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ON CONVOLUTIONS WITH THE MÖBIUS FUNCTION

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ABSTRACT. By using the results of [6], it is proved that for an extensive class of increasing functions h,

$$(*) \qquad \sum_{1 \leqq d \leqq x} \frac{\mu(d)}{d} h\left(\frac{x}{d}\right) \sim xh'(x) \quad \text{as } x \to \infty$$

where μ denotes the Möbius function. This result incidentally settles affirmatively Remark (iii) of [6], and refines the Tauberian Theorem 2 of that paper. It is also shown that one type of condition imposed in [6] is necessary to the truth of the cited Theorem 2, at least if some sort of quasi-Riemann hypothesis is true. Nevertheless, examples are given to show that on the one hand (*) may be true for functions not covered by the first theorem of this paper, and on the other that some sort of nonnaïve condition on a function h is necessary to ensure the truth of (*).

Much of this note, as will be evident, is in the nature of an interesting addendum to [6]; had I had the wit to notice it earlier, it should of course have been incorporated there.

Throughout, x denotes a real variable and $s=\sigma+it$ a complex variable, σ, t real. All sums begin at 1. Given a suitable function g, $\mathscr{I}$ will denote the operator defined by $\mathscr{I}g(y)=\int_1^y (g(x)/x)\,dx$, and $\mathscr{I}^r$ the rth iterate of $\mathscr{I}$. μ is the Möbius function and $N(x)=\sum_{n\leqq x}(\mu(n)/n)$. $\zeta(s)$ is the Riemann zeta-function, γ is Euler's constant. When $h(x)$ is constant, the right side of (*) is to be interpreted as equal to $o(1)$.

For convenience we state here the main results of [6] which will be used in the sequel:

THEOREM A. *Let $f(x)$ be any function bounded and integrable in every finite subinterval of* $[1, \infty)$ *which satisfies*

$$(1) \qquad \sum_{v \leqq x} f\left(\frac{x}{v}\right) = xg(x) + o(x^2 g'(x)),$$

where $g(x)$ is a positive, twice continuously differentiable function, defined

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on $[1, \infty)$ *such that*

(i) $g'(x)>0$ *for* $x \in (1, \infty)$,

(ii) $xg'(x)$ *is nonincreasing from some point on*,

(iii) *for some positive integer* k, $x(\log x)^k g'(x)=u(x)$ *is nondecreasing from some point on, and* $\liminf_{x\to\infty} u(x)=\infty$.

Then

$$\int_1^x \frac{f(t)}{t^2}\,dt = g(x) - \gamma \sum_{v\leq x} \frac{\mu(v)}{v}\, g\left(\frac{x}{v}\right) + o(xg'(x)) \tag{2}$$

as $x\to\infty$.

(This is Theorem 2 of [**6**].)

THEOREM B. *If f and g are as in Theorem* A, *then*

$$\int_1^x \frac{f(t)}{t^2 g'(t)}\,dt = x + o(x). \tag{3}$$

(This is equation (19) of [**6**]; if f is nondecreasing, one then easily deduces $f(x)\sim x^2 g'(x)$ as in Theorem 1 of [**6**].)

We now state

THEOREM 1. *Let* $h(x)$ *be a positive function which has the property that there exists a nonnegative integer* r *such that* $h \in C^{r+2}(1, \infty)$ *and* $h(x)=\mathscr{I}^r g(x)$ *where* g *satisfies* (i), (ii), (iii) *of Theorem* A; *then*

$$\sum_{d\leq x} \frac{\mu(d)}{d}\, h\left(\frac{x}{d}\right) \sim xh'(x) \quad as\ x \to \infty.$$

PROOF. By induction. For convenience, we define $g(x)=0$ for $x<1$. Suppose first the theorem were true for some k, $k\geqq 1$. Let $G(x)=\mathscr{I}^k g(x)$. Then (since $k\geqq 1$) G is clearly unbounded, and by hypothesis

$$\sum_{d\leq x} \frac{\mu(d)}{d}\, G\left(\frac{x}{d}\right) \sim xG'(x) \quad \text{as } x \to \infty,$$

and so

$$\sum_{d\leq x} \frac{\mu(d)}{d} \int_1^{x/d} \frac{G(t)}{t}\,dt = \int_1^x \frac{1}{t} \sum_{d\leq t} \frac{\mu(d)}{d}\, G\left(\frac{t}{d}\right) dt$$
$$= \int_1^x G'(t)(1 + o(1))\,dt = G(x) + o(G(x)),$$

which establishes the theorem for $r=k+1$. It remains to prove the theorem for $r=0$ and $r=1$. For $r=0$, $h(x)=\mathscr{I}^0 g(x)=g(x)$. Let

$$F(x) = \sum_{d\leq x} \frac{\mu(d)}{d}\, g\left(\frac{x}{d}\right).$$

Then

$$\sum_{d\leq x}\frac{x}{d}F\left(\frac{x}{d}\right) = xg(x)$$

and so, by Theorem A,

$$\int_1^x \frac{F(t)}{t}\,dt = g(x) - \gamma\sum_{v\leq x}\frac{\mu(v)}{v}\,g\left(\frac{x}{v}\right) + o(xg'(x)); \tag{4}$$

and, by Theorem B,

$$\int_1^x \frac{F(t)}{tg'(t)}\,dt = x + o(x). \tag{5}$$

(If $xF(x)$ were monotone, the conclusion, as remarked previously, would now follow almost immediately; unfortunately this need not be the case.)

