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# ON CONVERGENCE THEORY IN FUZZY TOPOLOGICAL SPACES AND ITS APPLICATIONS

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*Abstract.* In this paper we introduce and study new concepts of convergence and adherent points for fuzzy filters and fuzzy nets in the light of the $Q$-relation and the $Q$-neighborhood of fuzzy points due to Pu and Liu [28]. As applications of these concepts we give several new characterizations of the closure of fuzzy sets, fuzzy Hausdorff spaces, fuzzy continuous mappings and strong $Q$-compactness. We show that there is a relation between the convergence of fuzzy filters and the convergence of fuzzy nets similar to the one which exists between the convergence of filters and the convergence of nets in topological spaces.

*Keywords*: fuzzy points, $Q$-neighborhoods, fuzzy filters, fuzzy nets, limit, adherent and $Q$-adherent points of fuzzy filters and fuzzy nets, fuzzy continuity, strong $Q$-compactness

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## 1. Introduction

The notion of convergence is one of the basic notion in analysis. There are two different convergence theories used in general topology that lead to equivalent results. One of them is based on the notion of a net from 1922 due to Moore and Smith [25]; another one, which goes back to the work of Cartan [5] in 1937 and Bourbaki [4] in 1940, is based on the notion of a filter. In 1955 Bartle [1] studied the relation between filters and nets convergence theories, and he proved that they are equivalent. In 1979 Lowen [20] introduced and studied the theory of convergence for fuzzy filters (prefilters) in fuzzy topological spaces and applied the results to describe fuzzy compactness and fuzzy continuity. This was followed by an extensive study of the convergence theory of fuzzy filters in fuzzy topological spaces by several authors [7], [9]–[12], [21], [22], [26], [27], [31] from different standpoints. However, the concepts of $Q$-relation and $Q$-neighborhood of fuzzy points have not been used in these approaches. In 1980 Pu and Liu [28], by means of the concept of a $Q$-neighborhood of a fuzzy point in a fuzzy topological space, gave the notion of convergence for fuzzy nets. The relationship between the two types of convergence was studied by Lowen in [21]. This was also followed by an extensive study of the convergence theory in fuzzy topological spaces by several authors [7], [8], [13], [21]–[24], [28]–[30] from different standpoints. In this paper, we will give new concepts of convergence and adherence for fuzzy filters (prefilters in the sense of Lowen [20]) and fuzzy nets in fuzzy topological spaces. These concepts depend on the notions of $Q$-relation and $Q$-neighborhood of a fuzzy point given by Liu and Pu [28], as these concepts seem to be extremely suitable for the fuzzy situation. Then several characterizations of some fuzzy topological concepts, namely: the closure of a fuzzy set, fuzzy Hausdorff spaces, fuzzy continuous mappings and strong $Q$-compactness are given by means of convergence of fuzzy filters and fuzzy nets. Finally, we show that there is a relation between the convergence of fuzzy nets and the convergence of fuzzy filters similar to the one which exists between the convegence of nets and filters in topological spaces.

## 2. Preliminaries

Let $X$ be an arbitrary nonempty set. A fuzzy set in $X$ is a mapping from $X$ to the closed unit interval $I=[0,1]$, that is, an element of $I^X$. A fuzzy point (singleton) $x_\alpha$ is a fuzzy set in $X$ defined by $x_\alpha(x)=\alpha$ and $x_\alpha(y)=0$ for all $y\neq x$, whose support is the single point $x$ and whose value is $\alpha\in(0,1]$. We denote by $\mathrm{FP}(X)$ the collection of all fuzzy points in $X$. A family $\mathscr{A}\subseteq I^X$ is said to have finite intersection property (FIP, for short) if any finite intersection of elements of $\mathscr{A}$ is nonempty.

Throughout this paper $X$, $Y$, $Z$, etc. denote ordinary sets while $\mu$, $\eta$, $\lambda$, $\varrho$, etc. denote fuzzy sets defined on an ordinary set. For $A\subseteq X$, by $1_A$ we mean the characteristic mapping of $X$ to $\{0,1\}$. Also, we use $F$ to denote fuzzy and fts to denote a fuzzy topological space. The fuzzy set theoretical and fuzzy topological concepts used in this paper are standard and can be found in Zadeh [32], Klir and Yuah [18], Chang [6], Belohlavek [2], Lowen [19] and Pu and Liu [28]. However, we feel it necessary to fix some notions and recall a few concepts from [2], [6], [17], [18], [20], [28], [30], [32].

**Definition 2.1** [18], [32]. Let $\mu,\eta\in I^X$. We define the following fuzzy sets:
(i) $\mu\wedge\eta\in I^X$, by $(\mu\wedge\eta)(x)=\min\{\mu(x),\eta(x)\}$ for each $x\in X$ (intersection).
(ii) $\mu\vee\eta\in I^X$, by $(\mu\vee\eta)(x)=\max\{\mu(x),\eta(x)\}$ for each $x\in X$ (union).
(iii) $\mu^c\in I^X$, by $\mu^c(x)=1-\mu(x)$ for each $x\in X$ (complement).

**Definition 2.2** [17]. Let $\mu\in I^X$. We define the following crisp subsets:
(i) $\mu_{w\alpha}=\{x\in X\colon\ \mu(x)\geqslant\alpha\}$, the weak $\alpha$-cut of $\mu$, $\alpha\in(0,1]$.
(ii) $\mathrm{supp}(\mu)=\{x\in X\colon\ \mu(x)>0\}$, the support of $\mu$.

**Definition 2.3** [6]. Let $\tau \subseteq I^X$ satisfy the following three conditions:

(i) $X, \emptyset \in \tau$.

(ii) If $\mu_1, \mu_2 \in \tau$, then $\mu_1 \wedge \mu_2 \in \tau$.

(iii) If $\{\mu_j\colon\ j \in J\} \subseteq \tau$, then $\bigvee\limits_{j\in J} \mu_j \in \tau$.

Then $\tau$ is called a fuzzy topology on $X$ and $(X,\tau)$ is called a fuzzy topological space (fts, for short). The elements of $\tau$ are called open fuzzy sets. A fuzzy set is called a closed fuzzy set if $\mu^c \in \tau$. We denote by $\tau'$ the collection of all closed fuzzy sets in $X$.

One may easily verify the following example.

**Example 2.4.** Let $(X, T)$ be a topological space, $\mu \in I^X$ and $A \subseteq X$. Then:

(i) The classes $\tau_1 = \{X, \emptyset\}$, $\tau_2 = I^X$, $\tau_3 = \{\mu \in I^X\colon\ \mathrm{supp}(\mu) \in T\}$ and $\tau_4 = \{1_A\colon\ A \in T\}$ are fuzzy topologies on $X$.

(ii) If $X$ is an infinite set, then the class $\tau_\infty = \{\mu \in I^X\colon\ \mathrm{supp}(\mu^c)\ \text{is finite set}\} \vee \{\emptyset\}$ is a fuzzy topology on $X$; the so-called co-finite fuzzy topology on $X$.

**Definition 2.5** [28]. Let $(X,\tau)$ be a fts and $\mu, \varrho \in I^X$. Then:

(i) $\mu$ is called a quasi-coincident with $\varrho$, denoted by $\mu q \varrho$, if there exists $x \in X$ such that $\mu(x) + \varrho(x) > 1$. If $\mu$ is not quasi-coincident with $\varrho$, then we write $\mu \bar{q} \varrho$.

(ii) $\mu$ is called $Q$-neighborhood of a fuzzy point $x_\alpha \in \mathrm{FP}(X)$ if there exists an open fuzzy set $\eta \in \tau$ such that $x_\alpha q \eta$ and $\eta \leqslant \mu$. The class of all open $Q$-neighborhoods of $x_\alpha$ is denoted by $N^Q_{x_\alpha}$.

(iii) $\varrho$ is called a $Q_\alpha$-neighborhood of a fuzzy set $\mu$ iff $\varrho$ is a $Q$-neighborhood of each fuzzy point $x_\alpha \leqslant \mu$. The class of all open $Q_\alpha$-neighborhoods of $\mu$ will be denoted by $N^{Q_\alpha}_\mu$.

**Definition 2.6** [2], [28]. Let $(X,\tau)$ be a fts, $x_\alpha \in \mathrm{FP}(X)$ and $\mu \in I^X$. The closure of $\mu$, denoted by $\mathrm{cl}(\mu)$, is defined by: $x_\alpha \leqslant \mathrm{cl}(\mu)$ iff $(\forall\, \eta \in N^Q_{x_\alpha})(\eta q \mu)$. The fuzzy set $\mu$ is closed if $\mu = \mathrm{cl}(\mu)$.

**Theorem 2.7** [28]. *Let $(X,\tau)$ be a* fts *and $\mu \in I^X$. Then:*

(i) *$\mu$ is open iff $(\forall\, x_\alpha q \mu)(\exists\, \eta \in N^Q_{x_\alpha})(\eta \leqslant \mu)$.*

(ii) *For each $\eta \in \tau$, $\eta q \mu$ iff $\eta q\, \mathrm{cl}(\mu)$.*

**Definition 2.8** [28]. A fts $(X,\tau)$ is called a fuzzy Hausdorff space ($\mathrm{FT}_2$, for short) iff $(\forall\, x_\varepsilon, y_\nu \in \mathrm{FP}(X), x \neq y)(\exists\, \mu \in N^Q_{x_\varepsilon})(\exists\, \eta \in N^Q_{y_\nu})(\mu \wedge \eta = \emptyset)$.

## 3. New convergence theory of fuzzy filters

In this section we will introduce and study the notions of convergence, adherence and $Q$-adherence for fuzzy filters and filter bases in fuzzy topological spaces by means of the concept of a $Q$-neighborhood of a fuzzy point given by Liu and Pu [28]. This will enable us to give some results about fuzzy Hausdorff spaces [18], [28], closure of a fuzzy set [18], [28], fuzzy continuity [14], [28] and about strong $Q$-compactness [33], similar to the ones which hold in topological spaces.

**Definition 3.1** [20]. A fuzzy filter on $X$ is a nonempty subset $\mathscr{F} \subseteq I^X$ such that

(i) $\emptyset \notin \mathscr{F}$.
(ii) If $\mu \in \mathscr{F}$ and $\mu \leqslant \eta$, then $\eta \in \mathscr{F}$.
(iii) If $\mu_1, \mu_2 \in \mathscr{F}$, then $\mu_1 \wedge \mu_2 \in \mathscr{F}$.

The class of all fuzzy filters on $X$ will be denoted by $\mathrm{FL}(I^X)$.

**Definition 3.2** [20]. A fuzzy filter base on $X$ is a nonempty subset $\beta \subseteq I^X$ such that:

(i) $\emptyset \notin \beta$.
(ii) If $\mu_1, \mu_2 \in \beta$, then $\exists\, \mu_3 \in \beta$ such that $\mu_3 \leqslant \mu_1 \wedge \mu_2$.

The class of all fuzzy filter bases on $X$ will be denoted by $\mathrm{FLB}(I^X)$.

The fuzzy filter $\mathscr{F}$ generated by $\beta$ is defined by $\mathscr{F} = \{\mu \in I^X :\ \eta \leqslant \mu \text{ for some } \eta \in \beta\}$ and is denoted by $\langle\beta\rangle$. A collection $\beta$ of subsets of $\mathscr{F}$ is a base for $\mathscr{F}$ if for each $\mu \in \mathscr{F}$ there is $\eta \in \beta$ such that $\eta \leqslant \mu$.

**Definition 3.3** [20]. Let $\mathscr{F}, \mathscr{H} \in \mathrm{FL}(I^X)$ (or $\mathrm{FLB}(I^X)$). Then we say that $\mathscr{F}$ is finer than $\mathscr{H}$, written as $\mathscr{H} < \mathscr{F}$, if $(\forall\, \mu \in \mathscr{H})(\exists\, \varrho \in \mathscr{F})(\varrho \leqslant \mu)$.

