On Coprime Modules and Comodules

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Abstract

Many observations about coalgebras were inspired by comparable situations for algebras. Despite the prominent role of prime algebras, the theory of a corresponding notion for coalgebras was not well understood so far. Coalgebras C over fields may be called *coprime* provided the dual algebra C^* is prime. This definition, however, is not intrinsic - it strongly depends on the base ring being a field. The purpose of the paper is to provide a better understanding of related notions for coalgebras over commutative rings by employing traditional methods from (co-)module theory, in particular (pre-)torsion theory.

Dualising classical primeness condition, *coprimeness* can be defined for modules and algebras. These notions are developed for modules and then applied to comodules. We consider prime and coprime, fully prime and fully coprime, strongly prime and strongly coprime modules and comodules. In particular we obtain various characterisations of prime and coprime coalgebras over rings and fields.

Key Words: (co-)prime modules and comodules, coprime coalgebras, dual algebras.

Contents:

1 Preliminaries

In the category of left R-modules there are various notions of prime objects which generalise the well known notion of a prime associative (commutative) ring R. For the notion of primeness of modules we refer to [18, 20].

Unless explicitely stated, in the first sections R is an associative ring with unit, M usually will be a left R-module. The category of left R-modules is denoted as ${}_R\mathbf{M}$. The morphisms are written on the right side of the module and if needed we use the $\diamond$ for the composition of mappings written on the right side. The usual composition is denoted by $\circ$ and thus $(u)f \diamond g$ is equal to $g \circ f(u)$ when writing the maps on the left side.

For two R-modules M, N, we say N is *subgenerated by M* if N is isomorphic to a submodule of an M-generated module. The full subcategory of ${}_R\mathbf{M}$ whose objects are the modules subgenerated by M is denoted by $\sigma[M]$. For a family $\{N_\lambda\}_\Lambda$ of modules in $\sigma[M]$, the product in $\sigma[M]$ exists and is given by $\prod_{\Lambda}^{M} N_\lambda := \mathrm{Tr}(\sigma[M], \prod_{\Lambda} N_\lambda)$. For more details on the notions of generators, cogenerators, subgenerators and $\sigma[M]$, see [19].

For modules over prime rings recall the following facts.

1.1. Proposition. *Let R, S be rings and M be an (R,S)-bimodule. Then the following assertions are equivalent:*

(a) $\overline{R} = R/\mathrm{Ann}_R(M)$ *is a prime ring;*

(b) *for any submodule K of M,* $\mathrm{Ann}_R(K) = \mathrm{Ann}_R(M)$ *or* $\mathrm{Ann}_R(M/K) = \mathrm{Ann}_R(M)$*;*

(c) *for any (R,S)-subbimodule K of M,* $\mathrm{Ann}_R(K) = \mathrm{Ann}_R(M)$ *or* $\mathrm{Ann}_R(M/K) = \mathrm{Ann}_R(M)$.

Let $S := \mathrm{End}_R(M)$, $K \subset M$ a submodule and $I \subset S$ a left ideal. Denoting by $\pi_K : M \to M/K$ the canonical projection, we put

$$\begin{aligned} \mathrm{Ann}_S(K) &:= \{f \in S \mid (K)f = 0\} = \pi_K \diamond \mathrm{Hom}_R(M/K, M), \\ \mathrm{Ker}\ I &:= \bigcap \{\mathrm{Ker}\ f \mid f \in I\}. \end{aligned}$$

We always have $K \subseteq \mathrm{Ker}\ \mathrm{Ann}_S(K)$ and $I \subseteq \mathrm{Ann}_S(\mathrm{Ker}\ I)$.

For equality extra conditions on a module M are needed (see [19, 28.1]).

1.2. Lemma. *Let M be a module and $S = \mathrm{End}_R(M)$.*

(i) *For any submodule $K \subseteq M$,*

$$\mathrm{Ker}\ \mathrm{Ann}_S(K) = \mathrm{Ker}\ \pi_K \diamond \mathrm{Hom}_R(M/K, M) = K$$

if and only if M is a self-cogenerator module.

(ii) *If M is self-injective, then for every finitely generated right ideal $I \subseteq S$,*

$$\mathrm{Hom}_R(M/\mathrm{Ker}\ I, M) = I.$$

1.3. $*$-conditions. We formulate the following conditions for an R-module M:

($*$) For any non-zero submodule K of M, $\mathrm{Ann}_R(M/K) \not\subset \mathrm{Ann}_R(M)$.