Furthermore the hypotheses on g imply (see Lemma 1 of **[6]** for the easy deduction)

$$\lim_{x\to\infty}\frac{xg''(x)}{g'(x)} = -1 \quad\text{and also that}$$
$$\frac{xg''(x)}{g'(x)} = \frac{(d/dx)(xg'(x))}{g'(x)} - 1 \leqq -1, \tag{6}$$

from some point on. Hence $xg''(x)/g'(x)$ is eventually nondecreasing and so eventually has a nonnegative derivative, a.e.

Now, integrating by parts in (5), after using (4), gives

$$\begin{aligned}\int_1^x \frac{F(t)}{tg'(t)}dt &= \frac{1}{g'(x)}\int_1^x\frac{F(t)}{t}dt + \int_1^x \frac{g''(t)}{(g'(t))^2}\int_1^t\frac{F(u)}{u}\,du\,dt\\ &= \frac{g(x)}{g'(x)} - \frac{\gamma F(x)}{g'(x)} + o(x) + \int_1^x\frac{g''(t)}{(g'(t))^2}(g(t)-\gamma F(t)+o(tg'(t)))\,dt \qquad (7)\\ &= x - \frac{\gamma F(x)}{g'(x)} - \gamma\int_1^x\frac{F(t)g''(t)}{(g'(t))^2}dt + o(x)\end{aligned}$$

(cf. treatment of equation (15) in **[6]**).

And so (5) yields

$$\frac{F(x)}{g'(x)} = -\int_1^x \frac{F(t)g''(t)}{(g'(t))^2}\,dt + o(x). \tag{8}$$

But

$$\begin{aligned}\int_1^x \frac{F(t)g''(t)}{(g'(t))^2}\,dt &= \int_1^x \frac{F(t)}{tg'(t)}\,\frac{tg''(t)}{g'(t)}\,dt\\ &= \frac{xg''(x)}{g'(x)}\int_1^x\frac{F(t)}{tg'(t)}\,dt - \int_1^x \frac{d}{dt}\left(\frac{tg''(t)}{g'(t)}\right)\int_1^t\frac{F(u)}{ug'(u)}\,du\,dt + O(1),\end{aligned}$$

and on substituting this in (8), and using (6) and (5), we get

$$\text{(9)}\qquad \begin{aligned}\frac{F(x)}{g'(x)} &= x + o(x) + \int_1^x \frac{d}{dt}\left(\frac{tg''(t)}{g'(t)}\right)(t + o(t))\,dt \\ &= x + o(x) + O\left(\int_1^x t\,d\left(\frac{tg''(t)}{g'(t)}\right)\right).\end{aligned}$$

However, by (6),

$$\begin{aligned}\int_1^x t\,d\left(\frac{tg''(t)}{g'(t)}\right) &= \frac{x^2g''(x)}{g'(x)} - \int_1^x \frac{tg''(t)}{g'(t)}\,dt + O(1) \\ &= -x + o(x) + \int_1^x (1 + o(1))\,dt + O(1) = o(x),\end{aligned}$$

and substituting this in (9) gives

$$\text{(10)}\qquad F(x) \sim xg'(x)$$

as claimed. This proves the case $r=0$. The case $r=1$ now follows on substituting (10) in (4) to obtain

$$\text{(11)}\qquad \int_1^x \frac{F(t)}{t}\,dt = g(x) - \gamma xg'(x) + o(xg'(x)),$$

and noting that

$$\sum_{d\leqq x} \frac{\mu(d)}{d}\int_1^{x/d} \frac{g(t)}{t}\,dt = \int_1^x \frac{F(t)}{t}\,dt,$$

and that the conditions on g imply $xg'(x)=o(g(x))$ as $x\to\infty$.

REMARKS. (a) To prove the results of **[6]** used above, it was necessary to invoke a fairly strong form of the prime number theorem: $N(x)=o((\log x)^{-k})$ for every $k>0$. In the other direction, for k a positive integer, $(\log x)^k$ satisfies the conditions placed on $h(x)$ in Theorem 1 (with $g(x)=\log x$), and so we get, from Theorem 1,

$$\sum_{d\leqq x} \frac{\mu(d)}{d}\left(\log \frac{x}{d}\right)^k \sim k(\log x)^{k-1}$$

for every positive integer k (which is, of course, also deducible directly from $N(x)=o((\log x)^{-k})$ for every $k>0$).

