ON COORDINATES IN MODULAR LATTICES WITH A HOMOGENEOUS BASIS

BY
BENNO ARTMANN

Introduction

In his lectures about continuous geometry, [13], John von Neumann gave a powerful extension of the so-called coordinatization theorem for projective spaces, which asserts that for every projective space P^n of dimension $n \geqq 3$, there exists a (skew-) field K, such that the lattice of subspaces of P^n is isomorphic to the lattice of subspaces of the (right-) vector space K^{n+1} (see, e.g., [5]). He assumed the existence of a homogeneous basis of order $n \geqq 4$ (defined below) in a complemented modular lattice L and showed that there exists a regular ring R, such that L is isomorphic to the lattice of finitely generated right-submodules of the R-right-module R^n. In the case of a homogeneous basis of order 3, projective planes are involved in the discussion (see, among others, [7] and [13].).

This paper deals with related questions for modular lattices with a homogeneous basis. Let L be a modular lattice with least element N and greatest element U. Then a family $a_1, \cdots, a_n$ of elements of L will be called a homogeneous basis of order n of L if the following conditions are satisfied:

(i) $a_1 \cup \cdots \cup a_n = U$,

(ii) $(a_1 \cup \cdots \cup a_i) \cap a_{i+1} = N$ for all $1 \leq i < n$,

(iii) there exists a common complement c_{ij} for a_i and a_j in $a_i \cup a_j$ for all $1 \leq i, j \leq n$, $i \neq j$.

The c_{ij} may be arranged in a certain way (see 1.6); we then speak of a normalized frame of order n for L and write L_n for L together with this frame. Examples of such lattices are easily given, namely, the lattice $L(R^n)$ of all right submodules of the free module R^n (where R is any associative ring with unit) is a modular lattice with the homogeneous basis

$$a_1 = (1, 0, \cdots, 0)R, \cdots, a_n = (0, \cdots, 0, 1)R.$$

The obvious question now is whether this is the general situation, or in other words, whether, for a given lattice L_n, there exists a ring R such that L_n is isomorphic to a sublattice of $L(R^n)$. The non-complemented case was first discussed by Baer [4] and Inaba [11], who both assumed that the sublattices $L(N, a_i)$ were finite chains. We know that the cases $n = 3, 4$ are typical for the situation in complemented modular lattices, and therefore, restrict our attention to these two cases.

In the first section, we deal with the construction of a normalized frame

Received April 17, 1967.

starting from a given homogeneous basis, and, in particular, prove Lemma 1.5 which is an analogon to the theorem of Desargues in a projective space. The next section is devoted to the construction of a ternary ring $K(L_3)$ from a lattice L_3 which uses the methods developed for projective planes [8], [15] and gives rise to the general definition of a ternary ring (Def. 2.5). This concept is different from another generalization of ternary fields given by Sandler [16]. A pseudo ternary as defined in [16] seems to be a ternary ring in our sense only if it is a ternary field. In Proposition 5.7 we show that the ternary ring K derived from the lattice $L(K^3)$ from the above example is isomorphic to R with the ternary operation $(u, x, v) \to ux + v$, hence essentially the same as R. In general the ring-axioms for K are not valid, even if $T(u, x, v) = ux + v$, as examples of projective planes show. Therefore in the third section we derive several algebraic properties of $K(L_3)$ from assumptions about the automorphism group of L_3. This method was first introduced by Baer [3] in the theory of projective planes and was used in the case of complemented modular lattices by Amemiya [1] and the author [2], see also Skornyakov [16]. From the existence of a sufficient number of automorphisms of a special type (Def. 3.1) we obtain all the ring axioms except the associativity of multiplication. This law, and the existence of all the automorphisms needed for the other ones, are proven in Section 4 under the assumption that L_3 is embedded in a lattice L_4 in a way that the frame of L_3 consists of elements of the frame of L_4 (see Def. 1.6 andTheorem 4.5). Following Cronhe im [6], in the last section we define a "parallel system" P consisting of a set A of "points", a set B of "lines", an incidence relation $| \subseteq A \times B$ and two equivalence relations $\|_A \subseteq A \times A$, $\|_B \subseteq B \times B$ subject to three axioms stated in 5.3. It turns out that parallel systems may be derived from either a lattice L_3 or a ternary ring K, called $P(L_3)$ and $P(K)$ respectively. In the first case, A is the set of all complements of $a_2 \cup a_3$ and B the set of all complements of a_3. $P(L_3)$ and $P(K)$ are isomorphic if K is the ternary ring defined from L_3 (introduction of coordinates in $P(L_3)$, Theorem 5.6). If K is a ring, then we may embed the paralled system $P(L_3)(\cong P(K))$ into the lattice $L(K^4)$, (Theorem 5.8), which is a partial converse of Theorem 4.5. Theorem 4.5 and Theorem 5.8 together contain the theorem that a projective plane is embeddable in a projective space of higher dimension if and only if the theorem of Desargues holds in the plane (Hilbert [10]).

Examples of lattices L_3 which are not complemented and not embeddable in the defined way in a lattice L_4 can be obtained from Hjelmslev planes and will be given in another paper. Also, there are results about relations between the multiplicative structure of $K(L_3)$ and the lattice $L(N, a_3)$ which will be discussed later.

0. Notations

0.1. We deal with modular lattices with a least and a greatest element exclusively. The least element of the lattice L is denoted by N, the greatest by U. $L(N, a)$ is the sublattice of elements $\leq a$ of L. $a \cup^\cdot b = c$ means $a \cup b = c$ and $a \cap b = N$. In order to save brackets, we write $a \cup b \cap c$ instead of $a \cup (b \cap c)$, that is, $\cap$ binds closer than $\cup$. The modular law is then written:

$$a \leq c \Rightarrow a \cup b \cap c = (a \cup b) \cap c.$$

If $a \cup^\cdot c = b \cup^\cdot c$ $(a, b, c \in L)$, we say a and b are perspective and c is the center of perspectivity. We call the mapping $\pi: x \rightarrow (c \cup x) \cap b$ for $x \leq a$ a projection with center c of $L(N, a)$ into $L(N, b)$. In modular lattices, projections are isomorphisms [13, p. 18].

"Ring" means always associative ring with unit.

0.2. *Lemma. Let L be a modular lattice, $A \in L$, r, $s \leq A$, $r \cap s = N$ and p, q two complements of A such that $r \cup s \cup p \geq q$. Then $z = (r \cup p) \cap (s \cup q)$ is a complement of A.*

The proof is a simple check:

$$\begin{aligned} z \cup A &= (r \cup p) \cap (s \cup q) \cup A \cup s \qquad (\text{as } s \leq A) \\ &= (r \cup s \cup p) \cap (s \cup q) \cup A \\ &= s \cup q \cup A = U \\ z \cap A &= (r \cup p) \cap (s \cup q \cap A) \\ &= (r \cup p) \cap s \\ &= (r \cup p) \cap A \cap s \\ &= r \cap s = N. \end{aligned}$$

In the most cases where we apply this Lemma we will have $A = r \cup s$, hence $r \cup s \cup p \geq q$ trivially.

1. The normalized frame of a modular lattice

1.1. DEFINITION. Let L be a modular lattice with least element N and greatest element U. A set of elements $a_1, \cdots, a_n \in L$ is said to be a homogeneous basis of order n of L, if the following conditions hold.

(I) $\bigcup_{i=1}^{n} a_i = U$.

(II) $(\bigcup_{i=1}^{k} a_i) \cap a_{k+1} = N$ for all $1 \leq k < n$.

(III) For each j, $1 < j \leq n$, there exists c_{1j} such that $a_1 \cup^\cdot c_{1j} = a_j \cup^\cdot c_{1j}$.

We note that according to [13, p. 9] (II) implies the independence of the a_i, e.g. $(\bigcup_{i \in I} a_i) \cap (\bigcup_{j \in J} a_j) = N$ for all disjoint subsets I, J of $\{1, \cdots, n\}$.