**Definition 3.4** [20]. Let $X$ be a nonempty set. Then:

(i) A fuzzy filter $\mathscr{F}$ on $X$ is called a prime if for all $\mu, \eta \in I^X$, $\mu \vee \eta \in \mathscr{F}$ implies that either $\mu \in \mathscr{F}$ or $\eta \in \mathscr{F}$.
(ii) A fuzzy filter $\mathscr{F}$ on $X$ is called a maximal fuzzy filter on $X$ iff $\mathscr{F}$ is finer than every fuzzy filter comparable with it.
(iii) A fuzzy filter base $\beta$ on $X$ is called a maximal fuzzy filter base on $X$ if it is a base for a maximal fuzzy filter on $X$.
(iv) A subfamily $\xi$ of fuzzy filter $\mathscr{F}$ on $X$ is said to be a subbase for $\mathscr{F}$ if the family of all finite intersections of members of $\xi$ is a base for $\mathscr{F}$. We say that $\xi$ generates $\mathscr{F}$.

If $\mathscr{P}(\mathscr{F})$ denotes the family of all prime fuzzy filters finer than $\mathscr{F}$, then it is easy to verify that $\mathscr{F} = \bigcap\limits_{\mathscr{P} \in \mathscr{P}(\mathscr{F})} \mathscr{P}$.

The class of all maximal fuzzy filters and maximal fuzzy filter bases on $X$ will be denoted by $\mathrm{MFL}(I^X)$ and $\mathrm{MFLB}(I^X)$, respectively.

**Theorem 3.5.** *If $\mathscr{C}$ is a nonempty family of fuzzy subsets of $X$ having the* FIP, *then there exists a fuzzy filter base on $X$ containing $\mathscr{C}$.*

Proof. Similar to the proof of the corresponding one in the crisp case [15]. □

**Theorem 3.6.** *Let $\xi$ be a family of nonempty fuzzy subsets of $X$ which has* FIP. *Then there exists a fuzzy filter $\mathscr{F}$ on $X$ having $\xi$ as a subbase iff $\xi$ has* FIP.

Proof. Follows immediately from Definition 3.4 (iv) and Theorem 3.5. □

**Theorem 3.7** [31]**.** *For an $\mathscr{F} \in \mathrm{MFL}(I^X)$, the following statements are true:*
(i) *$\mathscr{F}$ is a prime fuzzy filter.*
(ii) *For every $\mu \in I^X$, either $\mu \in \mathscr{F}$ or $1 - \mu \in \mathscr{F}$.*
(iii) *Let $\mu \in I^X$. If $\mu \notin \mathscr{F}$, then there is $\varrho \in \mathscr{F}$ such that $\mu \wedge \varrho = \emptyset$.*

**Theorem 3.8** [26]**.** *Let $f\colon X \to Y$ be a mapping.*
(i) *If $\beta$ is a fuzzy filter base on $X$, then so is $f(\beta) = \{f(\lambda)\colon \lambda \in \beta\}$ on $Y$.*
(ii) *If $\beta$ is a fuzzy filter base on $Y$ and $f$ is onto, then $f^{-1}(\beta) = \{f^{-1}(\lambda)\colon \lambda \in \beta\}$ is a fuzzy filter base on $X$.*

For further details on fuzzy filters and related concepts, see [20], [26].

**Definition 3.9.** A fuzzy filter (filter base) $\mathscr{F}$ on $X$ is called:
(i) an upper fuzzy filter (filter base) on $X$ if for every $\lambda \in \mathscr{F}$, $\lambda(x) > \frac{1}{2}$ for some $x \in X$;
(ii) on $\mu \in I^X$ if $\lambda \wedge \mu \neq \emptyset$ for each $\lambda \in \mathscr{F}$.

**Definition 3.10.** A fuzzy filter (filter base) $\mathscr{F}$ in a fts $(X, \tau)$ is said to be:
(i) convergent to a fuzzy point $x_\alpha$, denoted by $\mathscr{F} \to x_\alpha$, iff $(\forall \eta \in N^Q_{x_\alpha})$ $(\exists \varrho \in \mathscr{F})(\varrho \leqslant \eta)$. The limit of a fuzzy filter base $\mathscr{F}$ is defined by $\lim(\mathscr{F}) = \bigvee\{x_\alpha \in \mathrm{FP}(X)\colon \mathscr{F} \to x_\alpha\}$.
(ii) convergent to a fuzzy set $\mu$, denoted by $\mathscr{F} \to \mu$, iff for each open $Q_\alpha$-cover $\mathscr{U}$ of $\mu$ (see Definition 3.26 below), there exist a finite subcollection $\mathscr{U}_0$ of $\mathscr{U}$ and an $\lambda \in \mathscr{F}$ such that $\lambda \leqslant \bigvee\{\eta\colon \eta \in \mathscr{U}_0\}$.

**Definition 3.11.** Let $(X, \tau)$ be a fts and $\mathscr{F}$ a fuzzy filter (filter base) on $X$. A fuzzy point $x_\alpha$ is said to be an adherent ($Q$-adherent) point for a fuzzy filter (filter base) $\mathscr{F}$, denoted by $\mathscr{F} \propto x_\alpha$ ($\mathscr{F} \overset{Q}{\propto} x_\alpha$), iff for all $\eta \in N^Q_{x_\alpha}$ and for all $\varrho \in \mathscr{F}$ we have $\varrho \wedge \eta \neq \emptyset$ ($\varrho q \eta$). The adherence and $Q$-adherence of the fuzzy filter (filter base) $\mathscr{F}$ is defined by $\mathrm{adh}(\mathscr{F}) = \bigvee\{x_\alpha \in \mathrm{FP}(X)\colon \mathscr{F} \propto x_\alpha\}$ and $Q\text{-}\mathrm{adh}(\mathscr{F}) = \bigvee\{x_\alpha \in \mathrm{FP}(X)\colon \mathscr{F} \overset{Q}{\propto} x_\alpha\}$, respectively.

The proof of the following result is straightforward and therefore it is omitted.

**Theorem 3.12.** *Let $(X,\tau)$ be a fts and $\mathscr{F} \in \mathrm{FL}(I^X)$ (or $\mathrm{FLB}(I^X)$). Then*

(i) *$N^Q_{x_\alpha}$ is a fuzzy filter (filter base) on $X$ and $N^Q_{x_\alpha} \to x_\alpha$.*

(ii) *$\mathscr{F} \to x_\alpha$ iff $x_\alpha \leqslant \lim(\mathscr{F})$ iff $N^Q_{x_\alpha} < \mathscr{F}$.*

(iii) *$\mathscr{F} \propto x_\alpha$ ($\mathscr{F} \overset{Q}{\propto} x_\alpha$) iff $x_\alpha \leqslant \mathrm{adh}(\mathscr{F})$ ($x_\alpha \leqslant Q\text{-}\mathrm{adh}(\mathscr{F})$).*

(iv) *$\lim(\mathscr{F}) \leqslant \mathrm{adh}(\mathscr{F})$ and $Q\text{-}\mathrm{adh}(\mathscr{F}) \leqslant \mathrm{adh}(\mathscr{F})$ but $\lim(\mathscr{F}) \nleqslant Q\text{-}\mathrm{adh}(\mathscr{F})$. If $\mathscr{F}$ is an upper fuzzy filter (filter base), then $\lim(\mathscr{F}) \leqslant Q\text{-}\mathrm{adh}(\mathscr{F})$.*

(v) *If $\mathscr{F} < \mathscr{F}^*$, then $\lim(\mathscr{F}) \leqslant \lim(\mathscr{F}^*)$, $\mathrm{adh}(\mathscr{F}^*) \leqslant \mathrm{adh}(\mathscr{F})$ and $Q\text{-}\mathrm{adh}(\mathscr{F}^*) \leqslant Q\text{-}\mathrm{adh}(\mathscr{F})$.*

(vi) *If $\mathscr{F} \to \mu$, then $N^{Q_\alpha}_\mu < \mathscr{F}$, where $N^{Q_\alpha}_\mu$ is a fuzzy filter base on $X$.*

(vii) *If $\mathscr{F} \to x_\alpha$ ($\mathscr{F} \propto x_\alpha$, $\mathscr{F} \overset{Q}{\propto} x_\alpha$) and if $\nu \leqslant \alpha$, then $\mathscr{F} \to x_\nu$ ($\mathscr{F} \propto x_\nu$, $\mathscr{F} \overset{Q}{\propto} x_\nu$, respectively).*

(viii) *If $\mathscr{F} \to x_\alpha$ and $x_\alpha \leqslant \mu$, then $\mathscr{F} \to \mu$.*

(ix) *If $\mathscr{F} < \mathscr{F}^*$ and $\mathscr{F} \to \mu$, then $\mathscr{F}^* \to \mu$.*

(x) *If $\beta$ is a base for a fuzzy filter $\mathscr{F}$ on $X$, then $\beta \to x_\alpha$ ($\beta \propto x_\alpha$, $\beta \overset{Q}{\to} x_\alpha$, $\beta \to \mu$) iff $\mathscr{F} \to x_\alpha$ ($\mathscr{F} \propto x_\alpha$, $\mathscr{F} \overset{Q}{\propto} x_\alpha$, $\mathscr{F} \to \mu$, respectively).*

**Theorem 3.13.** *Let $(X,\tau)$ be a fts. The fuzzy point $x_\alpha$ is an adherent point for a fuzzy filter base $\beta$ on $X$ iff there exists a fuzzy filter base $\beta^*$ on $X$ finer than $\beta$ and converging to $x_\alpha$.*

P r o o f. If $x_\alpha$ is an adherent point of $\beta$, then $(\forall\, \eta \in N^Q_{x_\alpha})(\forall\, \varrho \in \beta)(\varrho \wedge \eta \neq \emptyset)$. Then the family $\xi = \beta \vee N^Q_{x_\alpha}$ forms a subbase for some fuzzy filter base $\beta^*$ on $X$ finer than $\beta$ and $\mathscr{B}^* \to x_\alpha$. Conversely, if there exists a fuzzy filter base $\beta^*$ on $X$ finer than $\beta$ and $\beta^* \to x_\alpha$, then by Theorem 3.12 (iv), $\beta^* \propto x_\alpha$ and so by Theorem 3.12 (v), $\beta \to x_\alpha$. □

**Theorem 3.14.** *Let $(X,\tau)$ be a fts and $\beta$ an upper fuzzy filter base on $X$. A fuzzy point $x_\alpha$ is a $Q$-adherent point for $\beta$ iff there exists an upper fuzzy filter base $\beta^*$ on $X$ finer than $\beta$ and converging to $x_\alpha$.*

P r o o f. It is similar to that of Theorem 3.13 taking into consideration that $\beta$ and $\beta^*$ are upper fuzzy filter bases. □

**Theorem 3.15.** *Let $(X,\tau)$ be a fts, $x_\alpha \in \mathrm{FP}(X)$ and $\mu \in I^X$. Then $x_\alpha \leqslant \mathrm{cl}(\mu)$ iff there exists a fuzzy filter $\mathscr{F}$ on $X$ with $\mathscr{F} \to x_\alpha$ and $\varrho q \mu$ for each $\varrho \in \mathscr{F}$.*

P r o o f. Let $x_\alpha \leqslant \mathrm{cl}(\mu)$. Then $\mu q \eta$ for each $\eta \in N^Q_{x_\alpha}$. So it suffices to take $\mathscr{F} = N^Q_{x_\alpha}$. Conversely, let $\mathscr{F} \to x_\alpha$ and $\varrho q \mu$ for each $\varrho \in \mathscr{F}$. Then $N^Q_{x_\alpha} \leqslant \mathscr{F}$ and so $x_\alpha \leqslant \mathrm{cl}(\mu)$. □

**Theorem 3.16.** *Let $(X,\tau)$ be a fts and $\mu \in I^X$. Then $\mu$ is open iff $\mu$ is a member of each fuzzy filter on $X$ converging to a fuzzy point quasi-concident with $\mu$.*

Proof. Suppose that $\mu \in I^X$ is open and let $\mathscr{F}$ be an arbitrary fuzzy filter on $X$ converging to $x_\alpha$ and $x_\alpha q\mu$. Since $\mu \in N^Q_{x_\alpha}$, there exists $\lambda \in \mathscr{F}$ such that $\lambda \leqslant \mu$. Hence $\mu \in \mathscr{F}$. Conversely, for each $x_\alpha q\mu$ we consider $\mathscr{F} = N^Q_{x_\alpha}$. Then $\mu \in N^Q_{x_\alpha}$ for each $x_\alpha q\mu$ and so $\mu \in \tau$. □

The set of fuzzy points to which a fuzzy filter converges is, in general, infinite. But if we put certain restrictions on the supports, we can obtain a result concerning the uniqueness of convergence.