(b) The conclusion of Theorem A can now be amended to read:

$$\text{(12)}\qquad \int_1^x \frac{f(t)}{t^2}\,dt = g(x) - \gamma xg'(x) + o(xg'(x)).$$

$g(x)$ was assumed for this theorem to be twice continuously differentiable.

In practice, however, g is usually analytic. Suppose we assume g is analytic; under what subsidiary conditions on g can (12) be replaced by a similar asymptotic expansion with a desired number of terms for

$$\int_1^x (F(t)/t^2)\,dt?$$

Condition (iii) of Theorem A seems somewhat unnatural; while it may be somewhat ameliorated by using stronger results from prime number theory than used in **[6]** (see Remark (iv) of that paper), the real question is whether any condition in addition to (i) and (ii) is necessary for the truth of Theorem A. Assuming a "quasi-Riemann hypothesis", the answer is "yes", as Theorem 2 below shows.

THEOREM 2. *Suppose some sort of quasi-Riemann hypothesis holds; i.e. suppose* $\zeta(s)\neq 0$ *for some strip* $b<\sigma\leqq 1$ $(b\geqq\frac{1}{2})$. *Then there is a function* $k(x)$ *satisfying* (i) *and* (ii) *of Theorem* A *which is analytic in* $(1, \infty)$ *and for which the conclusion of Theorem* A (*in the form of* (12)) *is false. A similar contradiction is even easier to obtain in Theorem* A'*s original form.*

PROOF. Let $a\in(b, 1)$. Define $k_a(x)=1-x^{a-1}$. Then $k_a(x)$ satisfies (i), (ii) as is easily checked. By the hypothesis on ζ we have that, as $x\to\infty$,

$$\int_1^x \frac{N(t)}{t}\,dt = -1 + o(x^{a-1}), \tag{13}$$

and

$$\int_1^x \frac{N(t)}{t^a}\,dt \text{ converges to } \frac{1}{(a-1)\zeta(a)}. \tag{14}$$

(These results arise from using partial summation on

$$\sum_{d\leqq x}\frac{\mu(d)\log d}{d} \quad\text{and}\quad \sum_{d\leqq x}\frac{\mu(d)}{d^a}=\sum_{d\leqq x}\frac{\mu(d)}{d}\,d^{1-a},$$

and taking note of results usually stated in the literature for $b=\frac{1}{2}$ (the Riemann hypothesis), but whose analogues for other values of b are immediate, see e.g. **[5]**, [7, p. 315].)

Let $f(x)=(1-a)x^a\int_1^x(N(t)/t^a)\,dt$. Then

$$\begin{aligned}\sum_{d\leqq x} f\left(\frac{x}{d}\right) &= (1-a)x^a\sum_{d\leqq x}\frac{1}{d^a}\int_1^{x/d}\frac{N(t)}{t^a}\,dt\\ &= (1-a)x^a\int_1^x u^{-a}\sum_{d\leqq u}\frac{1}{d}N\left(\frac{u}{d}\right)du = x(1-x^{a-1}) = xk_a(x),\end{aligned}$$

since $\sum_{d \leqq u} (1/d)N(u/d)=1$, and $N(x)=0$ for $x<1$. If the conclusion expressed by (12) held for all $k_a(x)$, then we would have

$$\int_1^x \frac{f(t)}{t^2}\,dt = (1-a)\int_1^x t^{a-2}\int_1^t \frac{N(u)}{u^a}\,du\,dt \tag{15}$$
$$= 1 - x^{a-1} - \gamma(1-a)x^{a-1} + o(x^{a-1}) \quad \text{for all } a,\ b < a < 1.$$

But integrating by parts in (15) gives

$$-x^{a-1}\int_1^x \frac{N(u)}{u^a}\,du - \int_1^x \frac{N(t)}{t}\,dt = 1 - (1+\gamma(1-a))x^{a-1} + o(x^{a-1})$$

and so by (13), as $x \to \infty$,

$$\int_1^x \frac{N(u)}{u^a}\,du = 1 + \gamma(1-a) + o(1) \quad \text{for all } a,\ \tfrac{1}{2} \leqq b < a < 1;$$

or by (14),

$$\frac{1}{(a-1)\zeta(a)} = 1 + \gamma(1-a) \quad \text{for all } a,\ \tfrac{1}{2} \leqq b < a < 1. \tag{16}$$