1.2. *Duality.* Let $a_1, \cdots, a_n$ be a homogeneous basis of L. If we define

$$A_j = \bigcup_{i=1, i \neq j}^{n} a_i, \qquad C_{1j} = c_{1j} \cup \bigcup_{i=2, i \neq j}^{n} a_i,$$

then from

(a) $\bigcap_{i=1}^{n} A_i = N$,

$$\text{(b)}\quad (\bigcap_{i=1}^{k} A_i) \cup A_{k+1} = (\bigcup_{i=k+1}^{n} a_i) \cup A_{k+1} = U,$$

$$\begin{aligned}
\text{(c)}\quad A_1 \cap C_{1j} &= (\bigcup_{i=2}^{n} a_i) \cap (c_{1j} \cup \bigcup_{i=2, i\neq j}^{n} a_i) \\
&= (\bigcup_{i=2}^{n} a_i) \cap c_{1j} \cup \bigcup_{i=2, i\neq j}^{n} a_i \\
&= (\bigcup_{i=2}^{n} a_i) \cap (a_1 \cup a_j) \cap c_{1j} \cup \bigcup_{i=2, i\neq j}^{n} a_i \\
&= ((\bigcup_{i=2}^{n} a_i) \cap a_1 \cup a_j) \cap c_{1j} \cup \bigcup_{i=2, i\neq j}^{n} a_i \\
&\qquad\qquad \text{(because of } 1 < j \text{ we have } a_j \leq \textstyle\bigcup_{i=2}^{n} a_i) \\
&= a_j \cap c_{1j} \cup \bigcup_{i=2, i\neq j}^{n} a_i \\
&\qquad\qquad \text{(because of the independence of the } a_i) \\
&= \bigcup_{i=2, i\neq j}^{n} a_i \\
&= a_1 \cap c_{1j} \cup \bigcup_{i=2, i\neq j}^{n} a_i \\
&= ((\bigcup_{i=1, i\neq j}^{n} a_i) \cap a_j \cup a_1) \cap c_{1j} \cup \bigcup_{i=2, i\neq j}^{n} a_i \\
&= (\bigcup_{i=1, i\neq j}^{n} a_i) \cap (a_1 \cup a_j) \cap c_{1j} \cup \bigcup_{i=2, i\neq j}^{n} a_i \\
&= (\bigcup_{i=1, i\neq j}^{n} a_i) \cap (c_{1j} \cup \bigcup_{i=2, i\neq j}^{n} a_i) \\
&= A_j \cap C_{1j} \\
A_1 \cup C_{1j} &= U = A_j \cup C_{1j}
\end{aligned}$$

we see that the A_i form a homogeneous basis of order n of the lattice $\tilde{L}$ dual to L. Therefore the concept of a modular lattice with a homogeneous basis of order n is self dual.

Starting from condition (III), we may construct common complements c_{ij} $(i \neq j)$ of a_i and a_j in $a_i \cup a_j$ for any pairs (i, j), which are connected in a special way:

1.3. Proposition. *Let $a_1, \cdots, a_n$ be a homogeneous basis for L. Then*

(IV) *for each pair (i, j) with $i \neq j$, there exists a common complement c_{ij} of a_i and a_j in $a_i \cup a_j$, such that*

$$c_{ij} = c_{ji} \quad \textit{and} \quad c_{ij} = (c_{ki} \cup c_{kj}) \cap (a_i \cup a_j)$$

for all distinct i, j, k, $1 \leq i, j, k \leq n$.

The proof of this proposition is given after two lemmas following the pattern of [13, p. 117–119].

We define L_{ik} to be the set of complements of a_k in $a_i \cup a_k$.

1.4. Lemma. *If $i \neq j \neq k \neq i$, $b_{ij} \in L_{ij}$, $b_{jk} \in L_{jk}$, then*

$$b_{ik} = (b_{ij} \cup b_{jk}) \cap (a_i \cup a_k) \in L_{ik}.$$

Proof.

$$\begin{aligned} b_{ik} \cup a_k &= (b_{ij} \cup b_{jk} \cup a_k) \cap (a_i \cup a_k) \\ &= (b_{ij} \cup a_j \cup a_k) \cap (a_i \cup a_k) \\ &= (a_i \cup a_j \cup a_k) \cap (a_i \cup a_k) \\ &= a_i \cup a_k \\ b_{ik} \cap a_k &= (b_{ij} \cup b_{jk}) \cap a_k \\ &= (b_{ij} \cup b_{jk}) \cap (a_j \cup a_k) \cap a_k \\ &= (b_{jk} \cup b_{ij} \cap (a_j \cup a_k)) \cap a_k \\ &= (b_{jk} \cup b_{ij} \cap (a_i \cup a_j) \cap (a_j \cup a_k)) \cap a_k \\ &= (b_{jk} \cup b_{ij} \cap a_j) \cap a_k \\ &= b_{jk} \cap a_k \\ &= N. \end{aligned}$$

1.5 LEMMA. *For distinct integers i, j, k, m and $b_{ij} \in L_{ij}$, $b_{jk} \in L_{jk}$, $b_{km} \in L_{km}$, the following equation holds:*

$$((b_{ij} \cup b_{jk}) \cap (a_i \cup a_k) \cup b_{km}) \cap (a_i \cup a_m)$$
$$= (b_{ij} \cup (b_{jk} \cup b_{km}) \cap (a_j \cup a_m)) \cap (a_i \cup a_m).$$

Proof.

$$\begin{aligned} &((b_{ij} \cup b_{jk}) \cap (a_i \cup a_k) \cup b_{km}) \cap (a_i \cup a_m) \\ &\quad = ((b_{ij} \cup b_{jk}) \cap (a_i \cup a_k \cup a_m) \cup b_{km}) \cap (a_i \cup a_m) \\ &\qquad \text{as } a_i \cup a_k = (a_i \cup a_k \cup a_m) \cap (a_i \cup a_k \cup a_j) \\ &\qquad \text{because } j \neq m, \quad \text{and} \quad b_{ij} \cup b_{jk} \leq a_i \cup a_k \cup a_j \\ &\quad = (b_{ij} \cup b_{jk} \cup b_{km}) \cap (a_i \cup a_m) \\ &\quad = (b_{ij} \cup (b_{jk} \cup b_{km}) \cap (a_j \cup a_m \cup a_i)) \cap (a_i \cup a_m) \\ &\quad = (b_{ij} \cup (b_{jk} \cup b_{km}) \cap (a_j \cup a_m)) \cap (a_i \cup a_m). \end{aligned}$$

Remark. In the case of a projective space, a_r, b_{st} points, this lemma describes a Desargues configuration with center b_{jk} and axis $a_i \cup a_m$.

Now we are ready to prove the assertion (IV) of Prop. 1.3. We may define $c_{j1} = c_{1j}$, because $a_1 \cup^{\cdot} c_{1j} = a_j \cup^{\cdot} c_{1j} = a_1 \cup a_j$, and as $c_{1j} \in L_{1j}$ and $c_{1j} \in L_{j1}$, we get from Lemma 1.4:

$$(c_{1i} \cup c_{1k}) \cap (a_i \cup a_k) \in L_{ik} \quad \text{and} \quad \in L_{ki}.$$

Therefore we define

$$c_{ik} = (c_{1i} \cup c_{1k}) \cap (a_i \cup a_k) = c_{ki} \qquad \text{for all} \quad i, k.$$

To establish the second property of the c_{ij}, we see that for $k = 1$ this holds by the definition of c_{ij}. So let $k \neq 1$. If $i = 1$, then $j \neq k, 1$:

$$\begin{aligned}(c_{1k} \cup c_{kj}) \cap (a_1 \cup a_j) &= (c_{1k} \cup (c_{1k} \cup c_{1j}) \cap (a_k \cup a_j)) \cap (a_1 \cup a_j)\\ &= (c_{1k} \cup c_{1j}) \cap (c_{1k} \cup a_k \cup a_j) \cap (a_1 \cup a_j)\\ &= (c_{1k} \cup c_{1j}) \cap (a_1 \cup a_j)\\ &= c_{1j},\end{aligned}$$

and the same reasoning holds for $j = 1$, $i \neq k, 1$. So we may assume $1 \neq i, j, k$, hence $1, i, j, k$ are all distinct.

$$\begin{aligned}(c_{ik} \cup c_{kj}) \cap (a_i \cup a_j) &= (c_{ik} \cup (c_{1k} \cap c_{1j}) \cap (a_k \cup a_j)) \cap (a_i \cup a_j)\\ &= ((c_{ik} \cup c_{1k}) \cap (a_i \cup a_1) \cup c_{1j}) \cap (a_i \cup a_j), && \text{by Lemma 1.5,}\\ &= (c_{1i} \cup c_{1j}) \cap (a_i \cup a_j), && \text{by first case,}\\ &= c_{ij}, && \text{by definition.}\end{aligned}$$

1.6. Definition. As we have $c_{ij} = c_{ji}$, we may restrict the indices by the condition $i < j$ and make the following definition:

The family $(a_1, \cdots, a_n, c_{12}, \cdots, c_{n-1,n})$ of elements of L is called a normalized frame of order n of L, if the conditions (I), (II) from 1.1 and (IV) from 1.3 hold for the a_i, c_{ij} $(i, j = 1, \cdots, n; i < j)$.

