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ON CONVEXITIES OF d-GROUPS

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The notion of a d-group was introduced by Kopytov and Dimitrov [4]; a d-group is defined to be a directed group with two additional operations $\wedge$ and $\vee$ satisfying certain conditions. This notion was investigated also in [3].

Convexities of lattices were defined by Fried in [5] (p. 255). In the same way we can define convexities also for other types of ordered algebraic structures.

In [5] the question was proposed what is the "number" of convexities of lattices. In [2] it was shown that no such number exists; convexities of lattices form a proper class.

In the present paper it will be proved that an analogous result is valid for convexities of d-groups. Namely, we shall construct an injective mapping of the class of all infinite cardinals into the collection $\mathscr{C}$ of all convexities of d-groups.

Next we shall investigate the properties of the partial order on $\mathscr{C}$ which is defined by inclusion.

Some relations between convexities of d-groups and varieties of lattice ordered groups will also be dealt with.

1. PRELIMINARIES

The group operation in a directed group G will be denoted additively. Let x and y be elements of G. If x and y are incomparable, then we write $x \parallel y$. The notation $x \prec y$ means that x is covered by y (i.e., $x < y$ and there is no z in G with $x < z < y$).

We recall the definition of a d-group (cf. [3]).

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1.1. Definition. Let G be a directed group with two additional binary commutative operations $\wedge$ and $\vee$ such that the following conditions are satisfied:

(i) If $x \leqslant y$, then $x \wedge y = x$ and $x \vee y = y$.
(ii) If $x \parallel y$, then $x \wedge y < t < x \vee y$ for each $t \in \{x, y\}$.
(iii) $x \wedge y = -((-x) \vee (-y))$, and dually.
(iv) $a + (x \vee y) + b = (a + x + b) \vee (a + y + b)$, and dually.

Under these assumption G is said to be a d-group.

(The expression "dually" means that the symbols $\wedge$ and $\vee$ are interchanged in the corresponding condition.)

For d-groups we define the notions of homomorphism, direct product and d-subgroup in the usual way.

Let $\mathscr{D}$ be the class of all d-groups.

1.2. Definition. A nonempty subset of $\mathscr{D}$ will be said to be a convexity of d-groups (or shortly, a convexity) if it is closed under homomorphic images, convex d-subgroups and direct products.

Let us denote by $\mathscr{C}$ the collection of all convexities of d-groups. We consider $\mathscr{C}$ to be partially ordered by inclusion. Then $\mathscr{D}$ is the largest element of $\mathscr{C}$; the least element of $\mathscr{C}$ is the class X_0 consisting of all one-element d-groups.

For $X \subseteq \mathscr{D}$ we denote by

HX —the class of all homomorphic images of elements of X;
CX —the class of all convex d-subgroups of elements of X;
PX —the class of all direct products of elements of X.

1.3. Lemma. *Let $\emptyset \neq X \subseteq \mathscr{D}$. Then $HCPX$ is the least convexity containing X as a subclass.*

The proof is routine; it will be omitted. For an analogous result concerning convexities of lattices cf. [6].

The convexity $HCPX$ will be said to be generated by X.

A d-subgroup K of a d-group G will be called a d-ideal of G if it is a kernel of a homomorphism. For conditions characterizing d-ideals cf. [5] and [3]. If K is a d-ideal of G, then the factor d-group G/K is defined in the usual way.

2. The d-groups H_α

Let α be an infinite cardinal. We construct a d-group H_α as follows.

There exists an abelian group G_α with $\operatorname{card} G_\alpha = \alpha$. Let G^1 be the additive group of all integers with the natural linear order. We consider G_α to be trivially partially ordered and let H_α be the partially ordered group $G^1 \circ G_\alpha$, where $\circ$ denotes the operation of the lexicographic product.

