# On convex Sobolev inequalities and the rate of convergence to equilibrium for Fokker-Planck type equations

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Anton Arnold[1], Peter Markowich[2], Giuseppe Toscani[3], Andreas Unterreiter[4]

[1]Fachbereich Mathematik, Universität des Saarlandes, D-66123 Saarbrücken, Germany

[2]Institut für Mathematik, Universität Wien, A-1090 Wien, Austria

[3]Dipartimento di Matematica, Università di Pavia, I-27100 Pavia, Italy

[4]Fachbereich Mathematik, Universität Kaiserslautern, D-67653 Kaiserslautern, Germany

arnold@num.uni-sb.de, Peter.Markowich@univie.ac.at,
toscani@dimat.unipv.it, unterreiter@mathematik.uni-kl.de

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**Abstract**

It is well known that the analysis of the large-time asymptotics of Fokker-Planck type equations by the entropy method is closely related to proving the validity of convex Sobolev inequalities. Here we highlight this connection from an applied PDE point of view.

In our unified presentation of the theory we present new results to the following topics: an elementary derivation of Bakry-Emery type conditions, results concerning perturbations of invariant measures with general admissible entropies, sharpness of convex Sobolev inequalities, applications to non-symmetric linear and certain non-linear Fokker-Planck type equations (Desai-Zwanzig model, drift-diffusion-Poisson model).

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# 1 Introduction

One of the fundamental problems in kinetic theory and, more generally, in thermodynamics, is the analysis of the time decay (rate) of solutions of IVP models towards their thermal equilibrium states. The maybe most often applied methodology in time asymptotics of PDE's modeling thermodynamical systems is the entropy method, where the convergence towards equilibrium is concluded using the time monotonicity of the physical entropy of the system. A classical example for this approach is provided by the spatially homogeneous Boltzmann equation, where convergence to the Maxwellian equilibrium state has been proven in this way (cf., e.g., [CeIlPu94], Theorem 6.4.1 and the references therein).

To illustrate the entropy approach we shall now present two simple (even explicitly solvable) kinetic model problems outlining the ideas which we will develop in this paper.

At first we consider the Bhatnagar-Gross-Krook (BGK) model of gas dynamics, which is a simplified version of the Boltzmann equation [BhGrKr54]. The IVP for the space homogeneous version reads:

$$\frac{\partial f}{\partial t} = J(f) := \nu\left(M^f - f\right), \qquad v \in I\!R^n,\ t \geq 0, \tag{1.1a}$$

$$f(v, t=0) = f_I(v), \qquad v \in I\!R^n \tag{1.1b}$$

(of course, only the case $n = 3$ is of physical interest), with the normalization $\int_{I\!R^n} f_I dv = 1$. Here $M^f = M^f(v)$ denotes the Maxwellian distribution function

$$M^f(v) = m\,(2\pi\Theta)^{-n/2} \exp\left(-\frac{|v-u|^2}{2\Theta}\right) \tag{1.2}$$

with mass

$$m = \int_{I\!R^n} f dv,$$

mean velocity

$$u = \int_{\mathbb{R}^n} v f dv/m$$

and temperature

$$\Theta = \int_{\mathbb{R}^n} (v-u)^2 f/nm.$$

The relaxation rate $\nu$ is assumed to be a positive constant. It is then an easy exercise to show that $m, u$ and $\Theta$ are left invariant under the temporal evolution of (1.1), such that these quantities can be computed from the initial data $f_I$, and $M^{f(t)} \equiv M^{f_I}$ follows.

The exponential $L^1(\mathbb{R}^n_v)$-convergence of $f(.,t)$ to its equilibrium state $M^{f_I}$ with rate $\nu$ can be easily verified by explicitly solving (1.1).

For carrying out the entropy approach we consider the physical entropy (Boltzmann's $H$-functional)

$$H(f) = \int_{\mathbb{R}^n} f \ln f dv \tag{1.3}$$

and the entropy dissipation

$$I(f) = \int_{\mathbb{R}^n} \ln f J(f) dv, \tag{1.4}$$

with the easily verifiable relation

$$\frac{d}{dt} H(f(t)) = I(f(t)). \tag{1.5}$$

Since $\ln M^f$ is quadratic in $v$ and due to the conservation of mass, mean velocity and temperature, we calculate

$$\begin{aligned} I(f(t)) &= \nu \int_{\mathbb{R}^n} \ln f(t) \left(M^{f(t)} - f(t)\right) dv \\ &= -\nu \int_{\mathbb{R}^n} \left(\ln f(t) - \ln M^{f(t)}\right) \left(f(t) - M^{f(t)}\right) dv \\ &= -\nu \int_{\mathbb{R}^n} \ln \left(\frac{f(t)}{M^{f(t)}}\right) f(t) dv - \nu \int_{\mathbb{R}^n} \ln \left(\frac{M^{f(t)}}{f(t)}\right) M^{f(t)} dv. \end{aligned}$$

Since $f(t)$ and $M^{f(t)}$ have the same mass, Jensen's inequality implies that both terms are nonpositive. Thus we obtain the stronger version of Boltzmann's $H$-Theorem

$$-I(f(t)) \geq \nu e(f(t)|M^{f(t)}) \tag{1.6}$$

where the so called relative (to the Maxwellian) entropy is defined by

$$e(f|M^f) := \int_{\mathbb{R}^n} f \ln \left(\frac{f}{M^f}\right) dv. \tag{1.7}$$

Note that $e(f|M^f) = 0$ iff $f = M^f$ (Gibb's Lemma). Again, since $\ln M^f$ is quadratic in $v$ and since $f(t)$ and $M^{f(t)}$ have the same mass, velocity and temperature, we have

$$e(f(t)|M^{f(t)}) = H(f(t)) - H(M^{f(t)}),$$

and since $M^{f(t)} \equiv M^{f_I}$ we conclude from (1.5), (1.6)

$$\frac{d}{dt} e\left(f(t)|M^{f(t)}\right) = \frac{d}{dt} H(f(t)) = I(f(t)) \leq -\nu e\left(f(t)|M^{f(t)}\right).$$

Exponential convergence to zero of the relative entropy with rate $\nu$ follows immediately (for initial data which have finite relative entropy)

$$e(f(t)|M^{f_I}) \leq e^{-\nu t} e(f_I|M^{f_I}), \quad t \geq 0. \tag{1.8}$$

The Csiszár-Kullback inequality [Csi63],[Kul59]

$$\|f - M^f\|^2_{L^1(\mathbb{R}^n)} \leq 2e(f|M^f) \tag{1.9}$$

then gives $L^1(\mathbb{R}^n)$-convergence to equilibrium (at the suboptimal rate $\frac{\nu}{2}$).

Summing up, we used the lower bound (1.6) of the (negative) entropy dissipation in terms of the relative entropy to explicitly control the convergence of the (relative and absolute) entropies and to control the strong convergence of the solution to its equilibrium state. The second model problem, which is a two-velocity radiative transfer model, demonstrates that the approach of the first example may not be general enough. We consider the ODE in $\mathbb{R}^2$:

$$\frac{d\Phi}{dt} = \lambda \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \Phi, \qquad t \geq 0 \tag{1.10a}$$

$$\Phi(0) = \begin{pmatrix} u_I \\ v_I \end{pmatrix} \tag{1.10b}$$

where $\Phi(t) = \begin{pmatrix} u(t) \\ v(t) \end{pmatrix}$ and $\lambda > 0$. Since we consider (1.10) as a kinetic model, we assume – for physical reasons only – $u_I, v_I \geq 0$ which implies $u(t), v(t) \geq 0$. Obviously, $u(t)$ and $v(t)$ tend exponentially with rate $2\lambda$ to their equilibrium state $w_\infty = (u_I + v_I)/2$. To apply the entropy approach let $\psi = \psi(s)$ be any smooth strictly convex function, defined on $\mathbb{R}^+$, with $\psi(1) = 0$. We introduce the relative entropy generated by $\psi$:

$$e_\psi(\Phi|\Phi_\infty) := \left(\psi(\frac{u}{w_\infty}) + \psi(\frac{v}{w_\infty})\right) w_\infty \tag{1.11}$$

where we denoted the steady state

$$\Phi_\infty = \begin{pmatrix} w_\infty \\ w_\infty \end{pmatrix}.$$

Differentiating (1.11) with respect to $t$ and using (1.10) gives the entropy equation

$$\frac{d}{dt} e_\psi(\Phi(t)|\Phi_\infty) = I_\psi(\Phi(t)|\Phi_\infty) \tag{1.12}$$

with the dissipation term

$$I_\psi(\Phi|\Phi_\infty) = -\lambda(u-v)\left(\psi'(\frac{u}{w_\infty}) - \psi'(\frac{v}{w_\infty})\right). \tag{1.13}$$

Since $\psi$ is strictly convex we have $I_\psi(\Phi|\Phi_\infty) \le 0$ with equality iff $\Phi = \Phi_\infty$. Now it is not immediate to bound the absolute value of the entropy dissipation (1.13) from below directly by a multiple of the relative entropy (1.11). Therefore we consider the time evolution of the entropy dissipation. Differentiating (1.13) gives

$$\frac{d}{dt} I_\psi(\Phi(t)|\Phi_\infty) = R_\psi(\Phi(t)|\Phi_\infty) \tag{1.14}$$

with the entropy dissipation rate

$$R_\psi(\Phi(t)|\Phi_\infty) = -2\lambda I_\psi(\Phi(t)|\Phi_\infty) + \lambda^2 \frac{(u(t)-v(t))^2}{w_\infty}\left(\psi''(\frac{u(t)}{w_\infty}) + \psi''(\frac{v(t)}{w_\infty})\right). \tag{1.15}$$

We can now bound the entropy dissipation rate from below by a multiple of the negative entropy dissipation:

$$R_\psi(\Phi(t)|\Phi_\infty) \ge -2\lambda I_\psi(\Phi(t)|\Phi_\infty)$$

and conclude exponential convergence with rate $2\lambda$ of the entropy dissipation from (1.14)

$$|I(\Phi(t)|\Phi_\infty)| \le |I(\Phi_I|\Phi_\infty)| e^{-2\lambda t}. \tag{1.16}$$

Inserting $I_\psi(\Phi(t)|\Phi_\infty)$ from (1.15) into (1.12) and using (1.14) gives after integration from $s = t$ to $s = \infty$ (using $\lim_{t\to\infty} e(\Phi(t)|\Phi_\infty) = \lim_{t\to\infty} I_\psi(\Phi(t)|\Phi_\infty) = 0$)

$$e_\psi(\Phi(t)|\Phi_\infty) + \int_t^\infty \frac{\lambda(u(s)-v(s))^2}{2w_\infty}\left(\psi''(\frac{u(s)}{w_\infty}) + \psi''(\frac{v(s)}{w_\infty})\right) ds = -\frac{I(\Phi(t)|\Phi_\infty)}{2\lambda}. \tag{1.17}$$

First of all, (1.17) gives the exponential decay with rate $2\lambda$ of the relative entropy $e_\psi$, and hence again the suboptimal decay rate $\lambda$ of $\Phi(t)$ towards $\Phi_\infty$. Moreover, (1.17) furnishes an inequality involving the strictly convex function $\psi$, i.e. we obtain the Sobolev-type inequality

$$\left(\psi(\frac{u_I}{w_\infty}) + \psi(\frac{v_I}{w_\infty})\right) w_\infty \le \frac{1}{2}(u_I - v_I)\left(\psi'(\frac{u_I}{w_\infty}) - \psi'(\frac{v_I}{w_\infty})\right) \tag{1.18}$$

for all $u_I, v_I \geq 0$ by setting $t = 0$ in (1.17). Also, (1.17) implies that equality in (1.18) holds iff $u(t) \equiv v(t)$ for all $t \geq 0$ which is equivalent to $u_I = v_I = w_\infty$.

The inequality (1.18) can also be obtained in a direct way (cp. Example 2.6 of [Gro93], [GaMaUn97]), however, the presented approach allows generalization to much more complex thermodynamical IVP's (as will be the subject of the next sections). Particularly, the second example shows that in some cases it is not immediate to derive exponential decay of the relative entropy (and equivalently, Sobolev-type inequalities) from the entropy equation only. It is often convenient for the computations to consider the evolution of the entropy dissipation and to find a lower bound of the entropy dissipation rate in terms of the absolute value of the entropy dissipation.

This leads directly to the main subject of this paper. In the following sections we shall consider the IVP for Fokker-Planck type equations (cf. (2.1)) and apply the methodology presented above.

More specifically, Section 2 will be concerned with the decay of the relative entropy as $t \to \infty$. There we shall obtain conditions on the entropy, the diffusion matrix and on the velocity field which allows for exponential convergence under weak conditions on the initial data (bounded relative entropy). Section 3 then is concerned with the derivation of various forms of Sobolev-type inequalities and with the analysis of conditions which guarantee that the inequalities 'saturate' non trivially. In Section 4 we apply the developed theory to nonlinear Fokker-Planck models.

We conclude this introduction with a brief (and incomplete) discussion of the relevant literature and of those issues where our approach differs from already existing ones [1]. Most importantly, our approach is based on the work by D. Bakry and M. Emery (cf. e.g., [BaEm84], [Bak91], [Bak94]), which provides a very general framework for hypercontractive semigroups and generalized Sobolev-type inequalities using the so-called iterated gradient ($\Gamma_2$) formalism [BaEm86]. We rephrase the $\Gamma_2$ approach in PDE-language and, in fact, some of our results are

[1] In the time period between the submission of the first version of this paper and the resubmission of the revised version, the entropy-entropy dissipation method for the large-time asymptotics of IVP's with thermodynamical content and the field of generalized Sobolev inequalities blossomed and many papers were authored, some of which use and extend the results and techniques of this work. For the convenience of the reader we give here an (incomplete) list of related works of the above-mentioned time period: nonlinear Fokker-Planck type equations (porous medium and fast diffusion type) [BiDo99], [BDM99] [CMU99], [CaTo98], [CJMTU99], [DoDe99], [Ott99]; linear Fokker-Planck equations on phase space without $x$-dissipation / friction (cp. Section 2.4) [DeVi99], and non-symmetric equations (cp. Section 2.4) [ArCa99]; dissipative systems with slower than exponential decay including linear Fokker-Planck type equations with non-uniformly convex (at $|x| = +\infty$) confining potentials [ToVi99]; necessary and sufficient conditions for a logarithmic Sobolev inequality in one dimension [BoGo99]. For a review including very recent results we refer to [MaVi99].

concretizations of the Bakry-Emery criterion (cf. the references cited above) to Fokker-Planck type equations. However, in the Bakry-Emery formalism an explicit control of the remainder term in the Sobolev inequalities is not straightforward. Actually, this computation is much simpler to carry out in our PDE-style approach. Let us explain this point in more detail. Careful reading of the radiative transfer-type example presented above shows that the analysis of the time decay of the entropy dissipation gives at the same time the "sharp" decay of the relative entropy as well as the remainder in (1.17). The knowledge of this remainder allows to identify in (1.18) the (unique) state saturating the Sobolev-type inequality. Hence, the entropy approach gives at the same time a proof of a "convex" inequality and all cases of equality. We believe that the latter is in many cases rather difficult to extract from the Bakry-Emery approach.

As far as the classical Gross logarithmic Sobolev inequality is concerned [Gro75], the identification of the cases of equality is due to Carlen [Car91] and, slightly later, to Ledoux [Led92]. Carlen's approach is based on a somewhat different method, which relies on information-type inequalities, and ultimately requires a deep investigation of properties of Gaussian functions. In the same paper, the connection between Gross' logarithmic Sobolev inequality and the linear Fokker-Planck equation (with quadratic confinement potential and the identity as diffusion matrix) , through the Fisher measure of information and the Blachman-Stam inequality [Bla65], [Sta59] has been fruitfully developed. The entropy-entropy dissipation approach for the same Fokker-Planck equation has been recently addressed in [Tos96b], [Tos97a]. There, the cases of equality follow easily from the identification of the remainder. Physically speaking, the entropy approach put in evidence that the remainder in this type of inequalities depends on the complete dynamics in time of the solution of the (mass-conserving) linear problem that generates the inequality itself. In other words, many Sobolev type inequalities can be interpreted as a bound for the relative entropy of the initial state with respect to the steady state of a linear mass conserving system (with thermodynamic content) in terms of the absolute value of the corresponding entropy dissipation.

An excellent background reference for logarithmic Sobolev inequalities is the overview paper [Gro93]. General linear homogeneous radiative transfer equations generating certain inequalities involving convex functions are analyzed in [GaMaUn97]. A calculation of the logarithmic Sobolev constant for non-symmetric ODEs in $\mathbb{R}^2$ (generalizing our second example above) is presented in the appendix of [DiSC96]. Entropy dissipation arguments for the decay of the solution of the heat equation in $\mathbb{R}^n$ towards the fundamental solution were used in [Tos96a]. A further application in kinetic theory is to be found in the asymptotic analysis of the spatially homogeneous Boltzmann equation [ToVi99], and in its so called 'grazing collisions limit' given by the Landau-Fokker-Planck equation [DeVi97a], [DeVi97b]. In particular, the paper [ToVi99] gives a satisfactory answer to the problem of finding a lower bound for the absolute value of the entropy dissipation associated with the spatially homogeneous Boltzmann equation.

# 2 Decay to the thermal equilibrium state as $t \to \infty$

We now consider the IVP for the *Fokker-Planck type equation*

$$\rho_t = \mathrm{div}(\mathbf{D}(\nabla\rho + \rho\nabla A)), \quad x \in \mathbb{R}^n,\ t > 0, \tag{2.1a}$$

$$\rho(t=0) = \rho_I \in L^1(\mathbb{R}^n). \tag{2.1b}$$

with the sufficiently regular confinement potential $A = A(x)$ (i.e. $A \in W^{2,\infty}_{loc}(\mathbb{R}^n;\mathbb{R})$ and $e^{-A} \in L^1(\mathbb{R}^n)$. We assume that the symmetric diffusion matrix $\mathbf{D} = \mathbf{D}(x) = (d^{ij}(x))$ is locally uniformly positive definite on $\mathbb{R}^n$ and $d^{ij} \in W^{2,\infty}_{loc}(\mathbb{R}^n;\mathbb{R})$, $i,j = 1,\dots,n$. Obviously we have the conservation property

$$\int_{\mathbb{R}^n} \rho(x,t)dx = \int_{\mathbb{R}^n} \rho_I(x)dx. \tag{2.2}$$

In a kinetic context, the independent variable $x$ in (2.1) stands for the velocity.

In this Section we assume (without restriction of generality)

$$\int_{\mathbb{R}^n} \rho_I(x)dx = 1.$$

One easily sees that (2.1a) has the steady state

$$\rho_\infty(x) = e^{-A(x)} \in L^1_+(\mathbb{R}^n), \tag{2.3}$$

assuming (w.r.o.g.) $A$ to be normalized as $\int_{\mathbb{R}^n} e^{-A(x)}dx = 1$. We remark that (by a simple minimum principle) $\rho_I(x) \geq 0$ (and $\rho_I$ not identically zero) implies $\rho(x,t) > 0$ for all $x \in \mathbb{R}^n, t > 0$.

In this Section we shall investigate the convergence of $\rho(t)$ towards the steady state $\rho_\infty$ in various norms and in relative entropy. In particular we are concerned with equations (2.1a) which exhibit an exponential decay to the steady state (2.3).

