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ON CONVERGENCE OF THE NONLINEAR ACTIVE DISTURBANCE REJECTION CONTROL FOR MIMO SYSTEMS*

BAO-ZHU GUO† AND ZHI-LIANG ZHAO‡

Abstract. In this paper, the global and semiglobal convergence of the nonlinear active disturbance rejection control (ADRC) for a class of multi-input multi-output nonlinear systems with large uncertainty that comes from both dynamical modeling and external disturbance are proved. As a result, a class of linear systems with external disturbance that can be dealt with by the ADRC is classified. The ADRC is then compared both analytically and numerically to the well-known internal model principle. A number of illustrative examples are presented to show the efficiency and advantage of the ADRC in dealing with unknown dynamics and in achieving fast tracking with lower overstriking.

Key words. nonlinear system, observer, output feedback, Lyapunov stability, stabilization, robust design

AMS subject classifications. 93C15, 93B52, 34D20, 93D15, 93B51

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1. Introduction.

The uncertainty that arises either from an unmodeled part of the system dynamics or from external disturbance is essentially the major concern in postmodern control theory. Many methodologies have been developed to deal with the uncertainty. Among them are two well-studied strategies: robust control and sliding mode control. Both consider the worst case of the uncertainty, thus leading to relatively excessive control effort for some particular operations of the control system. The traditional high-gain control method (see, e.g., [19]) is sound mathematically, but not practically since it uses the high gain not only in the observer but also in the feedback loop in order to suppress the uncertainty. On the other hand, control strategies based on the idea of online estimation and compensation have also been presented. The modified high-gain control is a typical strategy of this kind (see [5, 18]); more examples can be found in [1, 2, 3, 6, 7], to name just a few.

The active disturbance rejection control (ADRC), an unconventional design strategy, was first proposed by Han in his pioneer work [14]. It is now acknowledged to be an effective control strategy in dealing with the total uncertainty (unknown part of model dynamics and of external disturbance). A major feature of the ADRC lies in its ability to cancel the total uncertainty in the feedback loop after an estimation in real time. Its power was initially demonstrated by numerical simulations [13, 14] in the early stage, and later consolidated in many engineering practices such as motion control, tension control in web transport and strip precessing systems, DC-DC

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†Academy of Mathematics and Systems Science, Academia Sinica, Beijing 100190, China, School of Computational and Applied Mathematics, University of the Witwatersrand, Johannesburg, South Africa, and King Abdulaziz University, Jeddah, Saudi Arabia (bzguo@iss.ac.cn). This author was supported by the National Natural Science Foundation of China, the National Basic Research Program of China (2011CB808002), and the National Research Foundation of South Africa.

‡Corresponding author. School of Mathematical Sciences, University of Science and Technology of China, Hefei People's Republic of China, School of Mathematics and Statistics, Xi'an Jiaotong University, Xi'an, China, and School of Computational and Applied Mathematics, University of the Witwatersrand, Johannesburg, South Africa (gsdxzzl@mail.ustc.edu.cn).

power converts in power electronics, continuous stirred tank reactor in chemical and process control, micro-electro-mechanical systems gyroscope. More concrete examples can also be found in [15, 17, 20, 22, 23, 25] and the references therein. For a more practical perspective, we refer to a recent paper [24]. Compared to its success in applications, the theoretical study on the ADRC lags far behind. It is only very recently, that the global convergence of a closed loop of the nonlinear ADRC was proved in [9] for some class of nonlinear SISO systems where the zero dynamics is not taken into account. For linear ADRC, a semiglobal convergence is obtained for the stabilization of a kind of SISO system in [5] and generalized to a class of MIMO systems in [21].

It is the main contribution of this paper that we present both semiglobal and global convergence of the nonlinear ADRC for a class of general MIMO nonlinear systems. The system we are concerned with is the following partial exact feedback linearizable MIMO system ([16]) with large uncertainties:

$$
\begin{cases}
\dot{x}^i(t) = A_{n_i} x^i(t) + B_{n_i}\Big[f_i(x(t),\xi(t),w_i(t)) \\
\qquad\qquad\qquad\qquad + \displaystyle\sum_{j=1}^{m} a_{ij}(x(t),\xi(t),w(t))u_j(t)\Big], \\
y_i(t) = C_i x^i(t), i = 1,2,\ldots,m, \\
\dot{\xi}(t) = F_0(x(t),\xi(t),w(t)),
\end{cases} \tag{1.1}
$$

where $u \in \mathbb{R}^m$, $\xi \in \mathbb{R}^s$, $x = (x^1, x^2, \ldots, x^m) \in \mathbb{R}^n$, $n = n_1 + \cdots + n_m$, F_0, f_i, a_{ij}, w_i are C^1-functions with their arguments, respectively, the external disturbance $w = (w_1, w_2, \ldots, w_m)$ satisfies $\sup_{t\in[0,\infty)} \|(w,\dot{w})\| < \infty$, and

$$
A_{n_i} = \begin{pmatrix} 0 & I_{n_i-1} \\ 0 & 0 \end{pmatrix}_{n_i\times n_i}, \quad B_{n_i} = \big(0,\ldots,0,1\big)^\top_{n_i\times 1}, \quad C_{n_i} = \big(1,0,\ldots,0\big)_{1\times n_i}. \tag{1.2}
$$

In the study of the ADRC of the system (1.1), it is generally assumed that the unmodeled dynamics terms f_i's are unknown, the control parameters a_{ij}'s have some uncertainties, and the external disturbance w is completely unknown.

Let the reference input signal be v_i. We then construct a tracking differentiator with input v_i and output z^i as follows:

$$
\text{TD: } \dot{z}^i(t) = A_{n_i+1} z^i(t) + B_{n_i+1}\rho^{n_i+1}\psi_i\left(z_1^i - v_i, \frac{z_2^i}{\rho}, \ldots, \frac{z_{n_i+1}^i}{\rho^{n_i}}\right), \quad i = 1,2\ldots,m. \tag{1.3}
$$

Note that z_j^i can be used as an approximation of $(v_i)^{(j-1)}$, the $(j-1)$th derivative of v_i.

The control objective of the ADRC is to make the output x_1^i track the measured signal v_i, and x_j^i track z_j^i, $1 \le j \le n_i$, $1 \le i \le m$. Moreover, each error $e_j^i = x_j^i - z_i^j$ converges to zero in a way that the reference state x_j^{i*} also converges to zero, where x_j^{i*} satisfies the following reference differential equation:

$$
\text{Ref: } \begin{cases}
\dot{x}_1^{i*}(t) = x_2^{i*}(t), \\
\dot{x}_2^{i*}(t) = x_3^{i*}(t), \\
\qquad\vdots \\
\dot{x}_{n_i}^{i*}(t) = \phi_i(x_1^{i*}(t),\ldots,x_{n_i}^{i*}(t)),\ \phi_i(0,0,\ldots,0) = 0.
\end{cases} \tag{1.4}
$$

In other words, it is required that $e_j^i = x_j^i - z_i^j \approx x_j^{i*}$ and $x_j^{i*}(t) \to 0$ as $t \to \infty$.

It is worth pointing out that with the use of the tracking differentiator (1.3), it is possible to deal with the signal v_i whose high-order derivatives do not exist in the classical sense, as it is often the case in boundary measurement of the PDEs.

The convergence of (1.3) implies that z_j^i can be regarded as an approximation of $(v_i)^{(j-1)}$ in the sense of distribution ([8, 12]). This result is detailed below (see [12]).

LEMMA 1.1. *Consider system* (1.3). *Suppose that ψ_i is locally Lipschitz continuous with its arguments. If it is globally asymptotically stable with $\rho = 1, v \equiv 0$, and $\dot{v}_i$ (the first derivative of v_i with respect to t) is bounded, then for any initial value of* (1.3) *and constant $\tau > 0$, $\lim_{\rho\to\infty} |z_1^i(t) - v_i(t)| = 0$ uniformly for $t \in [\tau, \infty)$. Moreover, for any $j, 1 \le j \le n_i + 1$, z_j^i is uniformly bounded over $\mathbb{R}^+$.*

Remark 1.1. We remark that all z_j^i are ρ-dependent, and that z_j^i is regarded as an approximation of the $(i-1)$th order derivative of v_i in the sense of a generalized derivative (see Theorem 2.1 of [12]). If all $(v_i)^{(j-1)}$ exist in the classical sense, we may consider simply $z_j^i = (v_i)^{(j-1)}$ for $j = 2, 3, \dots, n_i$. In the latter case, the TD (1.3) does not need to be coupled into the ADRC.

Owing to Lemma 1.1, we can make the following assumption.

Assumption A1. $\|z(t)\| = \|(z^1(t), z^2(t), \dots, z^m(t))\| < C_1 \ \forall\, t > 0$, where z^i is the solution of (1.3), $z^i(t) = (z_1^i(t), z_2^i(t), \dots, z_{n_i}^i(t))$, and C_1 is a ρ-dependent positive constant.

An important constituent in the ADRC is the extended state observer (ESO) which is used to estimate not only the state but also the total uncertainty in system dynamics and external disturbance.

The ESO is designed as follows:

$$\text{ESO:} \begin{cases} \dot{\hat{x}}_1^i(t) = \hat{x}_2^i(t) + \varepsilon^{n_i-1} g_1^i(e_1^i(t)), \\ \dot{\hat{x}}_2^i(t) = \hat{x}_3^i(t) + \varepsilon^{n_i-2} g_2^i(e_1^i(t)), \\ \quad\vdots \\ \dot{\hat{x}}_{n_i}^i(t) = \hat{x}_{n_i+1}^i(t) + g_{n_i}^i(e_1^i(t)) + u_i^*(t), \\ \dot{\hat{x}}_{i,n_i+1}(t) = \dfrac{1}{\varepsilon} g_{n_i+1}^i(e_1^i(t)), \ i = 1, 2, \dots, m, \end{cases} \tag{1.5}$$

where $e_1^i = (x_1^i - \hat{x}_1^i)/\varepsilon^{n_i}$, the g_j^i are possibly some nonlinear functions, thus the term nonlinear ADRC. When all g_j^i are linear, it becomes the special case studied in [5]. The function of ESO is to estimate, in real time, both the state, and the total disturbance in the ith subsystem by the extended state $\hat{x}_{i,n_i+1}$ in (1.5).

The convergence of ESO (1.5) itself (without feedback), like many other observers in nonlinear systems, is an independent issue. And some results are obtained in [10, 11].

In order to show the convergence of ADRC, we need the following assumptions on ESO (1.5) (Assumption A2) and reference system (1.4) (Assumption A3).

Assumption A2. For every $i \le m$, $|g_j^i(r)| \le \Lambda_j^i r$ for all $r \in \mathbb{R}$. And there exist constants $\lambda_{11}^i, \lambda_{12}^i, \lambda_{13}^i, \lambda_{14}^i$, β_1^i, and positive definite continuous differentiable functions $V_1^i, W_1^i : \mathbb{R}^{n_i+1} \to \mathbb{R}$ such that

(1) $\lambda_{11}^i \|y\|^2 \le V_1^i(y) \le \lambda_{12}^i \|y\|^2, \quad \lambda_{13}^i \|y\|^2 \le W_1^i(y) \le \lambda_{14}^i \|y\|^2 \ \forall\, y \in \mathbb{R}^{n+1}$,

(2) $\displaystyle\sum_{j=1}^{n_i} (y_{j+1} - g_j^i(y_1)) \frac{\partial V_1^i}{\partial y_j}(y) - g_{n_i+1}^i(y_1) \frac{\partial V_1^i}{\partial y_{n_i+1}}(y) \le -W_1^i(y) \ \forall\, y \in \mathbb{R}^{n_i+1}$,

(3) $\max\left\{ \left| \dfrac{\partial V_1^i}{\partial y_{n_i}}(y) \right|, \ \left| \dfrac{\partial V_1^i}{\partial y_{n_i+1}}(y) \right| \right\} \le \beta_1^i \|y\| \ \forall\, y \in \mathbb{R}^{n_i+1}$.

Assumption A3. For every $1 \le i \le m$, ϕ_i is globally Lipschitz continuous with Lipschitz constant L_i: $|\phi_i(x) - \phi_i(y)| \le L_i\|x-y\|$ for all $x, y \in \mathbb{R}^{n_i}$. And there exist constants $\lambda_{21}^i, \lambda_{22}^i, \lambda_{23}^i, \lambda_{24}^i$, β_2^i, and positive definite continuous differentiable functions $V_2^i, W_2^i : \mathbb{R}^{n_i} \to \mathbb{R}$ such that

$$
\begin{aligned}
&(1)\quad \lambda_{21}^i\|y\|^2 \le V_2^i(y) \le \lambda_{22}^i\|y\|^2, \quad \lambda_{23}^i\|y\|^2 \le W_2^i(y) \le \lambda_{24}^i\|y\|^2,\\
&(2)\quad \sum_{j=1}^{n_i-1} y_{j+1}\frac{\partial V_2^i}{\partial y_j}(y) + \phi_i(y_1, y_2, \ldots, y_{n_i})\frac{\partial V_2^i}{\partial y_{n_i}}(y) \le -W_2^i(y),\\
&(3)\quad \left|\frac{\partial V_2^i}{\partial y_{n_i}}\right| \le \beta_2^i\|y\| \;\forall\; y = (y_1, y_2, \ldots, y_{n_i}) \in \mathbb{R}^{n_i}.
\end{aligned}
$$

Throughout the paper, the following notation will be used without specifying every time:

$$
\tilde{x}^i = (\hat{x}_1^i, \hat{x}_2^i, \ldots, \hat{x}_{n_i}^i)^\top,\; \hat{x}^i = (\hat{x}_1^i, \hat{x}_2^i, \ldots, \hat{x}_{n_i+1}^i)^\top,\; \tilde{x} = (\tilde{x}^{1\top}, \ldots, \tilde{x}^{m\top})^\top,
$$

$$
e_j^i(t) = \frac{x_j^i(t) - \hat{x}_j^i(t)}{\varepsilon^{n_i+1-j}}, \quad 1 \le j \le n_i+1, 1 \le i \le m, \tag{1.6}
$$

$$
e^i = (e_1^i, \ldots, e_{n_i+1}^i)^\top,\; e = (e^{1\top}, \ldots, e^{m\top})^\top,\; \eta = x - z,\; \eta^i = x^i - z^i;
$$

$$
V_1 : \mathbb{R}^{2n+m} \to \mathbb{R}, \quad V_1(e) = \sum_{i=1}^m V_1^i(e^i), \quad V_2 : \mathbb{R}^{2n} \to \mathbb{R}, \quad V_2(\eta) = \sum_{i=1}^m V_2^i(\eta^i). \tag{1.7}
$$

And $u_i^*, x_{n_i+1}^i$ will be specified later in different cases.

We proceed as follows. In section 2, we design, using a saturated function, the feedback control to insure that the system states stay in a prescribed compact set. A semiglobal convergence for the closed loop is presented. In section 3, we give a feedback control without the prior assumptions on the boundary of the initial values that are used in section 2. A global separation principle is established under stronger condition for the system functions than that used in section 2. In section 4, we present a special ESO and feedback control based on some information on system functions. The convergence proof is also presented. The last section is devoted to examples and numerical simulations that are used to illustrate the efficiency of ADRC. As a special case, the ADRC is applied to a class of linear MIMO, and is compared to the internal model principle.