Next we define a binary operation $\vee$ on H_α. Let $g \in H_\alpha$, $g = (x, y)$. We put $0 \vee 0 = 0$ and $0 \vee (0, y) = (1, y)$ if $y \neq 0$. For $x \neq 0$ we set $0 \vee g = \max\{0, g\}$. If $g' \in G$ and $g' \neq 0$, then we define $g' \vee g = g' + (0 \vee (g - g'))$.

Now we put $g_1 \wedge g_2 = -((-g_1) \vee (-g_2))$ for each g_1 and g_2 in H_α. Then H_α turns out to be a d-group.

If β is an infinite cardinal with $\beta < \alpha$, then without loss of generality we can assume that G_β is a subgroup of G_α. Hence H_β is a d-subgroup of H_α. Let us remark that H_β fails to be a convex subset of H_α.

Let I be a nonempty set of indices and for each $i \in I$ let $A_i = H_\alpha$. Put $A = \prod\limits_{i \in I} A_i$ (the direct product of d-groups A_i). Let B be a convex d-subgroup of A and K a d-ideal of B.

2.1. Lemma. *Let α and β be infinite cardinals, $\beta < \alpha$. Then (under the above introduced notation) H_β is not isomorphic to B/K.*

Proof. By way of contradiction, suppose that there exists an isomorphism φ of H_β onto B/K. For each $g \in B$ we denote $\overline{g} = g + K$.

If $f \in A$, $i \in I$ and $f(i) = (x, y)$ with $x \in G^1$ and $y \in G_\alpha$, then we denote $f(i^1) = x$ and $f(i^2) = y$.

In view of the isomorphism φ there exists $g \in B$ such that $\overline{g} \parallel \overline{0}$. Then $g \parallel 0$. Put $g \vee 0 = h_0$. Hence for each $i \in I$ we have either $h_0(i) = 0$ or $h_0(i^1) \geqslant 1$. There is $h \in A$ such that, for each $i \in I$,

$$h(i^1) = h_0(i^1) \quad \text{and} \quad h(i^2) = 0.$$

Then $0 < h < 2h_0$, whence $h \in B$.

Since $\overline{g} \vee \overline{0} = \overline{h_0}$, by applying the isomorphism φ we obtain that $\overline{h_0} \neq \overline{0}$. Further we have $0 < h_0 < 2h$ and thus $\overline{h} \neq \overline{0}$.

There are h' and h'' in A such that, for each $i \in I$,

$$h'(i^1) = 1 \text{ if } h(i^1) = 1, \text{ and } h'(i^1) = 0 \text{ otherwise},$$
$$h''(i^1) = h(i^1) \text{ if } h(i^1) > 1, \text{ and } h''(i^1) = 0 \text{ otherwise},$$
$$h'(i^2) = h''(i^2) = 0.$$

Clearly $0 \leqslant h' \leqslant h$ and $0 \leqslant h'' \leqslant h$. Hence h' and h'' belong to B. Next, $h' + h'' = h$, thus either $h' \notin K$ or $h'' \notin K$.

First suppose that h' belongs to K. Then h'' does not belong to K. There is $h^* \in A$ such that, for each $i \in I$, the following conditions are satisfied:

(a) if $h''(i) = 0$, then $h^*(i) = 0$,

(b) if $h''(i) \neq 0$, then $h^*(i^1)$ is the least positive integer which is greater than or equal to $\frac{1}{2}h''(i^1)$;

(c) $h^*(i^2) = 0$ for each $i \in I$.

Hence $0 < h < h^* < h''$ and thus $h^* \in B$.

In view of the isomorphism φ the relation $\overline{0} \prec \overline{h_0}$ is valid. This yields that $\overline{0} \prec \overline{h}$ also holds; therefore $\overline{0} \prec \overline{h''}$. Since $\overline{0} \leqslant \overline{h^*} \leqslant \overline{h''}$, we must have either

(a_1) $\overline{0} = \overline{h^*}$

or

(a_2) $\overline{h^*} = \overline{h''}$.

In view of the definition of h^* we infer that

(b_1) $h'' \leqslant 2h^*$,

(b_2) $h'' \leqslant 3(h'' - h^*)$.