## 2.1 Spectral gap of the symmetrized equation - "linear methods"

We transform equation (2.1) to symmetric form (on $L^2(\mathbb{R}^n)$). Therefore we set

$$z := \rho/\sqrt{\rho_\infty},$$

which satisfies the IVP

$$z_t = \mathrm{div}(\mathbf{D}\nabla z) - V(x)z, \quad x \in I\!R^n,\ t > 0,$$
$$z(t=0) = z_I := \rho_I/\sqrt{\rho_\infty} \qquad \text{on } I\!R^n. \tag{2.4}$$

Here, $V(x)$ denotes the potential

$$V(x) = -\frac{1}{2}[\mathrm{Tr}(\mathbf{D}\frac{\partial^2 A}{\partial x^2}) - \frac{1}{2}(\nabla A)^\top \mathbf{D}\nabla A + (\mathrm{div}\,\mathbf{D})\cdot\nabla A], \quad x \in I\!R^n,\ t > 0. \tag{2.5}$$

$\frac{\partial^2 A}{\partial x^2}$ is the Hessian of $A(x)$ and the superscript "$\top$" denotes transposition. Now define the Hamiltonian

$$Hz = -\mathrm{div}(\mathbf{D}\nabla z) + Vz \tag{2.6}$$

on the domain

$$\mathsf{D}_Q := \{z \in L^2(I\!R^n) | Q(z,z) < \infty\}$$

of the quadratic form $Q(z_1, z_2) := (Hz_1, z_2)_{L^2(I\!R^n)}$ given by

$$Q(z_1, z_2) := \int_{I\!R^n} \nabla^\top \left(\frac{z_1}{\sqrt{\rho_\infty}}\right) \mathbf{D}(x) \nabla \left(\frac{z_2}{\sqrt{\rho_\infty}}\right) \rho_\infty(dx)$$

(obtained from (2.6) by a simple calculation). For the following we shall assume that the diffusion matrix $\mathbf{D}(x)$ and the potential $A(x)$ are such that $H$ generates a $C^0$-semigroup on $L^2(I\!R^n)$ (for various sufficient conditions see [ReSi86]). Note that $H$ satisfies

$$Hz = \sqrt{\rho_\infty} N \left(\frac{z}{\sqrt{\rho_\infty}}\right) \quad , \quad z \in \mathsf{D}_Q,$$

where $N$ is the Dirichlet form-type operator on $L^2(I\!R^n, d\rho_\infty)$ defined by

$$(Nu, v)_{L^2(I\!R^n, d\rho_\infty)} := \int_{I\!R^n} \nabla^\top u \mathbf{D} \nabla v \, \rho_\infty(dx)$$

(see [Gro93]). For future reference we also remark that

$$-\int_{I\!R^n} \frac{\rho_2}{\rho_\infty} L\rho_1 \, dx = Q\left(\frac{\rho_1}{\sqrt{\rho_\infty}}, \frac{\rho_2}{\sqrt{\rho_\infty}}\right) \quad ; \quad \frac{\rho_1}{\sqrt{\rho_\infty}}, \frac{\rho_2}{\sqrt{\rho_\infty}} \in \mathsf{D}_Q. \tag{2.7}$$

Here $L$ denotes the appropriate extension of the Fokker-Planck type operator

$$L\rho := \mathrm{div}\left(\mathbf{D}\left(\nabla\rho + \rho\nabla A\right)\right).$$

Obviously the spectrum $\sigma(H)$ is contained in $[0, \infty)$. Since $Q(z) = 0$ iff $z = const.\sqrt{\rho_\infty}$ we conclude that the ground state $z_\infty = \exp(-A/2)$ of $H$ is non-degenerate and that $\rho_\infty$ is the unique normalized steady state of (2.1a).

The solution of (2.4) can be written as

$$z(t) = \sqrt{\rho_\infty} + \int_{(0,\infty)} e^{-\lambda t} d(P_\lambda \frac{\rho_I}{\sqrt{\rho_\infty}}), \tag{2.8}$$

where $P_\lambda$ is the projection valued spectral measure of $H$. Of course, we assume here $\rho_I/\sqrt{\rho_\infty} \in L^2(\mathbb{R}^n)$. From this spectral representation we immediately conclude the convergence of $z(t)$ to $\sqrt{\rho_\infty}$ in $L^2(\mathbb{R}^n)$ as $t \to \infty$. Let us now consider a Hamiltonian $H$ with a positive spectral gap $\lambda_0$ (i.e. distance of $\sigma(H) \setminus \{0\}$ from the eigenvalue 0). For the case $\mathbf{D}(x) \equiv \mathbf{I}$, the identity matrix, a simple sufficient condition for a positive spectral gap is given by $V \in L^1_{loc}(\mathbb{R}^n; \mathbb{R})$, $V(x)$ bounded below and $V(x) \to \infty$ for $|x| \to \infty$ (see e.g. [ReSi87a, Th. XIII.67]). In this case we obtain exponential convergence:

$$\|z(t) - \sqrt{\rho_\infty}\|_{L^2(\mathbb{R}^n)} \leq \|z_I - \sqrt{\rho_\infty}\|_{L^2(\mathbb{R}^n)} e^{-t\lambda_0}, \tag{2.9}$$

which implies exponential convergence of $\rho(t)$ to $\rho_\infty$ in $L^1(\mathbb{R}^n)$:

$$\begin{aligned} \int_{\mathbb{R}^n} |\rho(t) - \rho_\infty| dx &= \int_{\mathbb{R}^n} \sqrt{\rho_\infty} \frac{|\rho(t) - \rho_\infty|}{\sqrt{\rho_\infty}} dx \\ &\leq \|\rho_\infty\|^{\frac{1}{2}}_{L^1(\mathbb{R}^n)} \|z(t) - \sqrt{\rho_\infty}\|_{L^2(\mathbb{R}^n)} = O(e^{-t\lambda_0}). \end{aligned} \tag{2.10}$$

**Example 2.1.** *Take $A \in C^2(\mathbb{R}^n; \mathbb{R})$ such that, for some $\alpha > 0, c > 0, L > 0$*

$$A(x) = c|x|^{2\alpha}, \qquad |x| > L$$

*and*

$$\mathbf{D}(x) = \mathbf{I} \quad \text{on } \mathbb{R}^n.$$

*Then we compute*

$$V(x) = c^2\alpha^2|x|^{2(2\alpha-1)} - c\alpha\,(n + 2\alpha - 2)\,|x|^{2(\alpha-1)}, \quad |x| > L.$$

*Clearly, the above mentioned growth assumptions on $V$, which are sufficient for a positive spectral gap at 0, are satisfied if and only if $\alpha > 1/2$.*

*We shall later on see that, if the diffusion matrix is the identity, at least quadratic growth of $A(x)$ at $|x| = +\infty$ is necessary to carry out the entropy-entropy dissipation approach giving exponential convergence to zero of the relative entropy.*

The simple spectral approach of course only holds under the restrictive assumption $\rho_I/\sqrt{\rho_\infty} \in L^2(\mathbb{R}^n)$. Also, as can be seen from the above, conditions for a positive spectral gap at 0 of the Hamiltonian (2.6) are usually phrased in $\mathbf{D}$, $V$ and are by no means intuitive in terms of $\mathbf{D}$ and $A$. In order to remedy this issue and to better illustrate the convergence of $\rho(t)$ towards $\rho_\infty$ we below investigate the decay in relative entropy $e(\rho(t)|\rho_\infty)$ defined by (1.7). In fact, we shall not only consider this logarithmic "physical entropy", but a wider class of relative entropies which essentially lie 'between' this physical entropy and the quadratic functional $\|\rho(t)-\rho_\infty\|^2_{L^2(I\!R^n,\rho_\infty^{-1}(dx))}$. The entropy approach shall use a uniform convexity condition on the confining potential $A$, which is much easier to verify than typical conditions on the existence of a spectral gap of the Hamiltonian $H$.

For future reference we also consider a second symmetrization of (2.1) (on $L^2(I\!R^n, d\rho_\infty)$) which is more commonly used in the probabilistic literature on this subject. We set

$$\mu := \rho/\rho_\infty,$$

which satisfies the IVP

$$\mu_t = \rho_\infty^{-1}\mathrm{div}(\rho_\infty \mathbf{D}\nabla\mu) = \mathrm{div}(\mathbf{D}\nabla\mu) - (\nabla A)^\top \mathbf{D}\nabla\mu =: \tilde{L}\mu, \quad x \in I\!R^n,\ t>0,$$
$$\mu_I := \rho_I/\rho_\infty \in L^1(I\!R^n, d\rho_\infty). \tag{2.11}$$

In terms of this symmetrized problem we shall extend the allowed initial data $\mu_I$ from $L^2(I\!R^n, d\rho_\infty)$ to the Orlicz space $L^1\log L(I\!R^n, d\rho_\infty)$.

## 2.2 Admissible relative entropies and their generating functions

We now introduce the relative entropies, which we shall use in the sequel, and discuss their analytical properties.

**Definition 2.2.** *Let $J$ be either $I\!R$ or $I\!R^+ := (0,\infty)$ . Let $\psi \in C(\overline{J}) \cap C^4(J)$ satisfy the conditions*

$$\psi(1) = 0, \tag{2.12a}$$
$$\psi'' \geq 0, \quad \psi'' \not\equiv 0 \quad \text{on } J, \tag{2.12b}$$
$$(\psi''')^2 \leq \frac{1}{2}\psi''\psi^{IV} \quad \text{on } J. \tag{2.12c}$$

*Let $\rho_1 \in L^1(I\!R^n), \rho_2 \in L^1_+(I\!R^n)$ with $\int \rho_1 dx = \int \rho_2 dx = 1$ and $\rho_1/\rho_2 \in \overline{J}$ $\rho_2(dx)-$ a.e. Then*

$$e_\psi(\rho_1|\rho_2) := \int_{I\!R^n} \psi\left(\frac{\rho_1}{\rho_2}\right) \rho_2(dx) \tag{2.13}$$

*is called an* admissible relative entropy *(of $\rho_1$ with respect to $\rho_2$) with* generating function *$\psi$.*

Note that

$$e_\psi(\rho_1|\rho_2) \geq 0 \tag{2.14}$$

follows from Jensen's inequality.

**Remark 2.3.** *The condition (2.12c) is equivalent to*

$$\left(\frac{1}{\psi''}\right)'' \leq 0 \tag{2.15}$$

*whenever* $\psi'' > 0$. *Since (2.15) excludes positive poles of* $\frac{1}{\psi''}$ *we conclude* $\psi'' > 0$ *on* $J$. *Thus admissible entropies are generated by strictly convex functions* $\psi$.

**Remark 2.4.** *If* $\psi$ *satisfies the conditions (2.12a)–(2.12c), so does its "normalization"* $\widetilde{\psi}(\sigma) = \psi(\sigma) - \psi'(1)(\sigma - 1)$, *and they both generate the same relative entropy:* $e_{\widetilde{\psi}}(\rho_1|\rho_2) = e_\psi(\rho_1|\rho_2)$. *In the sequel we will therefore assume that the* generator $\psi$ *of* $e_\psi$ *be normalized as*

$$\psi'(1) = 0. \tag{2.13d}$$

*This implies that the relative entropies* $e_\psi$ *and their generators* $\psi$ *are bijectively related. Due to the convexity of* $\psi$ *we have*

$$\psi \geq 0 \qquad \text{on } J. \tag{2.16}$$

Our class of generating functions $\psi$ coincides with those considered in [BaEm84] (up to the normalizations (2.12a), (2.13d)).

We now list some typical examples of admissible relative entropies on $J = I\!R^+$: The physical relative entropy (1.7) is generated by $\chi_{ph}(\sigma) = \sigma \ln \sigma - \sigma + 1$ rather than by $\psi(\sigma) = \sigma \ln \sigma$. It is a special case of the admissible relative entropies generated by

$$\chi(\sigma) = \alpha(\sigma + \beta) \ln \frac{\sigma + \beta}{1 + \beta} - \alpha(\sigma - 1), \qquad \sigma > 0;\ \alpha > 0,\ \beta \geq 0, \tag{2.17a}$$

with continuous extension to $\sigma = 0$. For $1 < p < 2$

$$\xi_p(\sigma) = \alpha\left[(\sigma + \beta)^p - (1 + \beta)^p - p(1 + \beta)^{p-1}(\sigma - 1)\right], \quad \sigma \geq 0;\ \alpha > 0, \beta \geq 0, \tag{2.17b}$$

and for $p = 2$

$$\varphi(\sigma) = \alpha(\sigma - 1)^2, \quad \sigma \geq 0;\ \alpha > 0, \tag{2.17c}$$

generate admissible relative entropies. In the last example we clearly have $e_\varphi(\rho_1|\rho_2) = \alpha(\|\rho_1\|^2_{L^2(I\!R^n, \rho_2^{-1}(dx))} - 1)$.

In the above definition we excluded linear entropy functionals as they would be zero due to the assumed normalizations of $\rho_1$ and $\rho_2$.

In the following Lemmata we derive important properties of admissible entropies:

**Lemma 2.5.** *For* $J = I\!R$ *all admissible entropies are generated by (2.17c).*

PROOF: Let $g(\sigma) := \frac{1}{\psi''(\sigma)}, \quad \sigma \in I\!R$. $g'' \le 0$ and $g > 0$ on $I\!R$ imply $g(\sigma) = \text{const} > 0$, and the assertion follows. □

For the following we shall assume $J = I\!R^+$ and $\rho_I \ge 0$. Except being reasonable from a kinetic and probabilistic viewpoint this case allows for a richer mathematical structure since only quadratic admissible entropies exist for $J = I\!R$.

The above examples (2.17a), (2.17c) of admissible entropies include the two limiting cases for the asymptotic behavior (as $\sigma \to \infty$) of the generating function $\psi$. An admissible relative entropy $e_\psi(\rho_1|\rho_2)$ can be bounded below by a logarithmic *subentropy* $e_\chi$ and bounded above by a quadratic *superentropy* $e_\varphi$:

**Lemma 2.6.** *Let $\psi$ generate an admissible relative entropy with $J = I\!R^+$. Then there exists a logarithmic-type function $\chi$ (2.17a) and a quadratic function $\varphi$ (2.17c) such that*

$$\chi(\sigma) \le \psi(\sigma) \le \varphi(\sigma), \quad \sigma \in J, \tag{2.18a}$$

*and hence*

$$0 \le e_\chi(\rho_1|\rho_2) \le e_\psi(\rho_1|\rho_2) \le e_\varphi(\rho_1|\rho_2). \tag{2.18b}$$

*$\chi$ and $\varphi$ both satisfy (2.12) and thus generate, respectively, an admissible sub- and superentropy for $e_\psi$.*

PROOF: Since $J = I\!R^+$, the function $g$ from the proof of Lemma 2.5 satisfies

$$g > 0, \quad g' \ge 0, \quad g'' \le 0 \quad \text{on } J. \tag{2.19}$$

Now denote the derivatives of the given function $\psi$ by

$$\psi(1) = 0,\ \psi'(1) = 0,\ \psi''(1) =: \mu_2 > 0,\ \psi'''(1) =: \mu_3 \le 0. \tag{2.20}$$

From (2.19) we readily get the estimate

$$\left.\begin{array}{ll} \sigma\mu_2^{-1}, & 0 < \sigma < 1 \\ \mu_2^{-1}, & \sigma > 1 \end{array}\right\} \le g(\sigma) \le \gamma\sigma + \delta,\ \sigma > 0,$$

with $\gamma := -\mu_3\mu_2^{-2} \ge 0$, $\delta := (\mu_2 + \mu_3)\mu_2^{-2} \ge 0$. Integrating the corresponding estimate for $\psi'' = \frac{1}{g}$,

$$(\gamma\sigma + \delta)^{-1} \le \psi''(\sigma) \le \left\{\begin{array}{ll} \mu_2/\sigma, & 0 < \sigma < 1 \\ \mu_2, & \sigma > 1 \end{array}\right. \tag{2.21}$$

we obtain with (2.20) the upper bound for $\psi$:

$$\begin{aligned} \psi(\sigma) &\le \left\{\begin{array}{ll} \mu_2(\sigma \ln \sigma - \sigma + 1), & 0 < \sigma < 1 \\ \frac{\mu_2}{2}(\sigma - 1)^2, & \sigma > 1 \end{array}\right\} \\ &\le \mu_2(\sigma - 1)^2 =: \varphi(\sigma), \quad \sigma > 0. \end{aligned} \tag{2.22}$$

To derive the lower bound of $\psi$ one integrates (2.21) twice to show $\chi(\sigma) \leq \psi(\sigma)$. For $\gamma > 0$ the function $\chi(\sigma)$ is given by (2.17a) with $\alpha = \frac{1}{\gamma}$, $\beta = \frac{\delta}{\gamma}$.

If $\gamma = 0$ we set

$$\chi(\sigma) = \frac{\mu_2}{2}(\sigma - 1)^2. \tag{2.23}$$

□

The sub- and superentropies are certainly the most important ones from both the physical and mathematical point of view. Via their exponential decay they correspond, resp., to hypercontractivity and (strict) contractivity of the semigroup generated by (2.11). However, we shall here in a unified treatment consider all admissible entropies in order to clarify the mathematical picture by also deriving the interpolating Sobolev inequalities of Beckner (see (3.8) below) and decay results in Orlicz spaces between $L^1 \log L(\mathbb{R}^n, d\rho_\infty)$ and $L^2(\mathbb{R}^n, d\rho_\infty)$.

The well-known Csiszár-Kullback inequality ([Csi63], [Kul59]) shows that the logarithmic relative entropy (1.7) $e = e_{\chi_{ph}}$ is a 'measure' for the distance between two normalized $L^1_+(\mathbb{R}^n)$-functions $\rho_1, \rho_2$ with $\int_{\mathbb{R}^n} \rho_1 dx = \int_{\mathbb{R}^n} \rho_2 dx = 1$:

$$\frac{1}{2}\|\rho_1 - \rho_2\|^2_{L^1(\mathbb{R}^n)} \leq e(\rho_1|\rho_2). \tag{2.24}$$

In a simple calculation using the subentropy $e_{\chi_{ph}}$ and Lemma 2.6 this inequality can be extended to any admissible entropy $e_\psi$: For $\gamma > 0$, and $\chi$ hence given by (2.17a) we introduce the normalized function $\widetilde{\rho} := \frac{\gamma\rho_1 + \delta\rho_2}{\gamma+\delta} \geq 0$, and estimate using (2.24):

$$\frac{1}{2}\|\rho_1 - \rho_2\|^2_{L^1(\mathbb{R}^n)} = \frac{\mu_2^2}{2\mu_3^2}\|\widetilde{\rho} - \rho_2\|^2_{L^1(\mathbb{R}^n)} \leq \frac{\mu_2^2}{\mu_3^2} e(\widetilde{\rho}|\rho_2) = \frac{1}{\mu_2} e_\chi(\rho_1|\rho_2). \tag{2.25}$$

With (2.18b) this gives

$$\frac{1}{2}\|\rho_1 - \rho_2\|^2_{L^1(\mathbb{R}^n)} \leq \frac{1}{\mu_2} e_\psi(\rho_1|\rho_2), \tag{2.26}$$

with the notation $\mu_2 = \psi''(1) = \chi''(1)$. If $\gamma = 0$, (2.26) is easily derived with the estimate (2.10) and with (2.23). A detailed analysis of Csiszár-Kullback inequalities is the topic of [AMTU00].

**Remark 2.7.** *The physical relative entropy $e = e_{\chi_{ph}}$ can be written as difference*

$$e(\rho|\rho_\infty) = F(\rho|A) - F(\rho_\infty|A); \quad F(\rho|A) := \int_{\mathbb{R}^n} (\rho \ln \rho + A(x)\rho) dx.$$

*In thermodynamics ([Ell85]), $F(\rho|A)$ is referred to as* free energy *of the state $\rho$ under the action of the potential $A$. $F$ is the sum of the physical entropy and the (potential) energy.*

In our subsequent analysis we shall need the continuity of the relative entropy:

**Lemma 2.8.** *Let the sequence* $\rho_j \to \rho$ *(as* $j \to \infty$*) in* $L^2_+(\mathbb{R}^n, \rho_\infty^{-1}(dx))$ *with the normalization* $\int \rho_j dx = \int \rho dx = \int \rho_\infty dx = 1$*. Then for all admissible entropies* $\psi$ *with* $J = \mathbb{R}^+$*:*

$$e_\psi(\rho_j|\rho_\infty) \to e_\psi(\rho|\rho_\infty) \quad \text{as } j \to \infty. \tag{2.27}$$

Proof: Since $\psi \in C[0,1]$ there exists for any $\varepsilon > 0$ a positive $\delta = \delta(\varepsilon)$ such that

$$|\psi(\sigma_1) - \psi(\sigma_2)| < \varepsilon \quad \text{for } 0 \le \sigma_{1,2} \le \delta(\varepsilon). \tag{2.28}$$

Integrating (2.21) we readily obtain the estimate:

$$\exists\, C_1(\delta) \text{ such that } \; |\psi'(\sigma)| \le C_1(\delta)(1+\sigma) \quad \text{for } \sigma \ge \delta. \tag{2.29}$$

We shall first derive estimates on $|\psi(a) - \psi(b)|; a, b \in J$, for three different cases of $a$ and $b$. For $a, b \ge \delta$ we use the mean value theorem and the monotonicity of $\psi'$ to estimate:

$$|\psi(a) - \psi(b)| \le |a-b|(|\psi'(a)| + |\psi'(b)|) \le |a-b| C_1(\delta)(2+a+b). \tag{2.30}$$

For the next case assume $a \ge \delta$, $b < \delta$ (or vice versa). With (2.30) this yields

$$\begin{aligned} |\psi(a) - \psi(b)| &\le |\psi(a) - \psi(\delta)| + |\psi(\delta) - \psi(b)| \\ &\le (a-\delta) C_1(\delta)(2+a+\delta) + (\delta - b) C_2(\delta) \\ &\le |a-b| \left[ C_1(\delta)(2+a+b+\delta) + C_2(\delta) \right], \end{aligned} \tag{2.31}$$

with $C_2(\delta) := \sup_{c\in[0,\delta)} \frac{|\psi(\delta)-\psi(c)|}{\delta - c} < \infty$.