**Theorem 3.17.** *A* fts $(X,\tau)$ *is* $\mathrm{FT}_2$ *iff no fuzzy filter $\mathscr{F}$ on $X$ converges to two fuzzy points with different supports.*

Proof. Let $(X,\tau)$ be an $\mathrm{FT}_2$-space and $\mathscr{F}$ a fuzzy filter on $X$ such that $\mathscr{F} \to x_\alpha$ and $y_\alpha$ such that $x \neq y$. Since $\mathscr{F} \to x_\alpha$, we have $(\forall\, \mu \in N^Q_{x_\alpha})(\exists\, \varrho_1 \in \mathscr{F})(\varrho_1 \leqslant \mu)$. Also, since $\mathscr{F} \to y_\alpha$, we have $(\forall\, \eta \in N^Q_{y_\alpha})(\exists\, \varrho_2 \in \mathscr{F})(\varrho_2 \leqslant \eta)$. Since $\varrho_1, \varrho_2 \in \mathscr{F}$, then $\varrho_3 = \varrho_1 \wedge \varrho_2 \in \mathscr{F}$ such that $\varrho_3 \leqslant \mu \wedge \eta$ and so $\mu \wedge \eta \neq \emptyset$, a contradiction. Conversely, suppose that no fuzzy filter $\mathscr{F}$ on $X$ converges to two fuzzy points with different supports and $(X,\tau)$ is not an $\mathrm{FT}_2$-space, then $(\forall\, \mu \in N^Q_{x_\alpha})(\forall\, \eta \in N^Q_{y_\alpha})(\mu \wedge \eta \neq \emptyset)$. Then it is easy to verify that the family $\beta = \{\mu \wedge \eta \colon \mu \in N^Q_{x_\alpha} \text{ and } \eta \in N^Q_{y_\alpha}\}$ is a fuzzy filter base on $X$ converging to both $x_\alpha$ and $y_\alpha$. Let $\mathscr{F} = \langle \beta \rangle$. Then by Theorem 3.12 (x), $\mathscr{F}$ converges to both $x_\alpha$ and $y_\alpha$, a contradiction. □

**Theorem 3.18.** *Let $(X,\tau)$ be a* fts *and let $\mathscr{F}$ be a fuzzy filter on $X$. Then:*

(i) $\lim(\mathscr{F})$, $\mathrm{adh}(\mathscr{F})$ *and* $Q$-$\mathrm{adh}(\mathscr{F})$ *are closed fuzzy sets.*

(ii) *If $\mathscr{F}$ is a maximal fuzzy filter, then* $\lim(\mathscr{F}) = \mathrm{adh}(\mathscr{F})$.

(iii) *If $\mathscr{F}$ is a maximal upper fuzzy filter, then* $\lim(\mathscr{F}) = Q$-$\mathrm{adh}(\mathscr{F})$.

Proof. (i) Since $\lim(\mathscr{F}) \leqslant \mathrm{cl}(\lim(\mathscr{F}))$ it suffices to show that $\mathrm{cl}(\lim(\mathscr{F})) \leqslant \lim(\mathscr{F})$. Let $x_\alpha \leqslant \mathrm{cl}(\lim(\mathscr{F}))$ and $\eta \in N^Q_{x_\alpha}$. Then $\eta q \lim(\mathscr{F})$ and so there exists $y \in X$ such that $\eta(y) + \lim(\mathscr{F})(y) > 1$. Put $t = \lim(\mathscr{F})(y)$. Then $\eta \in N^Q_{y_t}$ and $y_t \leqslant \lim(\mathscr{F})$. Then for each $\varrho \in N^Q_{y_t}$, there exists $\lambda \in \mathscr{F}$ such that $\lambda \leqslant \varrho$. Thus $\lambda \leqslant \eta$ and so $x_\alpha \leqslant \lim(\mathscr{F})$. Hence $\mathrm{cl}(\lim(\mathscr{F})) \leqslant \lim(\mathscr{F})$. Thus $\lim(\mathscr{F})$ is a closed fuzzy set. Similarly, $\mathrm{adh}(\mathscr{F})$ and $Q$-$\mathrm{adh}(\mathscr{F})$ are closed fuzzy sets.

(ii) follows from Theorem 3.12 (iv) and Theorem 3.13.

(iii) follows from Theorem 3.12 (iv) and Theorem 3.14. □

**Theorem 3.19.** *If $\{\mathscr{F}_j\colon j \in J\}$ is a family of fuzzy filters on $X$ and $\mathscr{F} = \bigcap_{j\in J} \mathscr{F}_j$, then:*

(i) $\lim(\mathscr{F}) = \bigwedge_{j\in J} \lim(\mathscr{F}_j)$.

(ii) $\mathscr{F} \to \mu$ *iff* $\mathscr{F}_j \to \mu$ *for each* $j \in J$.

Proof. (i) Since $\mathscr{F} < \mathscr{F}_j$ for all $j \in J$, we have $\lim(\mathscr{F}) \leqslant \lim(\mathscr{F}_j)$ for all $j \in J$ and so $\lim(\mathscr{F}) \leqslant \bigwedge_{j\in J} \lim(\mathscr{F}_j)$. Conversely, if $x_\alpha \leqslant \bigwedge_{j\in J} \lim(\mathscr{F}_j)$, then $x_\alpha \leqslant \lim(\mathscr{F}_j)$ for all $j \in J$, then $(\forall\, \eta \in N^Q_{x_\alpha})(\eta \in \mathscr{F}_j)$ for all $j \in J$. So $\eta \in \mathscr{F}$ and hence $x_\alpha \leqslant \lim(\mathscr{F})$. Thus $\bigwedge_{j\in J} \lim(\mathscr{F}_j) \leqslant \lim(\mathscr{F})$.

(ii) Since $\mathscr{F} < \mathscr{F}_j$ for all $j \in J$, then by Theorem 3.12 (ix), $\mathscr{F} \to \mu$ implies $\mathscr{F}_j \to \mu$ for all $j \in J$. Conversely, if $\mathscr{F}_j \to \mu$ for each $j \in J$, then for each open $Q_\alpha$-cover $\mathscr{U}$ of $\mu$ there exist a finite subset $\mathscr{U}_0$ of $\mathscr{U}$ and a $\lambda_j \in \mathscr{F}_j$ such that $\lambda_j \leqslant \bigvee\{\eta\colon \eta \in \mathscr{U}_0\}$ for each $j \in J$. So $\bigwedge_{j\in J} \lambda_j \leqslant \bigvee\{\eta\colon \eta \in \mathscr{U}_0\}$. Put $\lambda = \bigwedge_{j\in J} \lambda_j$. Then $\lambda \in \mathscr{F}$ and $\lambda \leqslant \bigvee\{\eta\colon \eta \in \mathscr{U}_0\}$. Hence $\mathscr{F} \to \mu$. □

**Corollary 3.20.** *For a fuzzy filter $\mathscr{F}$ on $X$, we have $\mathscr{F} \to x_\alpha$ ($\mathscr{F} \to \mu$) iff $\mathscr{P} \to x_\alpha$ ($\mathscr{P} \to \mu$) for each $\mathscr{P} \in P(\mathscr{F})$.*

It is well known that continuity can be completely characterized by means of the convergence of filter bases. In the following, we characterize the fuzzy continuity between fuzzy topological spaces by means of convergent fuzzy filter bases.

**Definition 3.21** [14], [28]**.** A mapping $f\colon (X,\tau) \to (Y,\Delta)$ is called $F$-continuous if $(\forall\, x_\alpha \in \mathrm{FP}(X))(\forall\, \eta \in N^Q_{f(x_\alpha)})(\exists\, \mu \in N^Q_{x_\alpha})(f(\mu) \leqslant \eta)$.

**Theorem 3.22.** *Let $f\colon (X,\tau) \to (Y,\Delta)$ be a mapping. Then the following are equivalent:*

(i) *$f$ is $F$-continuous.*

(ii) *For each $x_\alpha \in \mathrm{FP}(X)$, $f(N^Q_{x_\alpha}) \to f(x_\alpha)$.*

(iii) *For each fuzzy filter base $\beta$ on $X$, $\beta \to \mu$ implies $f(\beta) \to f(\mu)$.*

(iv) *For each fuzzy filter base $\beta$ on $X$, $\beta \to x_\alpha$ implies $f(\beta) \to f(x_\alpha)$ for each $x_\alpha \in \mathrm{FP}(X)$.*

(v) *For each prime fuzzy filter $\mathscr{P}$ on $X$, $\mathscr{P} \to \mu$ implies $f(\mathscr{P}) \to f(\mu)$.*

(vi) *For each prime fuzzy filter $\mathscr{P}$ on $X$, $\mathscr{P} \to x_\alpha$ implies $f(\mathscr{P}) \to f(x_\alpha)$ for each $x_\alpha \in \mathrm{FP}(X)$.*

(vii) *For each fuzzy filter base $\beta$ on $X$, $f(\lim(\beta)) \leqslant \lim(f(\beta))$.*

(viii) *For each prime fuzzy filter $\mathscr{P}$ on $X$, $f(\lim(\mathscr{P})) \leqslant \lim(f(\mathscr{P}))$.*

P r o o f. (i) $\iff$ (ii): The statement that $f$ is $F$-continuous is $(\forall\, x_\alpha \in \mathrm{FP}(X))(\forall\, \eta \in N^Q_{f(x_\alpha)})(\exists\, \mu \in N^Q_{x_\alpha})(f(\mu) \leqslant \eta)$ and this is exactly the statement that the fuzzy filter base $f(N^Q_{x_\alpha}) \to f(x_\alpha)$.

(i) $\Longrightarrow$ (iii): Let $\mu \in I^X$ and let $\beta$ be a fuzzy filter base on $X$ such that $\beta \to \mu$. Let $\mathscr{U} = \{\eta_j\colon\, j \in J\}$ be an open $Q_\alpha$-cover of $f(\mu)$. Then $(\forall\, x \in X$ with $\mu(x) \geqslant \alpha)(\exists\, j \in J)(\eta_j \in N^Q_{f(x_\alpha)})$. By $F$-continuity of $f$, $(\exists\, \sigma_j \in N^Q_{x_\alpha})(f(\sigma_j) \leqslant \eta_j)$. Then the family $\{\sigma_j\colon\, j \in J\}$ is an open $Q_\alpha$-cover of $\mu$. Since $\beta \to \mu$, there exist a finite subset $J_0$ of $J$ and a $\lambda \in \beta$ such that $\lambda \leqslant \bigvee\{\sigma_j\colon\, j \in J_0\}$. Then $f(\lambda) \leqslant f\big(\bigvee\{\sigma_j\colon\, j \in J_0\}\big) \leqslant \bigvee\{f(\sigma_j)\colon\, j \in J_0\} \leqslant \bigvee\{\eta_j\colon\, j \in J_0\}$. Thus $f(\beta) \to f(\mu)$.

(iii) $\iff$ (iv) and (v) $\iff$ (vi). Immediate from the fact that for each $\mu \in I^X$, $\mu = \bigvee\{x_\alpha\colon\, x_\alpha \leqslant \mu\}$ and $f(\mu) = \bigvee\{f(x_\alpha)\colon\, x_\alpha \leqslant \mu\}$.

(iv) $\iff$ (vii) and (vi) $\iff$ (viii): Follow immediately from the definition.

(iii) $\Longrightarrow$ (vi): Obvious.