2. Semiglobal convergence of ADRC. In this section, we assume that the initial values of system (1.1) lie in a compact set. This information is used to construct a saturated feedback control to avoid the peaking problem caused by the high gain in the ESO.

Assumption AS1. There are constants C_1, C_2 such that $\|x(0)\| < C_2$, $\|(w(t), \dot{w}(t))\| < C_3$.

Let $C_1^* = \max_{\{y \in \mathbb{R}^n,\ \|y\| \le C_1 + C_2\}} V_2(y)$. The following assumption is to guarantee the input-to-state stablilty for zero dynamics (see [18]).

Assumption AS2. There exist positive definite functions $V_0, W_0 : \mathbb{R}^s \to \mathbb{R}$ such that $L_{F_0}V_0(\xi) \le -W_0(\xi)$ for all $\xi : \|\xi\| > \chi(x, w)$, where $\chi : \mathbb{R}^{n+m} \to \mathbb{R}$ is a wedge function, and $L_{F_0}V_0(\xi)$ denotes the Lie derivative along the zero dynamics in system (1.1).

Set

$$
\begin{aligned}
&\max\left\{\sup_{\|x\|\le C_1+(C_1^*+1)/(\min\lambda_{23}^i)+1,\|w\|\le C_3,}|\chi(x,w)|,\ \|\xi(0)\|\right\}\le C_4,\\
&M_1\ge 2\Bigg(1+M_2+C_1\\
&\qquad+\max_{1\le i\le m}\sup_{\|x\|\le C_1+(C_1^*+1)/(\min\lambda_{23}^i)+1,\|w\|\le C_3,\|\xi\|\le C_4}|f_i(x,\xi,w_i)|\Bigg),\\
&M_2\ge \max_{\|x\|\le C_1+(C_1^*+1)/(\min\lambda_{23}^i)+1}|\phi_i(x)|.
\end{aligned} \tag{2.1}
$$

The following assumption is for the control parameters.

Assumption AS3. For each $a_{ij}(x,\xi,w_i)$, there exists a nominal parameter function $b_{ij}(x)$ such that the following hold.

(i) The matrix with entries b_{ij} are globally invertible with inverse matrix given by

$$
\begin{pmatrix} b_{11}^*(x) & b_{12}^*(x) & \cdots & b_{1m}^*(x)\\ b_{21}^*(x) & b_{22}^*(x) & \cdots & b_{2m}^*(x)\\ \vdots & \vdots & \ddots & \vdots\\ b_{m1}^*(x) & b_{m2}^*(x) & \cdots & b_{mm}^*(x)\end{pmatrix}=\begin{pmatrix} b_{11}(x) & b_{12}(x) & \cdots & b_{1m}(x)\\ b_{21}(x) & b_{22}(x) & \cdots & b_{2m}(x)\\ \vdots & \vdots & \ddots & \vdots\\ b_{m1}(x) & b_{m2}(x) & \cdots & b_{mm}(x)\end{pmatrix}^{-1}. \tag{2.2}
$$

(ii) For every $1\le i,j\le m$, b_{ij}, b_{ij}^*, and all partial derivatives of b_{ij} and b_{ij}^* with respect to their arguments are globally bounded.

(iii)

$$
\begin{aligned}
\vartheta&=\max_{1\le i\le m}\sup_{\|x\|\le C_1+(C_1^*+1)/(\min\lambda_{23}^i)+1,\|\xi\|\le C_4,\|w\|\le C_3,\nu\in\mathbb{R}^n}|a_{ij}(x,\xi,w)-b_{ij}(x)|\left|b_{ij}^*(\nu)\right|\\
&<\min_{1\le i\le m}\left\{\frac{1}{2},\ \lambda_{13}^i\left(m\beta_1^i\Lambda_{n_i+1}^i\left(M_1+\frac{1}{2}\right)\right)^{-1}\right\}.
\end{aligned} \tag{2.3}
$$

Let $\mathrm{sat}_M:\mathbb{R}\to\mathbb{R}$ be an odd continuous differentiable saturated function defined as follows (see [5]):

$$
\mathrm{sat}_M(r)=\begin{cases} r, \quad 0\le r\le M,\\ -\frac{1}{2}r^2+(M+1)r-\dfrac{1}{2}M^2, \quad M<r\le M+1,\\ M+\frac{1}{2}, \quad r>M+1,\end{cases} \tag{2.4}
$$

where $M>0$ is some constant.

The feedback control is designed as:

$$
\text{ADRC(S):}\ \begin{cases} u_i^*=-\mathrm{sat}_{M_1}(\hat{x}_{n_i+1}^i)+\mathrm{sat}_{M_2}(\phi_i(\tilde{x}^i-z^i))+z_{n_i+1}^i,\\ u_i=\displaystyle\sum_{k=1}^m b_{ik}^*(\tilde{x})u_k^*.\end{cases} \tag{2.5}
$$

The roles played by the different terms in control design (2.5) are as follows: $\hat{x}_{n_i+1}^i$ is to compensate the total disturbance; $x_{n_i+1}^i=f_i(x,\xi,w_i)+\sum_{j=1}^m(a_{ij}(x,\xi,w_i)-b_{ij}(x))u_j$, $\phi_i(\tilde{x}^i-z^i)+z_{n_i+1}^i$ is to guarantee the output tracking; and $\hat{x}_{n_1+1}^i$, $\phi_i(\hat{x}^i-z^i)$ are bounded by using $\mathrm{sat}_{M_1}, \mathrm{sat}_{M_2}$, respectively, to limit the peak value in control signal. Since both the cancellation and estimation are proceeding online for a particular operation, the control signal in the ADRC does not need to be unnecessarily large. That means the ADRC would spend less energy in control in order to cancel the effect of the disturbance ([26])

Under the feedback (2.5), the closed loop of system (1.1) and ESO (1.5) is rewritten as

(2.6)
$$\begin{cases} \dot{x}^i(t) = A_{n_i}x^i(t) + B_{n_i}\Big[f_i(x(t),\xi(t),w_i(t)) + \displaystyle\sum_{j=1}^m a_{ij}(x(t),\xi(t),w(t))u_j(t)\Big], \\ \dot{\xi}(t) = F_0(x(t),\xi(t),w(t)), \\ \dot{\hat{x}}^i(t) = A_{n_i+1}\hat{x}^i(t) + \begin{pmatrix} B_{n_i} \\ 0 \end{pmatrix} u_i^*(t) + \begin{pmatrix} \varepsilon^{n_i-1}g_1^i(e_1^i(t)) \\ \vdots \\ \frac{1}{\varepsilon}g_{n_i+1}^i(e_1^i(t)) \end{pmatrix}, \\ u_i^*(t) = -\mathrm{sat}_{M_1}(\hat{x}^i_{n_i+1}(t)) + \mathrm{sat}_{M_2}(\phi_i(\tilde{x}^i(t) - z^i(t))) + z^i_{n_i+1}(t), \\ u_i(t) = \displaystyle\sum_{k=1}^m b_{ik}^*(\tilde{x}(t))u_k^*(t). \end{cases}$$

Our first main result is stated as Theorem 2.1 below.

THEOREM 2.1. *Assume that Assumptions* A1–A3 *and* AS1–AS3 *are satisfied. Let an ε-dependent solution of* (2.6) *be* $(x(t,\varepsilon),\hat{x}(t,\varepsilon))$. *Then for any $\sigma > 0$, there exists an $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0,\varepsilon_0)$, there exists an ε-independent constant $t_0 > 0$ such that*

$$|\tilde{x}(t,\varepsilon) - x(t,\varepsilon)| \le \sigma \text{ for all } t > t_0 \tag{2.7}$$

and

$$\overline{\lim}_{t\to\infty}\|x(t,\varepsilon) - z(t)\| \le \sigma. \tag{2.8}$$

The proof of Theorem 2.1 is based on the boundedness of the solution stated in Lemma 2.2.

LEMMA 2.2. *Assume that Assumptions* A1, A2, AS1, AS2 *are satisfied. Let* $\Omega_0 = \{y|V_2(y) \le C_1^*\}$, $\Omega_1 = \{y|V_2(y) \le C_1^*+1\}$. *Then there exists an $\varepsilon_1 > 0$ such that for any $\varepsilon \in (0,\varepsilon_1)$, and $t \in [0,\infty)$, $\eta(t,\varepsilon) \in \Omega_1$.*

Proof. First, we see that for any $\varepsilon > 0$,

$$\begin{aligned} &|\eta_j^i(t,\varepsilon)| \le |\eta_j^i(0)| + |\eta_{j+1}^i(t,\varepsilon)|t, \ 1 \le j \le n_i - 1, \ 1 \le i \le m, \\ &|\eta_{n_i}^i(t,\varepsilon)| \le |\eta_{n_i}^i(0)| + [C_1 + M_1 + mM_1^*(C_1 + M_1 + M_2)]t, \quad \eta(t,\varepsilon) \in \Omega_1, \end{aligned} \tag{2.9}$$

where

$$M_1^* = \max_{1\le i,j\le m} \sup_{\|x\|\le C_1 + (C_1^*+1)/(\min \lambda_{23}^i)+1, \|w\|\le \bar{C}_3, \|\xi\|\le \bar{C}_4} |a_{ij}(x,\xi,w)|.$$

Next, by an iteration process, we can show that all terms on the right-hand side of (2.9) are ε-independent. Since $\|\eta(0)\| < C_1 + C_2$, $\eta(0) \in \Omega_0$, there exists an ε-independent constant $t_0 > 0$ such that $\eta(t,\varepsilon) \in \Omega_0$ for all $t \in [0,t_0]$.

Lemma 2.2 is finally proved by a contradiction. Suppose that Lemma 2.2 is false. Then for any $\varepsilon > 0$, there exist an $\varepsilon^* \in (0, \varepsilon)$ and $t^* \in (0, \infty)$ such that

$$\eta(t^*, \varepsilon) \in \mathbb{R}^n - \Omega_1. \tag{2.10}$$

Since for any $t \in [0, t_0]$, $\eta(t, \varepsilon^*) \in \Omega_0$, and η is continuous in t, there exists a $t_1 \in (t_0, t_2)$ such that

$$\begin{aligned}
&\eta(t_1, \varepsilon) \in \partial\Omega_0 \text{ or } V_2(\eta(t_1, \varepsilon)) = C_1^*,\\
&\|\eta(t_2, \varepsilon)\| \in \Omega_1 - \Omega_0 \text{ or } C_1^* < V_2(\eta(t_2, \varepsilon)) \leq C_1^* + 1,\\
&\eta(t, \varepsilon) \in \Omega_1 - \Omega_0^\circ \ \forall\, t \in [t_1, t_2] \text{ or } C_1^* \leq V_2(\eta(t, \varepsilon)) \leq C_1^* + 1,\\
&\eta(t, \varepsilon) \in \Omega_1 \ \forall\, t \in [0, t_2].
\end{aligned} \tag{2.11}$$

By (1.1) and (1.6), it follows that the errors e^i in this case satisfy

$$\varepsilon \dot{e}^i(t) = A_{n_i+1} e^i(t) + \Delta_{i1} \begin{pmatrix} B_{n_i} \\ 0 \end{pmatrix} + \varepsilon \Delta_{i2} B_{n_i+1} - \begin{pmatrix} g_1^i(e_1^i(t)) \\ \vdots \\ g_{n_i+1}^i(e_1^i(t)) \end{pmatrix}, \ 1 \leq i \leq m, \tag{2.12}$$

where

$$\begin{aligned}
\Delta_{i1} &= \sum_{j=1}^m \big(b_{ij}(x) - b_{ij}(\tilde{x})\big) u_j,\\
\Delta_{i2} &= \frac{d}{dt}\left(f_i(x, \xi, w_i) + \sum_{j=1}^m \big(a_{ij}(x, \xi, w_i) - b_{ij}(x)\big) u_j \right)\Bigg|_{\text{along (2.6)}}.
\end{aligned} \tag{2.13}$$

Since all derivatives of b_{ij} are globally bounded, there exists a constant $N_0 > 0$ such that$|\Delta_{i1}| \leq \varepsilon N_0 \|e\|$.

We define two vector fields of x by

$$F_i(x^i) = \begin{pmatrix} x_2^i \\ x_3^i \\ \vdots \\ x_{n_i+1}^i + f_i(x, \xi, w_i) + \displaystyle\sum_{j=1}^m a_{ij}(x, \xi, w_j) u_j - u_i^* \end{pmatrix}, \tag{2.14}$$

$$F(x) = \big(F_1(x^1)^\top, F_2(x^2)^\top, \ldots, F_m(x^m)^\top\big)^\top; \tag{2.15}$$

$$\hat{F}_i(\tilde{x}^i) = \begin{pmatrix} \hat{x}_2^i + \varepsilon^{n_i-1} g_1^i(e_1^i) \\ \hat{x}_3^i + \varepsilon^{n_i-2} g_2^i(e_1^i) \\ \vdots \\ \hat{x}_{n_i+1}^i + g_{n_i}^i(e_1^i) + u_i^* \end{pmatrix}, \tag{2.16}$$

$$\hat{F}(\tilde{x}) = \left(\hat{F}_1(\tilde{x}^1)^\top, \hat{F}_2(\tilde{x}^2)^\top, \ldots, \hat{F}_m(\tilde{x}^m)^\top\right)^\top.$$

Considering the derivative of $x^i_{n_i+1}$ with respect to t in the interval $[t_1, t_2]$, we have

(2.17)

$$
\begin{aligned}
\Delta_{i2} &= \frac{d}{dt}\left(f_i(x,\xi,w_i) + \sum_{j=1}^{m}\Big(a_{ij}(x,\xi,w_j) - b_{ij}(x)\Big)u_j\right) \\
&= \left[L_{F(x)}f_i(x,\xi,w_i) + L_{F_0(\xi)}f_i(x,\xi,w_i) + \frac{\partial f_i}{\partial w_i}\dot{w}_i\right] \\
&\quad + \frac{d}{dt}\left(\sum_{j,l=1}^{m}\big(a_{ij}(x,\xi,w_j) - b_{ij}(x)\big)b^*_{jl}(\tilde{x})u^*_l\right) \\
&= \left[L_{F(x)}f_i(x,\xi,w_i) + L_{F_0(\xi)}f_i(x,\xi,w_i) + \frac{\partial f_i}{\partial w_i}\dot{w}_i\right] \\
&\quad + \sum_{j,l=1}^{m}\left(L_{F(x)}\Big(a_{ij}(x,\xi,w_j) - b_{ij}(x)\Big) + L_{F_0(\xi)}a_{ij}(x,\xi,w_j) + \frac{\partial a_{ij}}{\partial w_i}\dot{w}_i\right)b^*_{jl}(\tilde{x})u^*_l \\
&\quad + \sum_{j,l=1}^{m}\Big(a_{ij}(x,\xi,w_j) - b_{ij}(x)\Big)L_{\hat{F}(\tilde{x})}\big(b^*_{jl}(\tilde{x})\big)u^*_l \\
&\quad + \sum_{j,l=1}^{m}\Big(a_{ij}(x,\xi,w_j) - b_{ij}(x)\Big)b^*_{jl}(\tilde{x}) \\
&\qquad \times\left(-\frac{1}{\varepsilon}\dot{h}_{M_1}(\hat{x}^i_{n_i+1})g^i_{n_i+1}(e^i_1) + L_{\hat{F}_i(\tilde{x}^i)}\mathrm{sat}_{M_2}(\phi_i(\tilde{x}^i - z^i))\right. \\
&\qquad\quad \left. - \sum_{s=1}^{n_i} z_{s+1}\frac{\partial \mathrm{sat}_{M_2}\circ\phi_i}{\partial y_s}(\tilde{x}^i - z^i) + \dot{z}^i_{n_i+1}\right).
\end{aligned}
$$