($*fi$) For any non-zero fully invariant submodule K of M, $\mathrm{Ann}_R(M/K) \not\subset \mathrm{Ann}_R(M)$.

($**$) For any proper (fully invariant) submodule K of M, $\mathrm{Ann}_R(K) \not\subset \mathrm{Ann}_R(M)$.

1.4. (Co-)retractable modules. A module M is called *(fi-)retractable* if for any non-zero (fully invariant) submodule K of M and $S = \mathrm{End}_R(M)$, $\mathrm{Hom}_R(M,K) \neq 0$.

Dually, M is called *(fi-)coretractable* if for any proper (fully invariant) submodule K of M, $\mathrm{Hom}_R(M/K, M) \neq 0$.

Some relevance of these notions can be seen from the following observations which are easy to prove.

1.5. Proposition. *Let M be a coretractable R-module and $S = \mathrm{End}_R(M)$. The following are equivalent:*

(a) *S has no zero-divisor;*

(b) *for any proper submodule K of M,* $\mathrm{Hom}_R(M,K) = 0$*;*

(c) *for any* $0 \neq f \in S$, $(M)f = M$ *(any non-zero endomorphism is an epimorphism).*

1.6. Proposition. *Let M be a fi-coretractable R-module and denote $S = \mathrm{End}_R(M)$. The following are equivalent:*

(a) *S is a prime ring;*

(b) *for any proper fully invariant submodule K of M, $\mathrm{Hom}_R(M, K) = 0$;*

(c) *for any $0 \neq f \in S$, $(M)fS = M$;*

(d) *for any ideal $0 \neq I \subset S$, $MI = M$.*

1.7. Proposition. *Let M be a retractable R-module and $S = \mathrm{End}_R(M)$. The following are equivalent:*

(a) *S has no zero-divisor;*

(b) *for any non-zero submodule K of M, $\mathrm{Hom}_R(M/K, M) = 0$;*

(c) *for any $0 \neq f \in S$, $\mathrm{Ker}\ f = 0$ (any non-zero endomorphism is a monomorphism).*

1.8. Proposition. *Let M be a fi-retractable R-module and $S = \mathrm{End}_R(M)$. The following are equivalent :*

(a) *S is a prime ring.*

(b) *For any non-zero fully invariant submodule K of M, $\mathrm{Hom}_R(M/K, M) = 0$.*

(c) *For any $0 \neq f \in S$, $\mathrm{Ker}\ Sf = 0$.*

(d) *For any ideal $0 \neq I \subset S$, $\mathrm{Ker}\ I = 0$.*

1.9. Corollary. *Let M be a retractable and coretractable R-module, $S = \mathrm{End}_R(M)$. The following are equivalent:*

(a) *S has no zero-divisor.*

(b) *M is a simple module.*

(c) *S is a division ring.*

The following observation from [13, Lemma 17] will be useful.

1.10. Lemma. *Let M, N be R-modules and $f \in \mathrm{Hom}_R(M, N)$ an epimorphism.*

(i) *If $\mathrm{Ker}\ f$ is fully-invariant and L is a fully-invariant submodule of N, then $(L)f^{-1}$ is a fully-invariant submodule of M.*

(ii) *If M is self-projective and U is a fully-invariant submodule of M, then $(U)f$ is a fully-invariant submodule of N.*

2 Prime and coprime modules

A module M is called *prime* if for every non-zero fully-invariant submodule K of M, $\mathrm{Ann}_R(K) = \mathrm{Ann}_R(M)$. The following characterisations are well-known.

2.1. Prime modules. *For a module M the following are equivalent:*

(a) *M is a prime module;*

(b) $\mathrm{Ann}_R(K) = \mathrm{Ann}_R(M)$ *for any non-zero submodule* K *of* M*;*

(c) $R/\mathrm{Ann}_R(M)$ *is cogenerated by* K *for any non-zero submodule* K *of* M*;*

(d) $R/\mathrm{Ann}_R(M)$ *is cogenerated by* K *for any non-zero fully-invariant submodule* K *of* M*.*

The next result is an obvious modification of 13.1 of [20].