(vi) $\Longrightarrow$ (iii): Let $\beta$ be a fuzzy filter base on $X$ with $\beta \to \mu$ and let $\mathscr{U} = \{\eta_j\colon\, j \in J\}$ be an open $Q_\alpha$-cover of $f(\mu)$. Put $\mathscr{F} = \langle\beta\rangle$. Then $\mathscr{F}$ is a fuzzy filter on $X$ with $\mathscr{F} \to \mu$. By Corollary 3.20, for each $\mathscr{P} \in \mathscr{P}(\mathscr{F})$ we have $\mathscr{P} \to \mu$ and so by (vi), $f(\mathscr{P}) \to f(\mu)$. Then there exist a finite subset $J_0$ of $J$ and a $\lambda \in \mathscr{P}$ such that $f(\lambda) \leqslant \bigvee\{\eta_j\colon\, j \in J_0\}$. Since $\lambda \in \mathscr{P}$ for each $\mathscr{P} \in \mathscr{P}(\mathscr{F})$, we have $\lambda \in \mathscr{F}$ and so $f(\lambda) \in f(\mathscr{F})$. Thus $f(\mathscr{F}) \to f(\mu)$.

(iv) $\Longrightarrow$ (i): Let $x_\alpha \in \mathrm{FP}(X)$ and $\eta \in N^Q_{f(x_\alpha)}$. Since $\beta = N^Q_{x_\alpha} \to x_\alpha$, , we have $f(\beta) \to f(x_\alpha)$ by (iv) and so $(\exists\, \varrho \in \beta)(f(\varrho) \leqslant \eta)$. Hence $f$ is $F$-continuous. □

If $(X, \tau)$ and $(Y, \Delta)$ are topological spaces and $f\colon\, (X, \tau) \to (Y, \Delta)$, then $f$ is continuous iff for any filter $\mathscr{F}$ on $X$, we have $f(\mathrm{adh}(\mathscr{B})) \subseteq \mathrm{adh}(f(\mathscr{F}))$. This property is lost in the fuzzy case as the following theorem shows.

**Theorem 3.23.** *If a mapping $f\colon\, (X, \tau) \to (Y, \Delta)$ is $F$-continuous, then for every $\beta \in \mathrm{FLB}(I^X)$ we have $f(\mathrm{adh}(\beta)) \leqslant \mathrm{adh}(f(\beta))$ $(f(Q\text{-}\mathrm{adh}(\beta)) \leqslant Q\text{-}\mathrm{adh}(f(\beta)))$.*

P r o o f. Let $\beta$ be a fuzzy filter base on $X$, $x_\alpha \leqslant Q$-$\mathrm{adh}(\beta)$, $\varrho \in \beta$ and $\eta \in N^Q_{f(x_\alpha)}$. By hypothesis, there exists $\mu \in N^Q_{x_\alpha}$ such that $f(\mu) \leqslant \eta$. Since $\mu q \varrho$, we have $\eta q f(\varrho)$. So $f(x_\alpha) \leqslant Q$-$\mathrm{adh}(f(\beta))$. This shows that $f(Q\text{-}\mathrm{adh}(\beta)) \leqslant Q\text{-}\mathrm{adh}(f(\beta))$. The proof for the part in front of the parentheses is similar. □

**Definition 3.24** [19]. Let $\{(X_j, \tau_j)\colon\, j \in J\}$ be a family of fuzzy topological spaces, $X = \prod\limits_{j \in J} X_j$ and let $\tau$ be the fuzzy topology on $X$ generated by the subbase $\{P_j^{-1}(\eta_j)\colon\, \eta_j \in \tau_j,\, j \in J\}$, where $P_j\colon\, X \to X_j$ is the projection mapping for each $j \in J$. The pair $(X, \tau)$ is called the fuzzy product space of the family $\{(X_j, \tau_j)\colon\, j \in J\}$.

**Theorem 3.25.** *Let $(X,\tau)$ be the fuzzy product space of a family $\{(X_j,\tau_j)\colon j\in J\}$ of fuzzy topological spaces, $\beta$ a fuzzy filter base on $X$, $\mu\in I^X$, $x_\alpha\in \mathrm{FP}(X)$ and let $P_j\colon X\to X_j$ be the projection map. Then:*
(i) *$\beta\to x_\alpha$ iff $P_j(\beta)\to P_j(x_\alpha)$ for each $j\in J$.*
(ii) *$\beta\to\mu$ iff $P_j(\beta)\to P_j(\mu)$ for each $j\in J$.*
(iii) *If $\beta\propto x_\alpha$, then $P_j(\beta)\propto P_j(x_\alpha)$ for each $j\in J$.*
(iv) *If $\beta$ is upper and $\beta\overset{Q}{\propto} x_\alpha$, then $P_j(\beta)\overset{Q}{\propto} P_j(x_\alpha)$ for each $j\in J$.*

Proof. (i) Since each $P_j$ is $F$-continuous, by Theorem 3.22 we have that $\beta\to x_\alpha$ implies that $p_j(\mathscr{B})\to P_j(x_\alpha)$ for each $j\in J$. Conversely, suppose that $p_j(\beta)\to P_j(x_\alpha)$ for each $j\in J$. Let $\mu\in N^Q_{x_\alpha}$ in $(X,\tau)$. Then there exist $j_1,j_2,\dots,j_n$ in $J$ and $\mu_n\in N^Q_{x^{j_n}_\alpha}$ in $(X_{j_n},\tau_{j_n})$ such that $\varrho=\bigwedge\limits_{k=1}^{n}P^{-1}_{j_k}(\mu_k)\leqslant\mu$ and $x_\alpha q\varrho$. Since each $p_j(\beta)\to P_j(x_\alpha)$, $\mu_k\in P_{j_k}(\beta)$. Let $\varrho_k\in\mathscr{B}$ be such that $P_{j_k}(\varrho_k)=\mu_k$. Then $\varrho_1\wedge\varrho_2\ldots\wedge\varrho_n\leqslant\varrho\leqslant\mu$ and hence $\mu\in\beta$. This proves that $\beta\to x_\alpha$.

(ii) Since each $P_j$ is $F$-continuous, by Theorem 3.22 we have that $\beta\to\mu$ implies $P_j(\beta)\to P_j(\mu)$. Conversely, suppose that $P_j(\beta)\to P_j(\mu)$ and let $\mathscr{U}=\{\eta_j\colon j\in J\}$ be an open $Q_\alpha$-cover of $\mu$. Then $(\forall x\in X$ with $\mu(x)\geqslant\alpha)(\exists j_0\in J)(x_\alpha q\eta_{j_0})$. Since $\eta_{j_0}\in\tau$, there exist $j_1,j_2,\dots,j_n$ in $J$ and $\sigma_i\in\tau_{j_i}$ such that $\sigma_0=\bigwedge\limits_{i=1}^{n}P^{-1}_{j_i}(\sigma_i)\leqslant\eta_{j_0}$ and $\sigma_0 q x_\alpha$. For each $i$ we have $P_{j_i}(x_\alpha)q\sigma_i$ and thus there exists $\lambda_i\in\beta$ with $P_{j_i}(\lambda_i)\leqslant\sigma_i$. Let $\lambda\in\beta$ with $\lambda\leqslant\bigwedge\limits_{i=1}^{n}\lambda_i$. Since $\lambda_i\leqslant P^{-1}_{j_i}(\sigma_i)$, we have $\lambda\leqslant\sigma_0\leqslant\eta_{j_0}$. Thus $\lambda\leqslant\bigvee\{\eta_{j_i}\colon i=0,1,2,\dots,n\}$ and this proves that $\beta\to\mu$.

(iii) follows from (i) and Theorem 3.13.

(iv) follows from (i) and Theorem 3.14. □

The concept of $Q_\alpha$-compactness (strong $Q$-compactness) due to Zhongfu [33] provides the theory of fuzzy compactness which is applicable to arbitrary fuzzy subsets. So, Kerre, Nouh and Kandil in [16], [17] introduced and studied the concept of $\varphi_{1,2}$-$F$-$Q_\alpha$-compactness to unify and generalize several characterizations and properties of $Q_\alpha$-compactness due to Zhonghu [33] and their weaker and stronger forms. In this section we give new characterizations and properties of strong $Q$-compactness in terms of convergence of the fuzzy filter base and the fuzzy filter.

**Definition 3.26** [33]. Let $(X,\tau)$ be a fts, $\alpha\in(0,1]$ and $\mu\in I^X$. Then:
(i) The family $\mathscr{U}=\{\eta_j\colon j\in J\}\subseteq\tau$ is called an open $Q_\alpha$-cover of $\mu$ iff $(\forall x\in X$ with $\mu(x)\geqslant\alpha)(\exists j\in J)(x_\varepsilon q\eta_j)$.
(ii) A subfamily $\mathscr{U}_0$ of a $Q_\alpha$-cover $\mathscr{U}$ of $\mu$, which is also a $Q_\alpha$-cover of $\mu$, is called a $Q_\alpha$-subcover of $\mu$.
(iii) A fuzzy set $\mu$ is called $Q_\alpha$-compact if each open $Q_\alpha$-cover of $\mu$ has a finite $Q_\alpha$-subcover of $\mu$.

(iii) A fuzzy set $\mu \in I^X$ of a fts $(X,\tau)$ is said to be strong $Q$-compact if $\mu$ is $Q_\alpha$-compact for each $\alpha \in (0,1]$.
(iv) A fts $(X,\tau)$ is called $Q_\alpha$-compact (strong $Q$-compact) iff $X$ is $Q_\alpha$-compact (strong $Q$-compact).

**Theorem 3.27.** *Let $(X,\tau)$ be a* fts *and $\mu \in I^X$. Then $\mu$ is strong $Q$-compact iff for each family $\{\mu_j\colon j \in J\} \subseteq \tau'$ such that $\bigwedge\limits_{j\in J} \mu_j(x) < \alpha$ for each $x \in \mu_{w\alpha}$, there exists a finite subset $J_0$ of $J$ such that $\bigwedge\limits_{j\in J_0} \mu_j(x) < \alpha$ for each $x \in \mu_{w\alpha}$.*

Proof. Let $\{\mu_j\colon j \in J\} \subseteq \tau'$ be such that $\bigwedge\limits_{j\in J} \mu_j(x) < \alpha$ for each $x \in \mu_{w\alpha}$. Then $(\forall\, x \in X,\, \mu(x) \geqslant \alpha)\Bigl(x_\alpha q \bigvee\limits_{j\in J} \mu_j^c\Bigr)$ and so the family $\{\mu_j^c\colon j \in J\}$ is an open $Q_\alpha$-cover of $\mu$. Since $\mu$ is strong $Q$-compact, there exists a finite subset $J_0$ of $J$ such that the family $\{\mu_j^c\colon j \in J_0\}$ is a $Q_\alpha$-subcover of $\mu$. Then $(\forall\, x \in X,\, \mu(x) \geqslant \alpha)\bigl(x_\alpha q \bigvee\limits_{j\in J_0} \mu_j^c\bigr)$. Thus $\bigwedge\limits_{j\in J_0} \mu_j(x) < \alpha$ for each $x \in \mu_{w\alpha}$. Conversely, let $\{\eta_j\colon j \in J\}$ be an open $Q_\alpha$-cover of $\mu$. Then $(\forall\, x \in X,\, \mu(x) \geqslant \alpha)(\exists\, j \in J)(x_\alpha q \eta_j)$ and so $\bigwedge\limits_{j\in J} \eta_j^c(x) < \alpha$ for each $x \in \mu_{w\alpha}$. Since $\eta_j^c \in \tau'$, by hypothesis there exists a finite subset $J_0$ of $J$ such that $\bigwedge\limits_{j\in J_0} \eta_j^c(x) < \alpha$ for each $x \in \mu_{w\alpha}$. It follows that $\{\eta_j\colon j \in J_0\}$ is a $Q_\alpha$-subcover of $\mu$. Thus $\mu$ is $Q_\alpha$-compact for each $\alpha \in (0,1]$. Thus $\mu$ is strong $Q$-compact. □