Throughout this paper, we deal with the cases $n = 3$ or $n = 4$, and the notation L_n $(n = 3, 4)$ stands for a modular lattice L together with a fixed normalized frame of order n. By saying L_3 is a sublattice of L_4 (or L_3 is embedded in L_4), we mean that L_3 is the sublattice

$$L(N, a_1 \cup a_2 \cup a_3)$$

of L_4, where $(a_1, a_2, a_3, a_4, c_{12}, \cdots, c_{34})$ is the fixed normalized frame of L_4 and the frame of L_3 is $(a_1, a_2, a_3, c_{12}, c_{13}, c_{23})$, consisting of elements of the frame of L_4.

1.7. *Remark.* Let the elements $a_1, \cdots, a_n$ be atoms of the lattice L. Then, by [14, p. 78], L is a complemented modular lattice of finite length (or dimension), and any atom of L is perspective to at least one of the a_i. Now, the a_i are all perspective by centers c_{ij}, and as for any other atom $p \in L$ either $p = a_i$ or $p \cap a_i = N$, p is perspective to any of the a_i. Hence L is irreducible [14, p. 80], and is, therefore, isomorphic to the lattice of subspaces of a finite-dimensional projective space.

2. Construction of a tenary ring from a lattice L_3

2.1. Let $(a_1, a_2, a_3, c_{12}, c_{13}, c_{23})$ be the normalized frame of L_3. In order to get shorter formulas in the following calculations, we introduce the abbreviations

$$a_j \cup a_k = A_i \quad (j, k \neq i)$$

$$a_3 \cup c_{12} = E, \qquad a_2 \cup c_{13} = F, \qquad a_1 \cup E \cap F = D,$$

$$d = D \cap A_1 .$$

DEFINITION. For any $x \leq A_2$, let

$$x^* = ((x \cup a_2) \cap E \cup a_1) \cap A_1 ,$$

$$\bar{x} = ((x \cup a_2) \cap D \cup a_3) \cap A_3 .$$

2.2. LEMMA. *The mappings $x \to x^*$, $x \to \bar{x}$ are lattice isomorphisms of $L(N, A_2)$ onto $L(N, A_1)$ and of $L(N, A_2)$ onto $L(N, A_3)$, respectively.*

$$x^* = x \quad \textit{for all} \quad x \leq a_3 , \qquad \bar{x} = x \quad \textit{for all} \quad x \leq a_1 .$$

Proof. These mappings are isomorphisms as they are products of projections.

For $x \leq a_3$ we have

$$\begin{aligned} x^* &= ((x \cup a_2) \cap (a_3 \cup c_{12}) \cup a_1) \cap A_1 \\ &= (a_2 \cap (a_3 \cup c_{12}) \cup x \cup a_1) \cap A_1 \\ &= (x \cup a_1) \cap A_1 \\ &= x \cup a_1 \cap A_1 \\ &= x. \end{aligned}$$

For $x \leq a_1$, $\bar{x} = x$ follows similarly.

2.3. DEFINITION. Let $L_{13} = \{x \mid x \cup^{\cdot} a_3 = a_1 \cup^{\cdot} a_3\}$. We define a mapping T of $L_{13} \times L_{13} \times L_{13}$ into $L(N, A_2)$ by

$$T(u, x, v) = ((u^* \cup v) \cap (\bar{x} \cup a_3) \cup a_2) \cap A_2 \qquad \text{for } u, x, v \in L_{13} .$$

2.4. PROPOSITION. (a) *T is a ternary operation in L_{13}, i.e. $T(u, x, v) \in L_{13}$ for all $u, x, v \in L_{13}$.*

(b) (i) *$T(a_1, x, v) = v = T(x, a_1, v)$ for all $x, v \in L_{13}$.*

(ii) *$T(c_{13}, x, a_1) = x = T(x, c_{13}, a_1)$ for all $x \in L_{13}$.*

(iii) *For any $u, x, y \in L_{13}$, there exists a unique $v \in L_{13}$ such that $y = T(u, x, v)$.*

Proof. (a) We first note

$$\begin{aligned} a_1^* &= ((a_1 \cup a_2) \cap E \cup a_1) \cap A_1 \\ &= (a_1 \cup a_2) \cap A_1 \\ &= a_2 , \text{ and} \end{aligned}$$

$$\bar{a}_1 = a_1 \qquad \text{(Lemma 2.2).}$$

Because the mappings $w \to w^*$, $w \to \bar{w}$ are isomorphisms, we have $u^* \cup' a_3 = A_1$ and $\bar{x} \cup' a_2 = A_3$ for $u, x \in L_{13}$. Applying Lemma 0.2 with $A_1 = A$, $u^* = r$, $a_3 = s$, $v = p$ and $\bar{x} = q$ we see that $(u^* \cup v) \cap (\bar{x} \cup a_3)$ is a complement of A_1. Applying Lemma 0.2 a second time with $A_1 = A$, $a_3 = r$, $a_2 = s$, $a_1 = p$ and $(u^* \cup v) \cap (\bar{x} \cup a_3) = q$ shows that $T(u, x, v)$ is a complement of A_1. Furthermore by definition we have $T(u, x, v) \leq A_2$, hence $T(u, x, v) \in L_{13}$.

(b) (i)
$$\begin{aligned} T(a_1, x, v) &= ((a_1^* \cup v) \cap (\bar{x} \cup a_3) \cup a_2) \cap A_2 \\ &= ((a_2 \cup v) \cap (\bar{x} \cup a_3) \cup a_2) \cap A_2, \\ &\qquad\qquad \text{since } a_1^* = a_2 \text{ (see (a))}, \\ &= (a_2 \cap v) \cap (\bar{x} \cup a_3 \cup a_2) \cap A_2 \\ &= (a_2 \cup v) \cap A_2, \quad \text{as } \bar{x} \cup a_2 = a_1 \cup a_2, \text{ (see (a))} \\ &= v. \end{aligned}$$

and similarly $T(x, a_1, v) = v$ observing that $\bar{a}_1 = a_1$.

(ii)
$$\begin{aligned} T(c_{13}, x, a_1) &= ((c_{13}^* \cup a_1) \cap (\bar{x} \cup a_3) \cup a_2) \cap A_2 \\ &= ((D \cap A_1 \cup a_1) \cap (\bar{x} \cup a_3) \cup a_2) \cap A_2 \\ &= (D \cap (A_1 \cup a_1) \cap (\bar{x} \cup a_3) \cup a_2) \cap A_2, \quad \text{as } a_1 \leq D, \\ &= (D \cap (\bar{x} \cup a_3) \cup a_2) \cap A_2 \\ &= ((D \cap (((x \cup a_2) \cap D \cup a_3) \cap A_3 \cup a_3) \cup a_2) \cap A_2 \\ &= (D \cap ((x \cup a_2) \cap D \cup a_3) \cup a_2) \cap A_2 \\ &= ((x \cup a_2) \cap D \cup a_2) \cap A_2 \\ &= (x \cup a_2) \cap (D \cup a_2) \cap A_2 \\ &= (x \cup a_2) \cap A_2 \\ &= x, \end{aligned}$$

$$\begin{aligned} T(x, c_{13}, a_1) &= ((x^* \cup a_1) \cap (\bar{c}_{13} \cup a_3) \cap a_2) \cap A_2 \\ &= ((x^* \cup a_1) \cap (((c_{13} \cup a_2) \cap D \cup a_3) \cap A_3 \cup a_3) \cup a_2) \cap A_2 \\ &= ((x^* \cup a_1) \cap ((c_{13} \cup a_2) \cap D \cup a_3) \cup a_2) \cap A_2 \\ &= ((x^* \cup a_1) \cap E \cup a_2) \cap A_2 \\ &= ((((x \cup a_2) \cap E \cup a_1) \cap A_1 \cup a_1) \cap E \cup a_2) \cap A_2 \\ &= (((x \cup a_2) \cap E \cup a_1) \cap E \cup a_2) \cap A_2 \\ &= ((x \cup a_2) \cap E \cup a_2) \cap A_2 \\ &= (x \cup a_2) \cap (E \cup a_2) \cap A_2 \\ &= x. \end{aligned}$$

(iii) If there is any $v \in L_{13}$ with $y = T(u, x, v)$, then we may calculate (each line implies the next one):

$$y \cup a_2 = (u^* \cup v) \cap (\bar{x} \cup a_3) \cup a_2$$

$$(y \cup a_2) \cap (\bar{x} \cup a_3) = (u^* \cup v) \cap (\bar{x} \cup a_3)$$

$$(y \cup a_2) \cap (\bar{x} \cup a_3) \cup u^* = u^* \cup v$$

$$((y \cup a_2) \cap (\bar{x} \cup a_3) \cup u^*) \cap A_2 = v,$$

and hence v is unique.