If (a_1) holds, then (b_1) yields that $\overline{h''} = \overline{0}$, which is a contradiction. If (a_2) is valid, then applying (b_2) we again obtain the relation $\overline{h''} = \overline{0}$, which is impossible.

We conclude that h' does not belong to K. Thus $\overline{0} < \overline{h'} \leqslant \overline{h}$. Since $\overline{0} \prec \overline{h}$, the relation $\overline{h'} = \overline{h}$ holds. Therefore $\overline{0} \prec \overline{h'}$ and $\overline{h''} = \overline{0}$. Put $I_1 = \{i \in I : h'(i) \neq 0\}$. Thus $I_1 \neq \emptyset$.

We denote by Q the set of all $z \in H_\alpha$ with $z \parallel 0$. For each $z \in Q$ there exists t_z in A such that $t_z(i) = z$ if $i \in I_1$ and $t_z(i) = 0$ otherwise. Then $-h' < t_z < h'$, hence $t_z \in B$. Put $Q_0 = \{t_z : z \in Q\}$.

Let $t_z \in Q_0$. Then $0 \vee t_z = h'$ and hence $\overline{0} \vee \overline{t_z} = \overline{h'} = \overline{h}$. This yields that $\overline{t_z} \neq \overline{0}$ and that the relation $\overline{t_z} < \overline{0}$ cannot hold. It is clear that $-z \in Q$ and $t_{-z} = -t_z$. Since $\overline{t_{-z}} < \overline{0}$ cannot be valid we infer that $\overline{t_z}$ is not greater than $\overline{0}$. Therefore $\overline{t_z} \parallel \overline{0}$ for each $z \in Q$.

If $z(1)$ and $z(2)$ are distinct elements of Q, then $z = z(1) - z(2)$ belongs to Q as well; thus

$$\overline{t_{z(1)}} - \overline{t_{z(2)}} = \overline{t_{z(1)-z(2)}} = \overline{t_z} \neq \overline{0}.$$

Therefore the number of those elements of the d-group B/K which are incomparable with $\overline{0}$ is greater than or equal to α. Thus in view of the isomorphisms φ we arrived at a contradiction. □

For each infinite cardinal α let X_α be the convexity of d-groups which is generated by the one-element set $\{H_\alpha\}$.

2.2. Theorem. *The mapping ψ defined by $\psi(\alpha) = X_\alpha$ is an injection of the class of all infinite cardinals into the collection $\mathscr{C}$ of all convexities of d-groups.*

Proof. If α and β are infinite cardinals with $\beta < \alpha$, then in view of 2.1 the d-group H_β does not belong to the class $HCP\{H_\alpha\}$. Thus according to 1.3 the convexities X_α and X_β are distinct. □

The above result shows that the collection $\mathscr{C}$ is a proper class.

3. The lattice $\mathscr{C}$

As we already remarked above, we consider $\mathscr{C}$ to be partially ordered by inclusion. We shall apply to $\mathscr{C}$ the usual order-theoretic terminology (though $\mathscr{C}$ is a proper collection).

3.1. Lemma. *Let $\{X_i\}_{i\in I}$ be a nonempty subcollection of $\mathscr{C}$. Then the meet $\bigwedge_{i\in I} X_i$ in $\mathscr{C}$ is equal to $\bigcap_{i\in I} X_i$.*

This is an immediate consequence of the definition of $\mathscr{C}$.

