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ON CONVEX POWER SERIES OF A CONSERVATIVE MARKOV OPERATOR

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ABSTRACT. A. Brunel proved that a conservative Markov operator, P, has a finite invariant measure if and only if every operator $Q=\sum_{n=0}^{\infty}\alpha_n P^n$ where $\alpha_n\geqq 0$ and $\sum\alpha_n=1$ is conservative.
In this note we give *a* different proof and study related problems.

Introduction. We shall use the notation and definitions of [**3**]. Let us quote some basic results:

The operator P is conservative if and only if for every $0\leqq f\in L_\infty$ the sum $\sum_{n=0}^{\infty}P^n f$ assumes the values 0 or ∞ only.

The operator P is conservative if and only if whenever $0\leqq f\in L_\infty$ and $Pf\leqq f$ then $Pf=f$. See [**4**, Corollary 1].

If P is conservative and $Pf=f$, then f is $\Sigma_i(P)$ measurable where $\Sigma_i(P)=\{A:P1_A=1_A\}$. See [**3**, Theorem A of Chapter III].

We shall study operators of the form $Q=\sum_{n=0}^{\infty}\alpha_n P^n$ where $\alpha_n\geqq 0$ and $\sum\alpha_n=1$. Such operators will be called convex power series of P, and denoted by $A(P)$ where $A(z)=\sum_{n=0}^{\infty}\alpha_n z^n$.

1. Conditions for Q to be conservative.

THEOREM 1.1. *Let P be a conservative operator and $Q=A(P)$ a convex power series of P. If $\sum_{n=1}n\alpha_n<\infty$ then Q is conservative too.*

PROOF. Note first that

$$\sum_{n=0}^{\infty}(1-\alpha_0-\cdots-\alpha_n)=\sum_{n=0}^{\infty}\left(\sum_{k=n+1}^{\infty}\alpha_k\right)=\sum_{n=1}^{\infty}n\alpha_n.$$

Put $\gamma_n=1-(\alpha_0+\cdots+\alpha_n)$ then, by assumption, $\sum\gamma_n<\infty$. Define $K=\sum\gamma_n P^n$ then K is a positive bounded operator on L_1. An easy computation shows that $I-Q=(I-P)K$. Thus if $0\leqq f\in L_\infty$ and $(I-Q)f\geqq 0$ then $(I-P)Kf\geqq 0$ and $(I-P)Kf=0$ because P is conservative. By the characterization of conservative operators, given in the introduction, Q is conservative too.

REMARK. Every finite convex combination of P^k is conservative.

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For the rest of this section we shall assume that P is ergodic and conservative and study $\Sigma_i(Q)$. If $A \in \Sigma_i(Q)$ or $1_A = \sum \alpha_n P^n 1_A$ then whenever $\alpha_n \neq 0$ $P^n 1_A \leqq 1_A$ (since $P^n 1_A \leqq 1$) and thus $P^n 1_A = 1_A$ since P^n is conservative.

Let r be the maximal common divisor of n such that $\alpha_n \neq 0$. Then, on the one hand, $Q = \sum_{n=0}^{\infty} \alpha_{nr} P^{nr}$, and on the other hand there exist $n_1 \cdots n_j$ with $\alpha_{n_k} \neq 0$ and $nr = q_1 n_1 + \cdots + q_i n_j$ with q_i positive integers and $n \geqq N$. Thus, $1_A = P^{(N+1)r} 1_A = P^r P^N 1_A = P^r 1_A$. To summarize $\Sigma_i(Q) = \Sigma_i(P^r)$ for some integer r.

Lemma 1.2. *Let P be ergodic and conservative then $\Sigma_i(P^r)$ is atomic.*

Proof. Let us assume, to the contrary, that $A_n \in \Sigma_i(P^r)$ where $A_n \downarrow \varnothing$. Now $0 = (I - P^r) 1_{A_n} = (I-P)(I+P+\cdots+P^{r-1}) 1_{A_n}$. Since P is ergodic and conservative $(I+P+\cdots+P^{r-1}) 1_{A_n}$ is a constant but $1_{A_n}(x) = 1$ if $x \in A_n$. Thus $(I+P+\cdots+P^{r-1}) 1_{A_n} \geqq 1$ and this contradicts the continuity of P on L_1.