By the assumptions, all $\|(w,\dot{w})\|$, $\|x\|$, $\|\xi\|$, $\|z\|$, and $|z^i_{n_i+1}|$ are bounded in $[t_1, t_2]$, we conclude that there exists a positive ε-independent number N_i such that for all $t \in [t_1, t_2]$,

(2.18)

$$
\begin{aligned}
&\Bigg|L_{F(x)}f_i(x,\xi,w_i) + L_{F_0(\xi)}f_i(x,\xi,w_i) + \frac{\partial f_i}{\partial w_i}\dot{w}_i \\
&+ \sum_{j,l=1}^{m}\left(L_{F(x)}\Big(a_{ij}(x,\xi,w_j) - b_{ij}(x)\Big) + L_{F_0(\xi)}a_{ij}(x,\xi,w_j) + \frac{\partial a_{ij}}{\partial w_i}\dot{w}_i\right)\times b^*_{jl}(\tilde{x})u^*_l\Bigg| \le N_1,
\end{aligned}
$$

$$
\left|\sum_{j,l=1}^{m}\left(a_{ij}(x,\xi,w_j) - b_{ij}(x)\right)L_{\hat{F}(\tilde{x})}b^*_{jl}(\tilde{x})u^*_l\right| \le N_2\|e\| + N_3, \tag{2.19}
$$

(2.20)

$$
\begin{aligned}
&\Bigg|\sum_{j,l=1}^{m}\Big(a_{ij}(x,\xi,w_j) - b_{ij}(x)\Big)b^*_{jl}(\tilde{x})\Big(-\frac{1}{\varepsilon}\dot{h}_{M_1}(\hat{x}^i_{n_i+1})g^i_{n_i+1}(e^i_1) + L_{\hat{F}_i(\tilde{x}^i)}\mathrm{sat}_{M_2}(\phi_i(\tilde{x}^i - z^i)) \\
&\qquad\qquad - \sum_{s=1}^{n_i} z_{s+1}\frac{\partial \mathrm{sat}_{M_2}\circ\phi_i}{\partial y_s}(\tilde{x}^i - z^i) + \dot{z}^i_{n_i+1}\Big)\Bigg| \\
&\le \frac{N}{\varepsilon}\|e^i\| + N_4\|e\| + N_5,
\end{aligned}
$$

where

$$
\begin{aligned}
N = {} & \Lambda^i_{n_i+1}\left(M_1+\frac{1}{2}\right) \\
& \times \max\nolimits_{1\le i\le m}\sup\nolimits_{\|x\|\le C_1+(C_1^*+1)/(\min\lambda^i_{23})+1,\|\xi\|\le C_4,\|w\|\le C_3,\nu\in\mathbb{R}^n} \\
& \sum_{j,l=1}^{m}\left|a_{ij}(x,\xi,w)-b_{ij}(x)\right|\left|b^*_{jl}(\nu)\right| = \Lambda^i_{n_i+1}\left(M_1+\frac{1}{2}\right)\vartheta.
\end{aligned}
\tag{2.21}
$$

Finding the derivative of V_1 along system (2.12) with respect to t shows that for any $0<\varepsilon<\min_{1\le i\le m}(\lambda^i_{13}-N\beta^i_1)/((N_0+N_2+N_4)\max_{1\le i\le m}\beta^i_1)$ and $t\in[0,t_2]$,

$$
\begin{aligned}
& \left.\frac{dV_1(e)}{dt}\right|_{\text{along (2.12)}} \\
& \le \sum_{i=1}^{m}\left\{-\frac{1}{\varepsilon}W^i_1\left(e^i\right)+\beta^i_1\|e^i\|\left(N_0\|e\|+N_1+N_2\|e\|+N_3+N_5\right.\right. \\
& \qquad\qquad \left.\left. +N_4\|e\|+\frac{N}{\varepsilon}\|e^i\|\right)\right\}, \\
& \le -\left(\frac{1}{\varepsilon}\min_{1\le i\le m}\left(\lambda^i_{13}-N\beta^i_1\right)-(N_0+N_2+N_4)\max_{1\le i\le m}\left(\beta^i_1\right)\right)\|e\|^2 \\
& \quad + m\max\beta^i_1(N_1+N_3+N_5)\|e\| \\
& \le -\frac{1}{\max\{\lambda^i_{12}\}}\left(\frac{1}{\varepsilon}\min_{1\le i\le m}\left(\lambda^i_{13}-N\beta^i_1\right)-(N_0+N_2+N_4)\right. \\
& \qquad \left.\times\max_{1\le i\le m}\left(\beta^i_1\right)\right)V_1(e)+\frac{m\max\beta^i_1(N_1+N_3+N_5)}{\sqrt{\lambda^i_{12}}}\sqrt{V_1(e)}.
\end{aligned}
\tag{2.22}
$$

Hence for any $0<\varepsilon<\min_{1\le i\le m}(\lambda^i_{13}-N\beta^i_1)/((N_0+N_2+N_4)\max_{1\le i\le m}\beta^i_1)$ and $t\in[0,t_2]$, one has

$$
\frac{d}{dt}\sqrt{V_1(e)}\le-\left(\frac{\Pi_1}{\varepsilon}-\Pi_2\right)\sqrt{V_1(e)}+\Pi_3, \tag{2.23}
$$

where

$$
\begin{aligned}
& \Pi_1=\frac{\min\left(\lambda^i_{13}-N\beta^i_1\right)}{\max\{\lambda^i_{12}\}},\ \Pi_2=\frac{(N_0+N_2+N_4)\max\left(\beta^i_1\right)}{\max\{\lambda^i_{12}\}}, \\
& \Pi_3=\frac{m\max\beta^i_1(N_1+N_3+N_5)}{\sqrt{\lambda^i_{12}}}.
\end{aligned}
\tag{2.24}
$$

Therefore, for every $0<\varepsilon<\min_{1\le i\le m}(\lambda^i_{13}-N\beta^i_1)/((N_0+N_2+N_4)\max_{1\le i\le m}\beta^i_1)$ and $t\in[0,t_2]$, we have

$$
\begin{aligned}
\|e\| \le \frac{1}{\sqrt{\lambda^i_{11}}}\sqrt{V_1(e)} \le \frac{1}{\sqrt{\lambda^i_{11}}}\Bigg[& e^{(-\Pi_1/\varepsilon+\Pi_2)t}\sqrt{V_1(e(0))} \\
& +\Pi_3\int_0^t e^{(-\Pi_1/\varepsilon+\Pi_2)(t-s)}ds\Bigg].
\end{aligned}
\tag{2.25}
$$

Passing to the limit as $\varepsilon\to 0$ yields, for any $t\in[t_1,t_2]$, that

$$
\begin{aligned}
e^{(-\Pi_1/\varepsilon+\Pi_2)t}\sqrt{V_1(e(0))} \le {} & \frac{1}{\sqrt{\min\{\lambda^i_{11}\}}}e^{(-\Pi_1/\varepsilon+\Pi_2)t} \\
& \times\sum_{i=1}^{m}\left\|\left(\frac{e_{i1}}{\varepsilon^{n_i+1}},\frac{e_{i2}}{\varepsilon^{n_i}},\ldots,e_{i(n_i+1)}\right)\right\|\to 0.
\end{aligned}
\tag{2.26}
$$

Hence for any $\sigma \in (0, \min\{1/2, \lambda_{23}^i(C_1+C_2)/(mN_6)\})$, there exists an $\varepsilon_1 \in (0,1)$ such that $\|e\| \le \sigma$ for all $\varepsilon \in (0,\varepsilon_1)$ and $t \in [t_1,t_2]$, where $N_6 = \max_{1\le i\le m}\{\beta_2^i(1+\hat{L}_i)\}$ and $\hat{L}_i$ is the Lipschitz constant of ϕ_i.

Notice that for any $0<\varepsilon<\varepsilon_1$ and $t\in[t_1,t_2]$, $\eta\in\Omega_1$, $\|e\|\le\sigma$,

$$
\begin{aligned}
\|\tilde{x}^i - z^i\| &\le \|x-\tilde{x}^i\| + \|x^i - z^i\| \le (C_1^*+1)/\min\lambda_{23}^i + 1, |\phi_i(\tilde{x}^i - z^i)| \\
&\le M_2, \\
|\hat{x}_{n_i+1}^i| &\le |e_{n_i+1}^i| + |x_{n_i+1}^i| \le |e_{n_i+1}^i| \\
&\quad + \left| f_i(x,\xi,w_i)\sum_{j=1}^m (a_{ij}(x,\xi,w_i) - b_{ij}(x))u_i \right| \\
&\le |e_{n_i+1}^i| + |f_i(x,\xi,w_i)| + \vartheta(M_1+M_2+C_2) \\
&\le 1 + M_2 + C_1 + |f_i(x,\xi,w_i)| + \vartheta M_1 \le M_1.
\end{aligned} \tag{2.27}
$$

So u_i^* in (2.6) takes the form $u_i^* = \hat{x}_{n_i+1}^i + \phi_i(\tilde{x}^i - z^i) + z_{n_i+1}^i$ for all $t\in[t_1,t_2]$. With this u_i^*, the derivative of V_2, along system (2.6) with respect to t in interval $[t_1,t_2]$, satisfies

$$
\begin{aligned}
\frac{d}{dt}V_2(\eta) &= \sum_{i=1}^m \left(-W_2^i(\eta^i(t)) + N_6\sigma\|\eta^i\|\right) \\
&\le -\min_{1\le i\le m}\{\lambda_{23}^i\}\|\eta\|^2 + mN_6\|e\|\|\eta\| < 0,
\end{aligned} \tag{2.28}
$$

which contradicts (2.11). And the proof is complete. $\square$

Proof of Theorem 2.1. From Lemma 2.2, $\eta(t,\varepsilon)\in\Omega_1$ for all $\varepsilon\in(0,\varepsilon_1)$ and $t\in(0,\infty)$, it follows that (2.27) holds true for all $t\in[0,\infty)$. Therefore (2.28) and (2.22) also hold true for any $\varepsilon\in(0,\varepsilon_1)$ and $t\in[0,\infty)$.

For any $\sigma>0$, it follows from (2.28) that there exists a $\sigma_1\in(0,\sigma/2)$ such that $\overline{\lim}_{t\to\infty}\|\eta(t,\varepsilon)\| \le \sigma/2$ provided that $\|e(t,\varepsilon)\|\le\sigma_1$. From (2.22), for any $\tau>0$ and this determined $\sigma_1>0$, there exists an $\varepsilon_0\in(1,\varepsilon_1)$ such that $\|x(t,\varepsilon)-\hat{x}(t,\varepsilon)\|\le\sigma_1$ for any $\varepsilon\in(0,\varepsilon_0)$, $t>\tau$. This completes the proof. $\square$

Remark 2.1. From Theorem 2.1, we can deduce the conclusion of [5] where the output stabilization for a class of SISO system with linear ESO is used. This is just to let $m=1$ in (1.1), $g_i(r)=r$ in (1.5), and all the reference signals $v_i\equiv 0$.

3. Global convergence of ADRC. In the last section, we develop the semi-global convergence for nonlinear ADRC. The advantage of this result is that the peaking problem can be effectively alleviated by introducing the saturation function in the control. However, the saturation function depends on the bound of initial values. When this bound is not available, we need the global convergence. The price in this case is probably the peaking problem, and more restricted assumptions as well.

Assumption GA1. For every $1\le i\le m$, all partial derivatives of f_i are bounded over $\mathbb{R}^{n+m}$, where $n=n_1+\cdots+n_m$.

Assumption GA2. For every $1\le i,j\le m$, $a_{ij}(x,\xi,w_i)=a_{ij}(w_i)$ and there exist constant nominal parameters b_{ij} such that the matrix with entry b_{ij} is invertible:

$$
\begin{pmatrix}
b_{11}^* & b_{12}^* & \cdots & b_{1m}^* \\
b_{21}^* & b_{22}^* & \cdots & b_{2m}^* \\
\vdots & \vdots & \ddots & \vdots \\
b_{m1}^* & b_{m2}^* & \cdots & b_{mm}^*
\end{pmatrix}
=
\begin{pmatrix}
b_{11} & b_{12} & \cdots & b_{1m} \\
b_{21} & b_{22} & \cdots & b_{2m} \\
\vdots & \vdots & \ddots & \vdots \\
b_{m1} & b_{m2} & \cdots & b_{mm}
\end{pmatrix}^{-1}.
$$

Moreover,

$$\min\{\lambda_{13}^i\} - \sqrt{m} \sum_{i,k,l=1}^{m} \beta_1^i \sup_{t\in[0,\infty)} |a_{ik}(w_i(t)) - b_{ik}|b_{kl}^* \Lambda_{n_l+1}^l > 0 \;\; \forall\, t \in [0,\infty). \tag{3.1}$$

Assumption GA3. For zero dynamics, there exist positive constants K_1, K_2 such that $\|F_0(x,\xi,w)\| \le K_1 + K_2(\|x\| + \|w\|)$.

The observer based feedback control is then designed as

$$\text{ADRC(G)}: \begin{cases} u_i^* = \phi_i(\tilde{x}^i - z^i) + z_{n_i+1}^i - \hat{x}_{n_i+1}^i, \\ u_i = \displaystyle\sum_{j=1}^{m} b_{ij}^* u_j. \end{cases} \tag{3.2}$$

It is seen that in feedback control (3.2), $\hat{x}_{n_i+1}^i$ is used to compensate the uncertainty $x_{n_i+1}^i = f_i(x,\xi,w_i) + \sum_{j=1}^{m}(a_{ij}(w_i) - b_{ij})u_j$ and $\phi_i(\tilde{x}^i - z^i) + z_{n_1+1}^i$ is used to guarantee the output tracking.

The closed loop of system (1.1) under ESO (1.5) and ADRC (3.2) becomes

$$\text{Closed-loop}: \begin{cases} \dot{x}^i = A_{n_i} x^i + B_{n_i}\left(f_i(x,\xi,w_i) + \displaystyle\sum_{k=1}^{m} a_{ik} u_k\right), \\ \dot{\hat{x}}^i = A_{n_i+1}\hat{x}^i + \begin{pmatrix} \varepsilon^{n_i-1} g_1^i(e_1^i) \\ \vdots \\ \dfrac{1}{\varepsilon} g_{n_i+1}^i(e_1^i) \end{pmatrix} + \begin{pmatrix} B_{n_i}, \\ 0 \end{pmatrix} u_i^*, \\ u_i = \displaystyle\sum_{k=1}^{m} b_{ik}^* u_k^*, \quad u_i^* = \phi_i(\tilde{x}^i - z^i) + z_{n_i+1}^i - \hat{x}_{n_i+1}^i. \end{cases} \tag{3.3}$$

THEOREM 3.1. *Assume that Assumptions* A1–A3 *and* GA1–GA3 *are satisfied. Let $x(t,\varepsilon)$, $\hat{x}(t,\varepsilon)$ be the ε-dependent solutions of* (3.3). *Then there exists a constant $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0,\varepsilon_0)$, there exists an ε and the initial value dependent constant $t_\varepsilon > 0$ such that for all $t > t_\varepsilon$,*

$$|x_j^i(t,\varepsilon) - \hat{x}_j^i(t,\varepsilon)| \le \Gamma_1 \varepsilon^{n_i+2-j}, \quad 1 \le j \le n_i+1,\; 1 \le i \le m, \tag{3.4}$$

and

$$\|x_j^i - z_j^i\| \le \Gamma_2 \varepsilon, \quad 1 \le j \le n_i,\; 1 \le i \le m, \tag{3.5}$$

where Γ_1, Γ_2 are constants independent of ε and the initial value. However, they are dependent on the bound of $\|z^i\|$ and $\|(w,\dot{w})\|$.