**Corollary 3.28.** *A* fts *$(X,\tau)$ is strong $Q$-compact iff for each family $\{\mu_j\colon j \in J\} \subseteq \tau'$ such that $\bigwedge\limits_{j\in J} \mu_j(x) < \alpha$ for each $x \in X$ there exists a finite subset $J_0$ of $J$ such that $\bigwedge\limits_{j\in J_0} \mu_j(x) < \alpha$ for each $x \in X$.*

**Theorem 3.29.** *Let $(X,\tau)$ be a* fts *and $\mu \in I^X$. If $\mu$ is strong $Q$-compact, then every upper fuzzy filter base (fuzzy filter base) $\beta$ on $\mu$ is $Q$-adherent (adherent) to some $x_\alpha$ in $\mu$.*

Proof. The proofs for both parts are very similar; so, we only present the proof for the part not in the parentheses. Let $\mu \in I^X$ be strong $Q$-compact and suppose that $\beta = \{\eta_j\colon j \in J\}$ is an upper fuzzy filter base on $\mu$ having no $Q$-adherent point in $\mu$. Let $x_\alpha \leqslant \mu$. Corresponding to each $j \in J$ there exists $\varrho_{x_\alpha} \in N^Q_{x_\alpha}$ and $\eta_j(x_\alpha) \in \beta$ such that $\varrho_{x_\alpha} \bar{q} \eta_j(x_\alpha)$. Thus, the family $\mathscr{U} = \{\varrho_{x_\alpha}\colon x \in X,\, \mu(x) \geqslant \alpha\}$ is an open $Q_\alpha$-cover of $\mu$. Since $\mu$ is strong $Q$-compact, for each $\alpha \in (0,1]$ there exist finitely many members $\varrho_{x^1_\alpha}, \varrho_{x^2_\alpha}, \ldots, \varrho_{x^n_\alpha}$ from $\mathscr{U}$ for $x^1_\alpha, x^2_\alpha, \ldots, x^n_\alpha \leqslant \mu$ such that the family $\mathscr{U}_0 = \{\varrho_{x^k_\alpha}\colon k = 1,2,\ldots,n\}$ is a $Q_\alpha$-subcover of $\mu$. Since $\beta$ is a fuzzy filter base on $\mu$, there exists $\eta_{j_0} \in \beta$ such that $\eta_{j_0} \leqslant \bigwedge_{k=1}^{n} \eta_j(x_\alpha^k)$ and $\eta_{j_0} \wedge \mu \neq \emptyset$ and so there exists $y_\nu \leqslant \eta_{j_0} \wedge \mu$. Since the family $\{\varrho_{x_\alpha^k} \colon k = 1, 2, \ldots, n\}$ is a $Q_\alpha$-cover of $\mu$, we have $y_\nu q \varrho_{x_\alpha^k}$ for some $1 \leqslant k \leqslant n$ and so $\eta_{j_0} q \varrho_{x_\alpha^k}$ which implies $\varrho_{x_\alpha^k} q \eta_j(x_\alpha^k)$, a contradiction. □

**Corollary 3.30.** *If a* fts $(X, \tau)$ *is strong* $Q$*-compact, then every upper fuzzy filter base (fuzzy filter base)* $\beta$ *on* $X$ *is* $Q$*-adherent (adherent) to some* $x_\alpha$*.*

**Theorem 3.31.** *Let* $(X, \tau)$ *be a* fts *and* $\mu \in I^X$*. If* $\mu$ *is strong* $Q$*-compact, then every maximal upper fuzzy filter base (maximal fuzzy filter base)* $\beta$ *on* $\mu$ *converges to some* $x_\alpha$ *in* $\mu$*.*

Proof. The proofs for both parts are very similar; so, we only present the proof for the part not in the parentheses. Let $\beta$ be a maximal upper fuzzy filter base on $\mu$. By Theorem 3.29, we have $\beta \overset{Q}{\propto} x_\alpha$ in $\mu$ and then by Theorem 3.18 (iii), we have $\beta \to x_\alpha$ in $\mu$. □

**Corollary 3.32.** *If a* fts $(X, \tau)$ *is strong* $Q$*-compact, then every maximal upper fuzzy filter base (maximal fuzzy filter base)* $\beta$ *on* $X$ *converges to some* $x_\alpha$*.*

As an application of the above results we can give a proof of the fuzzy Tychonoff Theorem (Theorem 3.16 in [17]) without use of the fuzzy Alexander's subbase theorem (Theorem 3.12 in [17]).

**Theorem 3.33.** *Let* $\{(X_j, \tau_j) \colon j \in J\}$ *be a family of* fts*'s, let* $(X, \tau) = \left(\prod_{j\in J} X_j, \prod_{j \in J} \tau_j\right)$ *be a fuzzy product space and* $\mu = \prod_{j\in J} \mu_j \in I^X$*. Then* $\mu_j$ *is a strong* $Q$*-compact fuzzy set in* $(X_j, \tau_j)$ *for each* $j \in J$ *iff* $\mu$ *is a strong* $Q$*-compact fuzzy set in* $(X, \tau)$*.*

Proof. Let $P_j \colon X \to X_j$ be the projection mapping and $\mu \in I^X$ a strong $Q$-compact fuzzy set in $(X, \tau)$. Since $P_j$ is $F$-continuous, hence by Theorem 3.8 in [16], $P_j(\mu) = \mu_j \in I^{X_j}$ is a strong $Q$-compact fuzzy set in $(X_j, \tau_j)$ for each $j \in J$. Conversely, suppose that each $\mu_j \in I^{X_j}$ is strong $Q$-compact and let $\beta$ be a fuzzy filter base on $\mu$. Then $p_j(\beta) = \beta_j$ is a fuzzy filter base on $\mu_j$ for each $j \in J$. By Theorem 3.27, $\beta_j \to x_\alpha^j$ and $x_\alpha^j \leqslant \mu_j$ for each $j \in J$. Then by Theorem 3.25 (i), $\beta \to x_\alpha = \langle x_\alpha^j \colon j \in J \rangle$ and $x_\alpha \leqslant \mu$. Hence $\mu$ is strong $Q$-compact. □

## 4. New convergence theory of fuzzy nets

In 1980, Pu and Liu introduced the notion of fuzzy nets and a new concept of the so-called Q-neighborhood was given, which could reflect the features of the neighborhood structure in fuzzy topological spaces. By this new neighborhood structure the Moore-Smith convergence theory was established splendidly [28].

In this section we will give a new notion of convergence for fuzzy nets different from the one given by Pu and Liu in [28]. For more details on the difference we refer to the results in Section 5, which are valid only for our new notions. Our notion is such that the relationship between fuzzy nets and fuzzy filters and their convergence is the same as the relation between filters and nets and their convergence in topological spaces.

**Definition 4.1** [28]**.** A mapping $\mathscr{S}\colon D \to \mathrm{FP}(X)$ is called a fuzzy net in $X$ and is denoted by $\{S(n);\, n \in D\}$, where $D$ is a directed set. If $S(n) = x^n_{\alpha_n}$ for each $n \in D$, where $x \in X$, $n \in D$ and $\alpha_n \in (0,1]$ then the fuzzy net $\mathscr{S}$ is denoted as $\{x^n_{\alpha_n};\, n \in D\}$ or simply $\{x^n_{\alpha_n}\}$.

**Definition 4.2** [28]**.** A fuzzy net $\mathscr{T} = \{y^m_{\alpha_m};\, m \in E\}$ in $X$ is called a fuzzy subnet of a fuzzy net $\mathscr{S} = \{x^n_{\alpha_n};\, n \in D\}$ iff there is a mapping $f\colon E \to D$ such that:

(i) $\mathscr{T} = \mathscr{S} \circ f$, that is, $T_i = \mathscr{S}_{f(i)}$ for each $i \in E$.

(ii) For each $n \in D$ there exists some $m \in E$ such that, if $p \in E$ with $p \geqslant m$, then $f(p) \geqslant n$.

We shall denote a fuzzy subnet of a fuzzy net $\{x^n_{\alpha_n};\, n \in D\}$ by $\{x^{f(m)}_{\alpha_{f(m)}};\, m \in E\}$, the justification for which is clear from the definition.

**Definition 4.3.** Let $(X,\tau)$ be a fts and let $\mathscr{S} = \{x^n_{\alpha_n};\, n \in D\}$ be a fuzzy net in $X$ and $\mu \in I^X$. Then $\mathscr{S}$ is said to be :

(i) in $\mu$ iff $(\forall\, n \in D)(x^n_{\alpha_n} \leqslant \mu)$.

(ii) eventually with $\mu$ iff $(\exists\, m \in D)(\forall\, n \in D,\, n \geqslant m)(x^n_{\alpha_n} \leqslant \mu)$.

(iii) frequently with $\mu$ iff $(\forall\, m \in D)(\exists\, n \in D,\, n \geqslant m)(x^n_{\alpha_n} \leqslant \mu)$.

(iv) universal iff $\forall\, \eta, \varrho \in I^X$, if $\mathscr{S}$ is eventually with $\eta \vee \varrho$, then it is eventually with $\eta$ or $\varrho$.

**Lemma 4.4.** *Let $(X,\tau)$ be a* fts *and $\alpha \in [0,1)$, then the net $\{x^n\}$ is a universal net in $(X, \iota_\alpha(\tau))$ iff the fuzzy net $\{x^n_{1-\alpha}\}$ is universal in $(X,\tau)$.*

Proof. It is similar to that of Lemma 1.1 in [8] and hence omitted.

**Definition 4.5.** Let $(X,\tau)$ be a fts, $\mathscr{S} = \{x^n_{\alpha_n};\, n \in D\}$ a fuzzy net in $X$ and $x_\alpha \in \mathrm{FP}(X)$. Then $\mathscr{S}$ is said to be:

(i) convergent to $x_\alpha$, denoted by $\mathscr{S} \to x_\alpha$, if $(\forall \eta \in N^Q_{x_\alpha})(\exists m \in D)$ $(\forall n \in D$ with $n \geqslant m)(x^n_{\alpha_n} \leqslant \eta)$.

(ii) $Q$-convergent to $x_\alpha$ [28], denoted by $\mathscr{S} \overset{Q}{\to} x_\alpha$, if $(\forall \eta \in N^Q_{x_\alpha})$ $(\exists m \in D)(\forall n \in D$ with $n \geqslant m)(x^n_{\alpha_n} q \eta)$.

The limit and $Q$-limit of the fuzzy net $\mathscr{S}$ is defined by $\lim(S) = \bigvee\{x_\alpha \in \mathrm{FP}(X)\colon S \to x_\alpha\}$ and $Q\text{-}\lim(S) = \bigvee\{x_\alpha \in \mathrm{FP}(X)\colon \mathscr{S} \overset{Q}{\to} x_\alpha\}$, respectively.

**Lemma 4.6.** *Let $(X,\tau)$ be a fts and $\mathscr{S} = \{x^n_{\alpha_n};\, n \in D\}$ a fuzzy net in $X$. Then*

(i) *$x_\alpha \leqslant \lim(\mathscr{S})$ $(x_\alpha \leqslant Q\text{-}\lim(\mathscr{S}))$ iff $\mathscr{S} \to x_\alpha$ $(\mathscr{S} \overset{Q}{\to} x_\alpha)$.*

(ii) *$\lim(\mathscr{S})$ and $Q\text{-}\lim(\mathscr{S})$ are closed fuzzy sets.*

Proof. (i) follows immediately from Definition 4.5.