Since $\mathscr{C}$ possesses the greatest and the least element, 3.1 yields

3.2. Corollary. *$\mathscr{C}$ is a complete lattice.*

3.3. Lemma. *Let $\{X_i\}_{i\in I}$ be as in 3.1. Then the join $\bigvee_{i\in I} X_i$ in $\mathscr{C}$ is equal to $HCP \bigcup_{i\in I} X_i$.*

Proof. In view of 3.2, the join $\bigvee_{i\in I} X_i$ does exist in $\mathscr{C}$. Then from 1.3 we conclude that $\bigvee_{i\in I} X_i = HCP \bigcup_{i\in I} X_i$ is valid. □

3.4. Lemma. *Let $\{X\}_{i\in I}$ be as in 3.1 and let G be a d-group. Then the following conditions are equivalent:*

(a) *G belongs to $\bigvee_{i\in I} X_i$.*

(b) *There is a set $I(1) \subseteq I$ and d-groups $G_i (i \in I(1))$ with $G_i \in X_i$ such that $G \in HC \prod_{i\in I(1)} G_i$.*

Proof. This is a consequence of 1.3, 3.3 and of the fact that each X_i is closed under direct products. □

3.5. Lemma. *Let $\{X_i\}_{i\in I}$ be as in 3.1. Suppose that for each $i \in I$ there exists a d-group D_i such that X_i is generated by $\{D_i\}$. Let G be a d-group. Then the condition* (a) *from 3.4 is equivalent to the following condition:*

(b$_1$) *There is a set $I(1) \subseteq I$ such that $G \in HC \prod_{i\in I(1)} D_i$.*

Proof. Let $\mathscr{B}$ be the class of all d-groups satisfying the condition (b$_1$). Then in view of 3.4, $\mathscr{B} \subseteq \bigvee_{i\in I} X_i$. Next, $\mathscr{B}$ is closed under homomorphisms, convex d-subgroups and direct products; moreover, $D_i \in \mathscr{B}$ for each $i \in I$. Thus according to 1.3 the relation $\bigvee_{i\in I} X_i \subseteq \mathscr{B}$ holds. □

The direct product of two d-groups A and B will be denoted by $A \times B$. If φ is an isomorphism of a d-group G onto $A\times B$, then we denote $A^0 = \{g \in G : \varphi(g)(B) = 0\}$ and $B^0 = \{g \in G : \varphi(g)(A) = 0\}$. ($\varphi(g)(A)$ is the component of $\varphi(g)$ in A, and similarly for $\varphi(g)(B)$.)

We need some auxiliary results on direct products with two factors (Lemmas 3.5–3.9); their proofs are routine and will be omitted.

3.6. Lemma. *Let A, B and G be as above. Then A^0 and B^0 are convex d-subgroups of G and they satisfy the following conditions:*

(i) *For each $g \in G$ there exist uniquely determined elements $g_A \in A^0$ and $g_B \in B^0$ such that $g = g_A + g_B$.*

(ii) *If g and g' are elements of G, then $gtg = (g_Atg'_A) + (g_Btg'_B)$ for each $t \in \{+, \wedge, \vee\}$.*

(iii) *For g and g' in G the relation* (a) *$g \leqslant g'$ is equivalent to* (b) *$g_A \leqslant g'_A$ and $g_B \leqslant g'_B$.*

3.7. Lemma. *Let A^0 and B^0 be convex d-subgroups of a d-group G. Assume that the conditions* (i), (ii) *and* (iii) *from 3.5 are satisfied. For each $g \in G$ put $\varphi(g) = (g_A, g_B)$. Then φ is an isomorphism of G onto the direct product $A^0 \times B^0$.*

In view of 3.5 and 3.6 we often identify (when no misunderstanding can occur) the d-groups A and A^0, and similarly for B and B^0; in this sense we write $G = A^0 \times B^0$, or also $G = A \times B$.

In this connection the following natural question arises. For a d-group G we denote by G^* the corresponding directed group (we forget the operations $\wedge$ and $\vee$). Assume that a direct product decomposition $G^* = A \times B$ is given. Then we can ask whether A and B are d-subgroups of G such that $G = A \times B$ is valid. The answer is that this need not hold in general (cf. [3], Example in 3.6).

Let us remark that for a related question concerning direct product decompositions of a directed group $(G; +, \leqslant)$ and direct product decompositions of the directed set $(G; \leqslant)$ (we forget the group operation $+$) the answer is affirmative (cf. [1]).