Proof. Using the notation of η^i, e^i in (1.6), we get the error equation as follows:

$$\begin{cases} \dot{\eta}^i = A_{n_i}\eta^i + B_{n_i}[\phi_i(\eta^i) + e_{n_i+1}^i + (\phi_i(\tilde{x}^i - z^i) - \phi_i(x^i - z^i))], \\ \varepsilon \dot{e}^i = A_{n_i+1} e^i + \varepsilon \bar{\Delta}_i B_{n_i+1} - \begin{pmatrix} g_1^i(e_1^i) \\ \vdots \\ g_{n_i+1}^i(e_1^i) \end{pmatrix}. \end{cases} \tag{3.6}$$

Let

$$\bar{\Delta}_i = \frac{d}{dt}\bigg|_{\text{along (3.3)}} \left[f_i(x,\xi,w_i) + \sum_{k=1}^{m} a_{ik}(w_i)u_k - u_i^* \right] \\ = \frac{d}{dt}\bigg|_{\text{along (3.3)}} \left[f_i(x,\xi,w_i) + \sum_{k,l=1}^{m} (a_{ik}(w_i) - b_{ik}) b_{kl}^* \Big(\phi_l(\tilde{x}^l - z^l) + z_{n_l+1}^l - \hat{x}_{n_l+1}^l \Big) \right]. \tag{3.7}$$

A straightforward computation shows that

$$\begin{aligned} \bar{\Delta}_i = & \sum_{s=1}^{m} \sum_{j=1}^{n_s-1} x_{j+1}^s \frac{\partial f_i}{\partial x_j^s}(x,\xi,w_i) \\ & + \sum_{s,k,l=1}^{m} b_{ik} b_{kl}^* \left(\phi_l\left(\tilde{x}^l - z^l\right) + z_{n_l+1}^l - \hat{x}_{n_l+1}^l \right) \frac{\partial f_i}{\partial x_{n_s}^s}(x,w_i) \\ & + \dot{w}_i \frac{\partial f_i}{\partial w_i}(x,w_i) + L_{F_0(\xi)} f_i(x,\xi,w_i) \\ & + \sum_{k,l=1}^{m} (a_{ik} - b_{ik}) b_{kl}^* \left\{ \sum_{j=1}^{n_l} \left(\tilde{x}_{j+1}^l - z_{j+1}^l - \varepsilon^{n_l-j} g_j^l(e_1^l) \right) \frac{\partial \phi_l}{\partial y_j}(\tilde{x}^l - z^l) \right\} \\ & + \sum_{k,l=1}^{m} (a_{ik} - b_{ik}) b_{kl}^* \left\{ \dot{z}_{n_l+1}^l - \frac{1}{\varepsilon} g_{n_l+1}^l(e_1^l) \right\} \\ & + \sum_{k,l=1}^{m} \dot{a}_{ik}(w_i) \dot{w}_i b_{kl}^* \left(\phi_l(\tilde{x}^l - z^l) + z_{n_l+1}^l - \hat{x}_{n_l+1}^l \right). \end{aligned} \tag{3.8}$$

It follows that

$$\begin{aligned} & |\bar{\Delta}_i| \le \Xi_0^i + \Xi_1^i \|e\| + \Xi_2^j \|\eta\| + \frac{\Xi^i}{\varepsilon}\|e\|, \\ & \Xi^i = \sqrt{m} \sum_{k,l=1}^{m} \sup_{t\in[0,\infty} |a_{ik}(w_i(t)) - b_{ik}| b_{kl}^* \Lambda_{n_l+1}^l, \end{aligned} \tag{3.9}$$

where $\Xi_0^i, \Xi_1^i, \Xi_2^i$ are ε-independent positive constants.

Construct the Lyapunov function $V: \mathbb{R}^{2n_1+\cdots+2n_m+m} \to \mathbb{R}$ for error system (3.6) as

$$V(\eta^1,\ldots,\eta^m,e^1,\ldots,e^m) = \sum_{i=1}^{m} [V_1^i(e^i) + V_2^i(\eta^i)]. \tag{3.10}$$

The derivative of V along the solution of (3.6) is computed as

$$\begin{aligned} \frac{dV}{dt}\bigg|_{\text{along (3.6)}} = \sum_{i=1}^{m} \Bigg\{ & \frac{1}{\varepsilon} \left[\sum_{j=1}^{n_i} (e_{j+1}^i - g_j^i(e_1^i)) \frac{\partial V_1^i}{\partial e_j^i}(e^i) - g_{n_i+1}^i(e_1^i) \frac{\partial V_1^i}{\partial e_{n+1}}(e^i) \right] \\ & + \bar{\Delta}_i \frac{\partial V_1^i}{\partial e_{n_i+1}^i}(e^i) + \sum_{j=1}^{n_i-1} \eta_{j+1}^i \frac{\partial V_2^i}{\partial \eta_j^i}(\eta^i) \\ & + \{\phi_i(\eta^i) + e_{n_i+1}^i + [\phi_i(\tilde{x}^i - z^i) - \phi_i(x^i - z^i)]\} \frac{\partial V_2^i}{\partial x_{n_i}^i}(\eta^i) \Bigg\}. \end{aligned} \tag{3.11}$$

It follows from Assumptions A2 and A3 that

$$\left.\frac{dV}{dt}\right|_{\text{along (3.6)}} \le \sum_{i=1}^{m}\left\{-\frac{1}{\varepsilon}W_1^i(e^i)+\beta_1^i\|e^i\|\left(\Xi_0^i+\Xi_1^i\|e\|+\Xi_2^i\|\eta\|+\frac{\Xi^i}{\varepsilon}\|e\|\right)\right. \\ \left. - W_2^i(\eta^i)+\beta_2^i(L_i+1)\|e^i\|\|\eta^i\|\right\}. \tag{3.12}$$

This together with Assumptions A2 and A3 gives

$$\begin{aligned}\left.\frac{dV}{dt}\right|_{\text{along (3.6)}} &\le \sum_{i=1}^{m}\left\{-\frac{\lambda_{13}^i}{\varepsilon}\|e^i\|^2+\beta_1^i\|e^i\|\left(\Xi_0^i+\Xi_1^i\|e\|+\Xi_2^i\|\eta\|+\frac{\Xi^i}{\varepsilon}\|e\|\right)\right.\\ &\qquad\left. -\lambda_{23}^i\|\eta^i\|^2+\beta_2^i(L_i+1)\|e^i\|\|\eta^i\|\right\}\\ &\le -\left(\frac{1}{\varepsilon}\left(\min\{\lambda_{13}^i\}-\sum_{i=1}^{m}\beta_1^i\Xi^i\right)-\sum_{i=1}^{m}\beta_1^i\Xi_1^i\right)\|e\|^2\\ &\quad+\left(\sum_{i=1}^{m}\beta_1^i\Xi_0^i\right)\|e\|-\min\{\lambda_{23}^i\}\|\eta\|^2+\sum_{i=1}^{m}\beta_2^i(L_i+1)\|e\|\eta\|.\end{aligned} \tag{3.13}$$

For notational simplicity, we denote

$$\begin{aligned}&\Pi_1=\min\{\lambda_{13}^i\}-\sum_{i=1}^{m}\beta_1^i\Xi^i,\ \ \Pi_2=\sum_{i=1}^{m}\beta_1^i\Xi_1^i,\\ &\Pi_3=\sum_{i=1}^{m}\beta_1^i\Xi_0^i,\ \ \Pi_4=\sum_{i=1}^{m}\beta_2^i(L_i+1),\ \ \lambda=\min\{\lambda_{23}^i\},\end{aligned} \tag{3.14}$$

and rewrite inequality (3.13) as

$$\left.\frac{dV}{dt}\right|_{\text{along (3.6)}} \le -\left(\frac{\Pi_1}{\varepsilon}-\Pi_2\right)\|e\|^2+\Pi_3\|e\|-\lambda\|\eta\|^2+\Pi_4\|e\|\|\eta\|. \tag{3.15}$$

Let $\varepsilon_1=\Pi_1/(2\Pi_2)$. For every $\varepsilon\in(0,\varepsilon_1)$, $\Pi_2=\Pi_1/(2\varepsilon_1)\le\Pi_1/(2\varepsilon)$ and

$$\Pi_4\|e\|\|\eta\|=\sqrt{\frac{\Pi_1}{4\varepsilon}}\|e\|\sqrt{\frac{4\varepsilon}{\Pi_1}}\Pi_2\|\eta\|\le\frac{\Pi_1}{4\varepsilon}\|e\|^2+\frac{4\varepsilon\Pi_2^2}{\Pi_1}\|\eta\|^2. \tag{3.16}$$

Hence (3.15) can be estimated further as

$$\left.\frac{dV}{dt}\right|_{\text{along (3.6)}} \le -\frac{\Pi_1}{4\varepsilon}\|e\|^2+\Pi_3\|e\|-\left(\lambda-4\varepsilon\frac{\Pi_2^2}{\Pi_1}\right)\|\eta\|^2. \tag{3.17}$$

Now we show that the solution of (3.6) is bounded when ε is sufficiently small. To this purpose, let

$$R=\max\left\{2,\frac{2\Pi_3}{\lambda}\right\},\ \varepsilon_0=\min\left\{1,\varepsilon_1,\frac{\Pi_1}{4\Pi_3},\frac{\lambda\Pi_1}{8\Pi_2^2}\right\}. \tag{3.18}$$

For any $\varepsilon\in(0,\varepsilon_0)$ and $\|e,\eta\|\ge R$, we consider the derivative of V along the solution of (3.6) by two different cases.

Case 1. $\|e\| \geq R/2$. In this case, $\|e\| \geq 1$. A direct computation from (3.17), with the definition of ε_0 in (3.18), gives

$$\tag{3.19} \begin{aligned} \frac{dV}{dt}\bigg|_{\text{along (3.6)}} &\leq -\frac{\Pi_1}{4\varepsilon}\|e\|^2 + \Pi_3\|e\| \\ &\leq -\left(\frac{\Pi_1}{4\varepsilon} - \Pi_3\right)\|e\|^2 \leq -\left(\frac{\Pi_1}{4\varepsilon} - \Pi_3\right) < 0. \end{aligned}$$

Case 2. $\|e\| < R/2$. In this case, from $\|\eta\| + \|e\| \geq \|(e,\eta)\|$, $\|\eta| \geq \mathbb{R}/2$. By (3.12) and the definition of ε_0 in (3.18), we have

$$\tag{3.20} \begin{aligned} \frac{dV}{dt}\bigg|_{\text{along (3.6)}} &\leq \Pi_3\|e\| - \left(\lambda - 4\varepsilon\frac{\Pi_2^2}{\Pi_1}\right)\|\eta\|^2 \\ &\leq -\left(\lambda - 4\varepsilon\frac{\Pi_2^2}{\Pi_1}\right)R^2 + \Pi_3 R \leq -\left(\frac{\lambda}{2}R - \Pi_3\right)R \leq 0. \end{aligned}$$

Summarizing these two cases, we get that for each $\varepsilon \in (0,\varepsilon_0)$, there exist an ε and $\tau_\varepsilon > 0$ that depends on the initial value such that $\|(e,\eta)\| \leq R$ for all $t \in (T_\varepsilon,\infty)$. This together with (3.8) shows that $|\bar{\Delta}_i| \leq M_i + (\Xi_i/\varepsilon)\|e\|$ for all $t \in (T_\varepsilon,\infty)$, where M_i is an R-dependent constant.

Finding the derivative of V_1 along the solution of (3.6) with respect to t gives, for any $t > \tau_\varepsilon$, that

$$\tag{3.21} \begin{aligned} \frac{dV_1}{dt}\bigg|_{\text{along (3.6)}} &= \frac{1}{\varepsilon}\sum_{i=1}^{m}\Bigg\{\sum_{j=1}^{n_i}\left(e_{j+1}^i - g_j^i(e_1^i)\right)\frac{\partial V_1^i}{\partial e_j^i}\left(e^i\right) \\ &\qquad + \left(\varepsilon\bar{\Delta}_i - g_{n_i+1}^i\left(e_1^i\right)\right)\frac{\partial V_1^i}{\partial e_{n_i+1}^i}\left(e^i\right)\Bigg\} \\ &\leq -\frac{\Pi_1}{\varepsilon}\|e^i\|^2 + \sum_{i=1}^{m} M_i\beta_1^i\|e^i\| \\ &\leq -\frac{\Pi_1}{\varepsilon\max\{\lambda_{12}^i\}}V_1(e) + \frac{\sum_{i=1}^{m} M_i\beta_1^i}{\sqrt{\min\{\lambda_{11}^i\}}}\sqrt{V_1(e)}. \end{aligned}$$

Hence

$$\tag{3.22} \frac{d}{dt}\sqrt{V_1(e)}\bigg|_{\text{along (3.6)}} \leq -\frac{\Pi_1}{2\varepsilon\max\{\lambda_{12}^i\}}\sqrt{V_1(e)} + \frac{\sum_{i=1}^{m} M_i\beta_1^i}{2\sqrt{\min\{\lambda_{11}^i\}}}$$

for all $t > \tau_\varepsilon$.

By the comparison principle of ordinary differential equations, we get immediately, for all $t > \tau_\varepsilon$, that

$$\tag{3.23} \sqrt{V_1(e)} \leq e^{-\frac{\Pi_1}{2\varepsilon\max\{\lambda_{12}^i\}}(t-\tau_\varepsilon)} + \frac{\sum_{i=1}^{m} M_i\beta_1^i}{2\sqrt{\min\{\lambda_{11}^i\}}}\int_{\tau_\varepsilon}^{t} e^{-\frac{\Pi_1}{2\varepsilon\max\{\lambda_{12}^i\}}(t-s)}ds.$$

It is seen that the first term of the right-hand side of the above inequality is convergent to zero as $t \to \infty$, so we may assume that it is less than ε as $t > t_\varepsilon$ for some $t_\varepsilon > 0$.

For the second term, we have

$$\left|\int_{\tau_\varepsilon}^{t} e^{-\frac{\Pi_1}{2\varepsilon \max\{\lambda_{12}^i\}}(t-s)} ds\right| \le \frac{2\max\{\lambda_{12}^i\}}{\Pi_1}\varepsilon. \tag{3.24}$$

These together with Assumption A2 show that there exists a positive constant $\Gamma_1 > 0$ such that

$$|e_j^i| \le \sqrt{\frac{V_1(e)}{\min\{\lambda_{11}^i\}}} \le \Gamma_1\varepsilon,\ t > t_\varepsilon. \tag{3.25}$$

Equation (3.4) then follows by taking (1.6) into account.