(ii) Since $\lim(\mathscr{S}) \leqslant \mathrm{cl}(\lim(\mathscr{S}))$ it suffices to show that $\mathrm{cl}(\lim(\mathscr{S})) \leqslant \lim(\mathscr{S})$. Let $x_\alpha \leqslant \mathrm{cl}(\lim(\mathscr{S}))$ and $\eta \in N^Q_{x_\alpha}$. Then $\eta q \lim(\mathscr{S})$ and so there exists $y \in X$ such that $\eta(y) + \lim(\mathscr{S})(y) > 1$. Put $t = \lim(\mathscr{S})(y)$. Then $y_t q \eta$ and $y_t \leqslant \lim(\mathscr{S})$ and so $(\forall \varrho \in N^Q_{y_t})(\exists m \in D)(\forall n \in D,\, n \geqslant m)(x^n_{\alpha_n} \leqslant \varrho)$, which implies $x^n_{\alpha_n} \leqslant \eta$. Thus $x_\alpha \leqslant \lim(\mathscr{S})$. Hence $\mathrm{cl}(\lim(\mathscr{S})) \leqslant \lim(\mathscr{S})$. Thus $\lim(\mathscr{S})$ is a closed fuzzy set. The proof for $Q\text{-}\lim(\mathscr{S})$ is similar. □

**Theorem 4.7.** *Let $(X,\tau)$ be a fts, $x_\alpha \in \mathrm{FP}(X)$ and $\mu \in I^X$. Then $x_\alpha \leqslant \mathrm{cl}(\mu)$ iff there exists a fuzzy net $\mathscr{S} = \{x^n_{\alpha_n};\, n \in D\}$ convergent to $x_\alpha$ such that $x^n_{\alpha_n} q \mu$ for all‘ $n$.*

Proof. Let $\mathscr{S} \to x_\alpha$ and $x^n_{\alpha_n} q\mu$ for all $n$. If $\eta \in N^Q_{x_\alpha}$, then $x^n_{\alpha_n} \leqslant \eta$ for some $n$ and so $\eta q \mu$. Thus $x_\alpha \leqslant \mathrm{cl}(\mu)$. Conversely, let $x_\alpha \leqslant \mathrm{cl}(\mu)$. Then $(\forall \eta \in N^Q_{x_\alpha})(\eta q \mu)$. That is, there exist $y \in \mathrm{supp}(\eta)$ and a real number $t_\eta$ with $0 < t_\eta \leqslant \eta(y)$ such that $y^\eta_{t_\eta} \leqslant \eta$ and $y^\eta_{t_\eta} q\mu$. Let $D = N^Q_{x_\alpha}$. Then $(D, \leqslant)$ is directed under the inclusion relation. Then $\mathscr{S} = \{y^\eta_{t_\eta} \leqslant \eta\colon\, y^\eta_{t_\eta} q\mu,\, \eta \in D\}$ is a fuzzy net such that $\mathscr{S} \to x_\alpha$. □

**Lemma 4.8.** *Let $(X,\tau)$ be a fts, $x_\alpha \in \mathrm{FP}(X)$ and $\mathscr{S} = \{x^n_{x_\alpha};\, n \in D\}$. If $\mathscr{S} \to x_\alpha$, then every fuzzy subnet of $\mathscr{S}$ is also convergent to $x_\alpha$.*

Proof. Let $\mathscr{S} = \{x^n_{\alpha_n};\, n \in D\}$ be a fuzzy net in $X$ such that $\mathscr{S} \to x_\alpha$ and let $T = \{y^m_{t_m};\, m \in E\}$ be a fuzzy subnet of $\mathscr{S}$. Let $\eta \in N^Q_{x_\alpha}$. Then there exists $m \in D$ such that for every $n \in D$, $n \geqslant m$, we have $x^n_{\alpha_n} \leqslant \eta$. By definition of $T$, for this $m$ there exists $l \in E$ such that for each $p \in E$, $p \geqslant l$, we have $f(p) \geqslant m$, where $f\colon E \to D$. Now, $y^p_{t_p} = x^{f(p)}_{\alpha_{f(p)}}$. Then $y^p_{t_p} \leqslant \eta$ for all $p \geqslant l$ and so $\mathscr{T} \to x_\alpha$. □

**Theorem 4.9.** *The fuzzy net $\mathscr{S} = \{x^n_{\alpha_n};\, n \in D\}$ is convergent to $x_\alpha$ iff every universal fuzzy subnet of $\mathscr{S}$ is convergent to $x_\alpha$.*

Proof. If $\mathscr{S} \to x_\alpha$, then by Lemma 4.8 every universal fuzzy subnet of $\mathscr{S}$ is convergent to $x_\alpha$. Conversely, assume that the condition is satisfied and (by way of contradiction) that $\mathscr{S}$ is not convergent to $x_\alpha$. Then $(\exists\, \varrho \in N^Q_{x_\alpha})(\forall\, m \in D)(\exists\, n \in D,\, n \geqslant m)(\varrho(x^n) < x^n_{\alpha_n}(x^n))$. So we may assume that $\varrho(x^n) < x^n_{\alpha_n}(x^n)$ for all $n \in D$. But then, although there are universal subnets of $\mathscr{S}$, no such subnet could converge to $x_\alpha$. □

**Definition 4.10.** Let $(X, \tau)$ be a fts, $x_\alpha \in \mathrm{FP}(X)$ and let $\mathscr{S} = \{x^n_{\alpha_n};\, n \in D\}$ be a fuzzy net in $X$. Then $x_\alpha$ is called:

(i) an adherent point of a fuzzy net $\mathscr{S}$, denoted by $\mathscr{S} \propto x_\alpha$, if $(\forall\, \eta \in N^Q_{x_\alpha})$ $(\forall\, n \in D)(\exists\, m \in D,\, m \geqslant n)(x^m_{\alpha_m} \leqslant \eta)$.

(ii) a $Q$-adherent point of a fuzzy net $\mathscr{S}$ [28], denoted by $\mathscr{S} \overset{Q}{\propto} x_\alpha$, if $(\forall\, \eta \in N^Q_{x_\alpha})$ $(\forall\, n \in D)(\exists\, m \in D,\, m \geqslant n)(x^m_{\alpha_m} q \eta)$.

The adherence and $Q$-adherence of the fuzzy net $\mathscr{S}$ is defined by $\mathrm{adh}(\mathscr{S}) = \bigvee\{x_\alpha \in \mathrm{FP}(X)\colon\, \mathscr{S} \propto x_\alpha\}$ and $Q\text{-}\mathrm{adh}(\mathscr{S}) = \bigvee\{x_\alpha \in \mathrm{FP}(X)\colon\, \mathscr{S} \overset{Q}{\propto} x_\alpha\}$, respectively.

We now prove the expected relationship between adherent (Q-adherent) points and fuzzy subnets.

**Lemma 4.11.** *Let $(X, \tau)$ be a fts and $\mathscr{S} = \{x^n_{\alpha_n};\, n \in D\}$ a fuzzy net in $X$. Then:*

(i) $x_\alpha \leqslant \mathrm{adh}(\mathscr{S})$ *($x_\alpha \leqslant Q$-$\mathrm{adh}(\mathscr{S})$) iff $\mathscr{S} \propto x_\alpha$ ($\mathscr{S} \overset{Q}{\propto} x_\alpha$).*

(ii) $\lim(\mathscr{S}) \leqslant \mathrm{adh}(\mathscr{S})$ *and* $Q\text{-}\lim(\mathscr{S}) \leqslant Q\text{-}\mathrm{adh}(\mathscr{S})$.

(iii) $\mathrm{adh}(\mathscr{S})$ *and* $Q\text{-}\mathrm{adh}(\mathscr{S})$ *are closed fuzzy sets.*

Proof. It is similar to that of Lemma 4.6. □

**Theorem 4.12.** *A fuzzy point $x_\alpha$ is an adherent ($Q$-adherent) point of a fuzzy net $\mathscr{S} = \{x^n_{\alpha_n};\, n \in D\}$ iff there exists a fuzzy subnet $\mathscr{T}$ of $\mathscr{S}$ which converges ($Q$-converges) to $x_\alpha$.*

Proof. The proofs for both parts are very similar; so, we only present the proof for the part not in the parentheses. Let $x_\alpha \leqslant \mathrm{adh}(\mathscr{S})$. Then for any $\mu \in N^Q_{x_\alpha}$, there exists an element $x^n_{\alpha_n}$ of the fuzzy net $\mathscr{S}$ such that $x^n_{\alpha_n} \leqslant \mu$. Let $E = \{(n, \mu)\colon\, n \in D,\, \mu \in N^Q_{x_\alpha}$ and $x^n_{\alpha_n} \leqslant \mu\}$. Then $(E, \gg)$ is a directed set where $(m, \mu) \gg (n, \varrho)$ iff $m \geqslant n$ in $D$ and $\mu \leqslant \varrho$ in $N^Q_{x_\alpha}$. Then $\mathscr{T}\colon\, E \to \mathrm{FP}(X)$ given by $\mathscr{T}(m, \mu) = x^m_{\alpha_m}$ can be checked by a fuzzy subnet of‘ $\mathscr{S}$. To show that $\mathscr{T} \to x_\alpha$, let $\eta \in N^Q_{x_\alpha}$. Then there exists $n \in D$ such that $(n, \eta) \in E$ and then $x^n_{\alpha_n} \leqslant \eta$. Thus, for any $(m, \mu) \in E$ such that $(m, \mu) \gg (n, \eta)$, we have $\mathscr{T}(m, \mu) = x^m_{\alpha_m} \leqslant \mu \leqslant \eta$. Hence $\mathscr{T} \to x_\alpha$. The converse is clear. $\square$

In the following, we characterize fuzzy continuity between fuzzy topological spaces by means of convergent fuzzy nets.

**Theorem 4.13.** *Let $f\colon (X, \tau) \to (Y, \Delta)$ be a mapping. Then the following are equivalent:*

(i) *$f$ is $F$-continuous.*

(ii) *$\mathscr{S} \to x_\alpha$ implies $f(\mathscr{S}) \to f(x_\alpha)$ for each $x_\alpha \in \mathrm{FP}(X)$.*

(iii) *For each universal fuzzy net $\mathscr{S}$ in $X$, $\mathscr{S} \to x_\alpha$ implies $f(\mathscr{S}) \to f(x_\alpha)$ for each $x_\alpha \in \mathrm{FP}(X)$.*

(iv) *For each fuzzy net $\mathscr{S} = \{x^n_{\alpha_n};\, n \in D\}$ in $X$, $f(\lim(\mathscr{S})) \leqslant \lim(f(\mathscr{S}))$.*

(v) *For each universal fuzzy net $\mathscr{S}$ in $X$, $f(\lim(\mathscr{S})) \leqslant \lim(f(\mathscr{S}))$.*

(vi) *For each fuzzy net $\mathscr{S}$ in $X$, $f(\mathrm{adh}(\mathscr{S})) \leqslant \mathrm{adh}(f(\mathscr{S}))$.*

Proof. (i) $\Longrightarrow$ (ii). Let $\mathscr{S} = \{x^n_{\alpha_n};\, n \in D\}$ be a fuzzy net in $X$ such that $\mathscr{S} \to x_\alpha$. If $\eta \in N^Q_{f(x_\alpha)}$, then by (i), $f^{-1}(\eta) \in N^Q_{x_\alpha}$. Since $\mathscr{S} \to x_\alpha$, we have $(\exists\, m \in D)(\forall\, n \in D,\, n \geqslant m)(x^n_{\alpha_n} \leqslant f^{-1}(\eta))$ which implies that $f(x^n_{\alpha_n}) \leqslant \eta$. Thus $f(\mathscr{S}) \to f(x_\alpha)$.

(ii) $\Longrightarrow$ (i). Let $\eta \in I^Y$ be open and $x_\alpha \in \mathrm{FP}(X)$ be such that $x_\alpha q f^{-1}(\eta)$. Then $f(x_\alpha) q \eta$ and so $\eta \in N^Q_{f(x_\alpha)}$. Suppose that for each $\mu \in N^Q_{x_\alpha}$, we have $\mu \nleqslant f^{-1}(\eta)$. Then $\mu(x^\mu) > f^{-1}(\eta)(x^\mu)$ for some $x^\mu \in X$. Take $x^\mu_{\alpha(\mu)}$ to be a fuzzy point with $x^\mu_{\alpha(\mu)}(x^\mu) = \mu(x^\mu)$. Then $x^\mu_{\alpha(\mu)} \leqslant \mu$ and $\alpha(\mu) > f^{-1}(\eta)(x^\mu) = \eta(f(x^\mu))$. Let $D = (N^Q_{x_\alpha}, \leqslant)$. Then $\mathscr{S} = \{x^\mu_{\alpha(\mu)};\, \mu \in N^Q_{x_\alpha}\}$ is a fuzzy net in $X$. It is easy to verify that $\mathscr{S}$ converges to $x_\alpha$ but $f(\mathscr{S})$ does not converge to $f(x_\alpha)$, since $\eta \in N^Q_{f(x_\alpha)}$ and $f(x^\mu_{\alpha(\mu)}) \nleqslant \eta$. This contradiction shows that there exists some $\mu \in N^Q_{x_\alpha}$ with $\mu \leqslant f^{-1}(\eta)$. Thus $f^{-1}(\eta)$ is an open $Q$-neighborhood of $x_\alpha$ and so $f^{-1}(\eta)$ is an open fuzzy set. Thus $f$ is $F$-continuous.