Let us take an atom A, of $\Sigma_i(P^r)$. Put $P^j 1_A = f$, $1 \leqq j \leqq r-1$. Note that $f \in \Sigma_i(P^r)$ too. Put $B_\varepsilon = \{x : f(x) \geqq \varepsilon\}$, $B_0 = \{x : f(x) > 0\}$; both sets are in $\Sigma_i(P^r)$. Now $f \geqq \varepsilon 1_{B_\varepsilon}$ thus $1_A = P^{r-j} f \geqq \varepsilon P^{r-j} 1_{B_\varepsilon}$ hence $P^{r-j} 1_{B_\varepsilon} \leqq 1_A$. Let $\varepsilon \to 0$ to conclude $P^{r-j} 1_B \leqq 1_A$. Since A is an atom and $P^{r-j} 1_B$ is invariant under P^r, we must have $P^{r-j} 1_B = \text{const } 1_A$. Now $P^r 1_B = 1_B$ so the constant is one. Thus $P^{r-j} 1_B = P^{r-j} f$ but $1_B \geqq f$. Therefore, $\sum P^n (1_B - f) < \infty$ or $f = 1_B$.

Theorem 1.3. $\Sigma_i(P^r) = \{A_0, \cdots, A_{k-1}\}$ *where the sets A_i are disjoint, k divides r and* $1_{A_1} = P 1_{A_0}$, $1_{A_2} = P 1_{A_1}, \cdots, 1_{A_0} = P 1_{A_{k-1}}$.

Proof. Let A_0 be an atom of $\Sigma_i(P^r)$. By the above argument $P^j 1_{A_0} = 1_A$ and $A_j \in \Sigma_i(P^r)$. Let k be the first integer such that $P^k 1_{A_0} = 1_{A_0}$. Clearly k divides r and the sets A_j, $1 \leqq j \leqq k-1$, are disjoint: If $B \subset A_j$ and $B \in \Sigma_i(P^r)$ then $P^{k-j} 1_B \leqq 1_{A_0}$ and like the previous argument must be equal to 1_{A_0} or $P^{k-j} 1_B = P^{k-j} 1_{A_j}$ and $B = A_j$.

Now $\bigcup_{j=0}^{k-1} A_j$ is invariant under P and thus must be all of X. Clearly each A_i is an atom of $\Sigma_i(P^r)$ and $\Sigma_i(P^r) = \{A_0, A_1, \cdots, A_{k-1}\}$.

Remark. Note that if n divides m then $\Sigma_i(P^n) \subset \Sigma_i(P^m)$. Thus $\bigvee \Sigma_i(P^n) = \bigvee \Sigma_i(P^{n!})$ and $\Sigma_i(P^{n!})$ is monotone in n.

Theorem 1.3 was proved by Moy in [7, Theorem 1] for a more general case by a different method.

2. Conditions for Q to be dissipative.

Lemma 2.1. *Let P_1 and P_2 be commuting elements of a Banach algebra with $\|P_1\| = \|P_2\| = 1$. Let $Q = \alpha P_1 + \beta P_2$ where $0 < \alpha$, $\beta < 1$ and $\alpha + \beta = 1$. Then $\|Q^n(P_1 - P_2)\| \leqq (K/\sqrt{n}) \cdot \alpha \cdot \beta$ where K is a constant.*

PROOF. First let us reduce the problem to the case where $\alpha=\beta=\frac{1}{2}$: If $\alpha<\frac{1}{2}$ then $Q=\frac{1}{2}(P_1'+P_2)$ where $P_1'=2\alpha P_1+(1-2\alpha)P_2$ and $P_1'-P_2=2\alpha(P_1-P_2)$.

Following [8] we write

$$Q^n(P_1-P_2)=\frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}P_1^kP_2^{n-k}(P_1-P_2)$$
$$=\frac{1}{2^n}\sum_{k=1}^{n}\left[\binom{n}{k-1}-\binom{n}{k}\right]P_1^kP_2^{n-k+1}+\frac{1}{2^n}P_1^{n+1}-\frac{1}{2^n}P_2^{n+1}$$

thus

$$\|Q^n(P_1-P_2)\|\leqq\frac{1}{2^{n-1}}+\frac{1}{2^n}\sum_{k=1}^{n}\left|\binom{n}{k-1}-\binom{n}{k}\right|.$$

We may assume that n is even. Since $\binom{n}{k}$ increases as k increases from 0 to $n/2$ and then decreases as k goes from $n/2$ to n the sum of absolute values is bounded by $(2/2^n)\binom{n}{n/2}$ which is, by Stirling's formula, bounded by $K/\sqrt{n}$, and hence the lemma follows.