Finally, for $a, b < \delta$ we have $|\psi(a) - \psi(b)| < \varepsilon$ from (2.28).

Using these three estimates we can now control the difference of relative entropies.

For arbitrarily small $\varepsilon$ we estimate:

$$
\begin{aligned}
&|e_\psi(\rho_j|\rho_\infty) - e_\psi(\rho|\rho_\infty)| \\
&\leq \int\limits_{\{\frac{\rho_j}{\rho_\infty}\geq\delta \text{ or } \frac{\rho}{\rho_\infty}\geq\delta\}} \left|\psi\left(\frac{\rho_j}{\rho_\infty}\right) - \psi\left(\frac{\rho}{\rho_\infty}\right)\right| \rho_\infty dx \qquad (2.32)\\
&\quad + \int\limits_{\{\frac{\rho_j}{\rho_\infty},\frac{\rho}{\rho_\infty}<\delta\}} \left|\psi\left(\frac{\rho_j}{\rho_\infty}\right) - \psi\left(\frac{\rho}{\rho_\infty}\right)\right| \rho_\infty dx \\
&\leq [C_1(\delta)(2+\delta) + C_2(\delta)] \int\limits_{I\!R^n} |\rho_j - \rho| dx \qquad (2.33)\\
&\quad + C_1(\delta) \int\limits_{I\!R^n} \frac{|\rho_j - \rho|}{\sqrt{\rho_\infty}} \frac{\rho_j + \rho}{\sqrt{\rho_\infty}} dx + \varepsilon \int\limits_{I\!R^n} \rho_\infty dx \\
&\leq [C_1(\delta)(2+\delta) + C_2(\delta)] \, \|\rho_j - \rho\|_{L^1(I\!R^n)} \\
&\quad + C_1(\delta) \|\rho_j - \rho\|_{L^2(I\!R^n, \rho_\infty^{-1}(dx))} \|\rho_j + \rho\|_{L^2(I\!R^n, \rho_\infty^{-1}(dx))} + \varepsilon. \qquad (2.34)
\end{aligned}
$$

As $j \to \infty$ this last term converges to $\varepsilon$, since $L^2(I\!R^n, \rho_\infty^{-1}(dx))$-convergence implies $L^1(I\!R^n)$-convergence. This finishes the proof. □

For future reference we state the following elementary result for generators of admissible relative entropies:

**Lemma 2.9.** *The generator $\psi$ of every admissible relative entropy $e_\psi$ satisfies:*

*a)*

$$\frac{\psi(\sigma)}{\sigma} + \mu_2\left(1 - \frac{1}{\sigma}\right) \leq \psi'(\sigma) \leq \frac{2\psi(\sigma)}{\sigma} + \mu_2\left(1 - \frac{1}{\sigma}\right), \ \sigma > 0, \qquad (2.35)$$

*with the notation $\mu_2 = \psi''(1) > 0$,*

*b)*

$$
\begin{aligned}
\psi(\sigma) &\leq \psi(\sigma_0)\left(\frac{\sigma}{\sigma_0}\right)^2 + \mu_2\left(\frac{\sigma}{\sigma_0} - 1\right)(\sigma - 1), \ \sigma \geq \sigma_0 > 0, \quad (2.36)\\
\psi(\sigma) &\leq \psi(\sigma_0)\frac{\sigma}{\sigma_0} + \mu_2\left(\frac{\sigma}{\sigma_0} - 1\right)(\sigma - 1), \ \sigma_0 \geq \sigma > 0. \quad (2.37)
\end{aligned}
$$

Proof: a) For the right inequality of (2.35) we have to show that $G_1(\sigma) := 2\psi + \mu_2(\sigma - 1) - \sigma\psi' \geq 0$. But this follows from $G_1(1) = G_1'(1) = 0$, $G_1''(\sigma) = -\sigma\psi''' \geq 0, \sigma > 0$ (see (2.20)).

For the left inequality of (2.35) we have to show that $G_2(\sigma) := \psi + \mu_2(\sigma - 1) - \sigma\psi' \le 0$. Like before we have $G_2(1) = G_2'(1) = 0$ and it remains to show

$$G_2''(\sigma) = -\psi'' - \sigma\psi''' \le 0, \ \sigma > 0. \tag{2.38}$$

Since $\psi$ generates an admissible entropy we have $\psi'''(\sigma) \le 0, \ \psi^{IV}(\sigma) \ge 0$ (see (2.19), (2.12c)). Thus there exists a unique $\sigma_1 \in [0, \infty]$ such that $\psi'''(\sigma) < 0$ on $(0, \sigma_1)$ and $\psi'''(\sigma) = 0$ on $[\sigma_1, \infty)$.

In the case $\sigma \in [\sigma_1, \infty)$ we have $G_2''(\sigma) = -\psi''(\sigma) < 0$.

In the case $\sigma \in (0, \sigma_1)$ we rewrite (2.12c) as

$$1 \le -\left(\frac{\psi''}{\psi'''}\right)'$$

and integrate over the interval $(\varepsilon, \sigma), \ 0 < \varepsilon < \sigma_1$:

$$\sigma - \varepsilon \le -\frac{\psi''(\sigma)}{\psi'''(\sigma)} + \frac{\psi''(\varepsilon)}{\psi'''(\varepsilon)} \le -\frac{\psi''(\sigma)}{\psi'''(\sigma)}.$$

Letting tend $\varepsilon \to 0$ we obtain $G_2'' = -\psi'' - \sigma\psi''' \le 0$, and this finishes the proof of (2.35).

b) Applying the Gronwall lemma to the right inequality of (2.35) (with fixed $\sigma_0$) gives

$$\psi(\sigma) \le \psi(\sigma_0)\left(\frac{\sigma}{\sigma_0}\right)^2 + \mu_2\left(\frac{\sigma}{\sigma_0} - 1\right)\left(\sigma - \frac{\sigma/\sigma_0 + 1}{2}\right), \ \sigma \ge \sigma_0 > 0,$$

and (2.36) follows from $\frac{\sigma}{\sigma_0} \ge 1$.

For the proof of (2.37) we use the left inequality of (2.35) and estimate:

$$\psi'(\sigma) \ge \frac{\psi(\sigma)}{\sigma} + \mu_2\left(1 - \frac{1}{\sigma}\right) \ge \frac{\psi(\sigma)}{\sigma} + \mu_2\left(\frac{1}{\sigma} - \frac{1}{\sigma^2}\right), \ \sigma > 0. \tag{2.39}$$

Applying the Gronwall lemma to (2.39) gives

$$\psi(\sigma) \le \psi(\sigma_0)\frac{\sigma}{\sigma_0} + \mu_2(\sigma \ln\frac{\sigma}{\sigma_0} + 1 - \frac{\sigma}{\sigma_0}), \ 0 < \sigma \le \sigma_0,$$

and (2.37) follows with the estimate $\ln x \le x - 1$. □

**Remark 2.10.** *For future reference we state that, due to (2.38), the function* $\sigma\psi''(\sigma)$ *is increasing on* $[0, +\infty)$. *Therefore* $0 \le \lim_{\sigma\to 0+} \sigma\psi''(\sigma) < +\infty$.

## 2.3 Exponential decay of the entropy dissipation and the relative entropy

With our notion of superentropies we can now show the convergence of $\rho(t)$ to $\rho_\infty$ in relative entropy.

**Lemma 2.11.** *Let $e_\psi$ be an admissible relative entropy and assume $z_I = \rho_I/\sqrt{\rho_\infty} \in L^2(I\!R^n)$. Then $e_\psi(\rho(t)|\rho_\infty) \to 0$ as $t \to \infty$.*

PROOF: With the notation $\mu_2 = \psi''(1)$ we estimate using (2.18b), (2.22):

$$0 \le e_\psi(\rho(t)|\rho_\infty) \le e_\varphi(\rho(t)|\rho_\infty) = \int_{I\!R^n} \varphi\Big(\frac{\rho(t)}{\rho_\infty}\Big)\,\rho_\infty(dx) = \mu_2 \|z(t) - \sqrt{\rho_\infty}\|^2_{L^2(I\!R^n)}, \tag{2.40}$$

and the assertion follows from (2.8). □

If the Hamiltonian $H$ in (2.6) has a spectral gap $\lambda_0 > 0$, then the exponential decay from (2.9) of course carries over to the relative entropy.

In the above lemma, the assumption $\rho_I/\sqrt{\rho_\infty} \in L^2(I\!R^n)$ is unnaturally restrictive. The subsequent analysis (i.e. the entropy-entropy dissipation method) aims at considering initial data which only have finite relative entropy and at proving explicit decay bounds for the relative entropy in terms of the initial relative entropy. The 'price' for this extension will be a condition on the evolution problem (2.1a) (see (A3) below) that is stronger than assuming $H$ to have a spectral gap (as we shall see below).

We remark that the entropy-entropy dissipation method is -contrary to the spectral analysis carried out above- inherently nonlinear and, thus, sufficiently robust to be applied to nonlinear problems as in Section 4.

We now proceed similarly to [Tos96b] and to example 2 of the introduction. Consider the entropy dissipation

$$I_\psi(\rho(t)|\rho_\infty) := \frac{d}{dt} e_\psi(\rho(t)|\rho_\infty) \tag{2.41}$$

and the entropy dissipation rate

$$R_\psi(\rho(t)|\rho_\infty) := \frac{d}{dt} I_\psi(\rho(t)|\rho_\infty). \tag{2.42}$$

Eq. (2.41) is referred to as entropy equation. To facilitate the computations we rewrite (2.1a) in the following form:

$$\rho_t = \operatorname{div}(\rho_\infty \mathbf{D} u) \tag{2.43}$$

with the notation $u = \nabla(\frac{\rho}{\rho_\infty})$. Differentiating the relative entropy $e_\psi(\rho(t)|\rho_\infty)$ gives

$$I_\psi(\rho(t)|\rho_\infty) = \int_{\mathbb{R}^n} \psi'\Big(\frac{\rho}{\rho_\infty}\Big)\rho_t\,dx. \tag{2.44}$$

By using (2.43) we obtain after an integration by parts

$$I_\psi(\rho(t)|\rho_\infty) = -\int_{\mathbb{R}^n} \psi''\Big(\frac{\rho}{\rho_\infty}\Big)u^\top \mathbf{D} u \rho_\infty\,dx \le 0, \tag{2.45}$$

due to the positivity of $\mathbf{D}$. Using (2.43) we compute (2.42):

$$\begin{aligned} R_\psi(\rho(t)|\rho_\infty) \quad &= \quad -\int_{\mathbb{R}^n} \psi'''\Big(\frac{\rho}{\rho_\infty}\Big)\operatorname{div}(\mathbf{D}u\rho_\infty)u^\top \mathbf{D}u\,dx \\ &- \quad 2\int_{\mathbb{R}^n} \psi''\Big(\frac{\rho}{\rho_\infty}\Big)u^\top \mathbf{D}u_t\rho_\infty dx. \end{aligned} \tag{2.46}$$

Clearly, the computations which lead to (2.45) and (2.46) are formal. However, they can easily be justified for initial data $\rho_I \in L^2(\mathbb{R}^n;\rho_\infty^{-1}(dx))$ and for entropy generators without singularity at $\sigma = 0$ by taking into account the semigroup property of the Hamiltonian $H$, the partial integration formula (2.7) and the fact $\rho > 0$ on $\mathbb{R}^n, t > 0$. General admissible entropies can easily be dealt with by a local cut-off at $\sigma = 0$.

We now return to proving the exponential decay of $e_\psi(\rho(t)|\rho_\infty)$ under additional assumptions on $A$ and $\mathbf{D}$. At first we shall derive an exponential decay rate for the entropy dissipation $I_\psi$ by using the special form of the entropy dissipation rate (2.46).

**Remark 2.12.** *For the following we shall have to give a meaning to $I_\psi(\rho|\rho_\infty)$ even when $\rho$ becomes zero (which may be the case at the initial state). Since, for $f$ positive and differentiable:*

$$\psi''(f)(\nabla f)^\top \mathbf{D}\nabla f = \left(\nabla \int_1^f \sqrt{\psi''(s)}\,ds\right)^\top \mathbf{D}\nabla \int_1^f \sqrt{\psi''(s)}\,ds \tag{2.47}$$

*we set for $\rho \ge 0$*

$$I_\psi(\rho|\rho_\infty) = -\int_{\mathbb{R}^n} (\nabla w)^\top \mathbf{D}\nabla w \rho_\infty(dx), \quad w = F_\psi\left(\frac{\rho}{\rho_\infty}\right), \tag{2.48}$$

*if $w \in H^1_{loc}(\mathbb{R}^n)$ with*

$$F_\psi(\mu) = \int_1^\mu \sqrt{\psi''(s)}\,ds, \quad \mu > 0. \tag{2.49}$$

*Note that due to (2.21) $F_\psi$ is Hölder continuous with exponent $1/2$ locally at $\mu = 0$.*

At first we consider the case of a scalar diffusion, i.e. $\mathbf{D}(x) = \mathbf{I}D(x)$.

**Lemma 2.13.** *Let the initial condition $\rho_I \in L^2(I\!R^n; \rho_\infty^{-1}(dx))$ satisfy $|I_\psi(\rho_I|\rho_\infty)| < \infty$ for the admissible entropy $e_\psi$. Assume that the scalar coefficients $A(x)$ and $D(x)$ of (2.1a) satisfy the condition*

*(A1)* $\exists\, \lambda_1 > 0$ *such that*

$$\begin{aligned}\Big(\frac{1}{2}-\frac{n}{4}\Big)\frac{1}{D}\nabla D\otimes\nabla D+\frac{1}{2}(\Delta D-\nabla D\cdot\nabla A)\mathbf{I}&\\ +D\frac{\partial^2 A}{\partial x^2}+\frac{\nabla A\otimes\nabla D+\nabla D\otimes\nabla A}{2}-\frac{\partial^2 D}{\partial x^2}&\;\geq\;\lambda_1\mathbf{I}\end{aligned}$$

*(in the sense of positive definite matrices) $\forall x \in I\!R^n$. Then the entropy dissipation converges to 0 exponentially:*

$$|I_\psi(\rho(t)|\rho_\infty)| \leq e^{-2\lambda_1 t}|I_\psi(\rho_I|\rho_\infty)|, \quad t>0. \tag{2.50}$$

Proof: After an integration by parts (which can be justified as mentioned above) the first term of the entropy dissipation rate (2.46) reads

$$\begin{aligned}R_1 &= \int_{I\!R^n}\psi^{\mathrm{IV}}(e^A\rho)D^2|u|^4e^{-A}dx\\ &\quad +2\int_{I\!R^n}\psi'''(e^A\rho)D^2e^{-A}u^\top\frac{\partial u}{\partial x}udx+\int_{I\!R^n}\psi'''(e^A\rho)e^{-A}D|u|^2u\cdot\nabla Ddx.\end{aligned}$$

We set

$$u_t = \nabla(e^A\mathrm{div}(De^{-A}u))$$

in the second term of (2.46), which becomes

$$R_2=-2\int_{I\!R^n}\psi''(e^A\rho)De^{-A}[u\cdot\nabla(D\mathrm{div}u)+u^\top\nabla\otimes(\nabla D-\nabla AD)u+u^\top\frac{\partial u}{\partial x}(\nabla D-\nabla AD)]dx.$$

We differentiate $u\cdot\nabla(D\mathrm{div}u) = (u\cdot\nabla D)\mathrm{div}u + Du\cdot\nabla(\mathrm{div}u)$, and we express $Du\cdot\nabla(\mathrm{div}u)$ according to

$$\frac{1}{2}\Delta(D|u|^2)=\frac{1}{2}|u|^2\Delta D+2\nabla D^\top\frac{\partial u}{\partial x}u+D\sum_{i,j}\left(\frac{\partial u_i}{\partial x_j}\right)^2+Du\cdot\nabla(\mathrm{div}u).$$

We rewrite $R_2$ as $R_2 = S_1 + S_2$ with

$$\begin{aligned}S_1 &= -\int_{I\!R^n}\psi''(e^A\rho)\mathrm{div}[D\nabla(D|u|^2)e^{-A}]dx\\ &= \int_{I\!R^n}\psi'''(e^A\rho)\left(D^2u\cdot\nabla(|u|^2)+D\nabla D\cdot u|u|^2\right)e^{-A}dx,\end{aligned}$$

$$
\begin{aligned}
S_2 &= 2\int_{I\!R^n} \psi''(e^A\rho) D e^{-A}\Big[ u^\top \nabla \otimes (\nabla A D - \nabla D) u \\
&\quad + \frac{1}{2}\Delta D |u|^2 - \frac{1}{2}|u|^2 \nabla D \cdot \nabla A + \frac{1}{D}\frac{2-n}{4}(u\cdot\nabla D)^2 \Big] dx \\
&\quad + 2\int_{I\!R^n} \psi''(e^A\rho) e^{-A}\Big[ \frac{n-2}{4}(u\cdot\nabla D)^2 - D(u\cdot\nabla D)\mathrm{div} u \\
&\quad + 2D\nabla D^\top \frac{\partial u}{\partial x} u + D^2 \sum_{i,j}\left(\frac{\partial u_i}{\partial x_j}\right)^2 + \frac{1}{2}|u|^2|\nabla D|^2 \Big] dx \\
&= T_1 + T_2.
\end{aligned}
$$

The second integral in $S_2$ can be written as

$$
T_2 = 2\int_{I\!R^n} \psi''(e^A\rho) e^{-A} \sum_{i,j}\left( D\frac{\partial u_i}{\partial x_j} + \frac{1}{2}\frac{\partial D}{\partial x_i} u_j + \frac{1}{2} u_i \frac{\partial D}{\partial x_j} - \frac{1}{2}\delta_{ij} \nabla D \cdot u \right)^2 dx
$$

and (A1) allows to estimate the first term in $S_2$ by

$$
T_1 \geq 2\lambda_1 \int_{I\!R^n} \psi''(e^A\rho) D e^{-A} |u|^2 dx.
$$

All in all we have

$$
\begin{aligned}
R_1 + R_2 &= R_1 + S_1 + S_2 = (R_1 + S_1 + T_2) + T_1 \\
&\geq \int_{I\!R^n} \Big[ \psi^{\mathrm{IV}}(e^A\rho) D^2 |u|^4 + \psi'''(e^A\rho)(4D^2 u^\top \frac{\partial u}{\partial x} u + 2D|u|^2 u \cdot \nabla D) \\
&\quad + 2\psi''(e^A\rho) \sum_{i,j}\left( D\frac{\partial u_i}{\partial x_j} + \frac{1}{2}\frac{\partial D}{\partial x_i} u_j + \frac{1}{2} u_i \frac{\partial D}{\partial x_j} - \frac{1}{2}\delta_{ij}\nabla D \cdot u \right)^2 \Big] e^{-A} dx \\
&\quad + 2\lambda_1 \int_{I\!R^n} \psi''(e^A\rho) D e^{-A} |u|^2 dx.
\end{aligned}
$$

The first integral can be written as

$$
\int_{I\!R^n} \mathrm{tr}(XY) e^{-A} dx,
$$

where $X$ and $Y$ are the $2\times 2$-matrices

$$
X = \begin{pmatrix} 2\psi''(e^A\rho) & 2\psi'''(e^A\rho) \\ 2\psi'''(e^A\rho) & \psi^{\mathrm{IV}}(e^A\rho) \end{pmatrix}
$$

and, resp.,

$$
Y = \begin{pmatrix} \alpha & D^2 u^\top \frac{\partial u}{\partial x} u + \frac{1}{2} D|u|^2 u \cdot \nabla D \\ D^2 u^\top \frac{\partial u}{\partial x} u + \frac{1}{2} D|u|^2 u \cdot \nabla D & D^2 |u|^4 \end{pmatrix},
$$

with

$$\alpha = \sum_{i,j} \left( D \frac{\partial u_i}{\partial x_j} + \frac{1}{2} \frac{\partial D}{\partial x_i} u_j + \frac{1}{2} u_i \frac{\partial D}{\partial x_j} - \frac{1}{2} \delta_{ij} \nabla D \cdot u \right)^2 .$$

$X$ is non-negative definite since $\psi$ generates an admissible entropy (cf. Definition 2.2). A simple calculation shows that $Y$ is also nonnegative definite. Thus,

$$\int_{I\!R^n} \mathrm{tr}(XY) e^{-A} dx \geq 0$$

and we have for the entropy dissipation rate (2.46):

$$R_\psi(\rho(t)|\rho_\infty) \geq 2\lambda_1 \int_{I\!R^n} \psi''(e^A \rho) D e^{-A} |u|^2 dx = -2\lambda_1 I_\psi(\rho(t)|\rho_\infty).$$

The assertion now follows from

$$\frac{d}{dt} |I_\psi(\rho(t)|\rho_\infty)| \leq -2\lambda_1 |I_\psi(\rho(t)|\rho_\infty)|. \tag{2.51}$$

□

In a special case of equation (2.1a) the condition (A1) has a simple geometric interpretation:

**Remark 2.14.** *For* $\mathbf{D}(\mathbf{x}) \equiv \mathbf{I}$ *condition (A1) simply requires the uniform convexity of* $A(x)$ *on* $I\!R^n$*, i.e.*

*(A2)* $\exists\, \lambda_1 > 0$ *such that* $\left( \frac{\partial^2 A(x)}{\partial x_i \partial x_j} \right)_{i,j=1,\ldots,n} \geq \lambda_1 \mathbf{I} \quad \forall x \in I\!R^n.$

The condition (A1) is a special case of the well-known *Bakry-Emery condition* for logarithmic Sobolev-inequalities [BaEm84], [Bak91], [Bak94]. In fact, the proof of Lemma 2.13 is a concretization of the approach of Bakry and Emery and was given here mainly for the sake of clarity. For general (symmetric and uniformly positive definite) diffusion matrices $\mathbf{D}(x)$ an 'excursion' into basic differential geometry (see, e.g. [Boo75], §7, 8) is, however, in order to understand the Bakry-Emery condition. Therefore we consider the Riemannian manifold $\mathcal{M} = (I\!R^n; \mathbf{D}^{-1})$, with $\mathbf{D}(x)^{-1} =: (d_{ij}(x))$ as covariant metric tensor.