2.2. Proposition. *Let* M *be an* R*-module,* $S = \mathrm{End}_R(M)$*, and* $\overline{R} := R/\mathrm{Ann}_R(M)$*.*

(i) *If* M *is prime, then* $\overline{R}$ *is a prime ring.*

(ii) *If* $\overline{R}$ *is a prime ring and* M *satisfies* $(*fi)$*, then* M *is prime.*

(iii) *If* ${}_RM$ *is prime and satisfies* $(*fi)$*, then* M_S *is prime (and* S *is a prime ring).*

A faithful module over a prime ring need not be prime: $\bigoplus_{n\in\mathbb{N}} \mathbb{Z}/n\mathbb{Z}$ is a faithful $\mathbb{Z}$-module but clearly is not prime.

2.3. Lemma. *Let* $\{M_\lambda\}_\Lambda$ *be a family of faithful modules,* Λ *an index set. Then* $\prod_\Lambda M_\lambda$ *is prime if and only if each* M_λ *is prime.*

Proof. $(\Rightarrow)$ Let $\prod_\Lambda M_\lambda$ be prime. Consider any submodule $U \subset M_\mu \subset \prod_\Lambda M_\lambda$. Then $\mathrm{Ann}_R(U) = \mathrm{Ann}_R(\prod_\Lambda M_\lambda) = 0$.

$(\Leftarrow)$ Let each M_λ be prime and $V \subset \prod_\Lambda M_\lambda$. There is a non-zero canonical projection $(V)\pi_{\lambda_0} \subset M_{\lambda_0}$, for some $\lambda_0 \in \Lambda$. Since $(V)\pi_{\lambda_0}$ is faithful, V is also faithful. □

2.4. Corollary. *If* R *is a prime ring, then submodules of* R*-cogenerated modules are prime. In particular, every projective module is prime.*

Primeness of M implies the primeness of projective modules in $\sigma[M]$.

2.5. Proposition. *Let* M *be prime. Then*

(i) *every* M*-cogenerated module is prime;*

(ii) *every projective module* P *in* $\sigma[M]$ *is prime.*

Proof. (i) According to Lemma 2.3, M^Λ is prime and hence any M-cogenerated module is prime.

(ii) Any projective module P in $\sigma[M]$ is isomorphic to a submodule of some $M^{(\Lambda)}$, Λ an index set, hence P is prime. □

2.6. Proposition. *Let* M *be a projective module in* $\sigma[M]$*. If every non-zero submodule of* M *cogenerates* M*, then* $\mathrm{End}_R(M)$ *is a prime ring.*

Proof. Let $J \subset S := \mathrm{End}_R(M)$ be a finitely generated proper left ideal. By assumption, M is MJ-cogenerated, i.e., there is a short exact sequence $0 \to M \to (MJ)^\Lambda$. Applying $\mathrm{Hom}_R(P, -)$ to this exact sequence yields the commutative diagram

$$\begin{array}{ccccc}
0 & \longrightarrow & \mathrm{Hom}_R(P,P) & \longrightarrow & \mathrm{Hom}_R(P,(PJ)^\Lambda) \\
 & & \downarrow = & & \downarrow \simeq \\
0 & \longrightarrow & S & \longrightarrow & J^\Lambda,
\end{array}$$

since $\mathrm{Hom}_R(P,(PJ)^\Lambda) = \mathrm{Hom}_R(P,(PJ))^\Lambda$ and, by projectivity, $\mathrm{Hom}_R(P,PJ) = J$ (see [19, 18.4]). Thus J is a faithful left S-module, i.e., $\mathrm{End}_R(P)$ is a (left) prime ring. $\square$

As a consequence, we obtain Proposition 1.3 of [10] :

2.7. Corollary. *Let P be a projective R-module. If the ring R is prime, then $\mathrm{End}_R(P)$ is prime.*

2.8. Proposition. *Let M be a module with $\mathrm{Soc}(M) \neq 0$. If M is prime, then $\overline{R} := R/\mathrm{Ann}_R(M)$ is a left primitive ring. If, in addition, R is commutative, then $\overline{R}$ is a field.*

Proof. By assumption, there is a simple $\overline{R}$-submodule K of M which is faithful, thus $\overline{R}$ is left primitive. If $\overline{R}$ is commutative and primitive, then $\overline{R}$ is a field. $\square$

Dual to prime modules, a module M is called *coprime* if for every proper fully-invariant submodule K of M, $\mathrm{Ann}_R(M/K) = \mathrm{Ann}_R(M)$ (e.g., [2]).