Let P be an operator and $Q=\sum_{n=0}^{\infty}\alpha_nP^n$, $\alpha_i\geqq 0$, $\sum\alpha_i=1$. Assume that α_i and α_j $(i<j)$ are the first nonzero coefficients. Put $Q'=\sum\alpha_nP^{n-i}$ where $P^iQ'=Q'P^i=Q$. Choose $0<\gamma<\min\{\alpha_i,\alpha_j,\frac{1}{2}\}$. Then

$$Q'=\gamma(I+P^{j-i})+\sum\beta_nP^n=\gamma(I+P^{j-i})+Q_1.$$

Note $\sum\beta_n+2\gamma=1$, $\beta_n\geqq 0$. Thus $Q'=\frac{1}{2}[(2\gamma+Q_1)+(2\gamma P^{j-i}+Q_1)]$ and $\|Q'^n(I-P^{j-i})\|\to 0$. Therefore, $\|Q^n(I-P^{j-i})\|=\|P^{in}Q'^n(I-P^{j-i})\|\to 0$.

THEOREM 2.2. *Let P be an operator with no invariant measure. Let $Q=A(P)$ be a convex power series of P such that $A(z)$ has at least two nonzero coefficients. There exists a set A, $m(A)>0$, such that $\|Q^n1_A\|_\infty\to 0$.*

PROOF. Note first that, by a standard argument, P^{j-i} has no invariant measure. By [3, Chapter IV, (4.9)] there exists a set A with $m(A)>0$, such that if $\lambda P^{j-i}=\lambda$ then $\lambda(A)=0$ $(\lambda\in L_\infty^*)$. By the Hahn-Banach Theorem

$$1_A\in\overline{\text{Range}(I-P^{j-i})}.$$

Thus $\|Q^n1_A\|_\infty\to 0$.

REMARK. In [1, Lemma 1] Brunel proves that property for $Q=(1/e)\exp P$.

THEOREM 2.3. *Let Q be a Markov operator such that, for some $0\leqq h\in L_\infty$, $\|Q^nh\|\to 0$. Choose a sequence N_m such that $\sum_{m=0}^{\infty}\|Q^{N_m}h\|<\infty$. The operator $R=\sum\alpha_nQ^n$ is not conservative if $\sum_{m=0}^{\infty}(\sum_{n=0}^{N_m-1}\alpha_n)^m<\infty$.*

PROOF. Put

$$R=R_{1,m}+R_{2,m}=\sum_{n=0}^{N_m-1}\alpha_nQ^n+\sum_{n=N_m}^{\infty}\alpha_nQ^n.$$

Then $R^m = R_{1,m}^m + S_m Q^{N_m}$ where S_m is of the form $\sum \beta_n Q^n$, $\sum \beta_n \leqq 1$, $\beta_n \geqq 0$. Thus

$$R^m h = R_{1,m}^m h + S_m Q^{N_m} h \leqq R_{1,m}^m h + \|Q^{N_m} h\| .$$

Since $\sum_{m=0}^{\infty} \|Q^{N_m} h\| < \infty$ it is enough to consider the first term:

$$R_{1,m}^m h = \left(\sum_{n=0}^{N_m - 1} \alpha_n Q^n \right)^m h \leqq \|h\| \left(\sum_{n=0}^{N_m - 1} \alpha_n \right)^m$$

and the sum over m of the right-hand side converges by assumption.

THE BRUNEL EXAMPLE. Let $\|Q^n h\| \to 0$ and $\sum \|Q^{N_m} h\| < \infty$. Choose

$$\rho_n = (1/n^2)^{1/n}, \qquad n \geqq 3,$$

then $\rho_n \uparrow 1$ and $\sum \rho_n^n < \infty$. Choose $\alpha_n \geqq 0$ such that $\sum_{n=0}^{N_m - 1} \alpha_n < \rho_m$ and $\sum \alpha_n = 1$ and, by Theorem 2.3, $\sum \alpha_n Q^n$ is not conservative.

3. **Dissipating power series.** Let us call a power series $A(z) = \sum_1^\infty \alpha_n z^n$ dissipating if (1) $\alpha_n \geqq 0$, (2) $A(1) = 1$, and (3) there is some conservative operator P with $A(P)$ dissipative. Theorem 1.1 says simply that if $A'(1)$ is finite then A is not dissipating. The main purpose of this section is to establish a converse: namely if $A'(1)$ is infinite then A is dissipating. We first make a slight detour to discuss *renewal sequences*. Recall that $\{u_n\}_{n=1}^\infty$, $0 \leqq u_n \leqq 1$, is said to be a renewal sequence if there is a sequence $\{f_n\}_{n=1}^\infty$, $f_n \geqq 0$, $\sum_1^\infty f_n \leqq 1$ such that

$$(1) \quad u_n = f_n + f_{n-1} u_1 + f_{n-2} u_2 + \cdots + f_1 u_{n-1} \qquad (n = 1, 2, \cdots).$$

Equivalently, if $U(z) = 1 + \sum u_n z^n$, $F(z) = \sum f_n z_n$ then

$$U(z) = F(z) U(z) + 1 \quad \text{or} \quad U(z) = \frac{1}{1 - F(z)} .$$

If $P = (p_{ij})_{i,j=1}^\infty$ is a Markovian transition matrix with all states forming a single ergodic class, the condition for recurrence or conservativeness is simply $\sum_{n=0}^\infty p_{11}^{(n)} = +\infty$ where $p_{ij}^{(n)}$ is the ij entry in P^n. It is well known that $\{p_{11}^{(n)}\}_{n=1}^\infty$ forms a renewal sequence. Here the f_n of (1) represent the probability that first return to 1 takes place at time n. We shall need the simple converse.