The symmetrized Fokker–Planck operator in Equation (2.11) acting on $\mu = \rho/\rho_\infty$ can be decomposed as

$$\tilde{L} = \Delta^{\mathbf{D}} + X,$$

where

$$\Delta^{\mathbf{D}} \mu = (\det \mathbf{D})^{\frac{1}{2}} \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left( d^{ij} (\det \mathbf{D})^{-\frac{1}{2}} \frac{\partial \mu}{\partial x_j} \right)$$

is the Laplace–Beltrami operator on $\mathcal{M}$ (cf. [Cha84], §1).

$$X = \sum_{i=1}^{n} X^i \frac{\partial}{\partial x_i}$$

is a vector field (or, equivalently, a directional derivative) on $\mathcal{M}$, with the components

$$X^i(x) = -\sum_{j=1}^{n} d^{ij} \frac{\partial}{\partial x_j} \left( A(x) - \frac{1}{2} \ln \det \mathbf{D}(x) \right). \tag{2.52}$$

The Christoffel symbols are defined as the elements of the 3-tensor:

$$\Gamma_{ij}^{l} = \frac{1}{2} \sum_{k=1}^{n} d^{kl} \left( \frac{\partial d_{jk}}{\partial x_i} + \frac{\partial d_{ki}}{\partial x_j} - \frac{\partial d_{ij}}{\partial x_k} \right). \tag{2.53}$$

The *Riemann curvature tensor* of $\mathcal{M}$ then reads

$$R_{ki\ j}^{\ \ l} = \frac{\partial}{\partial x_i} \Gamma_{jk}^{l} - \frac{\partial}{\partial x_j} \Gamma_{ik}^{l} + \sum_{m=1}^{n} \Gamma_{im}^{l} \Gamma_{jk}^{m} - \sum_{m=1}^{n} \Gamma_{jm}^{l} \Gamma_{ik}^{m} \tag{2.54}$$

and the (symmetric) *Ricci-tensor* of $\mathcal{M}$ is (cf. [Tay96], §C.3)

$$\rho_{ij} = \sum_{k=1}^{n} R_{ik\ j}^{\ \ k}. \tag{2.55}$$

The covariant derivative of a vector field $X = (X^1, \dots, X^n)$ is given by

$$\nabla_i X^j = \frac{\partial X^j}{\partial x_i} + \sum_{k=1}^{n} \Gamma_{ik}^{j} X^k. \tag{2.56}$$

We define the symmetric covariant derivative (2-tensor) of $X$

$$\left(\nabla^S X\right)_{ij} = \frac{1}{2} \sum_{l=1}^{n} (d_{jl} \nabla_i X^l + d_{il} \nabla_j X^l). \tag{2.57}$$

The *Ricci tensor of the Fokker-Planck operator* is defined in [Bak94] as

$$Ric^{ij}(x) = \sum_{k,l=1}^{n} d^{ik} d^{jl} \left[ \rho_{kl} - \left(\nabla^S X\right)_{kl}(x) \right] \tag{2.59}$$

with the components of $X$ defined in (2.52). Then the Bakry-Emery condition for a general symmetric positive definite diffusion matrix reads (cf. Prop. 6.6 of [Bak94]):

$$\begin{aligned} &\exists\, \lambda_1 > 0 \text{ and } m \in [n, \infty] \text{ such that} \\ &\frac{1}{m} X(x) \otimes X(x) \leq \frac{m-n}{m} (\mathbf{Ric}(x) - \lambda_1 \mathbf{D}(x)) \quad \forall x \in I\!R^n. \end{aligned} \tag{2.60}$$

In the above cited reference the constants $\lambda_1$ and $m$ are called, respectively, the *curvature* and *dimension of the Fokker–Planck operator* $L$. Setting $m = \infty$ gives the simpler condition

(A3) $\qquad \exists\, \lambda_1 > 0$ such that $\mathbf{Ric}(x) \geq \lambda_1 \mathbf{D}(x) \quad \forall x \in I\!R^n,$

which in the case of a scalar diffusion reduces to (A1). From the r.h.s of (2.59) we see that there are two possible mechanisms responsible for satisfying (A3): the first one stems from the positive definiteness of the dual Ricci-tensor $(\rho^{kl})$ of $\mathcal{M}$ and the second one from the uniform convexity of the potential $A$ w.r.t. the geometry of $\mathcal{M}$. The interplay of these two effects is discussed in Theorem (3.4) for the 1D case.

If $\mathbf{D}(x) \equiv \mathbf{I}$, condition (2.60) simplifies to

$$\frac{1}{m}\nabla A \otimes \nabla A \leq \frac{m-n}{m}\left(\frac{\partial^2 A}{\partial x^2} - \lambda_1 \mathbf{I}\right) \quad \forall x \in I\!R^n. \tag{2.61}$$

This differential inequality only admits a global solution $A(x)$, $x \in I\!R^n$, if $m = \infty$, i.e. if (A2) holds.

We have

**Lemma 2.13'.** *If $A = A(x), \mathbf{D} = \mathbf{D}(x)$ satisfy (A3) then the estimate (2.50) holds.*

Proof: See [BaEm84], [Bak94].

**Remark 2.15.** *To better understand the Bakry-Emery condition (A3) we consider a transformation of the Fokker-Planck type equation under an $x - y$ diffeomorphism*

$$y = y(x) \iff x = x(y)$$

*on $I\!R^n$. We denote $J(y) = \det \frac{\partial x}{\partial y}(y)$, assume $J > 0$ on $I\!R^n$ and set*

$$\rho'(y,t) = J(y)\rho(x(y),t).$$

*A lengthy calculation (cf. also [Ris89]) gives*

$$\rho'_t = \operatorname{div}_y(\widetilde{\mathbf{D}}(\nabla_y \rho' + \nabla_y(\tilde{A} - \ln J)\rho')) \tag{2.62a}$$

*where*

$$\widetilde{\mathbf{D}}(y) = \frac{\partial y}{\partial x}(x(y))\mathbf{D}(x(y))\left(\frac{\partial y}{\partial x}(x(y))\right)^{\top}, \tag{2.62b}$$

$$\tilde{A}(y) = A(x(y)). \tag{2.62c}$$

*Assume now that the Riemann curvature tensor (2.54) vanishes. Then the geometry produced by* $\mathbf{D}$ *as contravariant metric tensor on* $\mathbb{R}^n$ *is Euclidean and there exists a transformation* $y = y(x)$ *such that* $\widetilde{\mathbf{D}}(y) \equiv \mathbf{I}$ *[Ris89]. The condition (A3) applied to (2.1a) reduces to (A2) applied to (2.62a), i.e. we need to assume the uniform convexity of* $\tilde{A} - \ln J$ *in the y-coordinates.*

*In one dimension (*$n = 1$*) the Riemann curvature tensor always vanishes and the transformation which yields a Fokker-Planck equation with diffusion coefficient 1 can be constructed explicitly. It is given by*

$$y(x) = \int_0^x \frac{dz}{\sqrt{D(z)}}, \quad x \in \mathbb{R}. \tag{2.63}$$

*Note that the range of* $y = y(x)$ *is* $\mathbb{R}$ *iff*

$$\int_{-\infty}^0 \frac{dz}{\sqrt{D(z)}} = \int_0^\infty \frac{dz}{\sqrt{D(z)}} = \infty.$$

*For arbitrary* $n > 1$ *an analogous transformation works, if* $\mathbf{D} = diag(d^1(x_1), ..., d^n(x_n))$ *or if* $\mathbf{D}$ *is a constant matrix. In the latter case we set*

$$y(x) = \left(\sqrt{\mathbf{D}}\right)^{-1} x. \tag{2.64}$$

*Then the transformed equation has the identity matrix as diffusion matrix.*

*In general (i.e. for a non-vanishing Riemann curvature tensor) the metric tensor cannot be transformed to the identity by a coordinate transformation. It can be transformed to the form* $\widetilde{\mathbf{D}}(y) = \mathbf{I}\widetilde{D}(y)$*, where* $\widetilde{D}$ *is a scalar function, if isothermal coordinates exist on the manifold corresponding to* $\mathbf{D}$*. Locally at least, this is always the case for* $n = 2$ *(cf. [BeGo87], p.421).*

From the exponential decay of the negative entropy dissipation $I_\psi$ (Lemma 2.13) we shall now derive the exponential decay of the relative entropy $e_\psi$:

**Theorem 2.16.** *Let* $e_\psi$ *be an admissible relative entropy and assume that* $e_\psi(\rho_I|\rho_\infty) < \infty$*. Let the coefficients* $A(x)$ *and* $\mathbf{D}(x)$ *satisfy condition (A3). Then the relative entropy converges to 0 exponentially:*

$$e_\psi(\rho(t)|\rho_\infty) \le e^{-2\lambda_1 t} e_\psi(\rho_I|\rho_\infty), \quad t > 0. \tag{2.65}$$

Proof: We proceed in two steps and first derive (2.65) for $\rho_I \in S := \{\rho \in L_+^2(\mathbb{R}^n, \rho_\infty^{-1}(dx)) \,\Big|\, |I_\psi(\rho|\rho_\infty)| < \infty\}$.

From the Lemmata 2.11, 2.13' we then know that $e_\psi(\rho(t)|\rho_\infty) \to 0$ and $I_\psi(\rho(t)|\rho_\infty) \to 0$ as $t \to \infty$. Hence, integrating (2.51) (which also holds under condition (A3) - see [BaEm84], [Bak94]) over $(t, \infty)$ gives

$$I_\psi(t) = \frac{d}{dt} e_\psi(t) \le -2\lambda_1 e_\psi(t), \quad t \ge 0 \tag{2.66}$$

which proves the assertion for sufficiently regular initial data.

For the general case we use a density argument to approximate $\rho_I$ in two steps: $\rho_\infty$ is given in $L^1_+(\mathbb{R}^n)$, and $\rho_I$ is a normalized (i.e. $\int \rho_I dx = \int \rho_\infty dx = 1$) $L^1_+(\mathbb{R}^n)$–function with finite relative entropy, $\rho_I \in \{\rho \in L^1_+(\mathbb{R}^n) \Big| e_\psi(\rho|\rho_\infty) < \infty\}$.

We first approximate $\rho_I$ by $\rho_N \in L^1_+(\mathbb{R}^n)$ with $\int \rho_N dx = 1 (N \in \mathbb{N})$:

$$\rho_N(x) := \alpha_N \, \rho_I(x) \mathbb{1}_{\{\rho_I/\rho_\infty \le N\}}$$

with the monotonously decreasing normalization constants $\alpha_N = \left[ \int\limits_{\{\rho_I/\rho_\infty \le N\}} \rho_I dx \right]^{-1}$ $\searrow 1$ for $N \to \infty$. By construction we have $|\frac{\rho_N}{\rho_\infty}| \le N\alpha_N$, which implies $\rho_N \in L^2(\mathbb{R}^n, \rho_\infty^{-1}(dx))$.

$\rho_N$ converges to $\rho_I$ a.e., and also in $L^1(\mathbb{R}^n)$:

$$\|\rho_I - \rho_N\|_{L^1(\mathbb{R}^n)} = |1 - \alpha_N| \int_{\{\rho_I/\rho_\infty \le N\}} \rho_I dx + \int_{\{\rho_I/\rho_\infty > N\}} \rho_I dx \overset{N\to\infty}{\longrightarrow} 0.$$

Now we construct a uniform bound for $\psi(\frac{\rho_N}{\rho_\infty})\rho_\infty, N \in \mathbb{N}$: we consider the convex, monotonously increasing function $\psi_R(\sigma)$, $\sigma > 0$ defined by $\psi_R(\sigma) = \psi(R)$ for $0 < \sigma \le R$ and $\psi_R(\sigma) = \psi(\sigma)$ for $R < \sigma$, with $R > 1$ sufficiently large such that $\psi_R \ge \psi$ on $\mathbb{R}^+$. This implies

$$\psi_R(\sigma) \le \psi(\sigma) + \psi(R), \quad \sigma > 0. \tag{2.67}$$

From Lemma 2.9b we easily conclude with $\frac{\sigma}{\sigma_0} = 2$:

$$\psi_R(2\sigma) \le 4\psi_R(\sigma) + 2\mu_2\sigma, \quad \sigma \ge 0. \tag{2.68}$$

Since $\alpha_N \to 1$ we have $\rho_N \le 2\rho_I$ (for large $N$) and we estimate using (2.68), (2.67):

$$\begin{aligned} &\psi(\frac{\rho_N}{\rho_\infty})\rho_\infty \le \psi_R(\frac{\rho_N}{\rho_\infty})\rho_\infty \\ \le \; &\psi_R(\frac{2\rho_I}{\rho_\infty})\rho_\infty \le 4\psi_R(\frac{\rho_I}{\rho_\infty})\rho_\infty + 2\mu_2\rho_I \le 4\psi(\frac{\rho_I}{\rho_\infty})\rho_\infty + 4\psi(R)\rho_\infty + 2\mu_2\rho_I =: \zeta(\sigma). \end{aligned} \tag{2.69}$$

Our assumptions on $\rho_I$ show $\int_{I\!R^n} \zeta(\sigma) d\sigma < \infty$, and Lebesgue's dominated convergence theorem gives

$$\lim_{N\to\infty} e_\psi(\rho_N|\rho_\infty) = e_\psi(\rho_I|\rho_\infty). \tag{2.70}$$

In the second approximation step we shall approximate $\rho_N$ in $L^2_+(I\!R^n, \rho_\infty^{-1}(dx))$ by the normalized (i.e. $\int \rho_{N,M} dx = 1$) sequence $\{\rho_{N,M}\}_{M\in I\!N}$, having finite entropy dissipation. We choose $\rho_{N,M} \in C^\infty_{0+}(I\!R^n)$ which denotes non-negative $C^\infty$–functions with compact support in $I\!R^n$.

We shall now show that

$$C^\infty_{0+}(I\!R^n) \subseteq S = \{\sqrt{\rho} \in L^4_+(I\!R^n, \rho_\infty^{-1}(dx)) \Big| \, |I_\psi(\rho|\rho_\infty)| < \infty\}. \tag{2.71}$$

We consider $\rho$ with $\sqrt{\rho} \in C^\infty_{0+}(I\!R^n)$ and compact support $\bar\Omega = \text{supp}\rho \subseteq I\!R^n$. Since $A \in L^\infty_{loc}(I\!R^n)$, so is $1/\sqrt{\rho_\infty} = e^{A/2}$ and hence $\rho/\sqrt{\rho_\infty} \in L^2(I\!R^n)$ follows.

Using

$$\psi''(\sigma) \le \mu_2(1 + \frac{1}{\sigma}), \quad \sigma > 0,$$

which follows from (2.21), we estimate the entropy dissipation (2.48):

$$\begin{aligned} & |I_\psi(\rho|\rho_\infty)| & (2.72)\\ \le \;& 4\mu_2 \|\mathbf{D}\|_{L^\infty(\Omega)} \int_\Omega (\rho + \rho_\infty) \left|\nabla \sqrt{\frac{\rho}{\rho_\infty}}\right|^2 dx \\ \le \;& 2\mu_2 \|\mathbf{D}\|_{L^\infty(\Omega)} \int_\Omega (1 + \frac{\rho}{\rho_\infty})(4|\nabla\sqrt{\rho}|^2 + \rho|\nabla A|^2) dx, & (2.73)\end{aligned}$$

which is finite since $\mathbf{D}(x), \rho_\infty^{-1}(x)$ and $\nabla A(x) \in L^\infty_{loc}(I\!R^n)$. Note that we used the l.h.s. of (2.47) in the entropy dissipation on $\Omega$, where $\rho > 0$. Hence $\sqrt{\rho} \in S$.

Since $C^\infty_0(I\!R^n)$ is dense in $L^4(I\!R^n, \rho_\infty^{-1}(dx))$, $\rho_N$ can indeed be approximated by $\{\rho_{N,M}\} \subseteq S$ in $L^2(I\!R^n, \rho_\infty^{-1}(dx))$. Lemma 2.8 then shows

$$\lim_{M\to\infty} e_\psi(\rho_{N,M}|\rho_\infty) = e_\psi(\rho_N|\rho_\infty). \tag{2.74}$$

From these two approximations we extract a normalized 'diagonal' sequence $\{\rho_{N,M(N)}\} \subseteq S$ with

$$\rho_{N,M(N)} \stackrel{N\to\infty}{\longrightarrow} \rho_I \text{ in } L^1(I\!R^n), \tag{2.75a}$$

$$\lim_{N\to\infty} e_\psi(\rho_{N,M(N)}|\rho_\infty) = e_\psi(\rho_I|\rho_\infty). \tag{2.75b}$$

For the approximations $\rho_{N,M(N)}$ we can apply (2.65):

$$e_\psi(\rho_{N,M(N)}(t)|\rho_\infty) \le e^{-2\lambda_1 t} e_\psi(\rho_{N,M(N)}|\rho_\infty), \quad t > 0, \tag{2.76}$$

where $\rho_{N,M(N)}(t)$ denotes the solution of (2.1a) with initial data $\rho_{N,M(N)}$.

From the entropy bound (2.76) and from the Dunford-Pettis theorem [Ger87] we easily conclude

$$\frac{\rho_{N,M(N)}(t)}{\rho_\infty} \overset{N\to\infty}{\longrightarrow} \frac{\rho(t)}{\rho_\infty} \quad \text{in} \quad L^1(I\!R^n, \rho_\infty(dx)) \quad \text{weakly.}$$

We now use the lower semi-continuity of $e_\psi(.|\rho_\infty)$ to finish the proof:

$$\begin{aligned}\int_{I\!R^n} \psi\left(\frac{\rho(t)}{\rho_\infty}\right)\rho_\infty(dx) &\le \liminf_{N\to\infty} \int_{I\!R^n} \psi\left(\frac{\rho_{N,M(N)}(t)}{\rho_\infty}\right)\rho_\infty(dx) \\ &\le e^{-2\lambda_1 t} e_\psi(\rho_I|\rho_\infty), \quad t>0.\end{aligned}$$

□

Due to the above density argument we do not have to make the "algebra hypothesis" of [BaEm84] and §2 of [Bak94]: there one assumes that there exists a core for $L$ that is stable under the evolution $e^{Lt}$ and under composition with $C^\infty$ functions. Using different techniques such a density argument was also given in §6 of [DeSt84].