3.8. Lemma. *Let A, B and G be d-groups such that $G = A \times B$. Let C be a convex d-subgroup of G. Then $C = (C \cap A) \times (C \cap B)$.*

3.9. Lemma. *Let A, B and G be as in 3.8 and let K be a d-ideal of G. Then $K \cap A$ and $K \cap B$ are d-ideals of A or of B, respectively; moreover, $G/K = (A/K \cap A) \times (B/K \cap B)$.*

Let us remark that Lemmas 3.6 and 3.7 can be generalized to direct products with any number (i.e., also infinite number) of direct factors. On the other hand, neither 3.8 nor 3.9 are valid, in general, for direct product decompositions with an infinite number of direct factors.

3.10. Theorem. *The lattice $\mathscr{C}$ is distributive.*

Proof. Let X_1, X_2 and Y be elements of $\mathscr{C}$. We have to verify that $Y \wedge (X_1 \vee X_2) = (Y \wedge X_1) \vee (Y \wedge X_2)$ is valid. It suffices to verify that the relation

$$Y \wedge (X_1 \vee X_2) \subseteq (Y \wedge X_1) \vee (Y \wedge X_2)$$

holds.

Let $G \in Y \wedge (X_1 \vee X_2)$. Then $G \in Y$ and $G \in X_1 \vee X_2$. Thus in view of 3.4 there are $G_1 \in X_1$ and $G_2 \in X_2$ such that $G \in HC(G_1 \times G_2)$. Therefore we can assume that there are (i) a convex d-subgroup B of $G_1 \times G_2$, and (ii) a d-ideal K of B such that G is isomorphic to B/K.

According to 3.8 the relation $B = (B \cap G_1) \times (B \cap G_2)$ is valid. Put $B \cap G_i = G_i'$ $(i = 1, 2)$. Then G_i' is a convex d-subgroup of G_i, hence $G_i' \in X_i$ for $i = 1, 2$. In view of 3.9,

$$B/K = (G_1'/G_1' \cap K) \times (G_2'/G_2' \cap K).$$

Clearly $G_i'/G_i' \cap K \in X_i$ for $i = 1, 2$. Let $i \in \{1, 2\}$. Because $G_i'/G_i' \cap K \in HC\{G\}$, we infer that $G_i'/G_i' \cap K$ belongs to Y; hence it belongs to $Y \wedge X_i$ as well. Thus (cf. 3.4) B/K is an element of the convexity $(Y \wedge X_1) \vee (Y \wedge X_2)$. Hence B belongs to this convexity as well, which completes the proof. □

The question whether $\mathscr{C}$ satisfies the infinite distributive law

$$Y \wedge \left(\bigvee_{i \in I} X_i \right) = \bigvee_{i \in I} (Y \wedge X_i)$$

remains open.

4. Further properties of $\mathscr{C}$

We already observed in Section 2 that $\mathscr{C}$ is a proper collection. The question arises what can be said in this direction on chains in $\mathscr{C}$; i.e., we can ask whether there exists a subcollection $\mathscr{C}_1$ of $\mathscr{C}$ such that (i) $\mathscr{C}_1$ is a chain in the lattice $\mathscr{C}$, and (ii) $\mathscr{C}_1$ is a proper collection.

We shall construct a subcollection $\mathscr{C}_1$ of $\mathscr{C}$ having these properties.

For each infinite cardinal α let H_α be as in Section 2. Next let M_i be the class of all infinite cardinals.

As above we can suppose that if α and β are elements of M_i with $\beta < \alpha$, then H_β is a d-subgroup of H_α.

For $\alpha \in M_i$ let X_α be as in Section 2. We put $Y_\alpha = \bigvee\limits_{\beta \geqslant \alpha} X_\alpha$.

Let $\alpha, \beta \in M_i, \beta < \alpha$. Let I be a nonempty set of indices and for each $i \in I$ let A_i be a d-group which is equal to some H_β with $\beta \geqslant \alpha$. Put $A = \prod\limits_{i \in I} A_i$. Assume that B is a convex d-subgroup of A and that K is a d-ideal of B.

