X, d) be as above, H and δ be as in Theorem 3.2. Let F: $X \to N(X)$ fulfil the conditions:

(3.1) there exists $U \subset X$ such that

(a) $F(U) \in CN(X)$,

(b) $F\colon F(U) \to CN(F(U))$,

(c) diam $F(U) < \infty$;

(3.2) $(F(U), d)$ is an F-orbitally complete subspace of (X, d);

(3.3) for every orbit $(x_n) \in O(F, u)$, $u \in F(U)$ and for each $n, m \in N$, $m \neq n$, the inequality holds:

$$H(Fx_{n-1}, Fx_{m-1}) \leqslant a^{i(m,n)}\big(d(x_{n-1}, x_{m-1}), d(x_{n-1}, x_n), d(x_{m-1}, x_m), d(x_{n-1}, x_m), d(x_n, x_{m-1})\big),$$

where

$$i(m, n) = \begin{cases} 1 & \text{for } n < m, \\ 2 & \text{for } n > m, \end{cases}$$

and $a^i\colon ((R_+^\#)^n)^5 \to (R_+^\#)^n$ are such that

(a) the restriction of a^i to $(R_+^n)^5$ fulfils assumptions (A) and (B) of Theorem 3.2, $i = 1, 2$,

(b) for each $q \in R_+^n$ there exists a maximal solution $m_i(q) \in R_+^n$ of the equation $q + a^i(b) = b$ for $i = 1, 2$;

(3.4) for every convergent orbit $(x_n) \in O(F, u)$, $u \in F(U)$, with $\lim x_n = \bar{x}$ and for each $k, l \in N$, $k \neq l$, the inequality holds:

$$H(Fx_{k-1}, F\bar{x}) \leqslant a^3\big(d(x_{k-1}, \bar{x}), \delta(x_{k-1}, Fx_{k-1}), \delta(\bar{x}, F\bar{x}), \delta(x_{k-1}, F\bar{x}), \delta(\bar{x}, Fx_{k-1})\big),$$

where $a^3\colon ((R_+^\#)^n)^5 \to (R_+^\#)^n$ is a non-decreasing and monotonically continuous mapping such that $g = 0$ is the unique solution of the equation $g = a^3(0, 0, g, g, 0)$ or $g = a^3(0, g, 0, 0, g)$ in R_+^n.

THEOREM 3.8. *If assumptions* (3.1)–(3.4) *are fulfilled, then there exists* $x \in F(U)$ *and* $x \in Fx$.

This assertion follows from Theorem 3.2.

CONCLUSION 3.2 (a). If (X, d), H and δ are as above and (3.1)–(3.2) are fulfilled for $F\colon X \to N(X)$ and if, moreover, the inequality

$$(3.5)\qquad H(Fx, Fy) \leqslant a^i\big(d(x, y), \delta(x, Fx), \delta(y, Fy), \delta(x, Fy), \delta(y, Fx)\big)$$

holds for each $x, y \in X$, where a^i, $i = 1, 2$, are as in Theorem 3.1, then the assertion of this theorem is true.

(b) In particular, if

$$a^i(g_1, g_2, g_3, g_4, g_5) = A \cdot g_1 + B_1 \cdot g_2 + B_2 \cdot g_3 + C_1 \cdot g_4 + C_2 \cdot g_5$$

for $(g_1, g_2, g_3, g_4, g_5) \in (R_+^n)^5$, $i = 1, 2$, where A, B_1, B_2, C_1, C_2 are non-negative $n \times n$-matrices such that

(3.6) there exists either the product

$$L_1 = (I - B_2 - C_1)^{-1}(A + B_1 + C_1)$$

or

$$L_2 = (I - B_1 - C_2)^{-1}(A + B_2 + C_2)$$

satisfying $r(L_1) < 1$ and $r(B_2 + C_1) < 1$ [$r(L_2) < 1$ and $r(B_1 + C_2) < 1$ respectively], where $r(A)$ denotes the spectral radius of A, then the assertion of Theorem 3.1 is true.

Remark 3.4. If in Conclusion 3.1 (b) $F: X \to X$, (X, d) is a complete metric space, $a^1(g_1, g_2, g_3, 0, 0) = \psi(g_1, g_2, g_3)$, where ψ is continuous and such that $\psi(0, 0, g_3) \geqslant g_3$ implies $g_3 = 0$, then we obtain the Rus type theorem ([37], Theorem 3).

Now let (X, d) be a complete metric space and H be as above. Let the continuous function δ: $X \times N(X) \to (R_+)^n$ be such that the restriction of δ to $X \times B(X)$ ($B(X)$ denotes the class of all non-empty and bounded subsets of X) has the properties:

(δ'_1) $x \in A$ implies $\delta(x, A) = 0$ for each $A \in B(X)$ and $x \in X$,

(δ'_2) $\delta(x, A) \leqslant d(x, y) + \delta(y, B) + H(A, B)$ for each $x, y \in X$ and each A, $B \in B(X)$,

(δ'_3) if $A \in C(X)$, where $C(X)$ denotes the family of all non-empty and compact subsets of X, then $\delta(x, A) = 0$ implies $x \in A$.