Finding the derivative of V_2 along the solution of (3.6) with respect to t gives, for all $t > t_\varepsilon$, that

$$\begin{aligned} \left.\frac{dV_2}{dt}\right|_{\text{along (3.6)}} &= \sum_{i=1}^{m} \left\{ \sum_{j=1}^{n_i-1} \eta_{j+1}^i \frac{\partial V_2^i}{\partial \eta_j^i}(\eta^i) \right. \\ &\qquad \left. + \{\phi_i(\eta^i) + e_{n_i+1}^i + [\phi_i(\hat{x}^i - z^i) - \phi_i(x^i - z^i)]\} \frac{\partial V_2^i}{\partial x_{n_i}^i}(\eta^i) \right\} \\ &\le \sum_{i=1}^{m} \left\{ -W_2^i(\eta^i) + \beta_2^i(L_i+1)\|e^i\|\|\eta^i\| \right\} \\ &\le \sum_{i=1}^{m} \left\{ -W_2^i(\eta^i) + \beta_2^i(L_i+1)\Gamma_1\varepsilon\|\eta^i\| \right\}. \end{aligned} \tag{3.26}$$

By Assumption A3, we have, for any $t > t_\varepsilon$, that

$$\left.\frac{dV_2}{dt}\right|_{\text{along (3.6)}} \le -\min\left\{\lambda_{23}^i/\lambda_{22}^i\right\} V_2(\eta) + \frac{\sum_{i=1}^{m} \beta_2^i(L_i+1)\Gamma_1\varepsilon}{\sqrt{\min\{\lambda_{21}^i\}}} \sqrt{V_2(\eta)}. \tag{3.27}$$

It follows that for all $t > t_\varepsilon$,

$$\left.\frac{d}{dt}\sqrt{V_2(\eta)}\right|_{\text{along (3.6)}} \le -\frac{\min\left\{\lambda_{23}^i/\lambda_{22}^i\right\}}{2}\sqrt{V_2(\eta)} + \frac{\sum_{i=1}^{m} \beta_2^i(L_i+1)\Gamma_1\varepsilon}{2\sqrt{\min\{\lambda_{21}^i\}}}. \tag{3.28}$$

Applying the comparison principle in ordinary differential equations again, we get, for all $t > t_\varepsilon$, that

$$\begin{aligned} \sqrt{V(\eta)} &\le e^{-\frac{\min\left\{\lambda_{23}^i/\lambda_{22}^i\right\}}{2}(t-t_\varepsilon)} \sqrt{V(\eta(t_\varepsilon))} + \frac{\sum_{i=1}^{m} \beta_2^i(L_i+1)\Gamma_1\varepsilon}{2\sqrt{\min\{\lambda_{21}^i\}}} \\ &\quad \times \int_{t_\varepsilon}^{t} e^{-\frac{\min\left\{\lambda_{23}^i/\lambda_{22}^i\right\}}{2}(t-s)} ds. \end{aligned} \tag{3.29}$$

Noting (1.6), we finally get that there exist $T_\varepsilon > t_\varepsilon$ and $\Gamma_2 > 0$ such that (3.5) holds true. The proof is complete. $\square$

4. ADRC for external disturbance and parameter mismatch of control only. In this section, a special case of ADRC is considered where the functions in dynamics are known in the sense that for any $1 \le i \le m$, $f_i(x, \xi, w_i) = \tilde{f}_i(x) + \bar{f}(\xi, w_i)$, where $\tilde{f}_i$ is known. In other words, the uncertainty comes from external disturbances, zero dynamics, and parameter mismatch in control only. In this case, we try to use the known information in the design of the ESO.

In this spirit, for each output $y_i = x_1^i (i = 1, 2, \ldots, m)$, the ESO is designed below as ESO(f) to estimate $x_j^i (j = 1, 2, \ldots, n_i)$ and $x_{n_i+1}^i = \bar{f}_i(\xi, w_i) + \sum_{k=1}^m a_{ik}(w_i)u_i - u_i^*$:

$$\text{ESO(f)}: \begin{cases} \dot{\hat{x}}_1^i = \hat{x}_2^i + \varepsilon^{n_i-1} g_1^i(e_1^i), \\ \dot{\hat{x}}_2^i = \hat{x}_3^i + \varepsilon^{n_i-2} g_2^i(e_1^i), \\ \qquad \vdots \\ \dot{\hat{x}}_{n_i}^i = \hat{x}_{n_i+1}^i + g_{n_i}^i(e_1^i) + \tilde{f}(\tilde{x}) + u_i^*, \\ \dot{\hat{x}}_{n_i+1}^i = \dfrac{1}{\varepsilon} g_{n_i+1}^i(e_1^i), i = 1, 2, \ldots, m, \end{cases} \tag{4.1}$$

and the observer based feedback control is designed as

$$\text{ADRC(f)}: \begin{cases} u_i^* = -\tilde{f}(\tilde{x}) + \phi_i(\tilde{x}^i - z^i) + z_{n_i+1}^i - \hat{x}_{n_i+1}^i, \\ u_i = \displaystyle\sum_{i=1}^m b_{ij}^* u_j, \end{cases} \tag{4.2}$$

where the b_{ij}^* are the same as that in Assumption GA2.

The closed loop system is now composed of system (1.1), ESO(f) (4.1), ADRC(f) (4.2):

$$\text{Closed-loop(f)}: \begin{cases} \dot{x}^i = A_{n_i} x^i + B_{n_i} \left(f_i(x, \xi, w_i) + \displaystyle\sum_{k=1}^m a_{ik} u_k \right), \\ \dot{\hat{x}}^i = A_{n_i+1} \hat{x}^i + \begin{pmatrix} \varepsilon^{n_i-1} g_1^i(e_1^i) \\ \vdots \\ \dfrac{1}{\varepsilon} g_{n_i+1}^i(e_1^i) \end{pmatrix} + \begin{pmatrix} B_{n_i} \\ 0 \end{pmatrix} (\tilde{f}(\tilde{x}) + u_i^*), \\ u_i = \displaystyle\sum_{k=1}^m b_{ik}^* u_k^*, \\ u_i^* = -\tilde{f}(\tilde{x}) + \phi_i(\tilde{x}^i - z^i) + z_{(n_i+1)}^i - \hat{x}_{(n_i+1)}^i. \end{cases} \tag{4.3}$$

Assumption A4. All partial derivatives of $\tilde{f}_i, \bar{f}_i$ are bounded by a constant $\tilde{L}_i$.

THEOREM 4.1. *Let $x_j^i(t, \varepsilon)(1 \le j \le n_i, 1 \le i \le m)$ and $\hat{x}_j^i(t, \varepsilon)(1 \le j \le n_i + 1, 1 \le i \le m)$ be the solutions of the closed-loop system* (4.3), *$x_{n_i+1}^i = \bar{f}(\xi, w_i) + \sum_{k=1}^m a_{ik}(w_i)u_i - u_i^*$ be the extended state. Assume Assumptions* A1–A4, GA2–GA3 *are satisfied. In addition, we assume that* (3.1) *in Assumption* GA2 *is replaced by*

$$\min\{\lambda_{13}^i\} - \sum_{i=1}^m \beta_1^i \tilde{L}_i - \sqrt{m} \sum_{i,k,l=1}^m \beta_1^i \sup_{t \in [0,\infty)} |a_{ik}(w_i(t)) - b_{ik}| b_{kl}^* \Lambda_{n_l+1}^l > 0. \tag{4.4}$$

Then there is a constant $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$, there exists a $t_\varepsilon > 0$ such that for all $t > t_\varepsilon$,

$$|x_j^i(t, \varepsilon) - \hat{x}_j^i(t, \varepsilon)| \le \Gamma_1 \varepsilon^{n_i+2-j}, \ 1 \le j \le n_i + 1, \ 1 \le i \le m \tag{4.5}$$

and

$$|x_j^i(t,\varepsilon) - z_j^i(t,\varepsilon)| \le \Gamma_2\varepsilon,\ i = 1,2,\dots,n, \tag{4.6}$$

where Γ_1, Γ_2 *are* ε *and the initial value independent constants (again they depend on the bound of* $\|z^i\|$ *and* $\|(w,\dot{w})\|$*).*

Proof. The error equation in this case is

$$\begin{cases} \dot{\eta}^i = A_{n_i}\eta^i + B_{n_i}[\phi_i(\tilde{x}^i - z^i) + e^i_{n_i+1} + \tilde{f}_i(x) - \tilde{f}_i(\tilde{x})], \\ \varepsilon\dot{e}^i = A_{n_i+1}e^i + \begin{pmatrix} B_{n_i} \\ 0 \end{pmatrix}(\tilde{f}_i(x) - \tilde{f}_i(\tilde{x})) + \varepsilon\hbar_i B_{n_i+1} - \begin{pmatrix} g_1^i(e_1^i) \\ \vdots \\ g^i_{n_i+1}(e_1^i), \end{pmatrix}, \end{cases} \tag{4.7}$$

where

$$\begin{aligned} \hbar_i &= \frac{d}{dt}\bigg|_{\text{along (4.3)}}\left[\bar{f}(\xi,w_i) + \sum_{k=1}^m a_{ik}(w_i)u_k - u_i^*\right] \\ &= \frac{d}{dt}\bigg|_{\text{along (4.3)}}\left[\bar{f}(\xi,w_i) + \sum_{k,l=1}^m (a_{ik}(w_i) - b_{ik})b^*_{kl}(\phi_l(\tilde{x}^l - z^l) + z^l_{n_l+1} - \hat{x}^l_{n_l+1})\right]. \end{aligned} \tag{4.8}$$

A direct computation shows that

$$\begin{aligned} \hbar_i &= L_{F_0(\xi)}\bar{f}_i(\xi,w_i) + \dot{w}_i\frac{\partial\bar{f}_i}{\partial w_i}(\xi,w_i) \\ &\quad + \sum_{k,l=1}^m (a_{ik}(w_i) - b_{ik})b^*_{kl}\left\{\sum_{j=1}^{n_l}\left(\tilde{x}^l_{j+1} - z^l_{j+1} - \varepsilon^{n_l-j}g_j^l(e_1^l)\right)\frac{\partial\phi_l}{\partial y_j}(\tilde{x}^l - z^l)\right\} \\ &\quad + \sum_{k,l=1}^m (a_{ik}(w_i) - b_{ik})b^*_{kl}\left\{\dot{z}^l_{n_l+1} - \frac{1}{\varepsilon}g^l_{n_l+1}(e_1^l)\right\} \\ &\quad + \sum_{k,l=1}^m \dot{a}_{i,k}(w_i)\dot{w}_i b^*_{kl}\left(\phi_l(\tilde{x}^l - z^l) + z^l_{n_l+1} - \hat{x}^l_{n_l+1}\right). \end{aligned} \tag{4.9}$$

It follows that

$$\begin{aligned} &|\hbar_i| \le \Theta_0^i + \Theta_1^i\|e\| + \Theta_2^j\|\eta\| + \frac{\Theta^i}{\varepsilon}\|e\|, \\ &\Theta^i = \sqrt{m}\sum_{k,l=1}^m \sup_{t\in[0,\infty)}|a_{ik}(w_i(t)) - b_{ik}|b^*_{kl}\Lambda^l_{n_l+1} \end{aligned} \tag{4.10}$$

for some ε-independent positive constants $\Theta_0^i, \Theta_1^i, \Theta_2^i$.

Find the derivative of V along the solution of (4.7) to get

$$\begin{aligned}\frac{dV}{dt}\bigg|_{\text{along (4.7)}} = \sum_{i=1}^{m}\Bigg\{\frac{1}{\varepsilon}\Bigg[&\sum_{j=1}^{n_i}(e_{j+1}^i - g_j^i(e_1^i))\frac{\partial V_1^i}{\partial e_j^i}(e^i) - g_{n_i+1}^i(e_1^i)\frac{\partial V_1^i}{\partial e_{n+1}}(e^i)\\ &+(\tilde{f}_i(x)-\tilde{f}_i(\tilde{x}))\frac{\partial V_1^i}{\partial V_{n_i}^i}(e^i)\Bigg]\\ &+\hbar_i\frac{\partial V_1^i}{\partial e_{n_i+1}^i}(e^i)+\sum_{j=1}^{n_i-1}\eta_{j+1}^i\frac{\partial V_2^i}{\partial \eta_j^i}(\eta^i)\\ &+\Big\{\phi_i(\eta^i)+e_{n_i+1}^i+[\phi_i(\hat{x}^i-z^i)-\phi_i(x^i-z^i)]\\ &+(\tilde{f}_i(x)-\tilde{f}_i(\tilde{x}))\Big\}\frac{\partial V_2^i}{\partial x_{n_i}^i}(\eta^i)\Bigg\}.\end{aligned}\tag{4.11}$$

By Assumptions A2 and A3, we get

$$\begin{aligned}\frac{dV}{dt}\bigg|_{\text{along (4.7)}} \le \sum_{i=1}^{m}\Bigg\{&-\frac{1}{\varepsilon}W_1^i(e^i)+\frac{\tilde{L}_i\beta_1^i}{\varepsilon}\|e\|^2\\ &+\beta_1^i\|e^i\|\left(\Theta_0^i+\Theta_1^i\|e\|+\Theta_2^i\|\eta\|+\frac{\Theta^i}{\varepsilon}\|e\|\right)\\ &-W_2^i(\eta^i)+\beta_2^i(L_i+\tilde{L}_i+1)\|e\|\|\eta^i\|\Bigg\}.\end{aligned}\tag{4.12}$$

This together with Assumptions A2 and A3 again gives

$$\begin{aligned}\frac{dV}{dt}\bigg|_{\text{along (4.7)}} &\le \sum_{i=1}^{m}\Bigg\{-\frac{\lambda_{13}^i}{\varepsilon}\|e^i\|^2+\frac{\tilde{L}_i\beta_1^i}{\varepsilon}\|e\|^2\\ &\qquad+\beta_1^i\|e^i\|\left(\Theta_0^i+\Theta_1^i\|e\|+\Theta_2^i\|\eta\|+\frac{\Theta^i}{\varepsilon}\|e\|\right)\\ &\qquad-\lambda_{23}^i\|\eta^i\|^2+\beta_2^i(L_i+\tilde{L}_i+1)\|e\|\|\eta^i\|\Bigg\}\\ &\le -\left(\frac{1}{\varepsilon}\left(\min\{\lambda_{13}^i\}-\sum_{i=1}^{m}\beta_1^i\Theta^i-\sum_{i=1}^{m}\beta_1^i\tilde{L}_i\right)-\sum_{i=1}^{m}\beta_1^i\Theta_1^i\right)\|e\|^2\\ &\quad+\left(\sum_{i=1}^{m}\beta_1^i\Theta_0^i\right)\|e\|-\min\{\lambda_{23}^i\}\|\eta\|^2+\sum_{i=1}^{m}\beta_2^i(L_i+\tilde{L}_i+1)\|e\|\|\eta\|.\end{aligned}\tag{4.13}$$

For simplicity, we introduce the following symbols to represent the parameters in (4.13):

$$\begin{aligned}&\$_1=\min\{\lambda_{13}^i\}-\sum_{i=1}^{m}\beta_1^i\Theta^i-\sum_{i=1}^{m}\beta_1^i\tilde{L}_i,\ \ \$_2=\sum_{i=1}^{m}\beta_1^i\Theta_1^i,\\ &\$_3=\sum_{i=1}^{m}\beta_1^i\Theta_0^i,\ \ \$_4=\sum_{i=1}^{m}\beta_2^i(L_i+\tilde{L}_i+1),\ \ \lambda=\min\{\lambda_{23}^i\},\end{aligned}\tag{4.14}$$

and rewrite inequality (4.13) as follows:

$$\frac{dV}{dt}\bigg|_{\text{along (4.7)}} \le -\left(\frac{\$_1}{\varepsilon}-\$_2\right)\|e\|^2+\$_3\|e\|-\lambda\|\eta\|^2+\$_4\|e\|\|\eta\|.\tag{4.15}$$

It is seen that (4.15) and (3.15) are quite similar. In fact if set $\Pi_i = \$_i$, then (4.15) is just (3.15). So the boundedness of the solution to (4.7) can be obtained following the corresponding part of the proof of Theorem 3.1 that for any $\varepsilon \in (0, \varepsilon_0)$, there is a $\tau_\varepsilon > 0$ such that $\|(e,\eta)\| \le \bar{R}$ for all $t \in (T_\varepsilon, \infty)$. This together with (4.9) yields that $|\hbar_i| \le \bar{M}_i + (\Theta_i/\varepsilon)\|e\|$ for all $t \in (T_\varepsilon, \infty)$, where $\bar{M}_i$ is an $\bar{R}$-dependent positive constant.