On the other hand, if we define v by the last equation, then from Lemma 0.2 we get $v \in L_{13}$ similar to (a), and

$$\begin{aligned} T(u, x, v) &= ((u^* \cup (y \cup a_2) \cap (\bar{x} \cup a_3)) \cap (\bar{x} \cup a_3) \cup a_2) \cap A_2 \\ &= (u^* \cap (\bar{x} \cup a_3) \cup (y \cup a_2) \cap (\bar{x} \cup a_3) \cup a_2) \cap A_2 \\ &= (y \cup a_2) \cap A_2 \\ &= y, \end{aligned}$$

so v is a solution of the given equation.

2.5. The last proposition leads to the following

DEFINITION. An algebraic system $(K, T, 0, 1)$, where K is a set, T a ternary operation on K and 0, 1 are two distinct elements of K, is called a ternary ring, if the following axioms are satisfied:

(T 0) $\quad T(0, x, v) = v = T(x, 0, v) \quad$ for all $x, v \in K$.

(T 1) $\quad T(1, x, 0) = x = T(x, 1, 0) \quad$ for all $x \in K$.

(T 2) Given any $u, x \in K$. Then the mapping $v \to T(u, x, v)$ is bijective from K into K.

Because of (T 0), (T 1), 0 is called the zero element of K and 1 is called the unit element of K. Prop. 2.4 shows, that (L_{13}, T, a_1, c_{13}) with the operation T defined in 2.3, is a ternary ring. We denote this ring by $K(L_3)$ and call it the ternary ring of L_3. In general, $K(L_3)$ depends on both L and the normalized frame chosen for L. From (T 0), (T 1) we feel free to write $a_1 = 0$, $c_{13} = 1$. In connection with the lattice operations, however, we keep the old notations a_1, c_{13}.

2.6. *Other examples of ternary rings.* (a) Let R be a ring, and define $T(u, x, v) = ux + v$ for $u, x, v \in R$. (T 0), (T 1), (T 2) are easy consequences of the ring axioms. In Theorem 4.5 we show that the ternary ring of a lattice L_3 which is embeddable in a lattice L_4 is of this type. Furthermore, if R is a ring and L_3 happens to be the lattice $L(R^3)$ of all right-submodules of R^3 with the normalized frame $((1, 0, 0)R, (0, 1, 0)R, (0, 0, 1)R, (1, 1, 0)R,$

$(1, 0, 1)R$, $(0, 1, -1)R)$, then it is proved in Prop. 5.7 that

$$(R, T) \cong K(L(R^3)).$$

(b) Let K be a ternary field as defined in [15, p. 36]. Then K is a ternary ring. (In fact, here a projective plane serves as L_3.)

(c) The cartesian product of a set of ternary rings $K_\lambda (\lambda \in \Lambda)$ becomes a ternary ring if we define the operation T component-wise. The sequences (0_λ) and (1_λ) are zero respectively unit element of $\prod_{\lambda \in \Lambda} K_\lambda$.

(d) From a ternary ring (K, T) we may derive a ternary ring of matrices (K_2, T_2) by defining

$$K_2 = \left\{ \binom{ab}{cd} \,\middle|\, a, b, c, d \in K \right\},$$

$$T_2\left(\binom{a,b}{c,d}, \binom{p,q}{r,s}, \binom{v,w}{x,y}\right) = \binom{T(a,p,T(b,r,v)),T(a,q,T(b,s,w))}{T(c,p,T(d,r,x)),T(c,q,T(d,s,y))}.$$

It is easy to see that $\binom{00}{00}$ and $\binom{10}{01}$ are zero and one of K_2. We check $(T2)$: Let $A = \binom{ab}{cd}$ and $P = \binom{pq}{rs}$ are given. For a given matrix $H = \binom{hi}{jk}$ we show that there is one and only one matrix $V = \binom{vw}{xy}$ such that

$$T_2(A, P, V) = H.$$

From the property $(T2)$ of the ternary ring (K, T) we get the existence of a unique $z \in K$ with

$$T(a, p, z) = h,$$

and again by the same axiom a unique $v \in K$ such that

$$T(b, r, v) = z.$$

Hence there is a unique v such that

$$T(a, p, T(b, r, v)) = u.$$

In the same manner the existence and uniqueness of the other entries of the matrix V are shown.

(e) From the preceding example, we may take the subring of matrices $\binom{ab}{0a}$.

We leave now the general concept of a ternary ring until Section 5 and return to the ring $K(L_3)$ derived from the lattice L_3, which we denote by K in this and the following two sections.

2.7. As it is suggested by example (a), we try to split the operation T in K into addition and multiplication.

Fixing the first variable in $T(u, x, v)$ to $c_{13} = 1$, we define an addition by

$$x + v = T(1, x, v) = ((d \cup v) \cap (\bar{x} \cup a_3) \cup a_2) \cap A_2 .$$

(See 2.1 for the definition of d.)

2.8 PROPOSITION. *$(K, +)$ is a loop with neutral element* $0 = a_1$.

Proof. From Prop. 2.4(a) we get that K is additively closed, the parts

(i) and (ii), of the same Prop. show that 0 is neutral with respect to $+$. From part (iii) we get a unique solution v of the equation $x + v = y$. For given v, y, the element

$$((y \cup a_2) \cap (v \cup d) \cup a_3) \cap D \cup a_2) \cap A_2 \in K$$

is a unique solution for $x + v = y$ (proof as in Prop. 2.4(iii)).

2.9. Multiplication in K is defined by

$$ux = T(u, x, 0) = ((u^* \cup a_1) \cap (\bar{x} \cup a_3) \cup a_2) \cap A_2 .$$

2.10. *Remark.* Similar to 2.8, one can prove the proposition: Let K^x be the set of common complements of a_1 and a_3 in $a_1 \cup a_3$. Then $(K^x, \cdot)$ is a loop with neutral element $1 = c_{13}$. We leave out the proof as we do not need this proposition during the rest of this paper.

3. Automorphisms of L_3 and properties of $K(L_3)$

3.1. DEFINITION. Let L be a modular lattice, a, $A \in L$ and ϕ an automorphism of L such that

$$x \geq a \quad \text{implies} \quad x^\phi = x \quad \text{and} \quad x \leq A \quad \text{implies} \quad x^\phi = x.$$

We call ϕ an (a, A)-automorphism of L and say a is a center and A an axis of ϕ. (This notation is different from the notation used in [1], [17, p. 17], where (a, A)-automorphisms with $a \leq A$ are considered, for us, too, the only important case. Our notation is in line with [15].) The group of all (a, A)-automorphisms will be denoted by $G(a, A)$. We say L is (a, A)-transitive, if for any pair p, q of complements of A with $p \cap a = q \cap a = N$, $p \cup a = q \cup a$, there exists a $\phi \in G(a, A)$ with $p^\phi = q$.