Assume that there exists a function M: $[N(X)]^2 \to (R_+)^n$ such that

(3.6) (a) for each A, $B \in B(X)$, $H(A, B) \leqslant M(A, B)$,

(b) $d(x, y) \leqslant M(A, B)$ for each A, $B \in B(X)$ and $x \in A$, $y \in B$.

Let F: $X \to N(X)$ and suppose that there exists $U \in N(X)$ such that

(3.7) F: $F(U) \to B(X)$, $F(U) \in B(X)$ and $F(F(U))$ is a compact set,

(3.8) for all distinct elements $x, y \in F(U)$ the inequality holds:

$$M(Fx, Fy) < a\big(d(x, y)\, \delta(x, Fx), \delta(y, Fy), \delta(x, Fy), \delta(y, Fx)\big),$$

where a: $(R^n_+)^5 \to R^n_+$ is a non-decreasing function with $a(g, g, g, 2g, 2g) \leqslant g$ for each $g \in R^n_+$.

THEOREM 3.9. *If assumptions* (3.6)–(3.8) *are fulfilled, then there exists a unique fixed point of* F *in* $F(U)$.

It is easy to see that $\varphi(X)$ has the minimal element, $\varphi(x) = \delta(x, Fx)$, $x \in \overline{F(U)}$, where δ is as in Theorem 3.6. Therefore the above assertion is a conclusion of Theorem 3.6.

Remark 3.5. In particular, we can take

$$a(g_1, g_2, g_3, g_4, g_5) = Ag_1 + B_1 g_2 + B_2 g_3 + C(g_4 + g_5),$$

where A, B_1, B_2 and C are non-negative matrices such that $r(A+B_1+B_2+2C) < 1$. If $F\colon X \to X$, $B_1 = B_2 = 0$, then assumptions (3.6)–(3.8) are fulfilled.

Remark 3.6. To emphasize the importance of the generalizations stated in Section 3, we now show how they work in the context of the results of the recent papers of Kwapisz and Turo (see [28], [29]). We take here $(G_+, \leqslant) = (C_0(D, R_+), \leqslant)$, where $C_0(D, R_+)$ is the class of all upper semicontinuous functions from D to R_+ and $z_1 \leqslant z_2 \Leftrightarrow z_1(x, y) \leqslant z_2(x, y)$ for all $(x, y) \in D$; $D = [0, a] \times [0, b]$; $a, b > 0$; and let $d\colon [C(D, E)]^2 \to G_+$, where $d(z_1, z_2)(x, y) := \|z_1(x, y) - z_2(x, y)\|$, $(E, \|\ \|)$ is a Banach space. Suppose that the operator $f\colon D \times [C(D, E)]^4 \times E \to E$ satisfies the inequality

$$\begin{aligned}\|Fu^1 - Fu^2\|(x, y) &:= \|Fu^1(x, y) - Fu^2(x, y)\| \\ &\leqslant a_i\big(x, y, \|u^1 - u^2\|(\cdot,\cdot), \|u^1 - Fu^1\|(\cdot,\cdot), \|u^2 - Fu^2\|(\cdot,\cdot), \\ &\qquad \|u^1 - Fu^2\|(\cdot,\cdot), \|u^2 - Fu^1\|(\cdot,\cdot)\big) \\ &\qquad\qquad \text{for } u^1, u^2 \in C(D, E),\end{aligned}$$

where F and a_i are some integro-functional operators defined with the aid of f and Ω_i, respectively. Some additional assumptions imposed on f and Ω_i give a possibility of proving the generalizations of Theorems 1–4 of [29].

Remark 3.7. Let us consider two generalized metric spaces: (X, d) and (X, e). Assume that

(a) (X, e) is an $(\mathscr{F}, w)$-orbitally complete space, where $\mathscr{F}$ is the sequence of mappings $F_n\colon X \to X$, $n = 1, 2, \ldots$,

(b) every Cauchy orbit of $\mathscr{F}$ in (X, d) is a Cauchy orbit in (X, e),

(c) assumptions (2.5)–(2.6) of Theorem 3.2 are fulfilled with respect to the metric d except (2.5) (b),

(d) each mapping F_i, $i = 1, 2, \ldots$, is $(\mathscr{F}, w)$-orbitally continuous in the space (X, e).

Then there exists $\bar{x} = \lim x_n$ such that $\bar{x} \in V_n \bar{x}$, $n = 1, 2, \ldots$, where (x_n) is the orbit as in Theorem 3.2 and the limes-operator "lim" is generated by the metric e.

On the other hand, if instead of condition (d) we assume (2.7) of Theorem 3.2 for the metric e, then the common fixed point of F_i, $i = 1, 2, \ldots$, is unique. In particular, we can take $e(x, y) \leqslant kd(x, y)$, $x, y \in X$, $k > 0$, and condition (2.7) for the metric d (see for example Iseki [19] and Singh [38]).

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Reçu par la Rédaction le 22. 5. 1977