Finding the derivative of V_1 along the solution of (4.7) with respect to t gives, for all $t > \tau_\varepsilon$, that

(4.16)
$$\begin{aligned}\left.\frac{dV_1}{dt}\right|_{\text{along (4.7)}} &= \frac{1}{\varepsilon}\sum_{i=1}^{m}\Bigg\{\sum_{j=1}^{n_i}\left(e_{j+1}^i - g_j^i(e_1^i)\right)\frac{\partial V_1^i}{\partial e_j^i}\left(e^i\right) + (\tilde{f}(x) - \tilde{f}(\tilde{x}))\frac{\partial V_1^i}{\partial e_{n_i}^i}(e^i)\\ &\qquad + \left(\varepsilon\bar{\Delta}_i - g_{n_i+1}^i\left(e_1^i\right)\right)\frac{\partial V_1^i}{\partial e_{n_i+1}^i}\left(e^i\right)\Bigg\}\\ &\le -\frac{\Pi_1}{\varepsilon}\|e^i\|^2 + \sum_{i=1}^{m}\bar{M}_i\beta_1^i\|e^i\|\\ &\le -\frac{\Pi_1}{\varepsilon\max\{\lambda_{12}^i\}}V_1(e) + \frac{\sum_{i=1}^{m}\bar{M}_i\beta_1^i}{\sqrt{\min\{\lambda_{11}^i\}}}\sqrt{V_1(e)}.\end{aligned}$$

It follows that

$$\left.\frac{d}{dt}\sqrt{V_1(e)}\right|_{\text{along (4.7)}} \le -\frac{\Theta_1}{2\varepsilon\max\{\lambda_{12}^i\}}\sqrt{V_1(e)} + \frac{\sum_{i=1}^{m}\tilde{M}_i\beta_1^i}{2\sqrt{\min\{\lambda_{11}^i\}}}\ \forall\, t > \tau_\varepsilon. \tag{4.17}$$

Finding the derivative of V_2 along the solution of (4.7) with respect to t gives

(4.18)
$$\begin{aligned}\left.\frac{dV_2}{dt}\right|_{\text{along (4.7)}} &= \sum_{i=1}^{m}\Bigg\{\sum_{j=1}^{n_i-1}\eta_{j+1}^i\frac{\partial V_2^i}{\partial \eta_j^i}(\eta^i) + \Big\{\varphi_i(\eta^i) + e_{n_i+1}^i + (\tilde{f}_i(x) - \tilde{f}(\tilde{x}))\\ &\qquad + [\varphi_i(\hat{x}^i - z^i) - \varphi_i(x^i - z^i)]\Big\}\frac{\partial V_2^i}{\partial x_{n_i}^i}(\eta^i)\Bigg\}\\ &\le \sum_{i=1}^{m}\left\{-W_2^i(\eta^i) + \beta_2^i(L_i + \tilde{L}_i + 1)\|e^i\|\|\eta^i\|\right\}\\ &\le \sum_{i=1}^{m}\left\{-W_2^i(\eta^i) + \beta_2^i(L_i + \tilde{L}_i + 1)\Gamma_1\varepsilon\|\eta^i\|\right\}\ \forall\, t > t_\varepsilon.\end{aligned}$$

By Assumption A3, we have

$$\begin{aligned}\left.\frac{dV_2}{dt}\right|_{\text{along (4.7)}} &\le -\min\left\{\lambda_{23}^i/\lambda_{22}^i\right\}V_2(\eta)\\ &\quad + \frac{\sum_{i=1}^{m}\beta_2^i(L_i + \tilde{L}_i + 1)\Gamma_1\varepsilon}{\sqrt{\min\{\lambda_{21}^i\}}}\sqrt{V_2(\eta)}\ \forall\, t > t_\varepsilon.\end{aligned} \tag{4.19}$$

It follows that

$$\frac{d}{dt}\sqrt{V_2(\eta)}\Big|_{\text{along (4.7)}} \le -\frac{\min\{\lambda_{23}^i/\lambda_{22}^i\}}{2}\sqrt{V_2(\eta)} + \frac{\sum_{i=1}^m \beta_2^i(L_i+\tilde{L}_i+1)\Gamma_1\varepsilon}{2\sqrt{\min\{\lambda_{21}^i\}}} \;\forall\, t>t_\varepsilon. \tag{4.20}$$

It is seen that (4.17) and (4.20) are very similar to (3.22) and (3.28), respectively. Using similar arguments, we obtain Theorem 4.1. The details are omitted. ☐

5. Examples and numerical simulations.

5.1. Nonlinear ADRC for total disturbance.

Example 5.1. Consider the following MIMO system:

$$\begin{cases} \dot{x}_1^1 = x_2^1, \quad \dot{x}_2^1 = f_1(x,\zeta,w_1) + a_{11}u_1 + a_{12}u_2, \\ \dot{x}_1^2 = x_2^2, \quad \dot{x}_2^2 = f_2(x,\zeta,w_2) + a_{21}u_1 + a_{22}u_2, \\ \dot{\zeta} = x_2^1 + x_1^2 + \sin\zeta + \sin t, \\ y_1 = x_1^1, y_2 = x_1^2, \end{cases} \tag{5.1}$$

where y_1, y_2 are the outputs, and $u_1,\ u_2$ are inputs, and

$$\begin{cases} f_1(x_1^1,x_2^1,x_1^2,x_2^2,\zeta,w_1) = x_1^1 + x_1^2 + \zeta + \sin(x_2^1+x_2^2)w_1, \\ f_2(x_1^1,x_2^1,x_1^2,x_2^2,\zeta,w_2) = x_2^1 + x_2^2 + \zeta + \cos(x_1^1+x_1^2)w_2, \\ a_{11} = 1+\frac{1}{10}\sin t, \;\; a_{12} = 1+\frac{1}{10}\cos t, \;\; a_{21} = 1+\frac{1}{10}2^{-t}, \;\; a_{22} = -1, \end{cases} \tag{5.2}$$

are unknown functions.

Suppose that external disturbances w_1, w_2, and the reference signals v^1, v^2 are as follows:

$$w_1 = 1+\sin t,\; w_2 = 2^{-t}\cos t,\; v^1 = \sin t,\; v^2 = \cos t. \tag{5.3}$$

Let $\phi_1 = \phi_2 = \phi : \mathbb{R}^2 \to \mathbb{R}$ be defined by $\phi(r_1,r_2) = -9r_1 - 6r_2$. The objective is to design an observer based feedback control so that $x_1^i - z_1^i$ and $x_2^i - z_2^i$ converge to zero as time goes to infinity in the way of the following global asymptotic stable system converging to zero,

$$\begin{cases} \dot{x}_1^* = x_2^*, \\ \dot{x}_2^* = \phi_i(x_1^*, x_2^*), \end{cases} \tag{5.4}$$

where z_1^i, z_2^i, z_3^i are the states of the tracking differentiator (TD) to estimate the derivatives of v_i. For simplicity, we use the same TD for v_1 and v_2 as follows:

$$\begin{cases} \dot{z}_1^i = z_2^i, \\ \dot{z}_2^i = z_3^i, \\ \dot{z}_3^i = -\rho^3(z_1^i - v_i) - 3\rho^2 z_2^i - 3\rho z_3^i, i = 1,2. \end{cases} \tag{5.5}$$

Since most of the functions in (5.1) are unknown, the ESO design relies on very little information of the system, and the total disturbance should be estimated. This is quite different to the problem in [18] where only the uncertain constant nominal value of control is estimated. Here we need the approximate values b_{ij} of a_{ij}:

$$b_{11} = b_{12} = b_{13} = 1, \quad b_{22} = -1. \tag{5.6}$$

b^*_{ij} is found to be

$$\begin{pmatrix} b^*_{11} & b^*_{12} \\ b^*_{21} & b^*_{22} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}. \tag{5.7}$$

By Theorem 3.1, we design the nonlinear ESO for system (5.1) as

$$\begin{cases} \dot{\hat{x}}_1^1 = \hat{x}_2^1 + \dfrac{6}{\varepsilon}(y_1 - \hat{x}_1^1) - \varepsilon\Phi\left(\dfrac{y_1 - \hat{x}_1^1}{\varepsilon^2}\right), \\ \dot{\hat{x}}_2^1 = \hat{x}_3^1 + \dfrac{11}{\varepsilon^2}(y_1 - \hat{x}_1^1) + u_1^*, \\ \dot{\hat{x}}_3^1 = \dfrac{6}{\varepsilon^3}(y_1 - \hat{x}_1^1), \end{cases}$$
$$\begin{cases} \dot{\hat{x}}_1^2 = \hat{x}_2^2 + \dfrac{6}{\varepsilon}(y_2 - \hat{x}_1^2), \\ \dot{\hat{x}}_2^2 = \hat{x}_3^2 + \dfrac{11}{\varepsilon^2}(y_2 - \hat{x}_1^2) + u_2^*, \\ \dot{\hat{x}}_3^2 = \dfrac{6}{\varepsilon^3}(y_2 - \hat{x}_1^2), \end{cases} \tag{5.8}$$
$$u_1^* = \phi_1\left(\hat{x}_1^1 - z_1^1, \hat{x}_2^1 - z_2^1\right) + z_3^1 - \hat{x}_3^1,$$
$$u_2^* = \phi_2\left(\hat{x}_1^2 - z_1^2, \hat{x}_2^2 - z_2^2\right) + z_3^2 - \hat{x}_3^2,$$

where $\Phi : \mathbb{R} \to \mathbb{R}$ is given by

$$\Phi(r) = \begin{cases} -\dfrac{1}{4}, & r \in \left(-\infty, -\dfrac{\pi}{2}\right), \\ \dfrac{1}{4}\sin r, & r \in \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right), \\ \dfrac{1}{4}, & r \in \left(\dfrac{\pi}{2}, -\infty\right). \end{cases} \tag{5.9}$$

The ADRC for this example is the observer based feedback given by

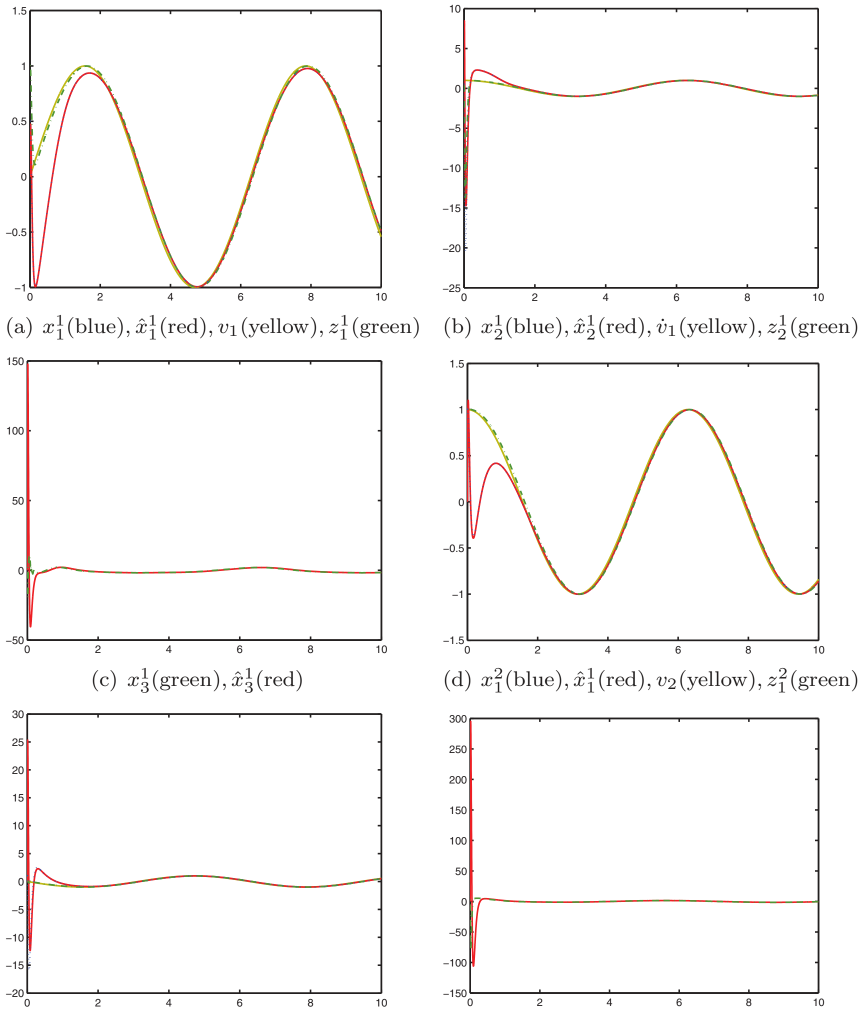
$$u_1 = \frac{1}{2}\left(u_1^* + u_2^*\right), \quad u_2 = \frac{1}{2}\left(u_1^* - u_2^*\right). \tag{5.10}$$

We take the initial values and parameters as follows:

$$x(0) = (0.5, 0.5, 1, 1),\ \hat{x}(0) = (0, 0, 0, 0, 0, 0),\ z(0) = (1, 1, 1, 1, 1, 1),\ \rho = 50,$$
$$\varepsilon = 0.05,\ h = 0.001,$$

where h is the integral step. Using the Euler method, the numerical results for system (5.1)–(5.3) under (5.5), (5.4), (5.8), and (5.10) are plotted in Figure 5.1.

Figures 5.1(a), 5.1(b), 5.1(d), and 5.1(e) indicate that for every $i, j = 1, 2$, $\hat{x}_j^i$ tracks x_j^i, z_j^i tracks $(v_i)^{(j-1)}$, and x_j^i tracks $(v_i)^{(j-1)}$ very satisfactorily. In addition, from Figures 5.1(c) and 5.1(f), we see that $\hat{x}_3^i$ tracks satisfactorily the extended state or total disturbance $x_3^i = f_i + a_{i1}u_1 + a_{i2}u_2 - u_i^*$.