3.2. LEMMA. *Let $a \leqq A, A = a \cup' b$ and $\phi \in G(a, A)$. If there exist complements p, q of A such that $q \cap (p \cup a) = N$, then for any x comparable with a complement of A, x^ϕ is determined by p^ϕ.*

Proof. First let r be a complement of A such that $r \leq q \cup a$. This implies

$$r \cap (p \cup a) = r \cap (q \cup a) \cap (p \cup a) = r \cap a = N$$

and

$$((r \cup p) \cap A \cup p) \cap (q \cup a) = (r \cup p) \cap (q \cup a) = r.$$

So we have

$$\begin{aligned} r^\phi &= ((r \cup p) \cap A \cup p)^\phi \cap (q \cup a)^\phi \\ &= ((r \cup p) \cap A)^\phi \cup p^\phi) \cap (q \cup a)^\phi \\ &= ((r \cup p) \cap A \cup p^\phi) \cap (q \cup a), \end{aligned}$$

as $(r \cup p) \cap A$ and $q \cup a$ are fixed by ϕ, hence r^ϕ is determined by p^ϕ.

Now let t be any complement of A. By Lemma 0.2, $(b \cup t) \cap (a \cup q) = s$

is a complement of A, and as $s \leq a \cup q$, $t = (b \cup s) \cap (a \cup t)$, we have $t^{\phi} = (b \cup s^{\phi}) \cap (a \cup t)$, hence t^{ϕ} is determined by p^{ϕ}.

For any $x \geq t$, $x = t \cup x \cap A$, hence $x^{\phi} = t^{\phi} \cup x \cap A$, and for $x \leq t$, $x = (x \cup a) \cap t$, hence $x^{\phi} = (x \cup a) \cap t^{\phi}$.

3.3. Lemma. *Let $a \leq A$, $A = a \cup' b$ and r be a fixed complement of A. If, for any complement s of A with $s \leq a \cup r$, there exists a $\phi \in G(a, A)$ with $r^{\phi} = s$, then L is (a, A)-transitive.*

Proof. The proof uses constructions similar to the preceding ones: Let p, q be any two complements of A with $a \cup p = a \cup q$. Then

$$v = (b \cup p) \cap (a \cup r) \quad \text{and} \quad w = (b \cup q) \cap (a \cup r)$$

are complements of A by Lemma 0.2 and $v, w \leq a \cup r$. Hence there are $\phi, \psi \in G(a, A)$ with $r^{\phi} = v$, $r^{\psi} = w$, and by $p = (b \cup v) \cap (a \cup p)$, $q = (b \cup w) \cap (a \cup q)$, we get $p^{\phi^{-1}\psi} = q$.

During the following considerations, we will always have $A = a_i \cup a_j$ (a_i, a_j of the normalized frame of L_3), and $a = a_i$ for some i, j.

We are now going to derive algebraic properties of $K(L_3)$ from assumptions about the transitivity of certain groups $G(a, A)$. The proofs of these propositions are omitted because they may be taken verbatim from [2] (where L_3 is assumed to be complemented).

3.4. Proposition. *If L_3 is (a_3, A_1)-transitive, then*
(i) *$(K, +)$ is a group,*
(ii) *$T(u, x, v) = ux + v$ for all $u, x, v \in K$,*
(iii) *$(K, +)$ is a homomorphic image of $G(a_3, A_1)$.*

Proof. (i) and (ii) are to be proved as in [2, p. 29] by showing $T(u, x, v)^{\phi} = T(u, x, v^{\phi})$. From this equation we get with $0^{\phi} = y$:

$$x^{\phi} = T(1, x, 0)^{\phi} = T(1, x, 0^{\phi}) = x + y,$$

and with $0^{\psi} = x$:

$$0^{\psi\phi} = x^{\phi} = x + y = 0^{\psi} + 0^{\phi}.$$

Therefore the mapping $\phi \rightarrow 0^{\phi}$ $(0 = a_1)$ is a homomorphism of $G(a_3, A_1)$ into $(K, +)$, which is onto if L_3 is (a_3, A_1)-transitive. The kernel H of this homomorphism consists of all ϕ with $0^{\phi} = 0$, which is true only for the identity of $G(a_3, A_1)$ in the case that L_3 is complemented (Lemma 3.2). Let P denote the set of all complements of A_1. As, by Lemma 3.2, the restriction $\phi \mid P$ is determined by a_1^{ϕ}, we observe that $(K, +)$ is isomorphic to the group of restricted automorphisms, which we may denote by $G(a_3, A_1)/H$.

3.5. Proposition. *If L_3 is (a_2, A_1)-transitive, then*
(i) *$(K, +)$ is a group,*

(ii) $T(u, x, uc) = u(x + c)$.

Proof. [2, pp. 48–49].

3.6. PROPOSITION. *If L_3 is (a_3, A_2)-transitive, then*

$$T(u, x, cx) = T(T(u, 1, c), x, 0).$$

Proof. [2, p. 53].

3.7. THEOREM. *If L_3 is (a_3, A_1)-, (a_2, A_1)- and (a_3, A_2)-transitive, then $(K, +, \cdot)$ is a not necessarily associative ring with unit, and $T(u, x, v) = ux + v$ for all $u, x, v \in K$.*

Proof. By Prop. 3.4 $(K, +)$ is a group and $T(u, x, v) = ux + v$. Hence from Prop. 3.5. we have

$$ux + uc = T(u, x, uc) = u(x + c)$$

and from Prop. 3.6. we get

$$ux + cx = T(u, x, cx) = T(u + c, x, 0) = (u + c)x.$$

As $1 = c_{13}$ is a unit of K by $(T\ 1)$, it remains to show that $(K, +)$ is commutative. From the distributive laws we derive

$$\begin{aligned} 1 + a + b + ab &= (1 + a) + (1 + a)b \\ &= (1 + a)\ (1 + b) \\ &= (1 + b) + a(1 + b) \\ &= 1 + b + a + ab \qquad \text{for all} \quad a,\ b \in K, \end{aligned}$$

and this implies $a + b = b + a$, as $(K, +)$ is a group.

4. The existence of automorphisms and the associative law of multiplication in $K(L_3)$ from the embedding of L_3 in L_4

4.1. THEOREM. *Let L_3 be embedded in L_4 as defined in* 1.6. *Then L_3 is (a_i, A_k)-transitive for $i, k = 1, 2, 3; i \neq k$.*

Proof. By Lemma 3.3., we have to show: For any a with $a \cup a_i = a_k \cup a_i$, there exists $a\ \phi \in G(a_i, A_k)$, such that $a_k^\phi = a$. In order to get shorter formulas, we use in this proof the abbreviations

$$\begin{aligned} U_3 &= A_k \cup a_k = a_1 \cup a_2 \cup a_3, \\ V &= A_k \cup a_4 = a_i \cup a_j \cup a_4, \\ W &= A_k \cup c_{k4} = a_i \cup a_j \cup c_{k4}. \end{aligned}$$

We are going to construct ϕ as a product of two projections in L_4.
If we put $r = (a \cup c_{k4}) \cap V$,

then
$$\begin{aligned} r \cup A_k &= (a \cup c_{k4} \cup a_i) \cap V \cup a_j \\ &= (a_k \cup c_{k4} \cup a_i) \cap V \cup a_j \\ &= (a_k \cup a_4 \cup a_i) \cap V \cup a_j \\ &= a_i \cup a_4 \cup a_j = V \\ r \cap A_k &= (a \cup c_{k4}) \cap A_k \\ &= (a \cup c_{k4}) \cap W \cap A_k \\ &= c_{k4} \cap A_k \\ &= N. \end{aligned}$$

That is, $r \cup^{\cdot} A_k = a_4 \cup^{\cdot} A_k$, and, therefore, r is a common complement of U_3 and W.

Now the mapping

$$\pi : x \longrightarrow (x \cup a_4) \cap W \qquad \text{for} \quad x \in L_3$$

is a projection (hence an isomorphism) with center a_4 from L_3 onto the sub-lattice $L(N. W)$, and the mapping

$$\rho : x^\pi \longrightarrow (x^\pi \cup r) \cap U_3$$

is a projection with center r from $L(N, W)$ onto L_3. Hence their product $\pi\rho = \phi$ is an automorphism of L_3. We check the desired properties of ϕ:

If $x \leq A_k$, then $x^\pi = x = x^\rho$, hence $x^\phi = x$.