2.9. Coprime modules. *For a module M the following are equivalent:*

(a) *M is a coprime module.*

(b) *$\mathrm{Ann}_R(M/K) = \mathrm{Ann}_R(M)$ for any proper submodule K of M.*

(c) *$R/\mathrm{Ann}_R(M)$ is cogenerated by M/K for any proper submodule K of M.*

(d) *$R/\mathrm{Ann}_R(M)$ is cogenerated by M/K for any proper fully-invariant submodule K of M.*

Proof. (a) $\Leftrightarrow$ (b) Let K be a submodule of M and assume $I := \mathrm{Ann}_R(M/K) \not\subset \mathrm{Ann}_R(M)$. Then $0 \neq IM \subset K$, and IM is fully invariant since $(IM)S = I(MS) = IM$. By (a), $I \subset \mathrm{Ann}_R(M/IM) = \mathrm{Ann}_R(M)$, a contradiction.

(b) $\Leftrightarrow$ (c) and (a) $\Leftrightarrow$ (d) are obvious. $\square$

Any module which has no proper fully invariant submodule is coprime.

2.10. Lemma. *Let M be an R-module.*

(i) *If M is coprime, then $\overline{R}$ is prime.*

(ii) *If $\overline{R}$ is prime and M satisfies $(**)$, then M is coprime.*

Proof. (i) Assume M to be coprime, i.e., $\mathrm{Ann}_R(M/K) = \mathrm{Ann}_R(M)$ for every proper (fully-invariant) submodule $K \subset M$. Then by Proposition 1.1, $\overline{R}$ is prime.

(ii) By $(**)$, $\mathrm{Ann}_R(K) \neq \mathrm{Ann}_R(M)$ for any proper (fully-invariant) submodule $K \subset M$. Now $\mathrm{Ann}_R(K)\mathrm{Ann}_R(M/K) \subseteq \mathrm{Ann}_R(M)$ and $\overline{R}$ being prime implies $\mathrm{Ann}_R(M/K) = \mathrm{Ann}_R(M)$, hence M is coprime. $\square$

The example $M = {}_{\mathbb{Z}}\mathbb{Z}$ illustrates that without condition $(**)$, assertion (ii) in the lemma above does not hold.

2.11. Prüfer groups. For any prime number p, the p-component of $\mathbb{Q}/\mathbb{Z}$ is the Prüfer group $\mathbb{Z}_{p^\infty}$. Any nonzero factor module $\mathbb{Z}_{p^\infty}/K$ of $\mathbb{Z}_{p^\infty}$ is isomorphic to $\mathbb{Z}_{p^\infty}$ itself. Thus $\mathbb{Z}_{p^\infty}$ is coprime. Moreover, a proper submodule $K \subset \mathbb{Z}_{p^\infty}$ is of the form $K = \mathbb{Z}\{\frac{1}{p^k} + \mathbb{Z}\}$ for some $k \in \mathbb{N}$, thus it is not faithful. Hence $\mathbb{Z}_{p^\infty}$ is not prime.

The coprimeness of a module is preserved by some factor module.

2.12. Proposition. *If M is coprime and K is a proper fully invariant submodule of M, then M/K is coprime.*

Proof. Take any proper fully invariant submodule U/K of M/K, where $K \subset U \subset M$ are proper submodules and K is fully invariant. By Lemma 1.10, U is fully invariant in M. Thus $(M/K)/(U/K) \simeq M/U$ is faithful, i.e.,

$$\mathrm{Ann}_R((M/K)/(U/K)) = \mathrm{Ann}_R(M/U) = \mathrm{Ann}_R(M) = \mathrm{Ann}_R(M/K).$$

□

The following observation is dual to Proposition 2.8 for prime modules.

2.13. Proposition. *If M is coprime and $\mathrm{Rad}(M) \neq M$, then:*

(i) *$\overline{R} := R/\mathrm{Ann}_R(M)$ is a left primitive ring.*

(ii) *If R is commutative, then $\overline{R}$ is a field.*

Proof. (i) By assumption there is a maximal submodule K in M. Consider the fully invariant submodule $\mathrm{Rej}(M, M/K) \subset K \neq M$. By coprimeness of M, $M/\mathrm{Rej}(M, M/K)$ is a faithful $\overline{R}$-module and is cogenerated by the simple module M/K (see 14.5 of [19]). Thus M/K is also a faithful $\overline{R}$-module. Thus $\overline{R}$ has a faithful simple left module, i.e., $\overline{R}$ is a left primitive ring.