(a) x_1^1(blue), $\hat{x}_1^1$(red), v_1(yellow), z_1^1(green) (b) x_2^1(blue), $\hat{x}_2^1$(red), $\dot{v}_1$(yellow), z_2^1(green)

(c) x_3^1(green), $\hat{x}_3^1$(red) (d) x_1^2(blue), $\hat{x}_1^1$(red), v_2(yellow), z_1^2(green)

(e) x_2^2(blue), $\hat{x}_2^2$(red), $\dot{v}_2$(yellow), z_2^2(green) (f) x_3^2(green), $\hat{x}_3^2$(red)

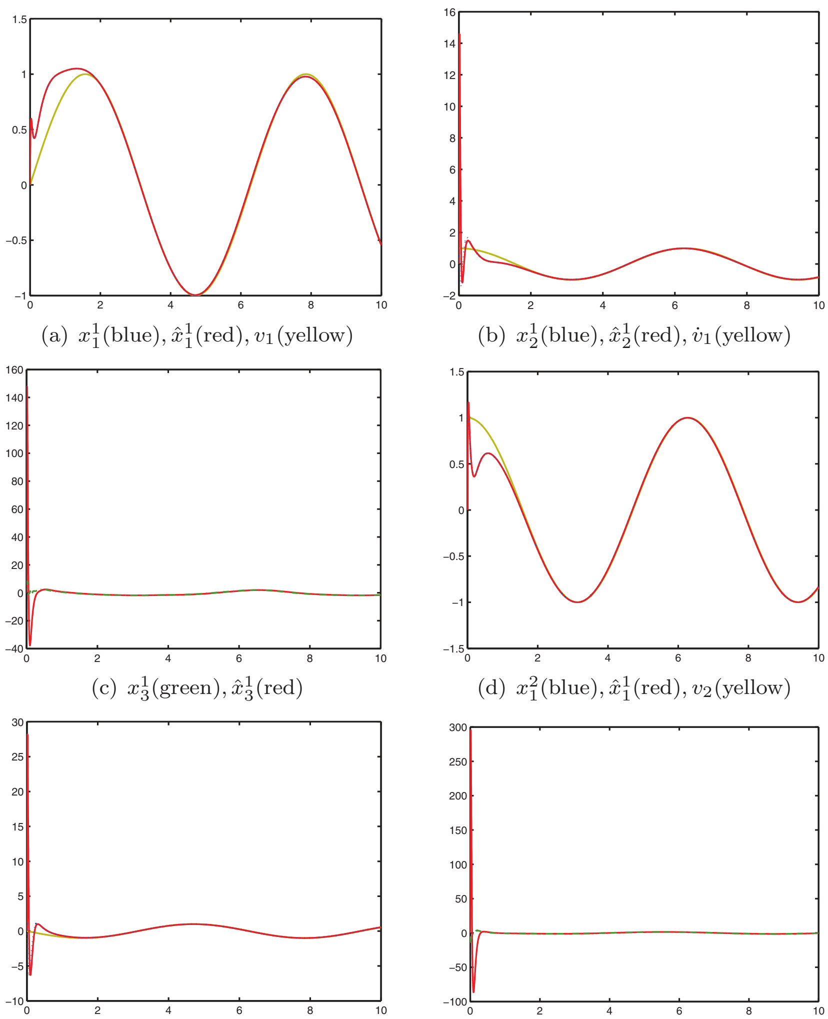
FIG. 5.1. *Numerical results of ADRC for system* (5.1), (5.5), (5.8), *and* (5.10) *with total disturbance (for interpretation of the references to color of the figure's legend in this section, we refer to the PDF version of this article).*

The peaking phenomena of the system states x_2^1, x_2^2 plotted in Figures 5.1(b) and 5.1(e) with the blue curve, and the extended states x_3^1, x_3^2 in Figures 5.1(c) and 5.1(f) with the green curve are observed.

To overcome the peaking problem, we use the saturated feedback control in Theorem 3.1 by replacing u_1^*, u_2^* with

$$\begin{aligned} u_1^* &= \mathrm{sat}_{20}\Big(\phi_1\left(\hat{x}_1^1 - z_1^1, \hat{x}_2^1 - z_2^1\right)\Big) + z_3^1 - \mathrm{sat}_{20}(\hat{x}_3^1), \\ u_2^* &= \mathrm{sat}_{20}\Big(\phi_2\left(\hat{x}_1^2 - z_1^2, \hat{x}_2^2 - z_2^2\right)\Big) + z_3^2 - \mathrm{sat}_{20}(\hat{x}_3^2). \end{aligned} \tag{5.11}$$

For simplicity, we use the exact values of $(v_i)^{(j-1)}$ instead of z_j^i. Under the same parameters as that in Figure 5.1, the numerical results under control (5.11) are plotted in



(a) $x_1^1(\text{blue}), \hat{x}_1^1(\text{red}), v_1(\text{yellow})$ (b) $x_2^1(\text{blue}), \hat{x}_2^1(\text{red}), \dot{v}_1(\text{yellow})$

(c) $x_3^1(\text{green}), \hat{x}_3^1(\text{red})$ (d) $x_1^2(\text{blue}), \hat{x}_1^1(\text{red}), v_2(\text{yellow})$

(e) $x_2^2(\text{blue}), \hat{x}_2^2(\text{red}), \dot{v}_2(\text{yellow})$ (f) $x_3^2(\text{green}), \hat{x}_3^2(\text{red})$

FIG. 5.2. *Numerical results of ADRC for system* (5.1), (5.5), (5.8), *and* (5.11) *with total disturbance using saturated feedback control (for interpretation of the references to color of the figure's legend in this section, we refer to the PDF version of this article).*

Figure 5.2. It is seen that the peak values of the states x_2^1, x_2^2 plotted in Figures 5.2(b) and 5.2(e) with the blue curve, and the extended state x_3^1, x_3^2 plotted in Figures 5.2(c) and 5.2(f) with the green curve are reduced significantly.

5.2. Output regulation: ADRC vs. IMP. The internal model principle (IMP) deals with the general output regulation problem of the following:

$$\begin{cases} \dot{x} = Ax + Bu + Pw, \\ \dot{w} = Sw, \\ e = Cx - Qw, \end{cases} \tag{5.12}$$

where x is the state, w the external signal, B the control matrix, C the observation matrix. The control purpose is to design the error feedback so that the error $e \to 0$

as $t \to \infty$, and at the same time, all internal systems are stable. The following Proposition 5.1 comes from [4].

PROPOSITION 5.1. *Suppose that* (A, B) *is stabilizable and the pair*

$$(C, -Q), \quad \begin{pmatrix} A & P \\ 0 & S \end{pmatrix} \tag{5.13}$$

is detectable. Then the output regulation is solvable if and only if the linear matrix equations

$$\begin{aligned} \tilde{M}_1 S &= A\tilde{M}_1 + P + B\tilde{M}_2, \\ 0 &= C\tilde{M}_1 - Q, \end{aligned} \tag{5.14}$$

have solutions $\tilde{M}_1$ *and* $\tilde{M}_2$.

It is well known that when the output regulation problem (5.12) is solvable, then the observer based feedback u is given by

$$u = \tilde{K}(\hat{x} - \tilde{M}_1\hat{w}) + \tilde{M}_2\hat{w}, \tag{5.15}$$

where $(\hat{x}, \hat{w})$ is the observer for (x, w):

$$\begin{pmatrix} \dot{\hat{x}} \\ \dot{\hat{w}} \end{pmatrix} = \begin{pmatrix} A & P \\ 0 & S \end{pmatrix} \begin{pmatrix} \hat{x} \\ \hat{w} \end{pmatrix} + \begin{pmatrix} \tilde{N}_1 \\ \tilde{N}_2 \end{pmatrix} (C\hat{x} - Q\hat{w} - e) + \begin{pmatrix} B \\ 0 \end{pmatrix} u. \tag{5.16}$$

The closed loop system, with error states $\tilde{x} = \hat{x} - x$ and $\tilde{w} = \hat{w} - w$, is

$$\begin{cases} \dot{x} = (A + B\tilde{K})x + Pw - BK\tilde{M}_1 w + B\tilde{M}_2 w + BK\tilde{x} + B(\tilde{M}_2 - K\tilde{M}_1)\tilde{w}, \\ \begin{pmatrix} \dot{\tilde{x}} \\ \dot{\tilde{w}} \end{pmatrix} = \begin{pmatrix} A + \tilde{N}_1 C & P - \tilde{N}_1 Q \\ \tilde{N}_2 C & S - \tilde{N}_2 Q \end{pmatrix} \begin{pmatrix} \tilde{x} \\ \tilde{w} \end{pmatrix}, \\ \dot{w} = Sw, \end{cases} \tag{5.17}$$

where $\tilde{K}$ is chosen so that $A + B\tilde{K}$ is Hurwitz, $\tilde{M}_1, \tilde{M}_2$ satisfy matrix equations (5.14), and the matrices $\tilde{N}_1, \tilde{N}_2$ are chosen so that the internal systems are stable, in other words

$$\begin{pmatrix} A + \tilde{N}_1 C & P - \tilde{N}_1 Q \\ \tilde{N}_2 C & S - \tilde{N}_2 Q \end{pmatrix} \tag{5.18}$$

is Hurwitz which is equivalent to that (5.13) is detectable. It should be pointed out that if w is unbounded, the control (5.15) may be unbounded.

We show that ADRC can be used to solve the output regulation for a class of MIMO systems in a very different way. Consider the following system:

$$\begin{cases} \dot{x} = Ax + Bu + Pw, \\ e = y - Qw, \quad y = Cx, \end{cases} \tag{5.19}$$

where $x \in \mathbb{R}^l, y, u, w \in \mathbb{R}^m$, A is an $l \times l$ matrix, B is $l \times m$, and P is $l \times m$. Notice that as opposed to (5.12), here we do not need the known dynamic of the disturbance w.

DEFINITION 5.2. *We say that the output regulation problem* (5.19) *is solvable by ADRC if there is an output feedback control so that for any given* $\sigma > 0$, *there exists* *a $t_0 > 0$ such that $\|e\| \leq \sigma$ for all $t \geq t_0$. Meanwhile, all internal systems including control are bounded.*

In order to apply ADRC to solve the regulation problem (5.19), we need TD to recover all derivatives of each $Q_i w$ up to $r_i + 1$, where Q_i denotes the ith row of Q, and the ESO to estimate the state and the external disturbance by the output y. Among them TD is actually an independent link of ADRC.

For simplicity and comparison with IMP, where all loops are linear, here we also use linear TD for all $1 \leq i \leq m$, a special case of (1.3) as follows:

(5.20)

$$\text{LTD: } \dot{z}^i(t) = A_{r_i+2} z^i(t) + B_{r_i+2}\rho^{r_i+2}\left(d_{i1}(z_1^i - Q_i w), d_{i2}\frac{z_2^i}{\rho}, \ldots, d_{i(r_i+2)}\frac{z^i_{i(r_i+2)}}{\rho^{r_i+1}}\right).$$

The ESO we used here is also linear; that is a special case of (1.5):

$$\text{LESO: } \dot{\hat{x}}^i = A_{r_i+2}\hat{x}^i + \begin{pmatrix} B_{r_i+1} \\ 0 \end{pmatrix} u_i^* + \begin{pmatrix} \frac{k_{i1}}{\varepsilon}(c_i x - \hat{x}_1^i)) \\ \vdots \\ \frac{k_{i(r_i+2)}}{\varepsilon^{r_i+2}}(c_i x - \hat{x}_1^i) \end{pmatrix}. \tag{5.21}$$

In addition, the ADRC (control u^*) also takes the linear form of the following:

$$u_i^* = -\hat{x}^i_{r_i+2} + z^i_{r_i+2} + \sum_{j=1}^{r_i+1} h_{ij}\left(\hat{x}_j^i - z_j^i\right), \tag{5.22}$$

where constants d_{ij}, k_{ij}, h_{ij} are to be specified in Proposition 5.3 below.

PROPOSITION 5.3. *Assume that the following matrices are Hurwitz:*

$$\begin{aligned} &D_i = \begin{pmatrix} 0 & & I_{r_i+1} \\ d_{i1} & \cdots & d_{i(r_i+2)} \end{pmatrix}, \quad K_i = \begin{pmatrix} -k_{i1} & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -k_{i(r_i+2)} & 0 & \cdots & 0 \end{pmatrix}, \\ &H_i = \begin{pmatrix} 0 & & I_{r_i} \\ h_{i1} & \cdots & h_{i(r_i+1)} \end{pmatrix}. \end{aligned} \tag{5.23}$$

The disturbance is assumed to satisfy $\|(w, \dot{w})\| < \infty$, and there exists a matrix P^f such that $P = BP^f$. Suppose that the triple (A, B, C) is decoupling with relative degree $\{r_1, r_2, \ldots, r_m\}$, that is equivalent to the following matrix, is invertible:

$$E = \begin{pmatrix} c_1 A^{r_1} B \\ c_2 A^{r_2} B \\ \vdots \\ c_m A^{r_m} B \end{pmatrix}, \tag{5.24}$$

where c_i denotes the ith row of C. The output regulation can be solved by ADRC under control $u = E^{-1}u^*$ if one of the following two conditions is satisfied:

(I) *$n = r_1 + r_2 + \cdots + r_m = l$ and the following matrix T_1 is invertible:*

$$T_1 = \begin{pmatrix} c_1 A \\ \vdots \\ c_1 A^{r_i} \\ \vdots \\ c_m A^{r_m} \end{pmatrix}_{n \times l}. \tag{5.25}$$

(II) $n < l$ *and there exists an* $(l-n) \times l$ *matrix* T_0 *such that the following matrix* T_2 *is invertible:*

$$T_2 = \begin{pmatrix} T_1 \\ T_0 \end{pmatrix}_{l \times l}, \tag{5.26}$$

and $T_0 A T_2^{-1} = (\tilde{A}_{(l-n)\times n},\ \bar{A}_{(l-n)\times(l-n)})$, *where* $\bar{A}$ *is Hurwitz and* $T_0 B = 0$.

Proof. By assumption, the triple (A, B, C) has relative degree $\{r_1, \dots, r_m\}$, so

$$c_i A^k B = 0 \ \forall\ 0 \le k \le r_i - 1, \quad c_i A^{r_i} B \neq 0. \tag{5.27}$$

Let

$$\bar{x}^i_j = c_i A^{j-1} x,\ j = 1, \dots, r_i + 1,\ i = 1, 2, \dots, m. \tag{5.28}$$

For $i = 1, 2, \dots, m$, finding the derivative of $\bar{x}^i_j$, we get

$$\begin{cases} \dot{\bar{x}}^i_j = c_i A^j x + c_i A^{j-1} B u + c_i A^{j-1} B P^f w = c_i A^j x = \bar{x}^i_{j+1}, \\ \hfill j = 1, 2, \dots, r_i, \\ \dot{\bar{x}}^i_{r_i+1} = c_i A^{r_i+1} x + c_i A^{r_i} B u + c_i A^{r_i} P w. \end{cases} \tag{5.29}$$

(I) $r_1 + r_2 + \cdots + r_m + m = l$. In this case, under the coordinate transformation $\bar{x} = T_1 x$, system (5.19) is transformed into

$$\begin{cases} \dot{\bar{x}}^i_1 = \bar{x}^i_2, \\ \dot{\bar{x}}^i_2 = \bar{x}^i_3, \\ \quad \vdots \\ \dot{\bar{x}}^i_{r_i+1} = c_i A^{r_i+1} T_1^{-1} \bar{x} + c_i A^{r_i} B u + c_i A^{r_i} P w. \end{cases} \tag{5.30}$$

It is obvious that (5.30) has the form of (1.1) without zero dynamic.