If $x > a_i$, then
$$\begin{aligned} x^\pi &= (x \cup a_4) \cap (a_i \cup a_j \cup c_{k4}) \\ &= (x \cup a_4) \cap (a_j \cup c_{k4}) \cup a_i \end{aligned}$$
and
$$\begin{aligned} x^\pi \cup r &= (x \cup a_4) \cap (a_j \cup c_{k4}) \cup a_i \cup (a \cup c_{k4}) \cap V \\ &= (x \cup a_4) \cap (a_j \cup c_{k4}) \cup (a_i \cup a \cup c_{k4}) \cap V \\ &= (x \cup a_4) \cap (a_j \cup c_{k4}) \cup (a_i \cup a_k \cup a_4) \cap V \\ &= (x \cup a_4) \cap (a_j \cup c_{k4}) \cup a_i \cup a_4 \\ &= (x \cup a_4) \cap (a_j \cup c_{k4} \cup a_i \cup a_4) \\ &= x \cup a_4 . \end{aligned}$$
Therefore we know
$$x^\phi = (x^\pi \cup r) \cap U_3 = (x \cup a_4) \cap U_3 = x.$$
Finally we have
$$a_k^\pi = c_{k4}, \qquad c_{k4} \cup r = c_{k4} \cup (a \cup c_{k4}) \cap V = a \cup c_{k4},$$
hence
$$a_k^\phi = (c_{k4} \cup r) \cap U_3 = a.$$

4.2. In order to prove the associativity of multiplication, we first make the following definition:

For $x, y \in L_{13}$, let

$$P(x, y) = ((x \cup c_{23}) \cap A_3 \cup (y \cup c_{12}) \cap A_1) \cap A_2 .$$

We note, that this is the definition of multiplication used in [13, p. 132] and in [7], and in fact, is dual to our one given in 2.9.

4.3. PROPOSITION. *If L_3 is (a_3, A_2)-transitive, then $P(x, y) = yx$ (the product defined in K) for all $x, y \in L_{13}$.*

Proof. (The proof is entirely the same as in [2, p. 56–57].)

As L_3 is assumed to be (a_3, A_2)-transitive, there exists $\varphi \in G(a_3, A_2)$ such that $c_{23}^{\varphi} = a_2$. For this φ we have

$$\begin{aligned}(c_{23} \cup c_{13})^{\varphi} &= c_{23}^{\varphi} \cup c_{13}, && \text{because } c_{13} \leq A_2,\\ &= a_2 \cup c_{13}\\ &= F,\\ E^{\varphi} &= E, && \text{because } a_3 \leq E,\\ c_{12}^{\varphi} &= (E \cap (c_{23} \cup c_{13}))^{\varphi}\\ &= E \cap F,\\ A_3^{\varphi} &= (a_1 \cup c_{12})^{\varphi}\\ &= a_1 \cup E \cap F, && \text{because } a_1 \leq A_2,\\ &= D.\end{aligned}$$

Therefore we have for any $x \in L_{13}$

$$\begin{aligned}((x \cup c_{23}) \cap A_3)^{\varphi} &= (x^{\varphi} \cup c_{23}^{\varphi}) \cap A_3^{\varphi}\\ &= (x \cup a_2) \cap D, \quad \text{because } x \leq A_2 .\end{aligned}$$

Recalling the definition of

$$\bar{x} = ((x \cup a_2) \cap D \cup a_3) \cap A_3$$

we get

$$((x \cup c_{23}) \cap A_3 \cup a_3)^{\varphi} = (x \cup c_{23}) \cap A_3 \cup a_3$$

on the one hand, because a_3 is the center of φ, and

$$\begin{aligned}((x \cup c_{23}) \cap A_3 \cup a_3)^{\varphi} &= (x \cup a_2) \cap D \cup a_3\\ &= \bar{x} \cup a_3, \qquad \text{on the other hand,}\end{aligned}$$

hence

$$\begin{aligned}\bar{x} &= (\bar{x} \cup a_3) \cap A_3\\ &= ((x \cup c_{23}) \cap A_3 \cup a_3) \cap A_3\\ &= (x \cup c_{23}) \cap A_3 .\end{aligned}$$

From this formula we derive

$$P(x, y) = (\bar{x} \cup (y \cup c_{12}) \cap A_1) \cap A_2 .$$

Now let ψ be an element of $G(a_3, A_2)$ such that

$$c_{12}^{\psi} = (y \cup a_2) \cap E.$$

Then we have at first

$$\begin{aligned} A_3^{\psi} &= (a_1 \cup c_{12})^{\psi} \\ &= a_1 \cup c_{12}^{\psi} \\ &= a_1 \cup (y \cup a_2) \cap E \\ &= a_1 \cup ((y \cup a_2) \cap E \cup a_1 \cap A_1 \\ &= a_1 \cup y^*, \qquad \text{recalling the definition of } y^*, \end{aligned}$$

hence

$$\bar{x}^{\psi} = (A_3 \cap (\bar{x} \cup a_3))^{\psi} = (a_1 \cup y^*) \cap (\bar{x} \cup a_3).$$

Secondly

$$\begin{aligned} ((y \cup c_{12}) \cap A_1)^{\psi} &= (y \cup c_{12}^{\psi}) \cap A_1 , \quad \text{as } y \le A_2 , a_3 \le A_1 , \\ &= (y \cup (y \cup a_2) \cap E) \cap A_1 \\ &= (y \cup a_2) \cap A_1 \\ &= a_2 . \end{aligned}$$

By definition, $P(x, y)$ is $\le A_2$, hence

$$P(x, y)^{\psi} = P(x, y).$$

On the other hand we may calculate

$$\begin{aligned} P(x, y)^{\psi} &= ((\bar{x} \cup (y \cup c_{12}) \cap A_1) \cap A_2)^{\psi} \\ &= (\bar{x}^{\psi} \cup ((y \cup c_{12}) \cap A_1)^{\psi}) \cap A_2 \\ &= ((a_1 \cup y^*) \cap (a_3 \cup \bar{x}) \cup a_2) \cap A_2 , \qquad \text{by the above formulas,} \\ &= yx, \qquad \text{as defined in 2.9.} \end{aligned}$$

This proves $P(x, y) = yx$.

4.4. Proposition. *If L_3 is embedded in L_4 (as in 1.6), then the multiplication in K is associative.*

Proof. If we know that the von Neumann-multiplication P is associative, so will be our one by the last proposition. In [13, p. 132] the associativity of P is proved in a complemented modular lattice with normalized frame of order 4, using as principal tool the Lemma 5.2 of [13, p. 117]. But this is our Lemma 1.5, and the whole proof carries over.

If we agree to use the phrase "$K(L_3)$ is a ring" for "$(K, +, \cdot)$ is a ring and $T(u, x, v) = ux + v$", then we may state as the general result of this section:

4.5. THEOREM. *If L_3 is embeddable in L_4, then $K(L_3)$ is a ring.*

5. Parallel systems and coordinates

In order to get coordinates at least for some elements of L_3, we derive the following type of 'geometric' structure from L_3:

5.1. DEFINITION. *Let L_3 be a modular lattice with normalized frame $(a_1, a_2, a_3, c_{12}, c_{13}, c_{23})$ and $A_1 = a_2 \cup a_3$. We define*

$$A = \{a \mid a \cup^{\cdot} A_1 = U\}$$

$$B = \{b \mid b \cup^{\cdot} a_3 = U\}$$

$$\mid\; = \{(a, b) \mid a \in A, b \in B, a \leq b\}$$

$$\|_A = \{(a, a') \mid a, a' \in A, a \cup a_3 = a' \cup a_3\}$$

$$\|_B = \{(b, b') \mid b, b' \in B, b \cap A_1 = b' \cap A_1\}.$$

As usual, we use the notation $a \mid b$, $a \|_A a'$, $b \|_B b'$ for the relations $|$, $\|_A$, $\|_B$. Instead of $(A, B, |, \|_A, \|_B)$ we write $P(L_3)$. Obviously, $\|_A$ and $\|_B$ are equivalence relations in A respectively B. Now the system $P(L_3)$ has the properties:

5.2. PROPOSITION. *Let $a \in A$, $b \in B$ be given.*
($P1$) *There exists one and only one $a' \in A$ such that $a \|_A a'$, $a' \mid b$.*
($P2$) *There exists one and only one $b' \in B$ such that $b \|_B b'$, $a \mid b'$.*
(3) *For any $p \in A$ with $p \|_A c_{12}$, there exists one and only one $b \in B$ such that $a_1 \mid b$ and $p \mid b$.*
(4) *For any $q \in B$ with $q \|_B A_3$, there exists one and only one $a \in A$ such that $a \mid D$, $a \mid q$.* ($A_3 = a_1 \cup a_2$, $D = a_1 \cup (c_{12} \cup a_3) \cap (c_{13} \cup a_2)$ in L_3)

Proof. ($P1$) Let $a' = (a \cup a_3) \cap b$. First we show $a' \in A$:

$$\begin{aligned} a' \cup A_1 &= (a \cup a_3) \cap b \cup a_2 \cup a_3 \\ &= (a \cup a_3) \cap (b \cup a_3) \cup a_2 \\ &= a \cup a_3 \cup a_2 \\ &= U, \\ a' \cap A_1 &= (a \cup a_3) \cap b \cap A_1 \\ &= a_3 \cap b \\ &= N. \end{aligned}$$

Furthermore we have

$$a' \cup a_3 = (a \cup a_3) \cap (b \cup a_3) = a \cup a_3,$$

so that indeed $a' \parallel_A a$. If $a'' \parallel_A a$, $a'' \mid b$, then $a'' = (a'' \cup a_3) \cap b = (a \cup a_3) \cap b = a'$, hence a' is unique.