The desired $L^1$–convergence of $\rho(t)$ to $\rho_\infty$ is now a direct consequence of Theorem 2.16 and the Csiszár-Kullback inequality (2.24), (2.26):

**Corollary 2.17.** *Let $e_\psi$ be an admissible relative entropy and assume that $e_\psi(\rho_I|\rho_\infty) < \infty$. Let the coefficients $A(x)$ and $\mathbf{D}(x)$ satisfy condition (A3). Then the solution of (2.1) satisfies*

$$\|\rho(t) - \rho_\infty\|_{L^1(I\!R^n)} \le e^{-\lambda_1 t}\sqrt{\frac{2}{\mu_2} e_\psi(\rho_I|\rho_\infty)}, \quad t>0, \tag{2.77}$$

*with the notation $\mu_2 = \psi''(1)$.*

Also, as a by-product of the $t \to \infty$ asymptotics of the Fokker-Planck type equation, we obtain the *entropy version* of a *convex Sobolev inequality*:

**Corollary 2.18.** *The inequality*

$$\int_{I\!R^n} \psi\left(\frac{\rho}{\rho_\infty}\right)\rho_\infty(dx) \le \frac{1}{2\lambda_1}\int_{I\!R^n} \nabla^\top F_\psi\left(\frac{\rho}{\rho_\infty}\right)\mathbf{D}\nabla F_\psi\left(\frac{\rho}{\rho_\infty}\right)\rho_\infty(dx) \tag{2.78a}$$

*holds*

$$\forall \rho \in L^1_+(I\!R^n) \quad \textit{with} \quad \int_{I\!R^n} \rho dx = \int_{I\!R^n} \rho_\infty dx. \tag{2.78b}$$

This inequality, of course, does not require our usual normalization $\int \rho(x)dx = 1$. The relation of (2.78) to other versions of this inequality will be discussed in §3.

Note that $L^1_+(\mathbb{R}^n)$ in (2.78b) can be replaced by $L^1(\mathbb{R}^n)$ if $\psi$ is quadratic.

PROOF: Evaluating (2.66) at $t = 0$ and setting $\rho = \rho_I$ gives (2.78a) under the assumptions

$$\rho \in L^2_+(\mathbb{R}^n, \rho_\infty^{-1}(dx)) \quad \text{and} \quad |I_\psi(\rho|\rho_\infty)| < \infty,$$

which were used in the proof of Theorem 2.16. We shall now employ a simple density argument to conclude (2.78) for all $\rho \geq 0$ with $|I_\psi(\rho|\rho_\infty)| < \infty$. Clearly, $F_\psi^{-1}$ (cf. Remark 2.12) exists since $\psi$ is strictly convex. We have

$$\int_{\mathbb{R}^n} w^2 \rho_\infty(dx) = \int_{\mathbb{R}^n} \left( \int_1^{\mu(x)} \sqrt{\psi''(s)} ds \right)^2 \rho_\infty(dx), \quad \mu = \frac{\rho}{\rho_\infty}$$

and estimate

$$\left( \int_1^\mu \sqrt{\psi''(s)} ds \right)^2 \leq \begin{cases} 4\mu_2(1-\mu^{1/2})^2 & , 0 \leq \mu \leq 1 \\ (\mu - 1)\psi'(\mu) & , \mu > 1 \end{cases}$$

using (2.21) for $0 \leq \mu \leq 1$ and the Cauchy-Schwartz inequality for $\mu > 1$. (2.35) then gives

$$\left( \int_1^\mu \sqrt{\psi''(s)} ds \right)^2 \leq 2\psi(\mu) + \text{const.}\,(\mu + 1)$$

and

$$\int_{\mathbb{R}^n} w^2 \rho_\infty(dx) \leq 2e_\psi(\rho|\rho_\infty) + \text{const.} < \infty$$

follows. Clearly $w = \alpha + \int_0^\mu \sqrt{\psi''(s)} ds \geq \alpha$ holds with $\alpha = -\int_0^1 \sqrt{\psi''(s)} ds \in \mathbb{R}$. Now let $\tilde{w}_k \in C^\infty(\mathbb{R}^n)$ be a sequence of functions with $\tilde{w}_k - \alpha \geq 0$ and $\tilde{w}_k - \alpha \in C_0^\infty(\mathbb{R}^n)$ such that $\tilde{w}_k \stackrel{k\to\infty}{\longrightarrow} w$ in the norm

$$|||w|||^2 := \int_{\mathbb{R}^n} w^2 \rho_\infty(dx) + \int_{\mathbb{R}^n} \nabla w^\top \mathbf{D}(x) \nabla w \rho_\infty(dx).$$

Obviously, $\tilde{\mu}_k := F_\psi^{-1}(\tilde{w}_k) \geq 0$ and $\mu_k$ has compact support in $\mathbb{R}^n$, which implies $\rho_\infty \tilde{\mu}_k \in L^2_+(\mathbb{R}^n; \rho_\infty^{-1}(dx))$. Now define

$$\mu_k := \frac{\tilde{\mu}_k}{\int_{\mathbb{R}^n} \tilde{\mu}_k \rho_\infty(dx)}$$

and $\rho_k := \rho_\infty \mu_k$. It is easy to show that also $w_k \stackrel{k\to\infty}{\longrightarrow} w$ in the $|||.|||$–norm (using the already proven properties of $\psi$ and (3.20)).

By convexity

$$\begin{aligned}
e_\psi(\rho|\rho_\infty) &\le \liminf_{k\to\infty} e_\psi(\rho_k|\rho_\infty) \\
&\le \liminf_{k\to\infty} \frac{1}{2\lambda_1}\int_{I\!R^n} \nabla w_k^\top \mathbf{D}(x)\nabla w_k \rho_\infty(dx) \\
&= \frac{1}{2\lambda_1}\int_{I\!R^n} \nabla w^\top \mathbf{D}(x)\nabla w \rho_\infty(dx) = \frac{1}{2\lambda_1}|I_\psi(\rho|\rho_\infty)|.
\end{aligned}$$

□

## 2.4 Non-symmetric Fokker-Planck equations

At the end of this Section we extend the above analysis to the following class of Fokker-Planck type equations with certain rotational perturbations to the conservative drift. We consider

$$\rho_t = \mathrm{div}(\mathbf{D}(\nabla\rho + \rho(\nabla A + \vec{F}))), \qquad x \in I\!R^n, t > 0, \tag{2.79}$$

$$\rho(t=0) = \rho_I, \tag{2.80}$$

with the above conditions on $\rho_I$, $\mathbf{D}$ and $A$. Additionally we assume for $\vec{F} = \vec{F}(x,t)$ sufficient local regularity and

$$\mathrm{div}(\mathbf{D}\vec{F}\rho_\infty) = 0 \qquad \text{on } I\!R^n \times (0,\infty), \tag{2.81}$$

such that $\rho_\infty = e^{-A}$ is still a stationary state of (2.79). Our objective is again to analyze the rate of convergence of $\rho(t)$ towards $\rho_\infty$. For the symmetric problem (2.1) we proceeded in §2.3 by deriving a differential inequality for the entropy dissipation (see (2.51)). In contrast, we shall here only apply the convex Sobolev inequality (2.78), which was obtained for the corresponding symmetric problem (i.e. (2.79) with $\vec{F} = 0$), to the entropy equation corresponding to (2.79). Therefore we remark that, in physical terms, the convex Sobolev inequality (2.78) represents an upper bound for the relative entropy in terms of the absolute value of the entropy dissipation. Thus, an a-priori knowledge of a convex Sobolev inequality allows to directly obtain the exponential decay of the relative entropy from the entropy equality (2.41) without involving the entropy dissipation rate. We shall use this fact below and in Section 4.

With the transformation $\mu = \rho/\rho_\infty$ (cf. (2.11)) (2.79) may be rewritten as

$$\mu_t = \rho_\infty^{-1}\mathrm{div}(\mathbf{D}\rho_\infty\nabla\mu) + \vec{F}^\top\mathbf{D}\nabla\mu,$$

where the second term of the r.h.s. is skew-symmetric in $L^2(I\!R^n, d\rho_\infty)$.

We first calculate:

$$\frac{d}{dt}e_\psi(\rho(t)|\rho_\infty) = I_\psi(\rho(t)|\rho_\infty) + T,$$

with $I_\psi$ as in (2.45) and

$$T = \int_{I\!R^n} \psi'\left(\frac{\rho}{\rho_\infty}\right) \mathrm{div}(\mathbf{D}\vec{F}\rho)dx.$$

We use (2.81) in the form

$$\mathrm{div}(\mathbf{D}\vec{F})\rho = -(\mathbf{D}\vec{F})\nabla\rho_\infty\frac{\rho}{\rho_\infty}$$

to obtain

$$T = \int_{I\!R^n} \psi'\left(\frac{\rho}{\rho_\infty}\right)(\mathbf{D}\vec{F})\nabla\left(\frac{\rho}{\rho_\infty}\right)\rho_\infty dx = \int_{I\!R^n} \nabla^\top\psi\left(\frac{\rho}{\rho_\infty}\right)\mathbf{D}\vec{F}\rho_\infty dx.$$

An integration by parts finally gives

$$T = -\int_{I\!R^n} \psi\left(\frac{\rho}{\rho_\infty}\right) \mathrm{div}(\mathbf{D}\vec{F}\rho_\infty)dx = 0.$$

By the convex Sobolev inequality (2.78) we obtain

$$\frac{d}{dt}e_\psi(\rho(t)|\rho_\infty) \leq -2\lambda_1 e_\psi(\rho(t)|\rho_\infty)$$

and

$$e_\psi(\rho(t)|\rho_\infty) \leq e^{-2\lambda_1 t}e_\psi(\rho_I|\rho_\infty)$$

follows.

As an example of (2.79) we consider the *Fokker-Planck-type equation on* $(x, v)$*-phase space* for the distribution function $f(x, v, t)$:

$$f_t + \{A, f\} = \mathrm{div}_{(x,v)}[e^{-A(x,v)}\nabla_{x,v}(e^{A(x,v)}f)], \quad t > 0, \tag{2.82a}$$
$$f(t = 0) = f_I \in L^1_+(I\!R^{2n}), \tag{2.82b}$$

with the position variable $x \in I\!R^n$ and the velocity variable $v \in I\!R^n$. Here

$$\{A, f\} := \nabla_v A \cdot \nabla_x f - \nabla_x A \cdot \nabla_v f$$

denotes the Poisson bracket. For the sake of simplicity we set $\mathbf{D} \equiv \mathbf{I}$. With the choice $\vec{F} = (\nabla_v A, -\nabla_x A)^\top$ the exponential convergence in relative entropy of $f(t)$ to its steady state $f_\infty(x, v) = e^{-A(x,v)}$ follows from the above calculations, after a density argument as in the proof of Theorem 2.16:

**Theorem 2.19.** *Let* $f_I \in L^1_+(I\!R^{2n})$, $A \in W^{2,\infty}_{loc}(I\!R^{2n})$ *and* $1 = \int_{I\!R^{2n}} f_I(x,v)dxdv = \int_{I\!R^{2n}} e^{-A(x,v)}dxdv$. *Let* $e_\psi$ *be an admissible relative entropy and assume that* $e_\psi(f_I|f_\infty) < \infty$. *Let* $A(x,v)$ *be strictly convex, i.e.*

$$\exists\, \lambda_1 > 0 \quad \textit{such that}\ \frac{\partial^2 A}{\partial(x,v)^2} \geq \lambda_1 \mathbf{I} \quad \forall x, v \in I\!R^n.$$

*Then the relative entropy converges to 0 exponentially:*

$$e_\psi(f(t)|f_\infty) \leq e^{-2\lambda_1 t} e_\psi(f_I|f_\infty), \quad t > 0. \tag{2.83}$$

**Example 2.20.** *In kinetic theory* $A(x,v) = \frac{|v|^2}{2} + V(x)$ *is a typical example for a phase-space potential (kinetic plus potential energy). Then the equation (2.82a) reads*

$$f_t + v \cdot \nabla_x f - \nabla_x V(x) \cdot \nabla_v f = \mathrm{div}_v(\nabla_v f + fv) + \mathrm{div}_x(\nabla_x f + f\nabla_x V(x)).$$

*This equation without the second term on the r.h.s. is the classical kinetic inhomogeneous Fokker–Planck equation, which is not covered by the theory in this paper.*

# 3 Sobolev Inequalities

## 3.1 Three versions of convex Sobolev inequalities

We shall now discuss in detail the convex Sobolev inequality (2.78). In particular we shall rewrite (2.78) in various equivalent forms and discuss its relation to other known inequalities. At first we set $\mu = \rho/\rho_\infty$. Then (2.78) becomes

$$\int_{I\!R^n} \psi(\mu)\rho_\infty(dx) \leq \frac{1}{2\lambda_1}\int_{I\!R^n} \nabla^\top F_\psi(\mu)\, \mathbf{D}\nabla F_\psi(\mu)\, \rho_\infty(dx) \tag{3.1a}$$

for all $\mu \in L^1_+(I\!R^n)$ ($L^1(I\!R^n)$ if $\psi$ is quadratic) which satisfy

$$\int_{I\!R^n} \mu\rho_\infty(dx) = \int_{I\!R^n} \rho_\infty(dx). \tag{3.1b}$$

Assume for the following $\int_{I\!R^n} \rho_\infty(dx) = 1$.

Hence setting $\mu = v/\int_{I\!R^n} v\rho_\infty(dx)$, we obtain

$$\int_{I\!R^n} \psi\left(\frac{v}{\int_{I\!R^n} v\rho_\infty(dx)}\right)\rho_\infty(dx)$$
$$\leq \frac{1}{2\lambda_1}\int_{I\!R^n} \nabla^\top F_\psi\left(\frac{v}{\int_{I\!R^n} v\rho_\infty(dx)}\right)\mathbf{D}\nabla F_\psi\left(\frac{v}{\int_{I\!R^n} v\rho_\infty(dx)}\right)\rho_\infty(dx) \tag{3.2}$$

for all nontrivial $v \in L^1_+(d\rho_\infty) := L^1_+(\mathbb{R}^n, \rho_\infty(dx))$ ($v \in L^1(d\rho_\infty)$ if $\psi$ is quadratic).

The most common form of convex Sobolev inequalities is obtained by setting $v = f^2$ in (3.2). This gives the so called *steady state measure version*:

$$\int_{\mathbb{R}^n} \psi\left(\frac{f^2}{\|f\|^2_{L^2(d\rho_\infty)}}\right) \rho_\infty(dx) \le \frac{2}{\lambda_1} \int_{\mathbb{R}^n} \frac{f^2}{\|f\|^4_{L^2(d\rho_\infty)}} \psi''\left(\frac{f^2}{\|f\|^2_{L^2(d\rho_\infty)}}\right) \nabla^\top f \mathbf{D} \nabla f \, \rho_\infty(dx) \tag{3.3}$$

for all nontrivial $f \in L^2(d\rho_\infty)$. Note that the right-hand side of (3.3) makes sense even if $f$ assumes the value 0 (cf. Remark 2.10).

The inequalities (3.1)–(3.3) hold for all functions $\rho_\infty = e^{-A}$ (with $L^1(dx)$-norm equal 1), $\rho_\infty > 0$ on $\mathbb{R}^n$ and symmetric positive definite matrices $\mathbf{D} = \mathbf{D}(x)$, which are sufficiently smooth (cf. Section 2) and which satisfy (A1) (if $\mathbf{D}(x) = D(x)\mathbf{I}$), or (A2) (if $\mathbf{D}(x) \equiv \mathbf{I}$), or (A3).

**Remark 3.1.** *Assume that $\mathbf{D}(x)$ is pointwise in $\mathbb{R}^n$ bounded below by a symmetric positive definite matrix $\mathbf{D}_1(x)$, i.e.*

$$\mathbf{D}_1(x) \le \mathbf{D}(x), \qquad x \in \mathbb{R}^n$$

*(in the sense of positive-definite matrices) and that the Fokker-Planck operator*

$$L_1(\rho) := \operatorname{div}(\mathbf{D}_1(\nabla\rho + \rho\nabla A))$$

*satisfies the Bakry-Emery condition (A3) (with $\mathbf{D}$ replaced by $\mathbf{D}_1$). Then the convex Sobolev inequality (2.78) holds with $\mathbf{D}$ replaced by $\mathbf{D}_1$. We have*

$$\begin{aligned} &\int_{\mathbb{R}^n} \nabla^\top F_\psi\left(\frac{\rho}{\rho_\infty}\right) \mathbf{D}_1 \nabla F_\psi\left(\frac{\rho}{\rho_\infty}\right) \rho_\infty(dx) \\ \le &\int_{\mathbb{R}^n} \nabla^\top F_\psi\left(\frac{\rho}{\rho_\infty}\right) \mathbf{D} \nabla F_\psi\left(\frac{\rho}{\rho_\infty}\right) \rho_\infty(dx) \end{aligned} \tag{3.4}$$

*and (2.78) follows. Since the convex Sobolev inequality applied directly to 2.41 gives exponential decay of the relative entropy, the statements in Lemma 2.13', Theorem 2.16 and Corollary 2.17 and the mass gap argument of Section 3.2 hold for L, too.*

*Note that this settles the case*

$$0 < D(x)\mathbf{I} \le \mathbf{D}(x), \qquad x \in \mathbb{R}^n,$$

*where $D(x)$ and $A(x)$ satisfy (A1). In particular, for a uniformly positive definite diffusion matrix*

$$d\mathbf{I} \le \mathbf{D}(x), \qquad x \in \mathbb{R}^n$$

*(with $d \in \mathbb{R}^+$) and a uniformly convex potential $A$ ((A2) holds) we obtain the Sobolev inequality (2.78), where $1/(2\lambda_1)$ has to be replaced by $1/(2d\lambda_1)$.*

If $\mathbf{D}(x) = \mathbf{I}$, $\rho_\infty(x) = M_a(x) := \frac{1}{(2\pi a)^{n/2}} \exp(-\frac{|x|^2}{2a})$ for some $a > 0$ then (A2) holds with $\lambda_1 = 1/a$ and we obtain the celebrated *Gross logarithmic Sobolev inequality* [Gro75]:

$$\int_{I\!\!R^n} f^2 \ln\left(\frac{f^2}{\|f\|^2_{L^2(dM_a)}}\right) M_a(dx) \leq 2a \int_{I\!\!R^n} |\nabla f|^2 M_a(dx) \tag{3.5}$$

for all $f \in L^2(dM_a)$, where we set $\psi(\sigma) = \sigma \ln \sigma - \sigma + 1$ (see also (3.15)).

A second example is provided by the same choices of $\rho_\infty$ and $\mathbf{D}$ setting $\psi(\sigma) = (\sigma - 1)^2$. We then have

$$\int_{I\!\!R^n} v^2 M_a(dx) - \left(\int_{I\!\!R^n} v M_a(dx)\right)^2 \leq a \int_{I\!\!R^n} |\nabla v|^2 M_a(dx) \tag{3.6}$$

for all $v \in L^1(dM_a)$. This is an old inequality, and as remarked by Beckner [Bec89] in one dimension was probably known to both mathematicians and physicists in the 1940's [Wey49]. It has been a useful tool in different subjects, like partial differential equations [Nas58] and statistics [Che81].

Finally let $\psi = \xi_p(\sigma) = \sigma^p - 1 - p(\sigma - 1)$, see (2.17b). Then we obtain the inequality

$$\int_{I\!\!R^n} v^p M_a(dx) - \left(\int_{I\!\!R^n} v M_a(dx)\right)^p \leq 2a \frac{p-1}{p} \int_{I\!\!R^n} |\nabla(v^{p/2})|^2 M_a(dx) \tag{3.7}$$

for all $v \in L^1_+(dM_a)$, $1 < p < 2$ ($v \in L^1(dM_a)$ for $p = 2$). Setting $|u| = v^{p/2}$ gives the generalized Poincaré-type inequality by Beckner [Bec89]:

$$\frac{p}{p-1}\left[\int_{I\!\!R^n} u^2 M_a(dx) - \left(\int_{I\!\!R^n} |u|^{2/p} M_a(dx)\right)^p\right] \leq 2a \int_{I\!\!R^n} |\nabla u|^2 M_a(dx) \tag{3.8}$$

for all $u \in L^{2/p}(dM_a)$, $1 < p \leq 2$.

We remark that the inequalities (3.8) interpolate in a very sharp way between the Poincaré-type inequality (3.6) and the logarithmic Sobolev inequality (3.5), which is obtained from (3.8) in the $p \to 1$ limit. (3.5) and (3.8) represent a hierarchy of convex Sobolev inequalities, with the logarithmic Sobolev inequality (3.5) being the 'strongest'. This, however, cannot be seen directly from (3.5), (3.8) and requires a more involved line of argument (see [Bak94], p.51, where the spectral gap inequality (3.6) is compared to the logarithmic Sobolev inequality (3.5)). In [Led92] this interpolation is discussed for the Ornstein–Uhlenbeck process on $I\!\!R^n$ and for the heat semigroup on spheres.