(II) $n < l$. In this case, let $\bar{x} = T_1 x$ and $\xi = T_0 x$. Then

$$\begin{cases} \dot{\bar{x}}^i_1 = \bar{x}^i_2, \\ \dot{\bar{x}}^i_2 = \bar{x}^i_3, \\ \quad \vdots \\ \dot{\bar{x}}^i_{r_i+1} = c_i A^{r_i+1} T_2^{-1} \begin{pmatrix} \bar{x} \\ \xi \end{pmatrix} + c_i A^{r_i} B u + c_i A^{r_i} P w, \\ \dot{\xi} = T_0 A T_2^{-1} \begin{pmatrix} \bar{x} \\ \xi \end{pmatrix} = \bar{A}\xi + \tilde{A}\bar{x}, \end{cases} \tag{5.31}$$

which is also has the form (1.1).

Now consider the zero dynamics in (5.31): $\dot{\xi} = \bar{A}\xi + \tilde{A}\bar{x}$. Since $\bar{A}$ is Hurwitz, there exists a solution $\hat{P}$ to the Lyapunov equation below:

$$\hat{P}\bar{A} + \bar{P}^\top \hat{P} = -I,$$

where I denotes the identity matrix. We claim that the zero dynamics is input-to-state stable. Actually, let Lyapunov function $V_0 : \mathbb{R}^{l-n} \to \mathbb{R}$ be $V_0(\xi) = \langle \hat{P}\xi, \xi \rangle$, and let $\chi(\bar{x}, w) = 2\lambda_{\max}(\bar{A}\tilde{A})\|\bar{x}\|^2$, where $\lambda_{\max}(\hat{P}\tilde{A})$ denotes the maximum eigenvalue of $(\hat{P}\tilde{A})(\hat{P}\tilde{A})^\top$.

Finding the derivative of V_0 along the zero dynamics gives

$$\begin{aligned}\frac{dV_0(\xi)}{dt} &= \xi^\top \bar{A}^\top \hat{P}\xi + \bar{x}^\top \tilde{A}^\top \hat{P}\xi + \xi^\top \hat{P}\bar{A}\xi + \xi^\top \hat{P}\tilde{A}\bar{x}\\ &\le -\|\xi\|^2 + 2\sqrt{\lambda_{\max}(\hat{P}\tilde{A})}\|\xi\|\|\bar{x}\|\\ &\le -\tfrac{1}{2}\|\xi\|^2 + \chi(\bar{x}).\end{aligned}$$

So, the zero dynamics is input-to-state stable.

Since all dynamics functions in (5.30) or (5.31) are linear, they are C^1 and globally Lipschitz continuous. All conditions for dynamics required in Theorem 3.1 are satisfied for systems (5.30) and (5.31). Meanwhile, since matrixes D_i, K_i, H_i are Hurwitz, all assumptions for LESO (5.21) and feedback control in Theorem 3.1 are satisfied for systems (5.30) and (5.31). It then follows directly from Theorem 3.1 that for any $\sigma > 0$, there exists $\rho_0 > 0$, $\varepsilon_0 > 0$, and ε-dependent $t_\varepsilon > 0$, such that for every $\rho > \rho_0$, $\varepsilon \in (0, \varepsilon_0)$, $\|e\| < \sigma$ for systems (5.30) and (5.31). Moreover, since all TD, ESO, and ADRC are convergent, all internal systems of systems (5.30) and (5.31) are bounded. The result then follows by the equivalence between systems (5.30), (5.31), and system (5.19) in the two different cases, respectively. □

Remark 5.1. In order to alleviate the peak value near the initial time, by Theorem 2.1, if the bound of all (r_i+2)th order derivatives of $v_i = Q_i w$ and the initial value of the system are known, we can use the saturation function in (2.4) to saturate as $\zeta_j^i = \mathrm{sat}_{M_0^i}(z_j^i)$, and u^* in (5.22) is modified as $u^* = -\mathrm{sat}_{M_1}(\hat{x}^i_{n_i+1}) + \mathrm{sat}_{M_2}(\phi_i(\tilde{x}^i - \zeta^i)) + \zeta^i_{n_i+1}$, where M_0^i is larger than the bound of all (r_i+2)th order derivatives of $v_i = Q_i w$, and M_1, M_2 are chosen according to (2.1).

Remark 5.2. We can compare the IMP over ADRC for the class of linear systems discussed in Proposition 5.1 as follows: (a) the IMP requires known dynamic S of exosystem, but ADRC does not; (b) in the design of the IMP, when the orders of internal and exosystem are high, it is very difficult to choose the corresponding matrices in (5.18), while ADRC does not need these and is relatively easy to design; (c) the IMP pursues disturbance injection while the ADRC pursues disturbance attenuation; (d) generally, two approaches deal with different classes of systems.

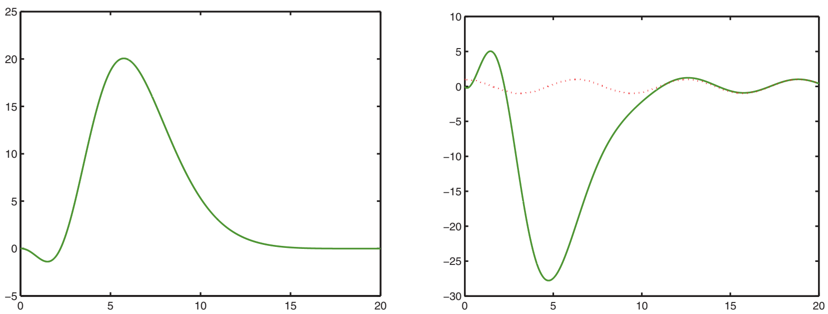
In order to have a more direct comparison of IMP and ADRC, we use a concrete example of the following that can be dealt with by two approaches.

Example 5.2. Consider the following MIMO system:

$$\begin{cases}\dot{x} = Ax + Bu + Pw, \quad y = Cx,\\ \dot{w} = Sw,\\ e = y - Qw,\end{cases} \tag{5.32}$$

where

$$\begin{aligned}&A = \begin{pmatrix}0 & 1 & 0\\ 1 & 1 & 1\\ 1 & -1 & 1\end{pmatrix}, \quad B = \begin{pmatrix}0 & 0\\ 1 & 1\\ 1 & -1\end{pmatrix}, \quad C = \begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 1\end{pmatrix},\\ &P = \begin{pmatrix}0 & 0\\ 1 & 0\\ 0 & 1\end{pmatrix}, \quad Q = \begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}, \quad S = \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}.\end{aligned} \tag{5.33}$$



(a) x_1(green) (b) x_3(green), $\cos t$(red),

FIG. 5.3. *Numerical results of IMP* (5.17) *for Example* (5.2) *(for interpretation of the references to color of the figure's legend in this section, we refer to the PDF version of this article).*

A direct verification shows that the following matrix $\tilde{K}$ makes $A + B\tilde{K}$ Hurwitz:

$$\tilde{K} = \begin{pmatrix} -5/2 & -4 & -2 \\ 5/2 & 4 & 2 \end{pmatrix}. \tag{5.34}$$

Solving the matrix equations (5.18), we get the solutions as follows:

$$\tilde{M}_1 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad \tilde{M}_2 = \begin{pmatrix} -1 & -3/2 \\ 0 & 1/2 \end{pmatrix}. \tag{5.35}$$

Furthermore, we find that the matrices N_1, N_2 below make the matrix in (5.18) Hurwitz:

$$\tilde{N}_1 = \begin{pmatrix} -7 & 0 \\ -22 & 0 \\ -53/5 & 0 \end{pmatrix}, \quad \tilde{N}_2 = \begin{pmatrix} -12/5 & 0 \\ 4/5 & 0 \end{pmatrix}. \tag{5.36}$$

Choose

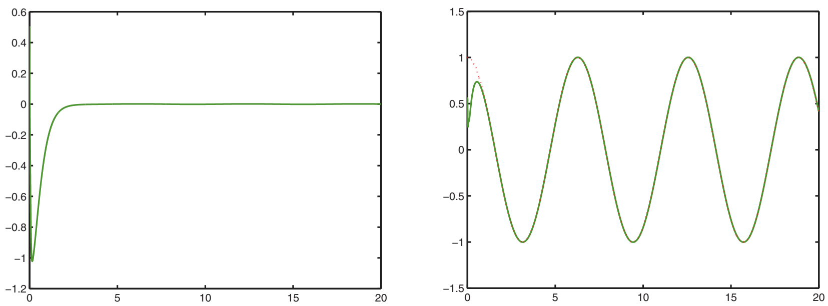
$$x(0) = (0,0,0),\ \tilde{x}(0) = (0.1, 0.5, 0.5),\ w(0) = (0,1),\ \tilde{w}_1(0) = (1,0),\ h = 0.001,$$

where h is the integral step. The numerical results for system (5.17) with specified matrices in (5.33), (5.34), (5.35), (5.36) are plotted in Figure 5.3. These are the whole process of applying IMP to system (5.32).

Now let us look at the design of ADRC for system (5.32). First, the TD given below is used to recover the derivatives of $Q_1 w$ and $Q_2 w$, where $Q = (Q_1, Q_2)^\top$:

$$\begin{cases} \dot{z}_1^1 = \dot{z}_2^1, \\ \dot{z}_2^1 = \dot{z}_3^1, \\ \dot{z}_3^1 = -\rho^3(z_1^1 - Q_1 w) - 3\rho^2 z_2^1 - 3\rho z_3^1, \end{cases} \\ \begin{cases} \dot{z}_1^2 = z_2^2, \\ \dot{z}_2^2 = -2\rho^2(z_1^2 - Q_2 w) - \rho z_2^2. \end{cases} \tag{5.37}$$

The ESO is designed below by injection of two outputs $c_1 x, c_2 x$, $(c_1, c_2)^\top = C$ into



(a) x_1(green) (b) x_3(green), $\cos t$(red),

FIG. 5.4. *Numerical results of ADRC for Example* (5.2) *(for interpretation of the references to color of the figure's legend in this section, we refer to the PDF version of this article).*

system (5.32):

$$\left\{\begin{array}{l}\dot{\hat{x}}_1^1=\hat{x}_2^1+\frac{6}{\varepsilon}(c_1x-\hat{x}_1^1),\\ \dot{\hat{x}}_2^1=\hat{x}_3^1+\frac{11}{\varepsilon^2}(c_1x-\hat{x}_1^1)+u_1+u_2,\\ \dot{\hat{x}}_3^1=\frac{6}{\varepsilon^3}(c_1x-\hat{x}_1^1),\end{array}\right. \qquad (5.38)$$
$$\left\{\begin{array}{l}\dot{\hat{x}}_1^2=\hat{x}_2^2+\frac{2}{\varepsilon}(c_2x-\hat{x}_1^2)+u_1-u_2,\\ \dot{\hat{x}}_2^2=\frac{1}{\varepsilon^2}(c_2x-\hat{x}_1^2).\end{array}\right.$$

The observer based feedback controls are designed by

$$\left\{\begin{array}{l}u_1^*=-9(\hat{x}_1^1-z_1^1)-6(\hat{x}_2^1-z_2^1)+z_3^1-\hat{x}_3^1,\\ u_2^*=-4(\hat{x}_1^2-z_1^2)+z_2^2-\hat{x}_2^2,\\ u_1=\frac{u_1^*+u_2^*}{2},\quad u_2=\frac{u_1^*-u_2^*}{2}.\end{array}\right. \qquad (5.39)$$

The numerical results by ADRC (5.37), (5.38), (5.39) for system (5.32) are plotted in Figure 5.4 with initial values $x(0)=(0.5,0.5,0.5)$, $\hat{x}^1(0)=(0,0,0)$, $\hat{x}^2(0)=(0,0)$, $z^1(0)=(1,1,1)$, $z^2(0)=(1,1)$, $\rho=50, \varepsilon=0.005$, and the integral step $h=0.001$.

Figures 5.3 and 5.4 witness the validity of both IMP and ADRC for the regulation problem Example 5.2. It is seen from Figures 5.3 and 5.4 that Figure 5.4 has the advantages of fast tracking and less overstriking.

Finally, we indicate the possible disadvantage of the peaking problem produced by the high gain in the ESO, which may lead to high energy in control. This phenomenon can be avoided effectively by saturated control as stated in Remark 5.1.

6. Concluding remarks. In this paper we established both the semiglobal convergence and global convergence of the nonlinear ADRC for a kind of MIMO system with large uncertainty. The key idea of ADRC is to use the ESO to estimate, in real time, both the state and the total disturbance (or extended state) which may arise from unknown system dynamics, external disturbance, and control parameters mismatch, and then cancel all the uncertainties in the feedback loop (Theorems 2.1, 3.1, and 4.1). As a result of this online estimation, the ADRC is expected to need less energy in control in comparison to other control strategies such as robust control, sliding mode control, and internal model principle. The nonlinear ADRC discussed in this paper extends the applicability of ADRC; it particularly covers as a special case the linear ESO for the stabilization of a nonlinear SISO with uncertainty used in [5] where the semi-global (local) convergence depends on the bound of the initial values. It also covers the ESO used in [18] for the stabilization of nonlinear SISO by estimating constant nominal value of control. For more discussions of the linear ADRC, we refer to [21].

The efficiency of ADRC in dealing with large uncertainly is demonstrated by simulations. Furthermore, for a class of linear MIMO systems, we compared the IMP and ADRC both analytically and numerically. We noted that the ADRC permits large uncertainty and does not require the dynamics of the exosystem in the control design. In some cases, simulation results suggest that the ADRC even leads to faster tracking and exhibits lower overstriking in the same time.

Roughly speaking, our results on the ADRC require that the unknown system functions are continuous differentiable for semiglobal convergence (Theorem 2.1), or Lipschitz continuous for global convergence (Theorem 3.1), and the external disturbances and their first order derivatives are bounded. However, these conditions are sufficient conditions. In practice, the ADRC may deal with more complicated disturbances. For these who use the ADRC to solve the engineering problems, our results suggest both theoretically and numerically that the ADRC is very effective for these uncertain systems with continuous differentiable or Lipschitz continuous unknown system functions, and the bounded disturbance with bounded first order derivatives. Moreover, if the bound of the initial state is known, the saturated feedback can avoid the peaking phenomenon very effectively. Otherwise, the high gain in the ESO should be small in the beginning and then increase gradually afterwards.

Finally, we indicate a potential application of the ADRC to more complicated systems like

$$\begin{cases} \dot{x}_1 = f_1(x_1, x_2), \\ \quad\vdots \\ \dot{x}_n = f_n(x, \xi, w) + b(x, \xi, w)u, \\ \dot{\xi} = F(x, \xi, w), \\ y = x_1, \end{cases}$$

or the MIMO system composed by the subsystems that are similar with above system. Our preliminary study shows that if $f_i, i = 1, \ldots, n-1$ are known and f_n is unknown, we can design a modified ESO and the associated ADRC to deal with the above system. This would be forthcoming work.

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