4.1. Lemma. *Under the above notation, H_β is not isomorphic to B/K.*

The proof is similar to that of 2.1, only minor modifications are required. It will be omitted.

4.2. Lemma. *Let α and β be as above. Then H_β does not belong to Y_α.*

Proof. This is a consequence of 4.1 and 3.5. □

4.3. Theorem. *For each $\beta \in M_i$ let $\varphi(\beta) = Y_\beta$. Then φ is a surjection of M_i onto $\mathscr{C}_1$. Moreover, if α, $\beta \in M_i$, $\beta < \alpha$, then $Y_\alpha < Y_\beta$.*

Proof. The first assertion is a consequence of 4.2; the second is obvious. □

The question whether $\bigwedge\limits_{\beta} Y_\beta$ ($\beta \in M_i$) is the least element of $\mathscr{C}$ remains open.

For a nonempty subclass X of $\mathscr{D}$ we denote by SX the class of all d-subgroups of elements of X.

If $\mathscr{V}$ is a variety of d-groups, then $HSP\mathscr{V} = \mathscr{V}$, hence, in particular, $HCP\mathscr{V} = \mathscr{V}$; thus $\mathscr{V} \in \mathscr{C}$.

4.4. Proposition. *Each variety of lattice ordered groups belongs to $\mathscr{C}$.*

P r o o f. Let $\mathscr{V}$ be a variety of lattice ordered groups. Then $P\mathscr{V} = \mathscr{V}$. Let $G \in \mathscr{V}$. If H_0 is a convex d-subgroup of G, then H_0 is a convex ℓ-subgroup of G; hence $C\mathscr{V} = \mathscr{V}$. Next let H_1 be a d-ideal of G. In [5] it has been shown that each d-ideal of a d-group is a convex normal d-subgroup. Therefore H_1 is an ℓ-ideal of G and so $G/H_1 \in \mathscr{V}$. Thus $\mathscr{V}$ is closed with respect to homomorphisms (which are considered as homomorphisms of d-groups). Summarizing, we conclude that $\mathscr{V}$ belongs to $\mathscr{C}$. □

For the particular case $\mathscr{V} = \mathscr{L}$ the relation $\mathscr{V} \in \mathscr{C}$ can be obtained also from [5], Theorem 4.3.

Let $\mathscr{A}$ be the variety of all abelian lattice ordered groups; X_0 denotes the class of all one-element d-groups. It is well-known that if $\mathscr{V}$ is a variety of lattice ordered groups with $\mathscr{V} \neq X_0$, then $\mathscr{A} \subseteq \mathscr{V}$.

In view of 3.4, $\mathscr{A} \in \mathscr{C}$. If $Y \in \mathscr{C}$ and $Y \neq X_0$, then the relation $\mathscr{A} \subseteq Y$ need not be valid. In fact, let R be the additive group of all reals and let G^1 be the subgroup of R consisting of all integers. We consider R and G^1 to be lattices (with the natural linear order). Put $Y = HCP\{R\}$. Then $X_0 \neq Y \in \mathscr{C}$ (and, at the same time, $Y \subseteq \mathscr{L}$). Since R is divisible, each element of Y is a divisible lattice ordered group. Thus G^1 does not belong to Y and therefore $\mathscr{A} \not\subseteq Y$.

The following consideration shows that the collection of elements Y of $\mathscr{C}$ which satisfy the condition $\mathscr{A} \not\subseteq Y$ is large.

4.5. Proposition. *Let G be a linearly ordered group. Assume that there exists $g_0 \in G$ such that $0 \prec g_0$. Then for each infinite cardinal α the relation $G \notin Y_\alpha$ is valid.*

P r o o f. By way of contradiction, assume that G belongs to Y_α for some $\alpha \in M_i$. Thus in view of 3.5 there exist d-groups $A_i (i \in I)$ such that for each $i \in I$ there is a cardinal $\alpha(i) \geqslant \alpha$ with $A_i = H_{\alpha(i)}$ and $G \in HC \prod_{i \in I} A_i$.