LEMMA 3.1. *If $\{u_n\}_{n=1}^\infty$ is a renewal sequence then there is an ergodic Markov matrix with $u_n = p_{11}^{(n)}$.*

PROOF. Let f_n be such that (1) holds and define $p_1 = f_1, \cdots, p_n = f_n/(1 - f_1 - f_2 - \cdots - f_{n-1})$. Set now

$$\begin{aligned} p_{ij} &= p_i && \text{if } j = 1, \\ &= 1 - p_i && \text{if } j = i + 1, \\ &= 0 && \text{otherwise.} \end{aligned}$$

The lemma now follows easily if one recalls the probabilistic interpretation of the f_j's, namely that they are the probability that the event occurs for the first time at time n. The structure of p_{ij} is such that one easily checks

$$\text{Prob}\{\text{return to 1 for the first time at time } n \mid \text{start at 1}\}$$
$$= (1-p_1)(1-p_2)\cdots(1-p_{n-1})p_n = f_n. \quad \square$$

The existence of a plentiful supply of renewal sequences is assured by Th. Kaluza's theorem [**5**] to the effect that if $1 \geqq u_n \geqq 0$ and

$$u_n/u_{n-1} \leqq u_{n+1}/u_n, \qquad n = 1, 2, \cdots \qquad (u_0 = 1),$$

then $\{u_n\}$ is a renewal sequence. Indeed as D. G. Kendall [**6**] has shown, these are precisely the "infinitely divisible" renewal sequences. We shall also need the following lemma, a proof of which may be found in [**1**].

LEMMA 3.2. *If x_j is a sequence of nonnegative numbers that tend to zero as $j \to \infty$ then there is a renewal sequence $\{u_n\}$, in fact, an infinitely divisible one, such that $\sum_1^\infty u_n = +\infty$ but $\sum_1^\infty u_n b_n < \infty$.*

THEOREM 3.1. *If $A(z) = \sum_1^\infty \alpha_n z^n$, $\alpha_n \geqq 0$, $A(1) = 1$ and $A'(1) = \infty$ then A is dissipating.*

PROOF. Let β_j be defined by

$$\sum_0^\infty \beta_j z^j = \frac{1}{1 - A(z)} = \sum_0^\infty A(z)^n.$$

Then since $A'(1) = 1$ by the renewal theorem (see [**2**, Chapter XIII.3]) we know that β_j tends to zero. Apply Lemma 3.2 to obtain a renewal sequence with $\sum_1^\infty u_n = \infty$ and $\sum_1^\infty u_j \beta_j < \infty$. By Lemma 3.1 there is a Markov matrix with $p_{11}^{(n)} = u_n$. Thus P is conservative. However, $A(P) = Q$ is dissipative since

$$\sum_0^\infty Q_{11}^{(n)} = \left(\sum_0^\infty Q^n\right)_{11} = \left(\sum_0^\infty A(P)^n\right)_{11} = \left(\sum_0^\infty \beta_n P^n\right)_{11}$$
$$= \sum_0^\infty \beta_n p_{11}^{(n)} = \sum_0^\infty \beta_n u_n < +\infty.$$

The formal interchanges of summations is justified since all the terms are nonnegative and the final result is a finite quantity. $\square$

It is worth remarking that even when a conservative operator P has no finite invariant measure there are dissipating power series $A(z)$ such that $A(P)$ is conservative. To see this it suffices to consider the special translation invariant Markov chains on the integers Z—the random walks defined by $\{p_j\}$, a probability distribution on Z. A necessary and sufficient condition for recurrence is known here in terms of $\varphi(\nu)=\sum_{-\infty}^{\infty} p_n e^{in\nu}$, the characteristic function of φ, namely

$$\int_{-\pi}^{+\pi} \operatorname{Re}\left(\frac{1}{1-\varphi(\nu)}\right) d\nu = +\infty \quad [\mathbf{9}, \text{Chapter II.8}].$$

Picking p_j with prescribed behavior at infinity and using a Tauberian theorem to relate the behavior of $\varphi(\nu)$ at $\nu=0$ one readily produces examples for the phenomenon described above.

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