Note that the inequalities (3.5), (3.6), (3.7), (3.8) also hold when the Gaussian measure $M_a$ is replaced by a general steady state $\rho_\infty = e^{-A}$, such that $(A, \mathbf{D})$

satisfies the Bakry–Emery condition (A3). The quadratic form on the r.h.s. is then replaced by $\frac{2}{\lambda_1} \int \nabla u^\top \mathbf{D} \nabla u \rho_\infty(dx)$.

Let us now discuss briefly some consequences of inequalities (3.5), (3.6), (3.7), (3.8).

In spite of the fact that the class of admissible entropies and steady state measures which generate inequalities (3.5), (3.6), (3.7), (3.8) is very wide, Sobolev inequalities with respect to the Lebesgue measure follow only if $\psi(\sigma) = \sigma \ln \sigma - \sigma + 1$. Actually, in this case the choice $g^2 = f^2 M_a$ leads to the inequality

$$\int_{I\!R^n} g^2 \ln \left( \frac{g^2}{\|g\|^2_{L^2(dx)}} \right) dx + \left( n + \frac{n}{2} \ln 2\pi a \right) \|g\|^2_{L^2(dx)} \le 2a \int_{I\!R^n} |\nabla g|^2 \, dx \qquad (3.9)$$

for all $g \in L^2(I\!R^n)$, $a > 0$ (cf. [Car91]). Inequality (3.9) is the logarithmic Sobolev inequality for the heat kernel $H_0 = -\Delta$. In Davies' book [Dav87] inequality (3.9) is contained in the more restrictive Sobolev inequalities framework of ultracontractive operators (see Theorem 2.2.3 and the rest of the Chapter). To obtain Davies' form, we rewrite (3.9) as a family of logarithmic Sobolev inequalities for $\epsilon > 0$:

$$\int_{I\!R^n} g^2 \ln g \, dx \le \epsilon \int_{I\!R^n} |\nabla g|^2 \, dx + M_G(\epsilon) \|g\|^2_{L^2(dx)} + \|g\|^2_{L^2(dx)} \ln \|g\|_{L^2(dx)} \qquad (3.10)$$

for all $0 \le g \in L^2(I\!R^n)$. As will be shown in Theorem 3.11 below $M_G(\epsilon) = -\frac{1}{2}\left(n + \frac{n}{2} \ln 2\pi\epsilon\right)$ is the sharp constant for all $\epsilon > 0$. Now, let us compare the function $M_G$ of (3.10) with the analogous one given by Davies' conditions. In his framework, inequality (3.10) follows if

$$\left\| e^{-H_0 t} g \right\|_{L^\infty(dx)} \le \|g\|_{L^2(dx)} e^{M(t)}, \quad t > 0, \qquad (3.11)$$

where $M(t)$ is a monotonically decreasing continuous function of $t$. In the present case, $H_0 = -\Delta$,

$$e^{-H_0 t} g = K_t * g, \qquad (3.12)$$

where $K_t(x)$ is the Gaussian $(4\pi t)^{-n/2} \exp\left\{ -\frac{|x|^2}{4t} \right\}$. Then Young's inequality gives

$$\left\| e^{-H_0 t} g \right\|_{L^\infty(dx)} \le \|g\|_{L^2(dx)} \|K_t\|_{L^2(dx)} \qquad (3.13)$$

and we obtain $M(t) = -\frac{n}{4} \ln 8\pi t$. Thus

$$M(\epsilon) - M_G(\epsilon) = \frac{n}{2}(1 - \ln 2) > 0, \qquad (3.14)$$

and the function $M(\epsilon)$ is not optimal.

In the above example we were able to rewrite (3.5) as (3.10) with arbitrarily small *principal coefficient* $\epsilon$. For general potentials $A(x)$ that satisfy (A2), the possibility of doing so is deeply connected to the difference between hypercontractivity and ultracontractivity (see §5 of [Gro93], §2 of [Dav87]). As the heat kernel example shows, the entropy approach of §2 could lead to better constants.

## 3.2 A convex Sobolev inequality implies a positive spectral gap

For the sake of (relative) completeness we shall in brevity present the well-know result saying that the Hamiltonian $H$ has a spectral gap if $A$ and $\mathbf{D}$ satisfy (A3) (which implies the logarithmic Sobolev inequalities (2.78) for any admissible relative entropy $e_\psi$). We start by rewriting the Sobolev-inequality (2.78) for $\psi(\sigma) = \sigma \ln \sigma - \sigma + 1$ by setting

$$\sqrt{\frac{\rho}{\rho_\infty}} = \frac{|g|}{\|g\|_{L^2(\mathbb{R}^n, \rho_\infty(dx))}}$$

(cf. also §3). We obtain after a simple calculation

$$\int_{\mathbb{R}^n} |g|^2 \ln |g| \rho_\infty(dx) \leq \frac{1}{\lambda_1} \int_{\mathbb{R}^n} \nabla g^\top \mathbf{D} \nabla g \rho_\infty(dx) + \|g\|^2_{L^2(\mathbb{R}^n, \rho_\infty(dx))} \ln \|g\|_{L^2(\mathbb{R}^n, \rho_\infty(dx))} \tag{3.15}$$

for all $g \in L^2(\mathbb{R}^n, \rho_\infty(dx))$. The well-known Rothaus–Simon mass gap theorem ([Rot81], [Sim76], [Gro93]) gives

$$\int_{\mathbb{R}^n} \nabla g^\top \mathbf{D} \nabla g \rho_\infty(dx) \geq \lambda_1 \|g\|^2_{L^2(\mathbb{R}^n, \rho_\infty(dx))} \tag{3.16a}$$

if

$$\int_{\mathbb{R}^n} g \rho_\infty(dx) = 0. \tag{3.16b}$$

Simple calculations show that

$$H = \frac{1}{\sqrt{\rho_\infty}} L \sqrt{\rho_\infty} \bullet : \mathsf{D}_Q \subseteq L^2(\mathbb{R}^n, dx) \to L^2(\mathbb{R}^n, dx)$$

where $H$ is given by (2.6) and

$$\int_{\mathbb{R}^n} \nabla g^\top \mathbf{D} \nabla g \rho_\infty(dx) = \int_{\mathbb{R}^n} \sqrt{\rho_\infty} g H(\sqrt{\rho_\infty} g) dx.$$

Using the spectral theorem it is easy to conclude that the spectral gap $\lambda_0$ of $H$ satisfies $\lambda_0 \geq \lambda_1$, i.e.:

$$\sigma(H) \cap (0, \lambda_1) = \emptyset.$$

In many cases the logarithmic Sobolev constant $\lambda_1$ is indeed smaller than the spectral gap $\lambda_0$ (see, e.g., example (1.10), §4 of [DiSC96], and the discussions in [Rot81], [Bak94], [DeSt90]).

## 3.3 Perturbation lemmata for the potential $A(x)$

In Section 2 we derived the convex Sobolev inequality (2.78) corresponding to Fokker-Planck operators $L\rho = -\mathrm{div}(\mathbf{D}(\nabla\rho + \rho\nabla A))$ that satisfy the Bakry-Emery condition (A3). Next we will present two perturbation results to extend this inequality to a larger class of operators.

First we shall consider bounded perturbations of the 'potential' $A(x)$. Our result generalizes the perturbation lemma of Holley and Stroock [HolSto87], [Gro90] from the logarithmic entropy to all admissible relative entropies $e_\psi$ from Definition 2.2:

**Theorem 3.2.** *Let* $\rho_\infty(x) = e^{-A(x)}$, $\widetilde{\rho_\infty}(x) = e^{-\widetilde{A}(x)} \in L^1_+(\mathbb{R}^n)$ *with* $\int_{\mathbb{R}^n} \rho_\infty dx = \int_{\mathbb{R}^n} \widetilde{\rho_\infty} dx = 1$ *and*

$$\begin{aligned} \widetilde{A}(x) &= A(x) + v(x), \\ 0 < a &\leq e^{-v(x)} \leq b < \infty, \qquad x \in \mathbb{R}^n. \end{aligned} \tag{3.17}$$

*Let the symmetric locally uniformly positive definite matrix* $\mathbf{D}(x)$ *be such that the convex Sobolev inequality (3.3) (with the admissible entropy generator* $\psi$*) holds for all* $f \in L^2(d\rho_\infty)$*.*

*Then, a convex Sobolev inequality also holds for the perturbed measure* $\widetilde{\rho_\infty}$ *:*

$$\begin{aligned} &\int_{\mathbb{R}^n} \psi\left(\frac{f^2}{\|f\|^2_{L^2(d\widetilde{\rho_\infty})}}\right) \widetilde{\rho_\infty}(dx) \qquad (3.18)\\ &\quad \leq \frac{2}{\lambda_1} \max\left(\frac{b}{a^2}, \frac{b^2}{a}\right) \int_{\mathbb{R}^n} \frac{f^2}{\|f\|^4_{L^2(d\widetilde{\rho_\infty})}} \psi''\left(\frac{f^2}{\|f\|^2_{L^2(d\widetilde{\rho_\infty})}}\right) \nabla^\top f \mathbf{D} \nabla f \, \widetilde{\rho_\infty}(dx) \end{aligned}$$

*for all nontrivial* $f \in L^2(d\widetilde{\rho_\infty}) = L^2(d\rho_\infty)$*.*

Proof: We introduce the notations

$$\sigma_0(x) := \frac{f^2(x)}{\|f\|^2_{L^2(d\rho_\infty)}}, \quad \sigma_1(x) := \frac{f^2(x)}{\|f\|^2_{L^2(d\widetilde{\rho_\infty})}}, \quad \alpha := \frac{\sigma_1}{\sigma_0} = \frac{\|f\|^2_{L^2(d\rho_\infty)}}{\|f\|^2_{L^2(d\widetilde{\rho_\infty})}},$$

and because of (3.17) we have

$$\frac{1}{b} \leq \alpha \leq \frac{1}{a}. \tag{3.19}$$

Below we shall need the estimate

$$\psi''(\sigma_0) \leq \begin{cases} \psi''(\sigma_1), & \sigma_1 < \sigma_0, \qquad (3.20a) \\ \psi''(\sigma_1)\frac{\sigma_1}{\sigma_0}, & \sigma_1 \geq \sigma_0. \qquad (3.20b) \end{cases}$$

The first line follows from $\psi''' \leq 0$ (see (2.20)), and the second line follows from (2.38).

We shall now first prove the assertion (3.18) for the case $\alpha \geq 1$. In the following chain of estimates we use, in this sequence, (2.36), (3.17), (3.3), (3.17), (3.20), (3.19):

$$\begin{aligned}
\int_{\mathbb{R}^n} \psi(\sigma_1)\widetilde{\rho_\infty}(dx) &\leq \int_{\mathbb{R}^n} \left[\psi(\sigma_0)\alpha^2 + \mu_2(\alpha-1)(\sigma-1)\right] \widetilde{\rho_\infty}(dx) \\
&= \alpha^2 \int_{\mathbb{R}^n} \psi(\sigma_0)\widetilde{\rho_\infty}(dx) \leq b\alpha^2 \int_{\mathbb{R}^n} \psi(\sigma_0)\rho_\infty(dx) \\
&\leq b\alpha^2 \frac{2}{\lambda_1} \int_{\mathbb{R}^n} \frac{f^2}{\|f\|^4_{L^2(d\rho_\infty)}} \psi''(\sigma_0)\nabla^\top f \mathbf{D}\nabla f \rho_\infty(dx) \quad (3.21) \\
&\leq \frac{b}{a}\frac{2}{\lambda_1} \int_{\mathbb{R}^n} \frac{f^2}{\|f\|^4_{L^2(d\widetilde{\rho_\infty})}} \psi''(\sigma_0)\nabla^\top f \mathbf{D}\nabla f \widetilde{\rho_\infty}(dx) \\
&\leq \frac{b}{a}\frac{2}{\lambda_1}\alpha \int_{\mathbb{R}^n} \frac{f^2}{\|f\|^4_{L^2(d\widetilde{\rho_\infty})}} \psi''(\sigma_1)\nabla^\top f \mathbf{D}\nabla f \widetilde{\rho_\infty}(dx) \\
&\leq \frac{b}{a^2}\frac{2}{\lambda_1} \int_{\mathbb{R}^n} \frac{f^2}{\|f\|^4_{L^2(d\widetilde{\rho_\infty})}} \psi''(\sigma_1)\nabla^\top f \mathbf{D}\nabla f \widetilde{\rho_\infty}(dx),
\end{aligned}$$

which is the result if $\sigma_1 \geq \sigma_0$.

In the case $\alpha < 1$ we proceed similarly and first apply (2.37) to obtain:

$$\int_{\mathbb{R}^n} \psi(\sigma_1)\widetilde{\rho_\infty}(dx) \leq \int_{\mathbb{R}^n} \left[\psi(\sigma_0)\alpha + \mu_2(\alpha-1)(\sigma-1)\right] \widetilde{\rho_\infty}(dx) = \alpha \int_{\mathbb{R}^n} \psi(\sigma_0)\widetilde{\rho_\infty}(dx). \quad (3.22)$$

Now we again use (3.17), (3.3), (3.17), (3.20a), (3.19) and finally obtain the result

$$\int_{\mathbb{R}^n} \psi(\sigma_1)\widetilde{\rho_\infty}(dx) \leq \frac{b^2}{a}\frac{2}{\lambda_1} \int_{\mathbb{R}^n} \frac{f^2}{\|f\|^4_{L^2(d\widetilde{\rho_\infty})}} \psi''(\sigma_1)\nabla^\top f \mathbf{D}\nabla f \widetilde{\rho_\infty}(dx). \quad (3.23)$$

□

We remark that the 'perturbation constant' in (3.18) is not as good as the one in the proof of [HolSto87] ($\max(\frac{b}{a^2}, \frac{b^2}{a})$ versus $\frac{b}{a}$). This is due to the fact that the proof of Holley and Stroock exploits homogeneity properties of $\psi$ and $\psi'$, in the case $\psi(\sigma) = \sigma \ln \sigma - \sigma + 1$. Thus, their original proof (with the constant $\frac{b}{a}$) can be extended to entropy generators of the form $\psi(\sigma) = \sigma^p - 1 - p(\sigma - 1),\ 1 < p \leq 2$, but not to the general case of Theorem 3.2.

**Example 3.3.** *Set* $\mathbf{D}(x) = \mathbf{I}$ *and consider the 'double well potential'* $A(x) = c_1|x|^4 - c_2|x|^2 (c_{1,2} > 0)$. *Then, the perturbation theorem 3.2 yields convex Sobolev inequalities as* $A(x)$ *can be written as a bounded perturbation of a uniformly convex potential that satisfies (A2).*

Next we shall specialize our discussion to the one-dimensional situation and derive convex Sobolev inequalities under very mild assumptions on the potential $A(x)$. In particular we shall prove that an appropriate choice of the diffusion coefficient $D$ compensates lack of convexity of the potential $A$.

**Theorem 3.4.** *Let $A \in W^{2,\infty}_{loc}(I\!R)$ be bounded below and satisfy $\rho_\infty(x) = e^{-A(x)} \in L^1_+(I\!R)$ with $\int_{I\!R} \rho_\infty(dx) = 1$, and let $\psi$ generate an admissible relative entropy. Then there exists a $\lambda_1 > 0$ and a function $D = D(x)$ with $D(x) \geq D_0$, $x \in I\!R$, for some constant $D_0 > 0$, such that:*

$$\int_{I\!R} \psi\left(\frac{\rho}{\rho_\infty}\right)\rho_\infty(dx) \leq \frac{1}{2\lambda_1}\int_{I\!R} \mathbf{D}\left|\left(F_\psi\left(\frac{\rho}{\rho_\infty}\right)\right)_x\right|^2 \rho_\infty(dx), \tag{3.24}$$

$$\forall \rho \in L^1_+(I\!R) \text{ with } \int_{I\!R} \rho dx = 1.$$

The idea of the proof is to construct a bounded–below function $D(x)$ such that $(A, D)$ satisfies (A1). To this end we need the following

**Lemma 3.5.** *Let $A(x)$ satisfy the conditions of Theorem 3.4. Then there exists a $\lambda > 0$ such that the ODE*

$$\frac{1}{4}\frac{D_x^2}{D} - \frac{1}{2}D_{xx} + \frac{1}{2}D_x A_x + D A_{xx} = \lambda \tag{3.25}$$

*admits at least one global solution that satisfies $D(x) \geq D_0$, $x \in I\!R$, for some constant $D_0 > 0$.*

Note that the left hand side of (3.25) is the $1-d$ version of the left hand side of (A1) such that Theorem 3.4 follows.

Proof: We transform $D(x) = y(x)^2$ in (3.25) and solve the IVP

$$y_{xx} = A_{xx}y + A_x y_x - \frac{\lambda}{y}, \qquad x \in I\!R, \tag{3.27}$$

$$y(0) = e^{A(0)}, \qquad y'(0) = A_x(0)e^{A(0)}, \tag{3.28}$$

where we added convenient initial conditions. The (possibly only local) solution of (3.26) satisfies

$$y(x) = e^{A(x)} - \lambda e^{A(x)} \int_0^x \left[e^{-A(z)}\int_0^z y(\xi)^{-1} d\xi\right] dz \leq e^{A(x)}. \tag{3.29}$$

Due to this upper bound, $y(x)$ can only break down at a finite $x_0$ if $y(x_0) = 0$. Locally, $y(x)$ can be obtained through a fixed point iteration starting with

$y_0(x) := e^{A(x)}$ :

$$
\begin{aligned}
y_{k+1}(x) &= e^{A(x)} - \lambda e^{A(x)} \int_0^x \left[ e^{-A(z)} \int_0^z y_k(\xi)^{-1} d\xi \right] dz \qquad (3.30) \\
&\geq e^{A(x)} \left( 1 - \lambda \| e^{-A} \|_{L^1(I\!R)} \left| \int_0^x y_k(\xi)^{-1} d\xi \right| \right), \qquad k \geq 0.
\end{aligned}
$$

Starting with $y_0(x) \geq e^{A(x)}$, an iterative application of the estimate (3.30) yields

$$
y(x) \geq \tau e^{A(x)} > 0 \tag{3.31}
$$

with

$$
\tau = 1 - \cfrac{\lambda}{1 - \cfrac{\lambda}{1 - \lambda \ddots}} = \frac{1}{2} + \sqrt{\frac{1}{4} - \lambda},
$$

whenever $\lambda \leq \frac{1}{4}$.

By induction one shows that $(y_k)$ is decreasing. Hence, (3.30) converges to $y(x) \quad \forall x \in I\!R$. From (3.31) we obtain $D(x) \geq \tau^2 e^{2A_0} =: D_0, \; x \in I\!R$, with $A_0$ denoting the lower bound of $A(x)$. □

Note that $D = e^{2A}$ satisfies (3.25) with $\lambda = 0$. Hence the couple $(A, D = e^{2A})$ violates the Bakry-Emery condition (A1), nevertheless the convex Sobolev inequality (3.24) holds with $D = e^{2A}$ and $\lambda_1 = \frac{1}{4}$ (due to the estimate (3.29)).

Obviously, for $A$ uniformly convex, $D$ can be chosen as a constant (cf. (A2)). We shall now show that for non uniformly convex $A$, in general (A1) cannot be satisfied with a uniformly bounded function $D$.

**Lemma 3.6.** *Let $A(x)$ satisfy the conditions of Theorem (3.4), and let $(A, D)$ satisfy the Bakry-Emery condition (A1).*

*a) If $0 < D(x) \leq D_1 < \infty, \quad x \in I\!R$, then $\overline{\lim}_{x \to \pm\infty} A_x(x) = \pm\infty$.*

*b) If $0 < D_0 \leq D(x) \leq D_1 < \infty, \quad x \in I\!R$, then $A(x)$ grows at least quadratically.*

PROOF: a) Using $D = y^2$ we rewrite the differential inequality (A1) as

$$
y_{xx} + \frac{\lambda}{y} + f = (A_x y)_x
$$

for some $f(x) \geq 0$. Solving for $A$ gives

$$
A_x(x) = \frac{1}{y(x)} \left[ C + \lambda \int_0^x \frac{dz}{y(z)} + \int_0^x f(z) dz + y_x(x) \right], \tag{3.32}
$$

for some $C \in \mathbb{R}$. Since $y$ is bounded on $\mathbb{R}$ we have $\overline{\lim}_{x\to\infty} y_x(x) > -\infty$. The result follows from

$$\int_0^x \frac{dz}{y(z)} \geq \frac{x}{\sqrt{D_1}}, \qquad x \geq 0.$$

b) Integrating (3.32) gives

$$A(x) = A(0) + \frac{\lambda}{2}\left(\int_0^x \frac{dz}{y(z)}\right)^2 + C\int_0^x \frac{dz}{y(z)} + \ln\frac{y(x)}{y(0)} + \int_0^x \frac{1}{y(z)}\int_0^z f(\xi)d\xi dz,$$

and the result follows from the boundedness assumptions on $D$. □

Closely related results have been obtained for the case that a logarithmic Sobolev inequality (i.e. (3.24) with $\psi(\sigma) = \sigma\ln\sigma - \sigma + 1$, $D \equiv 1$) holds. Then $A$ necessarily has to satisfy a certain growth condition, namely a *Herbst inequality* (see [GrRo97]).