(ii) $\Longrightarrow$ (vi): Let $x_\alpha \leqslant \mathrm{adh}(\mathscr{S})$. By Theorem 4.11, there exists a fuzzy subnet $T$ of $S$ with $T \to x_\alpha$. Then $f(T)$ is a fuzzy subnet of $f(S)$ and by (ii), $f(T) \to f(x_\alpha)$ which by Theorem 4.12 implies that $f(x_\alpha) \leqslant \mathrm{adh}(f(S))$. Thus $f(\mathrm{adh}(S)) \leqslant \mathrm{adh}(f(S))$.

(vi) $\Longrightarrow$ (ii): Suppose that there exists a fuzzy net $\mathscr{S} = \{x^n_{\alpha_n};\, n \in D\}$ with $\mathscr{S} \to x_\alpha$ while $f(S)$ does not converge to $f(x_\alpha)$. So $(\exists\, \eta \in N^Q_{f(x_\alpha)})(\forall\, m \in D)(\exists\, n \in D,\, n \geqslant m)(f(x^n_{\alpha_n}) \nleqslant \eta)$. So we may assume that $f(x^n_{\alpha_n}) \nleqslant \eta$ for all $n \in D$. Since $x_\alpha \leqslant \lim(S) \leqslant \mathrm{adh}(S)$, we have $f(x_\alpha) \leqslant \mathrm{adh}(f(S))$ by (vi). Thus by Theorem 4.12, there exists a fuzzy subnet of $f(S)$ which is convergent to $f(x_\alpha)$, which is impossible since $f(x^n_{\alpha_n}) \nleqslant \eta$ for all $n \in D$. Thus $f(\mathscr{S})$ must be convergent to $f(x_\alpha)$.

(iii) $\Longrightarrow$ (iv). It is obvious.

(iv) $\Longrightarrow$ (iii). Let $\mathscr{S} = \{x^n_{\alpha_n};\, n \in D\}$ be a fuzzy net in $X$ and suppose that $\mathscr{S} \to x_\alpha$ while $f(S)$ does not converge to $f(x_\alpha)$. Then $(\exists\, \eta \in N^Q_{f(x_\alpha)})(\forall\, m \in D)(\exists\, n \in D,\, n \geqslant m)(f(x^n_{\alpha_n}) \nleqslant \eta)$. So we may assume that $f(x^n_{\alpha_n}) \nleqslant \eta$ for all $n \in D$. If $T$ is a universal fuzzy subnet of $\mathscr{S}$, then $T \to x_\alpha$ but $f(T)$ is not convergent to $f(x_\alpha)$, a contradiction with (iv). □

**Theorem 4.14.** *Let $f\colon (X,\tau) \to (Y,\Delta)$ be a mapping. Then the following are equivalent:*

(i) *$f$ is $F$-continuous.*

(ii) *$\mathscr{S} \overset{Q}{\to} x_\alpha$ implies $f(\mathscr{S}) \overset{Q}{\to} f(x_\alpha)$ for each $x_\alpha \in \mathrm{FP}(X)$.*

(iii) *For each universal fuzzy net $\mathscr{S}$ in $X$, $\mathscr{S} \overset{Q}{\to} x_\alpha$ implies $f(\mathscr{S}) \overset{Q}{\to} f(x_\alpha)$ for each $x_\alpha \in \mathrm{FP}(X)$.*

(iv) *For each fuzzy net $\mathscr{S} = \{x^n_{\alpha_n};\, n \in D\}$ in $X$, $f(Q\text{-}\lim(\mathscr{S})) \leqslant Q\text{-}\lim(f(\mathscr{S}))$.*

(v) *For each universal fuzzy net $\mathscr{S}$ in $X$, $f(Q\text{-}\lim(\mathscr{S})) \leqslant Q\text{-}\lim(f(\mathscr{S}))$.*

(vi) *For each fuzzy net $\mathscr{S}$ in $X$, $f(Q\text{-adh}(\mathscr{S})) \leqslant Q\text{-adh}(f(\mathscr{S}))$.*

P r o o f. We omit the proof which follows a line similar to that of Theorem 4.13 with certain straightforward modifications. □

**Theorem 4.15.** *Let $\mathscr{S} = \{x^n_{\alpha_n};\, n \in D\}$ be a fuzzy net in the fuzzy product space $(X,\tau) = \Big(\prod\limits_{j\in J} X_j,\, \prod\limits_{j\in J} \tau_j\Big)$, $x_\alpha \in FP(X)$ and let $P_j\colon X \to X_j$ be the projection map for each $j \in J$. Then:*

(i) *$\mathscr{S} \to x_\alpha$ iff $P_j(\mathscr{S}) = \{P_j(x^n_{\alpha_n});\, n \in D\} \to P_j(x_\alpha)$ in $X_j$ for each $j \in J$.*

(ii) *If $\mathscr{S} \propto x_\alpha$, then $P_j(\mathscr{S}) = \{P_j(x^n_{\alpha_n});\, n \in D\} \propto P_j(x_\alpha)$ in $X_j$ for each $j \in J$.*

*(iii) If $\mathscr{S} \overset{Q}{\propto} x_\alpha$, then $P_j(\mathscr{S}) = \{P_j(x^n_{\alpha_n});\, n \in D\} \overset{Q}{\propto} P_j(x_\alpha)$ in $X_j$ for each $j \in J$.*

P r o o f. (i) Since $P_j$ is $F$-continuous for each $j \in J$, then by Theorem 4.13, $\mathscr{S} \to x_\alpha$ implies $P_j(\mathscr{S}) \to P_j(x_\alpha)$ for each $j \in J$. Conversely, suppose that $P_j(\mathscr{S}) \to P_j(x_\alpha)$ for each $j \in J$. Let $J_0$ be a finite subset of $J$ and $\beta^Q_{x_\alpha} = \Big\{\bigwedge\limits_{j\in J_0} P_j^{-1}(\mu_j)\colon x_\alpha q \bigwedge\limits_{j\in J_0} P_j^{-1}(\mu_j),\, \mu_j \in \tau_j\Big\}$. Then $\beta^Q_{x_\alpha}$ is a local base at $x_\alpha$ [28]. Let $\mu = \bigwedge\limits_{j\in J_0} P_j^{-1}(\mu_j) \in \beta^Q_{x_\alpha}$. Then $(\forall\, j \in J)(\exists\, n_j \in D,\, j \in J_0)(\forall\, n \in D,\, n \geqslant n_j)(P_j(x^n_{\alpha_n}) \leqslant \mu_j)$. Let $n_0 \geqslant n_j$ for each $j \in J_0$. Then, if $n \geqslant n_0$, we have $P_j(x^n_{\alpha_n}) \leqslant \mu_j$ for each $j \in J$ and so $x^n_{\alpha_n} \leqslant \bigwedge\limits_{j\in J_0} P_j^{-1}(\mu_j) = \mu$. Thus, $\mathscr{S} \to x_\alpha$.

(ii) and (iii) follow from (i) and Theorem 4.12. □

In the following results we give new characterizations and properties of strong $Q$-compactness in terms of convergence of fuzzy nets.

**Theorem 4.16.** *Let $(X,\tau)$ be a fts, $\alpha \in (0,1]$ and $\mu \in I^X$. Then $\mu$ is strong $Q$-compact iff each fuzzy net $\mathscr{S}$ in $\mu$ has a $Q$-adherent point in $\mu$.*

P r o o f. Let $(X,\tau)$ be a fts and let $\mu \in I^X$ be strong $Q$-compact and if possible, let $\mathscr{S} = \{x^n_{\alpha_n};\, n \in D\}$ be a fuzzy net in $\mu$ without any $Q$-adherent point in $\mu$. Then $(\forall\, x_\alpha \leqslant \mu)(\exists\, \eta_{x_\alpha} \in N^Q_{x_\alpha})(\exists\, n_{x_\alpha} \in D)(\forall\, m \in D,\, m \geqslant n_{x_\alpha})(x^m_{\alpha_m} \bar{q} \eta_{x_\alpha})$. Then the family $\mathscr{U} = \{\eta_{x_\alpha}\colon\, x \in X,\, \mu(x) \geqslant \alpha\}$ is an open $Q_\alpha$-cover of $\mu$. Since $\mu$ is strong $Q$-compact, for each $\alpha \in (0,1]$ there exist finitely many members $\eta_{x^1_\alpha}, \eta_{x^2_\alpha}, \ldots, \eta_{x^k_\alpha}$ from $\mathscr{U}$ for $x^1_\alpha, x^2_\alpha, \ldots, x^k_\alpha \leqslant \mu$ such that the family $\mathscr{U}_0 = \{\eta_{x^i_\alpha}\colon\, i = 1,2,\ldots,k\}$ is a $Q_\alpha$-subcover of $\mu$. Let the corresponding elements in $D$ be $n_{x^1_\alpha}, n_{x^2_\alpha}, \ldots, n_{x^k_\alpha}$. Because $D$ is a directed set, there exists $m \in D$ such that $m \geqslant n_{x^i_\alpha}$ for $i = 1,2,\ldots,k$. Then $x^m_{\alpha_m} q \eta_{x^i_\alpha}$ for some $1 \leqslant i \leqslant k$, a contradiction. Thus $\mathscr{S}$ must have a $Q$-adherent point $x_\alpha$ (say) in $\mu$. Conversely, let $\mathscr{A} = \{\mu_j\colon\, j \in J\} \subseteq \tau'$ such that for each finite subset $J_0$ of $J$ we have $\bigwedge\limits_{j \in J_0} \mu_j(x) \geqslant \alpha$ for some $x \in \mu_{w\alpha}$. Let $D = \Big\{\bigwedge\limits_{j \in J_0} \mu_j\colon\, \mu_j \in \mathscr{A},\, j \in J\Big\}$. Then $\mathscr{A} \subseteq D$. For each $\lambda_j \in D$, let us choose a fuzzy point $x^{\lambda_j}_\alpha$ and consider the fuzzy net $\mathscr{S} = \{x^{\lambda_j}_\alpha\colon\, \lambda_j \in D\}$ with the directed set $(D, \geqslant)$ where for $\lambda_1, \lambda_2 \in D$, $\lambda_1 \geqslant \lambda_2$ iff $\lambda_1 \leqslant \lambda_2$ in $X$. By hypothesis, there exists $x_\alpha \leqslant \mu \wedge Q\text{-adh}(\mathscr{S})$. Let $\eta \in N^Q_{x_\alpha}$ and $\mu_j \in \mathscr{A}$. Since $\mu_j \in D$, there is $\lambda \in D$ with $\lambda \geqslant \mu_j$ (that is, $\lambda \leqslant \mu_j$) such that $\lambda q \eta$ and hence $\mu_j q \eta$. Thus $x_\alpha \leqslant \mathrm{cl}(\mu_j) = \mu_j$ for each $j \in J$ and so $\bigwedge\limits_{j \in J} \mu_j(x) \geqslant \alpha$ for some $x \in \mu_{w\alpha}$, a contradiction. Consequently, by Theorem 3.27, we have that $\mu$ is strong $Q$-compact. □

**Corollary 4.17.** *A fts $(X,\tau)$ is strong $Q$-compact iff each fuzzy net in $X$ has a $Q$-adherent point.*

**Theorem 4.18.** *Let $(X,\tau)$ be a fts and $\mu \in I^X$. Then $\mu$ is strong $Q$-compact iff each fuzzy net in $\mu$ has a fuzzy subnet in $\mu$ $Q$-convergent to some fuzzy point in $\mu$.*

P r o o f. Immediate consequence of Theorems 4.12 and 4.16. □

**Corollary 4.19.** *A fts $(X,\tau)$ is strong $Q$-compact iff each fuzzy net has a fuzzy subnet convergent to some fuzzy point.*

**Definition 4.20.** Let $(X,\tau)$ be a fts, $(D,\geqslant)$ a directed set. A fuzzy point $x_\alpha$ is called an adherent point of the fuzzy net $\mathscr{S} = \{\lambda_n;\, n \in D\}$ of fuzzy sets in $X$ iff $(\forall\, n \in D)(\forall\, \eta \in N^Q_{x_\alpha})(\exists\, m \in D,\, m \geqslant n)(\eta q \lambda_m)$.