(ii) If $\overline{R}$ is commutative and primitive, then $\overline{R}$ is a field. □

Without the condition $\mathrm{Rad}(M) \neq M$, Proposition 2.13 does not hold. For example, the Prüfer group $\mathbb{Z}_{p^\infty}$ is a coprime $\mathbb{Z}$-module which has no maximal submodules.

3 Fully prime and coprime modules

We call a module M *fully prime* if for any non-zero fully invariant submodule K of M, M is K-cogenerated.

This notion can be described using a *product* of fully invariant submodules K, L of M studied by Raggi et al. in [13] which is defined by

$$K *_M L := K\mathrm{Hom}_R(M, L).$$

For not necessarily fully invariant submodules this is also considered in Bican et al. [4]. Their definition of "prime modules" is more restrictive than the one we consider here. However, the proof of Proposition 2.3 of [4] can be modified to yield:

3.1. Fully prime modules. *The following are equivalent for an R-module M:*

(a) *M is a fully prime module.*

(b) $\mathrm{Rej}(M,K)=0$ *for any non-zero fully-invariant submodule $K\subset M$.*

(c) $K*_M L\neq 0$ *for any non-zero fully-invariant submodules $K,L\subset M$.*

(d) $\mathrm{Rej}(-,M)=\mathrm{Rej}(-,K)$ *for any non-zero fully-invariant submodule K of M, i.e., any M-cogenerated module is also K-cogenerated.*

Based on the $*_M$-product we define

3.2. Fully prime submodules. A fully invariant submodule N of M is called *fully prime in M* if for any fully invariant submodules K,L of M,

$$K*_M L\subseteq N \text{ implies } K\subseteq N \text{ or } L\subseteq N.$$

Thus the module M is fully prime if the zero submodule is fully prime in M.

Proposition 18 of [13] provides a relationship between a fully prime submodule N of M and the factor module M/N. As a special case, consider R as a left R-module and let I,J be ideals of R. Then $I*_R J=IJ$. Since every ideal of R is a fully invariant R-submodule, we get:

3.3. Proposition. *The following are equivalent for a two-sided ideal I :*

(a) *R/I is a prime ring.*

(b) *I is a fully prime submodule in R.*

(c) *I is a prime ideal.*

In general prime modules need not be fully prime. For the following relationship the proof of [20, Proposition 13.2] can be adopted.

3.4. Proposition. *For an R-module M with $(*fi)$, the following are equivalent :*

(a) *M is prime and fi-retractable.*

(b) *M is fully prime.*

Notice that for any ring R, $\mathrm{End}_R(R)\simeq R$ and as a left R-module, R satisfies $(*fi)$ and is fi-retractable.

3.5. Corollary. *For the ring R the following assertions are equivalent:*

(a) *R is a prime ring.*

(b) *${}_RR$ is a prime module.*

(c) *${}_RR$ is a fully prime module.*

3.6. Proposition. *Let M be a module with* $\mathrm{Soc}(M)\neq 0$. *If M is fully prime, then*

(i) *M is cogenerated by a simple module.*

(ii) *$\overline{R}:=R/\mathrm{Ann}_R(M)$ is a left primitive ring.*

Proof. (i) Let K be a simple submodule of M. Then $\mathrm{Tr}(K,M)$ is a fully invariant submodule and hence M is $\mathrm{Tr}(K,M)$-cogenerated. $\mathrm{Tr}(K,M)$ is K-cogenerated, and hence M is K-cogenerated.

(ii) $\overline{R}$ is cogenerated by M and hence by the simple module K (from (i)). □

We call a module M *fully coprime* if for any proper fully invariant submodule K of M, M is M/K-generated.

To study these modules an *inner coproduct* of fully invariant submodules $K, L \subset M$, is defined by putting

$$\begin{aligned} K :_M L &:= \bigcap\{(L)f^{-1} \mid f \in \mathrm{End}_R(M), K \subseteq \mathrm{Ker} f\} \\ &= \mathrm{Ker}\ \pi_K \diamond \mathrm{Hom}_R(M/K, M) \diamond \pi_L, \end{aligned}$$

where $\pi_K : M \to M/K$ and $\pi_L : M \to M/L$ denote the canonical projections. $K :_M L$ is also a fully invariant submodule (see [14]).

Such a coproduct is considered in Bican et al. [4] for any pair of not necessary fully invariant submodules and they derive "coprime modules" from this coproduct.