For a given fixed value of a, (16) might be shown false by *ad hoc* computation, however, this would not suffice to prove the theorem for any quasi-Riemann hypothesis; instead, we simply argue as follows: (16) would imply

$$\frac{\zeta(a) - (1/(a-1)) - \gamma}{a-1} = \frac{\gamma^2}{1-\gamma(a-1)}, \quad \text{for all } a,\ \tfrac{1}{2} \leqq b < a < 1. \tag{17}$$

Letting $a \to 1$ in (17) would give

$$\lim_{a \to 1^-} \frac{\zeta(a) - (1/(a-1)) - \gamma}{a-1} = \gamma^2.$$

But the limit on the left is known to equal

$$-\gamma_1 \overset{\text{def}}{=} -\lim_{n\to\infty}\left(\sum_{v=1}^{n} \frac{\log v}{v} - \tfrac{1}{2}(\log n)^2\right) = 0.07281+ \neq \gamma^2$$

which is the desired contradiction. (Two proofs of the form taken by the coefficients of the Taylor expansion of $\zeta(s)-1/(s-1)$ about 1 may be found in **[1]**; the result was known to Hardy in 1912, and is no doubt much older still. The computation of $-\gamma_1$ was made by Wilton **[8]** from a different expression for that constant.)

We close with two examples further illuminating the relationship (*).

EXAMPLE 1. There is a nondecreasing function h for which (ii) does not hold but (*) still does.

PROOF. Take $h(x)=\int_1^x ((1+N(t))/t)\,dt$. It is easy to see that $|N(x)|\leqq 1$ for all $x\geqq 1$ [**4**, p. 583] whence h is nondecreasing; clearly $xh'(x)=1+N(x)$ is not monotone decreasing. However,

$$\begin{aligned}\sum_{d\leqq x}\frac{\mu(d)}{d}h\left(\frac{x}{d}\right)\\ &=\sum_{d\leqq x}\frac{\mu(d)}{d}\log\left(\frac{x}{d}\right)+\sum_{d\leqq x}\frac{\mu(d)}{d}\int_1^{x/d}\frac{N(t)}{t}\,dt\\ &=1+o(1)-\sum_{d\leqq x}\frac{\mu(d)}{d}\left(\sum_{m\leqq x/d}\frac{\mu(m)\log m}{m}-\sum_{m\leqq x/d}\frac{\mu(m)}{m}\log\left(\frac{x}{d}\right)\right)\\ &=1+o(1)-2\sum_{d\leqq x}\frac{\mu(d)}{d}\sum_{m\leqq x/d}\frac{\mu(m)\log m}{m}+\log x\sum_{d\leqq x}\frac{\mu(d)}{d}\sum_{m\leqq x/d}\frac{\mu(m)}{m}\\ &=1+o(1),\end{aligned}$$

since $\sum_{d\leqq x}(\mu(d)\log d/d)=-1+o(1)$ (e.g. [**4**, §158]),

$$\sum_{d\leqq x}\frac{\mu(d)}{d}\sum_{m\leqq x/d}\frac{\mu(m)\log m}{m}=o(1),\quad\text{and}\quad\sum_{d\leqq x}\frac{\mu(d)}{d}\sum_{m\leqq x/d}\frac{\mu(m)}{m}=o\left(\frac{1}{\log x}\right)$$

as $x\to\infty$; these last two being easily proved by a technique going back to Dirichlet and embodied in the argument in [**4**, p. 685].

EXAMPLE 2. There is a function h such that $h(x)\sim C$ as $x\to\infty$, C a constant, but $\sum_{d\leqq x}(\mu(d)/d)h(x/d)\neq o(1)$ as $x\to\infty$.

PROOF. It is known that there are (I)-summable series which are not convergent; i.e. there is a sequence $\{a_n\}$ such that $\lim_{x\to\infty}(1/x)\sum_{n\leqq x}\sum_{d|n}da_d$ exists but $\sum a_n$ does not converge. (This is stated in [**3**]; a proof may be obtained by applying the Silverman-Toeplitz Theorem to the inverse of the (I)-transformation.) It is also known that (I)-summability implies $(C, 1)$-summability [**3**]. Let

$$I(x)=\frac{1}{x}\sum_{n\leqq x}\sum_{d|n}da_d=\frac{1}{x}\sum_{d\leqq x}\sum_{m\leqq x/d}ma_m$$

where $\{a_n\}$ is such a sequence. Then

$$\sum_{d\leqq x}\frac{\mu(d)}{d}I\left(\frac{x}{d}\right)=\frac{1}{x}\sum_{m\leqq x}ma_m,$$

$\lim_{x\to\infty} I(x)$ exists, $\sum a_n$ is $(C, 1)$-summable but not convergent, and so $\sum_{m\leqq x} ma_m \neq o(x)$ as $x\to\infty$ (e.g. [**2**, Theorem 65]).

REMARK. No explicit example of such a sequence $\{a_n\}$ seems to be known.

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