**Theorem 4.21.** *Let $(X,\tau)$ be a fts and $\mu \in I^X$. Then $\mu$ is strong $Q$-compact iff every fuzzy net of fuzzy sets in $\mu$ has an adherent point with value $\alpha$ in $\mu$.*

P r o o f. Let $\mathscr{S} = \{\mu_n;\ n \in D\}$ be a fuzzy net in $\mu$ (that is, $\mu_n \wedge \mu \neq \emptyset$ for each $n \in D$) of fuzzy sets in $X$ and $\mu$ is strong $Q$-compact. For each $n \in D$, let $\lambda_n = \mathrm{cl}\bigl(\bigvee\{\mu_m\colon\ m \in D,\ m \geqslant n\}\bigr)$. Then $\mathscr{A} = \{\lambda_n\colon\ n \in D\}$ is a family of closed fuzzy sets with $\bigwedge\limits_{n\in D_0} \lambda_n(x) \geqslant \alpha$ for some $x \in \mu_{w\alpha}$ and every finite subset $D_0$ of $D$. By Theorem 3.27, we have $\bigwedge\limits_{n\in D} \lambda_n(x) \geqslant \alpha$ for some $x \in \mu_{w\alpha}$. Let $n \in D$ and $\eta \in N^Q_{x_\alpha}$. Then $\eta q\Bigl(\bigvee\limits_{m\geqslant n} \mu_m\Bigr)$. Then $(\exists\, m \in D,\ m \geqslant n)(\eta q \mu_m)$. Thus $x_\alpha \leqslant \mathrm{adh}(S)$. Conversely, let $\mathscr{A} = \{\mu_j\colon\ j \in J\}$ be a family of closed fuzzy sets with $\bigwedge\limits_{j\in J_0} \mu_j(x) \geqslant \alpha$ for some $x \in \mu_{w\alpha}$ and for each finite subset $J_0$ of $J$. Let $D = \Bigl\{\bigwedge\limits_{j\in J} \mu_j\colon\ j \in J_0\Bigr\}$. Then $(D,\leqslant)$ is a directed set, where $\leqslant$ denotes the inclusion relation on $D$. For each $\lambda \in D$, we have $\lambda(x) \geqslant \alpha$ for some $x \in \mu_{w\alpha}$. Then the fuzzy net $\mathscr{S} = \{\lambda\colon\ \lambda \in D\}$ of fuzzy sets in $\mu$ has an adherent point $x_\alpha$ (say) in $\mu$. Let $\mu_j \in \mathscr{A}$ and $\eta \in N^Q_{x_\alpha}$. Since $\mu_j \in D$, we have $(\exists\, \lambda \in D)(\lambda \leqslant \mu_j$ and $\lambda q \eta)$. Then $\mu_j q \eta$ and so $(\forall\, j \in J)(x_\alpha \leqslant \mathrm{cl}(\mu_j) = \mu_j)$. Thus $\bigwedge\limits_{j\in J} \mu_j(y) \geqslant \alpha$ for some $y \in \mu_{w\alpha}$. Consequently, by Theorem 3.27, we have that $\mu$ is strong $Q$-compact. □

**Corollary 4.22.** *A* fts $(X,\tau)$ *is strong $Q$-compact iff every fuzzy net of fuzzy sets in $X$ has an adherent point.*

## 5. The relation between fuzzy filter and fuzzy net convergence

Like in convergence theory in general topology, we can associate with each fuzzy net in $X$ a fuzzy filter on $X$ and conversely. In this section we define the concepts of fuzzy filter base generated by a fuzzy net and fuzzy net based on a fuzzy filter. Then we study the relation between their convergence.

**Definition 5.1.** Let $\mathscr{S} = \{x^n_{\alpha_n};\ n \in D\}$ be a fuzzy net in $X$. For each $n \in D$, let $\mu_n = \bigvee\{x^m_{\alpha_m}\colon\ m \in D,\ m \geqslant n\}$. Then the family $\beta_{\mathscr{S}} = \{\mu_n\colon\ n \in D\}$ forms a fuzzy filter base on $X$, called the fuzzy filter base generated by the fuzzy net $\mathscr{S}$.

**Theorem 5.2.** *Let $(X,\tau)$ be a* fts, *$x_\alpha \in \mathrm{FP}(X)$ and let $\mathscr{S}$ be a fuzzy net in $X$. Then*

(i) $\mathscr{S} \to x_\alpha$ *iff* $\beta_{\mathscr{S}} \to x_\alpha$.
(ii) $\mathscr{S} \propto x_\alpha$ *iff* $\beta_{\mathscr{S}} \propto x_\alpha$.

Proof. (i) Since $\beta_{\mathscr{S}} = \{\mu \in I^X\colon (\exists n \in D)(\mu = \bigvee\{x^m_{\alpha_m}\colon m \in D,\, m \geqslant n\})\}$, we have $\beta_{\mathscr{S}} \to x_\alpha$ iff $(\forall \eta \in N^Q_{x_\alpha})(\exists \mu \in \beta_{\mathscr{S}})(\mu \leqslant \eta)$ iff $(\forall \eta \in N^Q_{x_\alpha})(\exists n \in D)\Bigl(\mu = \bigvee\limits_{m \geqslant n} x^m_{\alpha_m} \leqslant \eta\Bigr)$ iff $(\forall \eta \in N^Q_{x_\alpha})(\exists n \in D)(\forall m \in D,\, m \geqslant n)(x^m_{\alpha_m} \leqslant \eta)$ iff $\mathscr{S} \to x_\alpha$.

(ii) Since $\beta_{\mathscr{S}} \propto x_\alpha$ iff $(\forall \eta \in N^Q_{x_\alpha})(\forall \mu \in \beta_{\mathscr{S}})(\mu \wedge \eta \neq \emptyset)$ iff $(\forall \eta \in N^Q_{x_\alpha})(\forall n \in D)\Bigl(\bigvee\limits_{m \geqslant n} x^m_{\alpha_m} \wedge \eta \neq \emptyset\Bigr)$ iff $(\forall \eta \in N^Q_{x_\alpha})(\forall n \in D)(\exists m \in D,\, m \geqslant n)(x^m_{\alpha_m} \leqslant \eta)$ iff $\mathscr{S} \propto x_\alpha$. □

**Definition 5.3.** Let $\beta$ be a fuzzy filter base on $X$, $x_\alpha \in \mathrm{FP}(X)$ and let $D_\beta = \{(\lambda, x_\alpha)\colon x_\alpha \leqslant \lambda \in \beta\}$. Let us define $\geqslant$ in $D_\beta$ by $(x^1_\alpha, \lambda_1) \geqslant (x^2_\alpha, \lambda_2)$ iff $\lambda_1 \leqslant \lambda_2$. Then $(D_\beta, \geqslant)$ is a directed set. Define $\mathscr{S}_\beta\colon D_\beta \to \mathrm{FP}(X)$ by $\mathscr{S}_\beta(x_\alpha, \lambda) = x_\alpha$. Then $\mathscr{S}_\beta$ is a fuzzy net in $X$, called the fuzzy net based on the fuzzy filter base $\beta$.

**Theorem 5.4.** *Let $(X, \tau)$ be a* fts, *$x_\alpha \in \mathrm{FP}(X)$ and let $\beta$ be a fuzzy filter base on $X$. Then:*
(i) *$\beta \to x_\alpha$ iff $\mathscr{S}_\beta \to x_\alpha$.*
(ii) *$\mathscr{S} \propto x_\alpha$ iff $\mathscr{S}_\beta \propto x_\alpha$.*

Proof. (i) Let $\beta \to x_\alpha$ and let $\mathscr{S}_\beta\colon D_\beta \to \mathrm{FP}(X)$ be the fuzzy net based on $\beta$, where $D_\beta = \{(x_\alpha, \lambda)\colon x_\alpha \leqslant \lambda \in \beta\}$ and $\mathscr{S}_\beta(x_\alpha, \lambda) = x_\alpha$. If $\mu \in N^Q_{x_\alpha}$, then there exists $\lambda \in \beta$ such that $\lambda \leqslant \mu$. Choose $y_t \leqslant \lambda$ such that $(y_t, \lambda) \in D_\beta$. If $(z_r, \lambda_1) \in D_\beta$ is such that $(z_r, \lambda_1) \geqslant (y_t, \lambda)$, then $\mathscr{S}_\beta(z_r, \lambda_1) = z_r \leqslant \lambda_1$. Since $\lambda_1 \leqslant \lambda \leqslant \mu$, we have $z_r \leqslant \mu$. Hence $\mathscr{S}_\beta(z_r, \lambda_1) \leqslant \mu$. Thus $\mathscr{S}_\beta \to x_\alpha$.

Conversely, let $\mathscr{S}_\beta \to x_\alpha$ and let $\mu \in N^Q_{x_\alpha}$. Then there is $(z_r, \lambda_1) \in D_\beta$ such that for each $(y_t, \lambda) \in D_\beta$ with $(y_t, \lambda) \geqslant (z_r, \lambda_1)$ we have $\mathscr{S}_\beta(y_t, \lambda) = y_t \leqslant \mu$. For each $w_\nu \leqslant \lambda_1$ we have $(w_\nu, \lambda_1) \geqslant (z_r, \lambda_1)$ and hence $\mathscr{S}_\beta(w_r, \lambda_1) = w_r \leqslant \mu$. Hence $\lambda_1 \leqslant \mu$. Thus $\beta \to x_\alpha$.

(ii) Let $\beta \propto x_\alpha$ and let $\mathscr{S}_\beta\colon D_\beta \to \mathrm{FP}(X)$ be the fuzzy net based on $\mathscr{B}$, where $D_\beta = \{(x_\alpha, \lambda)\colon x_\alpha \leqslant \lambda \in \beta\}$ and $\mathscr{S}_\beta(x_\alpha, \lambda) = x_\alpha$. Let $\mu \in N^Q_{x_\alpha}$ and $(y_t, \lambda) \in D_\beta$. Then $\lambda \wedge \mu \neq$. If $z_r \leqslant \lambda \wedge \mu$, then $(z_r, \lambda) \in D_\beta$ is such that $(z_r, \lambda) \geqslant (y_t, \lambda)$. Hence $\mathscr{S}_\beta(z_r, \lambda) = z_r \leqslant \mu$. Thus $\mathscr{S}_\beta \propto x_\alpha$. Conversely, let $\mathscr{S}_\beta \propto x_\alpha$ and let $\mu \in N^Q_{x_\alpha}$ and $\lambda \in \beta$. If $y_t \leqslant \lambda$, then $(y_t, \lambda) \in D_\beta$ and hence there is $(z_r, \lambda_1) \in D_\beta$ with $(z_r, \lambda_1) \geqslant (y_t, \lambda)$, and we have $\mathscr{S}_\beta(z_r, \lambda_1) = z_r \leqslant \mu$. Since $\lambda_1 \leqslant \lambda$ and $z_r \leqslant \lambda_1 \wedge \mu$, we have $z_r \leqslant \lambda \wedge \mu$ and so $\lambda \wedge \mu \neq \emptyset$. Thus $\beta \propto x_\alpha$. □

In the above Theorems 5.2 and 5.4, the concepts of convergence and adherence of a fuzzy net in our sense can not be replaced by $Q$-convergence and $Q$-adherence of a fuzzy net due to Pu and Liu [28].

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