($P2$) Let $b' = (b \cap A_1) \cup a$. The proof of ($P2$) is dual to the preceding one.

(3) Let $u = (p \cup a_2) \cap A_2$ (Lemma 0.2 shows $u \in A$), then $a_1 \cup p = a_1 \cup u^* \in B$, and for any $b' \in B$ with $b' \geq p, a_1$, we have $b' = p \cup a_1$ since both elements are complements of a_3 and $p \cup a_1 \leq b'$.

(4) Dual to (3).

5.3. This proposition shows, that $P(L_3)$ is a P-system (or parallel-system) in the sense of [6, p. 2], with the additional properties (3) and (4). The latter two imply 'regularity' of the P-system as defined in [6, pp. 17, 20]. To avoid another meaning of this term, we take (3) and (4) into the definition:

A P-system (or parallel-system, or incidence system with parallelism) $P = (A, B, |, \parallel_A, \parallel_B)$ consists of sets A, B, an incidence relation $| \subseteq A \times B$ and two equivalence relations $\parallel_A \subseteq A \times A$, $\parallel_B \subseteq B \times B$, such that the axioms ($P1$), ($P2$) as stated in Prop. 5.2 and the following one hold:
($P3$) (a) There exists a pair (a, a') of elements of A, such that for all $p \parallel_A a'$, there exists one and only one $b \in B$ with $p \mid b$, $a \mid b$.
(b) There exists a pair (b, b') of elements of B such that for all $q \parallel_B b'$, there exists one and only one $a \in A$ with $a \mid b$, $a \mid q$.

A typical example of a P-system is a semi-affine plane, that is an affine plane without one class of parallel lines (the "vertical" ones). This may be obtained in the manner of Def. 5.1 from a projective plane L_3, and a detailed discussion shows, that in this case it is possible to reconstruct L_3 from $P(L_3)$ in a unique way. Motivated by this example, one might interpret the equivalence relations $\parallel_A$ and $\parallel_B$ as parallelism for "points" (= elements of A) and "lines" (= elements of B), respectively and say "a and b are incident" when $a \mid b$.

5.4 *Definition* [6, pp. 2-3]. An isomorphism of a P-system $P = (A, B, |, \parallel_A, \parallel_B)$ onto a P-system $P' = (A', B', |', \parallel_{A'}, \parallel_{B'})$ is a pair of bijective mappings $(\phi, \bar{\phi}): \phi : A \to A', \bar{\phi} : B \to B'$, such that $\phi, \phi^{-1}, \bar{\phi}, \bar{\phi}^{-1}$ preserve incidence and parallelism.

Remark. If $\phi, \bar{\phi}$ preserve all relations, so do $\phi^{-1}, \bar{\phi}^{-1}$ [6, pp. 2–3].

5.5. Proposition. *Let K be a ternary ring. Then the system $P(K)$ as defined below is a P-system.*

Starting from K (Def. 2.5), we define

$$P(K) = (A, B, |, \parallel_A, \parallel_B) \quad \text{by} \quad A = K \times K = B$$

(In order to distinguish the elements of A and B we write $(x, y) \in A$ and

$[u, v] \in B$.)

$$| = \{((x, y), [u, v]) \mid y = T(u, x, v)\}$$

$$\|_A = \{((x, y), (r, s)) \mid x = r\}$$

$$\|_B = \{([u, v], [m, n]) \mid u = m\}.$$

Proof. Obviously, $\|_A$ and $\|_B$ are equivalence relations. For any two pairs $(x, y) \in A$, $[u, v] \in B$ we may calculate $T(u, x, v)$ and see that $(x, T(u, x, v))$ is a point parallel to (x, y) and incident with $[u, v]$. If also $(x, w) \mid [u, v]$, then by the definition of $|$ we know $w = T(u, x, v)$, hence the point $(x, T(u, x, v))$ is unique. On the other hand there is a unique z with $y = T(u, x, z)$ for given y, u, x, and with this z we get $[u, z] \|_B [u, v]$, $(x, y) \mid [u, z]$. Therefore we have ($P1$) and ($P2$).

($P3$) (a) We look at the points $(0, 0)$ and $(1, 0)$. For any point $(1, p) \|_A (1, 0)$ we have the line $[p, 0]$ incident with $(0, 0)$ because of $0 = T(p, 0, 0)$ and with $(1, p)$ because of $p = T(p, 1, 0)$. Let $[u, v]$ be another line incident with $(0, 0)$ and $(1, p)$. Then $0 = T(u, 0, v) = v$ and $p = T(u, 1, 0)$, hence $[u, v] = [p, 0]$. So the points $(0, 0)$ and $(1, 0)$ are as required for ($P3$) (a). The two lines $[1, 0]$ and $[0, 0]$ have the property (b): $[0, v]$ and $[1, 0]$ are both incident with (v, v) by $v = T(0, v, v)$ and $v = T(1, v, 0)$, and $(x, y) \mid [0, v]$ implies $y = T(0, x, v) = v$, $(x, v) \mid [1, 0]$ implies $v = T(1, x, 0) = x$, hence $(x, y) = (v, v)$, this point is unique.

5.6. Theorem. (Introduction of coordinates in $P(L_3)$). *If K is the ternary ring of L_3, then $P(K)$ and $P(L_3)$ are isomorphic.*

Proof. We write $A(K)$, $B(K)$ and $A(L_3)$, $B(L_3)$ to distinguish the P-system, but denote the relations with the same signs.

To $a \in A(L_3)$ we assign the "coordinates"

$$x(a) = ((a \cup a_3) \cap D \cup a_2) \cap A_2$$

$$y(a) = (a \cup a_2) \cap A_2 .$$

$$\begin{aligned} \bar{x}(a) \cup a_3 &= ((x(a) \cup a_2) \cap D \cup a_3) \cap A_3 \cup a_3 \\ &= (x(a) \cup a_2) \cap D \cup a_3 \\ &= (((a \cup a_3) \cap D \cup a_2) \cap A_2 \cup a_2) \cap D \cup a_3 \\ &= ((a \cup a_3) \cap D \cup a_2) \cap D \cup a_3 \\ &= a \cup a_3 . \end{aligned}$$

It is easily seen (with the help of Lemma 0.2) that ϕ: $a \to (x(a), y(a))$ is a bijection between $A(L_3)$ and $A(K)$.

Now let $b \in B(L_3)$. We find "coordinates" for b by defining

$$u(b) = ((b \cap A_1 \cup a_1) \cap (c_{12} \cup a_3) \cup a_2) \cap A_2 ,$$

that is,

$$u(b)^* = b \cap A_1 \quad \text{and} \quad v(b) = b \cap A_2 .$$

$\bar{\Phi} : b \to [u(b), v(b)]$ is a bijection between $B(L_3)$ and $B(K)$ by straightforward calculations.