Thus there is a convex d-subgroup B of $\prod_{i \in I} A_i$ and a d-ideal K of B such that G is isomorphic to B/K. Without loss of generality we can assume that $G = B/K$. As above, we denote $\overline{g} = g + K$ for each $g \in B$. By the assumption there exists $b \in B$ such that $\overline{0} \prec \overline{b}$.

We have $\overline{b \vee 0} = \overline{b} \vee \overline{0} = \overline{b}$, hence we can take $b \vee 0$ instead of b. Thus we can assume that $b(i^1) \geqslant 0$ for each $i \in I$. There is $b' \in A$ having the property that $b'(i^1) = b(i^1)$ and $b'(i^2) = 0$ for each $i \in I$. Then $0 \leqslant b' < 2b$, whence $b' \in B$. Moreover, $\overline{0} \leqslant \overline{b'} \leqslant 2\overline{b}$.

Since $b < 2b'$, the relation $\overline{b'} = \overline{0}$ would imply that $\overline{b} = \overline{0}$, which is impossible. Next, $b < 2b - b'$ and thus $2\overline{b} = \overline{b'}$. Since B/K is linearly ordered and $\overline{0} \prec \overline{b} \prec 2\overline{b}$, we must have $\overline{b'} = \overline{b}$. Therefore we can take b' instead of b. Hence $b(i^2) = 0$ for each $i \in I$.

Put $I_1 = \{i \in I : b(i) \neq 0\}$. Thus $I_1 \neq \emptyset$. Let $g \in A$ be such that $g(i) \parallel 0$ for each $i \in I_1$ and $g(i) = 0$ otherwise. Then $-b < g < b$, hence $g \in B$. Next, $g \parallel 0$ and $g < (g \vee 0) - g$. If $\overline{g} > \overline{0}$, then $\overline{g} \leqslant (\overline{g} \vee \overline{0}) - \overline{g} = \overline{0}$, which is a contradiction. Similarly, the relation $\overline{g} < \overline{0}$ cannot be valid. Thus $\overline{g} = \overline{0}$. Hence $\overline{g \vee 0} = \overline{0}$.

Let $h \in A$ such that $h(i^1) = 1$, $h(i^2) = 0$ for each $i \in I_1$, and $h(i) = 0$ otherwise. Then $0 < h \leqslant b$, whence $h \in B$. There exists $g \in B$ with the properties as above such that $h = g \vee 0$. Therefore $\overline{h} = \overline{0}$.

Let b_1 and b_2 be elements of A such that

$b_1(i) = b(i)$ if $b(i^1) = 1$, and $b_1(i) = 0$ otherwise,

$b_2(i) = b(i)$ if $b(i^1) > 1$, and $b_2(i) = 0$ otherwise.

It is obvious that b_1 and b_2 belong to B. Next, $b = b_1 + b_2$ and $0 \leqslant b_1 \leqslant h$, whence $\overline{b_1} = \overline{0}$. Thus $\overline{b_2} = \overline{b}$.

There exists $c \in A$ such that

(i) $c(i^2) = 0$ for each $i \in I$;

(ii) if $b_2(i) = 0$, then $c(i^1) = 0$;

(iii) if $b_2(i) \neq 0$, then $c(i^1)$ is the least positive integer which is greater than or equal to $\frac{1}{2} b_2(i^1)$.

Then clearly $c \in B$. Now by analogous steps as in the proof of 2.1 (cf. the relations (a_1) and (b_1), $i = 1, 2$) we arrive at a contradiction. □

4.6. Corollary. *Let $\alpha \in M_i$. The linearly ordered group G^1 does not belong to Y_α.*

4.7. Corollary. *Let $\alpha \in M_i$. Then $\mathscr{A}$ fails to be a subclass of Y_α.*

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