We now finish our discussion of perturbation results on convex Sobolev inequalities with an example. It demonstrates that the diffusion coefficient $D$ constructed in the proof of Lemma 3.5 in many cases has a much stronger growth at $x = \pm\infty$ than necessary to satisfy the Bakry-Emery condition (A1).

**Example 3.7.** *Let $A \in C^2(\mathbb{R})$ satisfy $A(x) = c|x|^{2\alpha}$ for $|x| > L$ and $0 < \alpha \leq 1$. Then a pair $(D, \lambda)$, $\lambda > 0$ satisfying (A1) can be constructed such that*

$$D(x) = \begin{cases} \left(c_1^+ + c_2^+ x^{1-\alpha}\right)^2 & , \quad x > L_1 \\ \left(c_1^- + c_2^- |x|^{1-\alpha}\right)^2 & , \quad x < -L_1 \end{cases}, \tag{3.33}$$

*for some $c_1^\pm \in \mathbb{R}$, $c_2^\pm > 0$, $L_1 > L$. The construction of $D$ proceeds as follows: On the interval $(-L_1, L_1)$, where $L_1$ will be determined later, we choose $D = y^2$, where $y$ is the solution (3.29) of the IVP (3.26). Outside of this interval we extend $D$ by (3.33) in a $C^1$-way, thus fixing $c_{1,2}^\pm$ (in dependence of $\lambda$ and $L_1$). By a perturbation argument around $\lambda = 0$ one easily sees that $\lambda > 0$ can be chosen small enough and $L_1$ large enough such that $\pm y_x(x)(\pm L_1) > 1$ and such that (A1) holds for $|x| > L_1$.*

## 3.4 Poincaré-type inequalities

As an application of Theorem 3.2 we shall now derive Poincaré-type inequalities.

**Remark 3.8.** *Poincaré-type inequalities on bounded, uniformly convex domains are readily obtained from convex Sobolev inequalities. For simplicity's sake, let $B = \{x \in \mathbb{R}^n | \, |x| < 1\}$ be the unit ball in $\mathbb{R}^n$ and let $\rho_0 = \rho_0(x)$, $\underline{\rho} \leq \rho_0(x) \leq \overline{\rho}$*

*on $B$, be the density of a probability measure on $B$. We define the $C^1$-function $\tilde{A}^\varepsilon = \tilde{A}^\varepsilon(|x|)$ on $I\!R^n$ for $\varepsilon > 0$ by*

$$\frac{d^2}{dr^2}\tilde{A}^\varepsilon(r) = \begin{cases} 1, & 0 < r < 1 \\ \frac{1}{\varepsilon}, & r > 1 \end{cases} \qquad , \tilde{A}^\varepsilon(0) = \frac{d}{dr}\tilde{A}^\varepsilon(0) = 0.$$

*We set*

$$A^\varepsilon(x) = \begin{cases} \tilde{A}^\varepsilon(|x|) - \frac{|x|^2}{2} - \ln \rho_0(x), & x \in B \\ \tilde{A}^\varepsilon(|x|), & x \notin B \end{cases}$$

*and*

$$\rho_\infty^\varepsilon(x) = \exp(-A^\varepsilon(x)) / \int_{I\!R^n} \exp(-A^\varepsilon(y))dy.$$

*Obviously,*

$$\rho_\infty^\varepsilon(x) \overset{\varepsilon \to 0}{\longrightarrow} \begin{cases} \rho_0(x), & x \in B \\ 0, & x \notin B \end{cases}.$$

*Since $-\ln \rho_\infty^\varepsilon$ is an $L^\infty$- perturbation (uniformly as $\varepsilon \to 0$) of the uniformly convex function $\tilde{A}^\varepsilon(|x|)$, which satisfies*

$$\frac{\partial^2 \tilde{A}^\varepsilon}{\partial x^2} \geq \mathbf{I} \qquad on\ I\!R^n,$$

*we can apply the perturbation result Theorem 3.2 and obtain the convex Sobolev inequality (with $\mathbf{D} \equiv \mathbf{I}$)*

$$\int_{I\!R^n} \psi \left( \frac{v}{\int_{I\!R^n} v d\rho_\infty^\varepsilon} \right) d\rho_\infty^\varepsilon \leq c \int_{I\!R^n} \left| \nabla F_\psi \left( \frac{v}{\int_{I\!R^n} v d\rho_\infty^\varepsilon} \right) \right|^2 d\rho_\infty^\varepsilon$$

*for all admissible entropies $\psi$ and all nontrivial functions $v \in L^1_+(I\!R^n, d\rho_\infty^\varepsilon)$ ($v \in L^1(I\!R^n, d\rho_\infty^\varepsilon)$ if $\psi$ is quadratic on $I\!R$). Here $c$ is independent of $\varepsilon$.*

*Passing to the limit $\varepsilon \to 0$ gives the Poincaré-type inequality*

$$\int_B \psi \left( \frac{v}{\int_B v d\rho_0} \right) d\rho_0 \leq c \int_B \left| \nabla F_\psi \left( \frac{v}{\int_B v d\rho_0} \right) \right|^2 d\rho_0$$

*for all nontrivial $v \in L^1_+(B, d\rho_0)$ ($v \in L^1(B, d\rho_0)$ if $\psi$ is quadratic on $I\!R$). In the latter case $\psi(\sigma) = (\sigma - 1)^2$ with $\rho_0 = \frac{1}{\text{vol}(B)}$ we obtain the classical Poincaré inequality*

$$\int_B \left( v - \frac{1}{\text{vol}(B)} \int_B v dx \right)^2 dx \leq 2c \int_B |\nabla v|^2 dx, \quad v \in L^1(B).$$

*Obviously, the above limit argument can easily be carried over to general uniformly convex domains.*

## 3.5 Sharpness results

We now turn to the analysis of the saturation of the convex Sobolev inequalities, i.e. we shall answer the question for which function $\rho$ the inequality (2.78) becomes an equality. In particular we shall find necessary and sufficient conditions on the entropy generator $\psi$ and on $\rho_\infty$, which imply the existence of an admissible function $\rho \neq \rho_\infty$ such that (2.78) (under the assumption $\mathbf{D} = \mathbf{I}$) becomes an equality. We remark that this question was completely answered in [Car91] for the Gross logarithmic Sobolev inequality (i.e. $\psi(\sigma) = \sigma \ln \sigma - \sigma + 1$, $\rho_\infty = M_a$) by a technique different from the one presented in the sequel. The same problem was treated in [Tos97a] and [Led92] using a method which we shall generalize below.

At first we observe that the derivation of the convex Sobolev inequalities given in Section 2 is based on writing the entropy equation

$$\frac{d}{dt} e_\psi(\rho(t)|\rho_\infty) = I_\psi(e(t)|\rho_\infty), \tag{3.34}$$

the equation for the entropy dissipation

$$\frac{d}{dt} I_\psi(\rho(t)|\rho_\infty) = -2\lambda_1 I_\psi(\rho(t)|e_\infty) + r_\psi(\rho(t)) \tag{3.35}$$

and proving $r_\psi(\rho(t)) \geq 0$. Explicitly, we obtain by inserting (3.34) into the right hand side of (3.35) and by integrating with respect to $t$:

$$e_\psi(\rho_I|\rho_\infty) = -\frac{1}{2\lambda_1} I_\psi(\rho_I|\rho_\infty) - \frac{1}{2\lambda_1}\int_0^\infty r_\psi(\rho(s))ds \tag{3.36}$$

where $\rho(t)$ in the Fokker-Planck trajectory which 'connects' the initial state $\rho_I$ with the steady state $\rho_\infty$. $r_\psi \geq 0$ then gives the convex Sobolev inequality

$$e_\psi(\rho_I|\rho_\infty) \leq \frac{1}{2\lambda_1} |I_\psi(\rho_I|\rho_\infty)|,$$

which becomes an equality iff

$$\int_0^\infty r_\psi(\rho(s))ds = 0 \iff r_\psi(\rho(t)) = 0 \quad \text{a.e. in } I\!R_t^+. \tag{3.37}$$

The precise form of the remainder $r_\psi$ can easily be extracted from the proof of Lemma 2.13. Assuming henceforth

$$\mathbf{D} \equiv \mathbf{I} \tag{3.38}$$

we obtain

$$
\begin{aligned}
r_\psi(\rho) &= \int_{\mathbb{R}^n} \left( \psi^{\mathrm{IV}}(e^A\rho)|u|^4 + 4\psi'''(e^A\rho)u^\top \frac{\partial u}{\partial x}u + 2\psi''(e^A\rho)\sum_{l,m=1}^n (\frac{\partial u_l}{\partial x_m})^2 \right) e^{-A}dx \\
&\quad +2\int_{\mathbb{R}^n} \psi''(e^A\rho)u^\top \left( \frac{\partial^2 A}{\partial x^2} - \lambda_1 \mathbf{I} \right) u\, e^{-A}dx, \qquad (3.39)
\end{aligned}
$$

where we recall $u = \nabla(e^A\rho)$. Obviously, $r_\psi(\rho) = 0$ if $u \equiv 0$, which (taking into account the normalization) implies $\rho \equiv \rho_\infty$ and gives the trivial case of equality in the convex Sobolev inequality. Thus, assume $u \not\equiv 0$. Then, using the admissibility conditions (2.12) for the entropy generator $\psi$ and (A2), we conclude that $r_\psi(\rho) = 0$ holds iff the subsequent four conditions are satisfied:

$$
\psi'''(e^A\rho)^2 = \frac{1}{2}\psi''(e^A\rho)\psi^{\mathrm{IV}}(e^A\rho), \qquad (3.40a)
$$

$$
|u|^2 \left( \sum_{l,m=1}^n (\frac{\partial u_l}{\partial x_m})^2 \right)^{\frac{1}{2}} = |u^\top \frac{\partial u}{\partial x} u|, \qquad (3.40b)
$$

$$
\psi''(e^A\rho)u^\top \frac{\partial u}{\partial x}u = -\psi'''(e^A\rho)|u|^4, \qquad (3.40c)
$$

$$
u^\top \left( \frac{\partial^2 A}{\partial x^2} - \lambda_1 \mathbf{I} \right) u = 0. \qquad (3.40d)
$$

Since $\mu(t) := e^A\rho(t)$ satisfies (2.11) we have by the maximum-minimum principle

$$
0 \le \inf_{\mathbb{R}^n} e^A\rho_I \le e^{A(x)}\rho(x,t) \le \sup_{\mathbb{R}^n} e^A\rho_I \le +\infty
$$

a.e. in $\mathbb{R}^n \times \mathbb{R}_t^+$. We conclude from (3.40a):

**Lemma 3.9.** *If the Sobolev inequality (2.78) with $\mathbf{D} \equiv \mathbf{I}$ becomes an equality for $\rho \not\equiv \rho_\infty$, then $\psi(\sigma) = \chi(\sigma)$ or $\psi(\sigma) = \varphi(\sigma)$ holds for $\sigma \in [\inf_{\mathbb{R}^n} e^A\rho_I, \sup_{\mathbb{R}^n} e^A\rho_I]$, where $\chi$ and $\varphi$ are given by (2.17a) and (2.17c) resp.*

Nontrivial saturation can therefore only occur for the "minimal" and "maximal" entropies.

At first we investigate the case of the maximal (quadratic) entropy. Note that now any $\rho \in L^1(\mathbb{R}^n, dx)$ is admissible. Positivity is not required.

**Theorem 3.10.** *The convex Sobolev inequality (2.78) with $\mathbf{D} \equiv \mathbf{I}$ and $\psi(\sigma) = (\sigma - 1)^2$ becomes an equality iff the following two conditions hold:*

*(i) there exist Cartesian coordinates $y = (y_1, \ldots, y_n)^\top = y(x)$ on $I\!R^n$ such that for some $\beta \in I\!R$*

$$A(x(y)) = \frac{\lambda_1}{2} y_1^2 + \beta y_1 + B(y_2, \ldots, y_n), \tag{3.41}$$

*(ii) $\rho$ satisfies for some $\xi \in I\!R$:*

$$\rho(x(y)) = (1 + \xi y_1) e^{-A(x(y))}. \tag{3.42}$$

PROOF: Since $\psi''' \equiv 0$ we conclude from (3.40b) and (3.40c):

$$\frac{\partial u_l}{\partial x_m} = 0; \qquad l, m = 1, \ldots, n \tag{3.43}$$

($u \not\equiv 0$ !). Thus $u(x,t) = \nabla_x (e^{A(x)} \rho(x,t)) \equiv C(t)$, where $C(t)$ is constant in $x$. Then $\rho(x,t) = (C(t) \cdot x + C_1(t))\, e^{-A(x)}$ follows with $C_1$ real valued. Inserting into the Fokker-Planck equation gives $\dot{C} \cdot x + \dot{C}_1 = -\nabla A \cdot C$ and $\dot{C} = -\frac{\partial^2 A}{\partial x^2} C$ follows. (3.40d) gives $C(t)^\top \left( \frac{\partial^2 A}{\partial x^2} - \lambda_1 \mathbf{I} \right) C(t) = 0$ and since $\frac{\partial^2 A}{\partial x^2} - \lambda_1 \mathbf{I} \geq 0$ we conclude $\frac{\partial^2 A}{\partial x^2} C(t) = \lambda_1 C(t)$. We obtain $\dot{C} = -\lambda_1 C$ and $C(t) = C_0 e^{-\lambda_1 t}$. Therefore $\left( \frac{\partial^2 A}{\partial x^2} - \lambda_1 \mathbf{I} \right) C_0 = 0$ and we have $\nabla A \cdot C_0 - \lambda_1 C_0 \cdot x \equiv \alpha \in I\!R$. We insert $C(t) = C_0 e^{-\lambda_1 t}$ into $\dot{C} \cdot x + \dot{C}_1 = -\nabla A \cdot C$ and find $C_1(t) = \frac{\alpha}{\lambda_1} e^{-\lambda_1 t} + C_2,\ C_2 \in I\!R$. This gives

$$\rho(x,t) = \left( C_0 \cdot x + \frac{\alpha}{\lambda_1} \right) e^{-\lambda t - A(x)} + C_2 e^{-A(x)}$$

and $t \to \infty$ implies $C_2 = 1$. Summing up, we found, for some $C_0 \in I\!R^n$, $\alpha \in I\!R$:

$$\rho(x,t) = \left( C_0 \cdot x + \frac{\alpha}{\lambda_1} \right) e^{-\lambda_1 t - A(x)} + e^{-A(x)} \tag{3.44a}$$

and

$$\nabla (A - \frac{\lambda_1}{2} |x|^2) \cdot C_0 = \alpha. \tag{3.44b}$$

Note that $C_0 = 0$ implies $\alpha = 0$ and $\rho(x,t) \equiv e^{-A(x)}$ follows. We therefore assume $C_0 \neq 0$ from now on.

Set $\omega_1 = \frac{C_0}{|C_0|}$, choose orthonormed vectors $\omega_2, \ldots, \omega_n \in \{\omega_1\}^\perp$ and define the change of coordinates $x \leftrightarrow y$ by $x = y_1 \omega_1 + \ldots + y_n \omega_n$. We have $\nabla_x f(x) \cdot C_0 = \frac{\partial f(x(y))}{\partial y_1} |C_0|$ and thus

$$A(x(y)) = \frac{\lambda_1}{2} y_1^2 + \frac{\alpha}{|C_0|} y_1 + B(y_2, \ldots, y_n)$$

follows from (3.44b), and this finishes the proof. □

Next we treat the "minimal" entropy case.

**Theorem 3.11.** *The convex Sobolev inequality (2.78) with* $\mathbf{D} = \mathbf{I}$ *and* $\psi(\sigma) = \sigma \ln \sigma - \sigma + 1$ *becomes an equality iff the following two conditions hold:*

*(i) there exist Cartesian coordinates* $y = (y_1, \ldots, y_n) = y(x)$ *on* $I\!R^n$ *such that for some* $\beta \in I\!R$

$$A(x(y)) = \frac{\lambda_1}{2} y_1^2 + \beta y_1 + B(y_2, \ldots, y_n),$$

*(ii)* $\rho$ *satisfies for some* $\xi \in I\!R$

$$\rho = \exp\left( -A(x(y)) + \xi y_1 - \frac{\xi^2}{2\lambda_1} + \frac{\beta\xi}{\lambda_1} \right). \tag{3.45}$$

Proof: We set $z = \nabla \ln(\rho e^A)$ in (3.39) and calculate (using the specific form of $\psi$):

$$r_\psi(\rho) = 2\int_{I\!R^n} \rho \sum_{l,m=1}^{n} \left( \frac{\partial z_l}{\partial x_m} \right)^2 dx + 2\int_{I\!R^n} \rho z^\top \left( \frac{\partial^2 A}{\partial x^2} - \lambda \mathbf{I} \right) z dx.$$

Assume $z \not\equiv 0$. Then $\frac{\partial z}{\partial x} \equiv 0$ follows and $z(x,t) \equiv C(t) \in I\!R^n$. This gives

$$\rho(x,t) = C_1(t) \exp(C(t) \cdot x - A(x))$$

with $C_1(t) > 0$. Inserting into the Fokker-Planck equation and proceeding as in the proof of the previous Theorem implies

$$\rho(x,t) = \exp\left( (C_0 \cdot x e^{-\lambda t} + \frac{\alpha}{\lambda}) e^{-\lambda t} - A(x) \right) \tag{3.46}$$

where $\alpha$ and $C_0$ satisfy (3.44b). Setting $t = 0$ in (3.46) proves the assertion. □

Note that $\rho$ in (3.45) is obtained by a shift in the $y_1$-coordinate of $\rho_\infty$ (cf. [Car91]).

In one dimension ($n = 1$) or for constant matrices $\mathbf{D}$ the analogous result to the Theorems 3.10 and 3.11 is obtained by the coordinate transformation of Remark 2.15. For $\mathbf{D}(x) \not\equiv \mathbf{I}$, however, nontrivial saturation of the convex Sobolev inequality (2.78) is not always possible for any $A(x)$: even for scalar diffusion $\mathbf{D}(x) = D(x)\mathbf{I}$, the analogue of (3.43) implies an integrability condition on $D(x)$, which is not satisfied in general.

# 4 Nonlinear Model Problems

We conclude this paper with an application of the theory presented in the previous Sections to nonlinear Fokker-Planck type equations. The first example is based on an application of the proof methods of Sections 2, 3 and the second example is a more direct application of the results of those Sections.

## 4.1 Desai-Zwanzig type models

Firstly, consider the following model describing the interaction of a system of coupled oscillators in the thermodynamic limit:

$$\rho_t = D\mathrm{div}_x(\nabla_x \rho + \nabla_x A(x,\xi;\rho(t))\rho), \quad t > 0, \tag{4.1a}$$

$$\rho(t=0) = \rho_I(x,\xi), \tag{4.1b}$$

where $x \in I\!R^n$ and the parameter vector $\xi = (\xi_1, \ldots, \xi_M) \in I\!R^M$. $D$ is a positive diffusion constant, $\xi$ presents noise in the oscillator interaction and the oscillator ensemble potential $A$ is given by

$$\begin{aligned} &A(x,\xi;\rho(t)) \qquad\qquad (4.1c)\\ = &\frac{1}{D}\left(W(x) + \frac{\Theta}{2}|z_\rho(t) - x|^2 - x \cdot \sum_{l,m=1}^{M} s_{\rho,m}(t) E_{lm} \xi_l + \frac{1}{2}\sum_{l,m=1}^{M} E_{lm} s_{\rho,l}(t) \cdot s_{\rho,m}(t)\right). \end{aligned}$$

Here $W$ is the single oscillator potential (which we assume to be purely deterministic, i. e. without noise), $\Theta$ is a nonnegative parameter (interaction strength) and $E = (E_{lm})_{l,m=1,\ldots,M}$ is a symmetric real matrix, on whose largest eigenvalue $e_0$ we will impose conditions later. We have

(C1) $$E \le e_0 \mathbf{I}$$

( in the sense of p.d. matrices). The nonlinearity in the model stems from the occurrence of the moments

$$z_\rho(t) := \int_{I\!R_\xi^M} \int_{I\!R_x^n} x\rho(x,\xi,t)\,dx\,dP(\xi), \tag{4.1d}$$

$$s_{\rho,m}(t) := \int_{I\!R_\xi^M} \int_{I\!R_x^n} \xi_m x \rho(x,\xi,t)\,dx\,dP(\xi). \tag{4.1e}$$

The probability measure $P$ represents the distribution of the noise $\xi$.