Now let $a \in A(L_3)$, $b \in B(L_3)$ and $a \mid b$. We have to show $y(a) = T(u(b), x(a), v(b))$.

$$T(u(b), x(a), v(b)) = ((u(b)^* \cup v(b)) \cap (\bar{x}(a) \cup a_3) \cup a_2) \cap A_2$$
$$= ((b \cap A_1 \cup b \cap A_2) \cap (a \cup a_3) \cup a_2) \cap A_2 ,$$

by the above noted formulas,

$$= (b \cap (a \cup a_3) \cup a_2) \cap A_2 ,$$

but $a \leq b$, hence

$$T(u(b), x(a), v(b)) = (a \cup b \cap a_3 \cup a_2) \cap A_2$$
$$= (a \cup a_2) \cap A_2$$
$$= y(a).$$

So $(\phi, \bar{\Phi})$ preserves incidence.

For $a, a' \in A(L_3)$; $a \parallel_A a'$ means $a \cup a_3 = a' \cup a_3$, hence $x(a) = x(a')$, the images in $A(K)$ are parallel.

For $b, b' \in B(L_3)$, $b \parallel_B b'$ means $b \cap A_1 = b' \cap A_1$, hence $u(b)^* = u(b')^*$, and the images in $B(K)$ are parallel.

By the remark following in Def. 5.4, the theorem is proved.

5.7 Proposition. *Let K be a ring and $L(K^3)$ the lattice of all right submodules of the K-right-module K^3 with the normalized frame* ($a_1 = (1, 0, 0)K$, $a_2 = (0, 1, 0)K$, $a_3 = (0, 0, 1)K$, $c_{12} = (1, 1, 0)K$, $c_{13} = (1, 0, 1)K$, $c_{23} = (0, 1, -1)K$) and let *the operation T_k in K be defined by $T_k(u, x, v) = ux + v$. Then we have $P(L(K^3)) \cong P(K)$ and $(K, T_k) \cong K(L(K^3))$.*

Proof. First we are going to describe the sets $A(L(K^3))$ and $B(L(K^3))$ in terms of generating vectors of K^3.

(i) If $p \in A(L(K^3))$, then there exists a unique pair $(x, y) \in K^2$ such that $p = (1, x, y)K$.

Proof. We have $p \cup A_1 = U$, hence

$$(1, 0, 0)K \leq p \cup A_1 \quad \text{or} \quad (1, 0, 0) \in p \cup A_1 .$$

That is, there exist $r, s, t, x, y \in K$ such that $(r, s, t) \in p$ and

$$(r, s, t) - (0, 1, 0)x - (0, 0, 1)y \text{ W } (1, 0, 0),$$

hence $(r, s, t) = (1, x, y)$.

If also $(1, x', y') \in p$, then $(0, x - x', y - y') \in p \cap A_1 = N$, therefore $(1, x, y)$ is unique.

If $(u, v, w) \in p$, then

$$(u, v, w) - (1, x, y)u = (0, v - xu, w - yu) \in p \cap A_1 = N,$$

hence $(u, v, w) = (1, x, y)u$ and $p = (1, x, y)K$.

(ii) If $b \in B(L(K^3))$, then there exists a unique pair $(u, v) \in K^2$ such that $b = (0, 1, u)K \cup (1, 0, v)K$.

Proof. Using $b \cup a_3 = U$, we get the existence of $(r, s, t) \in b$, $(x, y, z) \in b$ and $u, v \in K$ such that

$$(r, s, t) - (0, 0, v) = (1, 0, 0),$$

hence

$$(1, 0, v) \in b,$$

and

$$(x, y, z) - (0, 0, u) = (0, 1, 0),$$

hence

$$(0, 1, u) \in b.$$

For any $(d, e, f) \in b$ we have therefore

$$(d, e, f) - (1, 0, v)d - (0, 1, u)e = (0, 0, f - vd - ue) \in b \cap a_3 = N,$$

i.e.

$$(d, e, f) = (1, 0, v)d + (0, 1, u)e.$$

Finally,

$$(1, 0, v) - (1, 0, v') = (0, 0, v - v') \in b \cap a_3 = N$$

and

$$(0, 1, u) - (0, 1, u') = (0, 0, u - u') \in b \cap a_3 = N$$

show the uniqueness of v and u.

These generating vectors suggest the definitions for the first isomorphism: If $p \in A(L(K^3))$ is generated by $(1, x(p), y(p))$, then let $\phi(p) = (x(p), y(p))$; and if $b \in B(L(K^3))$ is generated by $(0, 1, u(b))$ and $(1, 0, v(b))$, let $\bar{\phi}(b) = [u(b), v(b)]$. From (i) and (ii) it is easy to see that these two mappings are bijections between $A(L(K^3))$, $B(L(K^3))$ and K^2 as required.

If $p \mid b$, then there exist $r, s \in K$ such that

$$(1, x(p), y(p)) = (1, 0, v)r + (0, 1, u)s,$$

hence

$$r = 1, s = x(p) \quad \text{and} \quad y(p) = u(b)x(p) + v(b),$$

that is $\phi(p) \mid \bar{\phi}(b)$ in $P(K)$.

If $p \parallel_A q$, that is $p \cup a_3 = q \cup a_3$, then there exists $z \in K$ such that

$$(1, x(p), y(p)) + (0, 0, z) = (1, x(q)y(q)),$$

hence

$$x(p) = x(q), \qquad \phi(p) \parallel_A \phi(q).$$

If $b \parallel_B c$, that is

$$b \cap A_1 = (0, 1, u(b))K = c \cap A_1 = (0, 1, u(c))K,$$

then

$$u(b) = u(c), \qquad \Phi(b) \parallel_B \Phi(c).$$

For the second isomorphism of the assertion, we map $x \in K$ onto $(1, 0, x)K \in L(K^3)$ and obtain

$$T((1, 0, u)K, (1, 0, x)K, (1, 0, v)K) = (1, 0, ux + v)K$$

in the following way: For $z \leq A_2$, z^* was defined by

$$z^* = ((z \cup a_2) \cap (a_3 \cup c_{12}) \cup a_1) \cap A_1 .$$

From this we find $((1, 0, u)K)^* = (0, 1, u)K$ by

$$((1, 0, u)K \cup (0, 1, 0)K) \cap ((0, 0, 1)K \cup (1, 1, 0)K) = (1, 1, u)K,$$
$$((1, 1, u)K \cup (1, 0, 0)K) \cap ((0, 1, 0)K \cup (0, 0, 1)K) = (0, 1, u)K.$$

Similarly from

$$\bar{z} = ((z \cup a_2) \cap D \cup a_3) \cap A_2$$

we have

$$\overline{(1, 0, x)K} = (1, x, 0)K.$$

Now we see that

$$((1, 0, u)K)^* \cup (1, 0, v)K = (0, 1, u)K \cup (1, 0, v)K$$

and

$$\overline{(1, 0, x)K} \cup (0, 0, 1)K = (0, 0, 1)K \cup (1, x, 0)K,$$

so that the intersection of these two submodules is $(1, x, ux + v)K$. By

$$((1, x, ux + v)K \cup (0, 1, 0)K) \cap A_2 = (1, 0, ux + v)K,$$

the desired equation holds.

If K is a ring, one can embed $L(K^3)$ in $L(K^4)$ such that $L(K^3)$ is (isomorphic to) the sublattice of elements of $L(K^4)$ which are less than or equal to

$$(1, 0, 0, 0)K \cup (0, 1, 0, 0)K \cup (0, 0, 1, 0)K.$$

This shows that the embedding is possible in the way that the bases fit together as required in Def. 1.6. Observing this and the first isomorphism of Prop. 5.7, we may state as a counterpart to Theorem 4.5:

5.8. THEOREM. *If the ternary ring $K(L_3)$ of the lattice L_3 is an associative ring, then there exists a lattice L_4 such that the P-system $P(L_3)$ is isomorphic to the P-system $P(L_3')$ of the sublattice $L_3' = L(N, a_1 \cup a_2 \cup a_3)$ of L_4.*

Expressed in a more informal way, this says: If $K(L_3)$ is a ring, then $P(L_3)$ is embeddable in a modular lattice L_4.

Remembering what was said about semi-affine planes in 5.3 and that in a projective plane L_3 the theorem of Desargues is equivalent to the fact that $K(L_3)$ is a field (which we could derive from Theorem 4.5 and Remark 2.10), we see that Theorem 4.5 and Theorem 5.8 together contain the theorem: A projective plane is embeddable in a projective space of dimension ≥ 3 if and only if the theorem of Desargues holds in the plane.

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University of Michigan
Ann Arbor, Michigan
Justus Liebig Universität
Giessen, Germany