For the following we shall assume

$$\rho_I \in L_+^1(I\!R_\xi^M \times I\!R_x^n; dx\,dP(\xi)) \text{ and } \int_{I\!R_\xi^M} \int_{I\!R_x^n} \rho_I\,dx\,dP = 1.$$

For more information on the physics of the problem, an extensive list of references and a preliminary asymptotic analysis we refer to [ArBoMa95].

We start the analysis by defining

$$\rho_0(x,\xi;\rho(t)) := \exp(-A(x,\xi;\rho(t))) \tag{4.2}$$

and rewrite (4.1a) in the usual way

$$\rho_t = D\mathrm{div}_x \left( \rho_0 \nabla_x (\frac{\rho}{\rho_0}) \right). \tag{4.3}$$

We set up the relative entropy-type functional

$$e(\rho|\rho_0) := \int_{I\!R^M_\xi} \int_{I\!R^n_x} \rho \ln(\frac{\rho}{\rho_0}) dx dP(\xi). \tag{4.4}$$

Note that, strictly applying Definition 2.2, $e(\rho|\rho_0)$ is not a relative entropy since $\rho_0$ is not necessarily normalized to 1 in $L^1(dxdP(\xi))$. Actually, the normalization of $\rho_0$ was chosen such that

$$\frac{d}{dt} e(\rho|\rho_0) = I(\rho|\rho_0) \tag{4.5}$$

with the entropy dissipation

$$I(\rho|\rho_0) = -4D \int_{I\!R^M_\xi} \int_{I\!R^n_x} \left| \nabla_x \sqrt{\frac{\rho}{\rho_0}} \right|^2 \rho_0(dx) dP(\xi) \leq 0 \tag{4.6}$$

(cf. [ArBoMa95]). We now proceed as in the linear case in Section 2 and compute the entropy dissipation rate

$$\begin{aligned} \frac{d}{dt} I(\rho|\rho_0) &= -2\lambda I(\rho|\rho_0) + 2D^2 \int_{I\!R^M} \int_{I\!R^n} \rho \sum_{l,m=1}^{M} \left( \frac{\partial z_l}{\partial x_m} \right)^2 dx dP \\ &\quad + 2D\Theta \left( \int_{I\!R^M} \int_{I\!R^n} \rho |z|^2 dx dP - | \int_{I\!R^M} \int_{I\!R^n} \rho z dx dP|^2 \right) \\ &\quad + 2D \int_{I\!R^M} \int_{I\!R^n} \rho z^\top \left( \frac{\partial^2 W}{\partial x^2} - \lambda \mathbf{I} \right) z dx dP \\ &\quad - 2D \sum_{l,m=1}^{M} E_{lm} \int_{I\!R^M} \int_{I\!R^n} \xi_l \rho z dx dP \cdot \int_{I\!R^M} \int_{I\!R^n} \xi_m \rho z dx dP, \end{aligned} \tag{4.7}$$

where $z := \nabla_x \ln(\frac{\rho}{\rho_0})$. We remark that this computation follows the lines of the proof of Lemma 2.13, where, in addition, the terms which come from the time dependence of $\rho_0$, have to be taken care of (cf. [ArBoMa95] for details). To determine $\lambda$ in (4.7) appropriately we assume that $W$ is uniformly convex

(C2) $$\exists a > 0 : \frac{\partial^2 W}{\partial x^2}(x) \geq a\mathbf{I} \quad \forall x \in I\!R^n.$$

Since

$$\left| \int_{I\!R^M} \int_{I\!R^n} \xi_l \rho z dx dP \right|^2 \leq \int_{I\!R^M} \xi_l^2 dP(\xi) \int_{I\!R^M} \int_{I\!R^n} \rho |z|^2 dx dP$$

we can bound the last term on the right hand side of (4.7) (from below) by

$$-2De_0\gamma \int_{\mathbb{R}^M}\int_{\mathbb{R}^n} \rho|z|^2 dxdP$$

assuming on the second moment of P:

(C3) $$\int_{\mathbb{R}^M} |\xi|^2 dP(\xi) = \gamma < \infty.$$

Altogether we have

$$\frac{d}{dt} I(\rho|\rho_0) = -2\lambda I(\rho|\rho_0) + f(t) \tag{4.8}$$

where $f(t) \geq 0$ if $\lambda$ exists such that

(C4) $$a - e_0\gamma \geq \lambda > 0.$$

Clearly, this is the case if either $e_0 = 0$ (i. e. $E$ is negative semi-definite) or, if $e_0 > 0$, then the product $e_0\gamma$ has to be smaller than the convexity bound $a$.

Assume now that $\lambda$ satisfies (C4). Then (4.8) implies

$$|I(\rho|\rho_0)(t)| \leq e^{-2\lambda t}|I(\rho_I|\rho_0(t=0))|. \tag{4.9}$$

Since

$$\dot{z}_\rho(t) = -D\int_{\mathbb{R}^M}\int_{\mathbb{R}^n} \rho z dxdP, \quad \dot{s}_{\rho,m}(t) = -D\int_{\mathbb{R}^M}\int_{\mathbb{R}^n} \xi_m \rho z dxdP$$

we estimate (using $I(\rho|\rho_0) = -D\int_{\mathbb{R}^M}\int_{\mathbb{R}^n}\rho|z|^2 dxdP$):

$$|\dot{z}_\rho(t)| \leq e^{-2\lambda t}|I(\rho_I|\rho_0(t=0))|, \tag{4.10a}$$

$$|\dot{s}_{\rho,m}(t)| \leq \gamma e^{-2\lambda t}|I(\rho_I|\rho_0(t=0))|. \tag{4.10b}$$

Now we normalize $\rho_0$:

$$\rho_\infty(x,\xi;\rho(t)) := \frac{\rho_0(x,\xi;\rho(t))}{\int_{\mathbb{R}^n}\rho_0(y,\xi;\rho(t))dy}\int_{\mathbb{R}^n}\rho_I(y,\xi)dy \tag{4.11}$$

assuming

(C5) $\int_{\mathbb{R}^M} e^{\delta|\xi|^2} dP(\xi) < \infty$ for some $\delta$ sufficiently large.

Obviously we have

$$I(\rho|\rho_0) = I(\rho|\rho_\infty).$$

Since $\int_{\mathbb{R}_x^n} \rho(x,\xi,t)dx$ is conserved by the equation (4.1a) we can apply the logarithmic Sobolev-inequality (2.78) (with $\psi(\sigma) = \sigma\ln\sigma - \sigma + 1$ and $\frac{a+\Theta}{D}$ as convexity constant):

$$\int_{\mathbb{R}^n} \rho \ln\left(\frac{\rho}{\rho_\infty}\right) dx \leq \frac{D}{2(a+\Theta)} \int_{\mathbb{R}^n} \frac{1}{\rho} |\nabla(\frac{\rho}{\rho_\infty})|^2 dx.$$

After integration with respect to $dP(\xi)$, $-I(\rho|\rho_\infty)$ appears on the right hand side and the relative entropy of $\rho(t)$ with respect to $\rho_\infty(t)$ on the left hand side. Thus

$$e(\rho(t)|\rho_\infty(t)) \leq \frac{De^{-2\lambda t}}{2(a+\Theta)} |I(\rho_I|\rho_0(t=0))| \tag{4.12}$$

follows and the Csiszár-Kullback inequality gives:

$$\|\rho(t) - \rho_\infty(t)\|_{L^1(dxdP(\xi))} \leq 2\left(\frac{D|I(\rho_I|\rho_0(t=0))|}{2(a+\Theta)}\right)^{\frac{1}{2}} e^{-\lambda t}. \tag{4.13}$$

The estimates (4.10) imply that $z_\rho(\infty) := \lim_{t\to\infty} z_\rho(t)$ and $s_{\rho,m}(\infty) := \lim_{t\to\infty} s_{\rho,m}(t)$ exist and

$$|z_\rho(t) - z_\rho(\infty)| \leq \frac{1}{2\lambda} |I(\rho_I|\rho_0(t=0))| e^{-2\lambda t}, \tag{4.14a}$$

$$|s_{\rho,m}(t) - s_{\rho,m}(\infty)| \leq \frac{\gamma}{2\lambda} |I(\rho_I|\rho_0(t=0))| e^{-2\lambda t}. \tag{4.14b}$$

Using these estimates we can 'eliminate' the $t$-dependence of $\rho_\infty(x,\xi;\rho(t))$ asymptotically and prove

**Theorem 4.1.** *Let (C1)–(C5) hold and assume* $|I(\rho_I|\rho_0(t=0))| < \infty$*. Then a steady state* $\tilde\rho_\infty \in L^1_+(dxdP(\xi))$ *of (4.1) exists and there is a constant* $K > 0$*, only depending on* $\rho_I$ *and on the parameters of the problem (4.1) such that*

$$\|\rho(t) - \tilde\rho_\infty\|_{L^1(dxdP(\xi))} \leq Ke^{-\lambda t} \quad \forall t > 0.$$

Clearly, $\tilde\rho_\infty$ is a solution of the equation

$$\tilde\rho_\infty(x,\xi) = \frac{\rho_0(x,\xi;\tilde\rho_\infty)}{\int_{\mathbb{R}^n} \rho_0(y,\xi;\tilde\rho_\infty)dy} \int_{\mathbb{R}^n} \rho_I(y,\xi)dy. \tag{4.15}$$

If $W(x) = W(-x)\ \forall x \in \mathbb{R}^n$ holds, then a solution of (4.15) can be easily constructed. We set $z_\rho = s_{\rho,m} = 0$ and

$$\tilde\rho_\infty(x,\xi) = N\exp\left(-\frac{1}{D}(W(x) + \frac{\Theta}{2}|x|^2)\right) \int_{\mathbb{R}^n} \rho_I(y,\xi)dy, \tag{4.16}$$

where $N = \left(\int_{I\!R^n} \exp\left(-\frac{1}{D}(W(y) + \frac{\Theta}{2}|y|^2)\right) dy\right)^{-1}$. It is immediately verified that this function solves the equation (4.15).

We remark that convergence of $\rho(t)$ to a steady state $\tilde{\rho}_\infty$ without a convergence rate can still be shown if the single oscillator potential $W(x)$ is not uniformly convex but satisfies only certain mild growth conditions (cf. [ArBoMa95]). Then it was shown in [ArBoMa95] that $I(t) \stackrel{t\to\infty}{\longrightarrow} 0$ still holds (without a rate). The method used above implies immediately

$$e(\rho(t)|\rho_\infty) \stackrel{t\to\infty}{\longrightarrow} 0$$

and $\rho(t) \stackrel{t\to\infty}{\longrightarrow} \tilde{\rho}_\infty$ in $L^1(dxdP(\xi))$ follows.

## 4.2 The drift-diffusion-Poisson model

Secondly, we consider the drift-diffusion model with Poisson-coupling for an electron gas [Mar86], [MaRiSc90]

$$\rho_t = \operatorname{div}\left(\nabla\rho + \nabla\left(\frac{|x|^2}{2} + V\right)\rho\right), \quad x \in I\!R^n, t > 0, \tag{4.17a}$$

$$\rho(t=0) = \rho_I \geq 0 \quad \text{on } I\!R^n, \qquad \int_{I\!R^n} \rho_I(x)dx = 1, \tag{4.17b}$$

$$-\Delta V = \rho. \tag{4.17c}$$

Here $\frac{|x|^2}{2}$ acts as a confining potential. For the sake of simplicity we only consider the case of at least 3 dimensions, i.e. $n \geq 3$ and take the Newtonian solution of (4.17c):

$$V(x,t) = \frac{1}{(n-2)S_n}\int_{I\!R^n} \frac{\rho(y,t)}{|x-y|^{n-2}}dy, \tag{4.17d}$$

with $S_n$ being the surface area of the unit sphere in $I\!R^n$. An existence-uniqueness result (for a global solution) of (4.17) is easily obtained by proceeding in analogy to the bounded domain case (cf. e.g. [Gaj85]). The steady state of (4.17) is the unique solution of the mean-field equation

$$\rho_\infty(x) := \exp\left(-\frac{|x|^2}{2} - V_\infty(x)\right) \Big/ \int_{I\!R^n} \exp\left(-\frac{|y|^2}{2} - V_\infty(y)\right) dy, \tag{4.18a}$$

$$V_\infty(x) = \frac{1}{(n-2)S_n}\int_{I\!R^n} \frac{\rho_\infty(y)}{|x-y|^{n-2}}dy. \tag{4.18b}$$

A detailed analysis of equations of the type (4.18) can be found in [Dol91].

We shall now show that $\rho(t)$ converges exponentially to $\rho_\infty$ by using logarithmic Sobolev inequalities. We start by defining the relative entropy type functional

$$e := \int_{I\!R^n} \rho \ln\left(\frac{\rho}{\rho_\infty}\right) dx + \frac{1}{2}\int_{I\!R^n} |\nabla(V - V_\infty)|^2 dx \tag{4.19}$$

(cf. [Gaj85]). A simple calculation shows that $e$ can be written as

$$e = F(\rho) - F(\rho_\infty),$$

where $F$ is the free energy of the electron gas:

$$F(\rho) = \int_{I\!R} \left(\rho \ln \rho + \frac{x^2}{2}\rho + \frac{V}{2}\rho\right) dx.$$

Note that the potential energy density of the external confinement potential $V_{ext} = |x|^2/2$ is $V_{ext}\rho$ (see Remark (2.7)), while it is $\frac{1}{2}V\rho$ for the self-consistent Coulomb potential $V(x,t)$. The time-derivative of $e$ is given by:

$$\frac{d}{dt}e(t) = -\int_{I\!R^n} \rho(t)|\nabla \ln\left(\frac{\rho(t)}{N(t)}\right)|^2 dx \tag{4.20}$$

where we denoted the $t$-local state

$$N(t) = \exp\left(-\frac{|x|^2}{2} - V(x,t)\right) \Big/ \int_{I\!R^n} \exp\left(-\frac{|y|^2}{2} - V(y,t)\right) dy. \tag{4.21}$$

We remark that the calculation which leads to (4.20) is completely analogous to [Gaj85] making appropriate use of the Poisson equations (4.17d), (4.18b).

Assume now that $V(t)$ is in $L^\infty(I\!R^n)$ uniformly in $t$, i.e. there is a $K > 0$ such that

$$\|V(t)\|_{L^\infty(I\!R^n)} \le K, \quad t \ge 0 \tag{4.22}$$

(which shall be proven later on under additional assumptions on $\rho_I$). Then the logarithmic Sobolev inequality (2.78) and the perturbation result of Theorem 3.2 imply the existence of $\lambda_1 > 0$ (independent of $t$) such that

$$\int \rho \ln\left(\frac{\rho}{N}\right) dx \le \frac{1}{2\lambda_1} \int \rho |\nabla \ln(\frac{\rho}{N})|^2 dx, \quad t \ge 0. \tag{4.23}$$

We use (4.23) in (4.20) and obtain

$$\frac{d}{dt}e(t) \le -2\lambda_1 \int_{I\!R^n} \rho \ln\left(\frac{\rho}{N}\right) dx = -2\lambda_1 \int_{I\!R^n} \rho \ln\left(\frac{\rho}{\rho_\infty}\right) dx - 2\lambda_1 \int_{I\!R^n} \rho \ln\left(\frac{\rho_\infty}{N}\right) dx.$$

Since

$$\ln\left(\frac{\rho_\infty}{N(t)}\right) = V(t) - V_\infty + \ln\left(\int_{I\!R^n} e^{V_\infty - V(t)} \rho_\infty dx\right)$$

we have

$$\frac{d}{dt}e(t) \le -2\lambda_1 \int_{I\!R^n} \rho \ln\left(\frac{\rho}{\rho_\infty}\right) dx - 2\lambda_1 \int_{I\!R^n} (V(t)-V_\infty)\rho dx + 2\lambda_1 \left(-\ln\left(\int_{I\!R^n} e^{V_\infty - V(t)} \rho_\infty dx\right)\right)$$

(using $\int_{I\!R^n} \rho(y,t)dy = 1 \quad \forall t > 0$). The convexity of $f(u) = -\ln u$ implies

$$-\ln\left(\int_{I\!R^n} e^{V_\infty - V(t)} \rho_\infty dx\right) \le \int_{I\!R^n} (V(t) - V_\infty)\rho_\infty dx$$

and

$$\frac{d}{dt}e(t) \le -2\lambda_1 \int_{I\!R^n} \rho \ln\left(\frac{\rho}{\rho_\infty}\right) dx - 2\lambda_1 \int_{I\!R^n} (V(t) - V_\infty)(\rho(t) - \rho_\infty) dx$$

follows. Now we use

$$\rho(t) - \rho_\infty = -\Delta(V(t) - V_\infty)$$

such that

$$\begin{aligned} \frac{d}{dt}e(t) &\le -2\lambda_1 \int_{I\!R^n} \rho(t) \ln\left(\frac{\rho(t)}{\rho_\infty}\right) dx - 2\lambda_1 \int |\nabla(V(t) - V_\infty)|^2 dx \\ &\le -2\lambda_1 e(t). \end{aligned}$$

We conclude exponential convergence in relative entropy:

$$\begin{aligned} &\int_{I\!R^n} \rho(t) \ln\left(\frac{\rho(t)}{\rho_\infty}\right) dx + \frac{1}{2}\int_{I\!R^n} |\nabla(V(t) - V_\infty)|^2 dx \\ &\le e^{-2\lambda_1 t} \left(\int_{I\!R^n} \rho_I \ln\left(\frac{\rho_I}{\rho_\infty}\right) dx + \frac{1}{2}\int_{I\!R^n} |\nabla(V_I - V_\infty)|^2 dx\right) \qquad (4.24) \end{aligned}$$

(where, obviously, $V_I$ is the Newtonian solution of $-\Delta V_I = \rho_I$) and exponential $L^1(I\!R^n)$-convergence of $\rho(t)$ to $\rho_\infty$ follows from the Csiszár-Kullback inequality.

We are left with proving the bound (4.22). Therefore we proceed as in [FaStr86] and multiply (4.17a) by $\rho^q$, $q \in I\!N$. Integration by parts and using (4.17c) gives

$$\frac{1}{q+1}\frac{d}{dt}\int_{I\!R^n} \rho^{q+1} dx = -\frac{4q}{q+1}\int_{I\!R^n} |\nabla(\rho^{\frac{q+1}{2}})|^2 dx + \frac{qn}{q+1}\int_{I\!R^n} \rho^{q+1} dx - \frac{q}{q+1}\int_{I\!R^n} \rho^{q+2} dx.$$

Applying the Nash-inequality [Nas58]

$$\left(\int_{I\!R^n} f^2 dx\right)^{1+\frac{2}{n}} \le a_n \left(\int_{I\!R^n} |f| dx\right)^{\frac{4}{n}} \int_{I\!R^n} |\nabla f|^2 dx$$

for some $a_n > 0$ to the first term on the right hand side (with $f = \rho^{\frac{q+1}{2}}$) gives

$$\frac{d}{dt} z_{q+1} \le -\frac{4q}{a_n} \frac{(z_{q+1})^{1+\frac{2}{n}}}{(z_{\frac{q+1}{2}})^{\frac{4}{n}}} + nq z_{q+1},$$

where we set

$$z_q := \int_{\mathbb{R}^n} \rho^q dx.$$

It is easy to conclude that $z_{q+1}(t)$ is uniformly bounded for $t > 0$ if $z_{q+1}(0) < \infty$. By interpolation we find $\rho(t) \in L^p_+(\mathbb{R}^n)$ uniformly for $t > 0$ if $\rho_I \in L^p_+(\mathbb{R}^n) \cap L^1_+(\mathbb{R}^n)$.

Since the Newtonian potential of a function in $L^p(\mathbb{R}^n)$ is in $L^\infty(\mathbb{R}^n)$ if $p > \frac{n}{2}$ we obtain

**Theorem 4.2.** *Let $\rho_I \in L^1_+(\mathbb{R}^n) \cap L^p_+(\mathbb{R}^n)$ for some $p > \frac{n}{2}$. Then there is a constant $\lambda_1 > 0$ such that the exponential convergence (4.24) of the relative entropy holds for the solution $(\rho(t), V(t))$ of (4.17).*

We refer to [ArMaTo98] for a detailed analysis of bipolar models of the form (4.17) (i.e. with two types of carriers).

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