We can characterise (our) fully coprime modules similarly to [4, Proposition 4.3].

3.7. Fully coprime modules. *The following are equivalent for an R-module M:*

(a) *M is a fully coprime module;*

(b) *If $K :_M L = M$, then $K = M$ or $L = M$, for any fully invariant submodules K, L of M;*

(c) *$K :_M L \neq M$ for any proper fully invariant submodules K, L of M;*

(d) $\mathrm{Tr}(M/K, -) = \mathrm{Tr}(M, -)$ *for any proper fully invariant submodules K of M, i.e. any M-generated module is also M/K-generated.*

Proof. (a)⇔(d) and (b)⇔(c) are trivial.

(c)⇒(a) Let $K \subset M$ be a proper fully invariant submodule such that

$$N = \mathrm{Tr}(M/K, M) = (M)\ \pi_K \diamond \mathrm{Hom}_R(M/K, M) \neq M.$$

Then $0 = (M)\ \pi_K \diamond \mathrm{Hom}_R(M/K, M) \diamond \pi_N$ and $K :_M N = M$.

(d)⇒(c) Let K, L be any proper fully invariant submodules of M and assume $(M)\ \pi_K \diamond \mathrm{Hom}_R(M/K, M) \diamond \pi_L = 0$. Then $M = \mathrm{Tr}(M, M) = \mathrm{Tr}(M/K, M) \subset L$. □

3.8. Fully coprime rings. *For the ring R the following are equivalent :*

(a) *${}_RR$ is coprime;*

(b) *${}_RR$ is fully coprime;*

(c) *R is a simple ring.*

3.9. Lemma. *Let M be fully coprime, $S = \mathrm{End}_R(M)$. Then M is indecomposable as (R, S)-bimodule.*

Proof. Assume $M = U \oplus V$ where U, V are nonzero (R, S)-subbimodules of M. Then $\mathrm{Hom}_R(V, U) = 0$. Since M is fully coprime, M is generated by $M/U \simeq V$. It means V also generates U contradicting $\mathrm{Hom}_R(V, U) = 0$. □

3.10. Corollary. *Let M be a fully coprime module. If M is semilocal, then M is homogeneous semisimple.*

Proof. $\mathrm{Rad}(M)$ is a fully invariant submodule of M, hence M is generated by $M/\mathrm{Rad}(M)$ which is semisimple. Thus M is semisimple and now apply Lemma 3.9. $\square$

A fully invariant submodule $N \subset M$ is called *fully coprime in* M if for any fully invariant submodules $K, L \subset M$, $N \subseteq K :_M L$ implies $N \subset K$ or $N \subset L$. By 3.7, M is fully coprime if and only if M is fully coprime in M. An immediate consequence of the definition is (compare with Proposition 3.4):

3.11. Proposition. *If a module M is fully coprime, then M is coprime and fi-coretractable.*

In view of later use for coalgebras we define another type of coproduct, the

3.12. Wedge product. For two proper fully invariant submodules $K, L \subset M$ put

$$\begin{aligned} K \wedge^M L &:= \mathrm{Ker}\ \pi_K \diamond \mathrm{Hom}_R(M/K, M) \diamond \pi_L \diamond \mathrm{Hom}_R(M/L, M) \\ &= \mathrm{Ker}\ (\mathrm{Ann}_S(K) \diamond \mathrm{Ann}_S(L)), \end{aligned}$$

a fully invariant submodule of M. Obviously, $K :_M L \subseteq K \wedge^M L$. If M is a self-cogenerator, then equality holds.

3.13. Proposition. *Consider the following assertions for a module M:*

(a) *M is a fully coprime module;*

(b) *if $K \wedge^M L = M$, then $K = M$ or $L = M$, for any fully invariant submodules K, L of M;*

(c) *$K \wedge^M L \neq M$ for any proper fully invariant submodules K, L of M;*

(d) *$\mathrm{Tr}(M/K, -) = \mathrm{Tr}(M, -)$ for any proper fully invariant submodules K of M, i.e., an M-generated module is also an M/K-generated module.*

Then we have (a)$\Leftrightarrow$(d), (b)$\Leftrightarrow$(c) and (c)$\Rightarrow$(a).

If M is a self-cogenerator, then (d)$\Rightarrow$(c) *and*

$$K :_M L = K \wedge^M L.$$

Proof. (c)$\Leftrightarrow$(b) is trivial and (a)$\Leftrightarrow$(d) is known (see 3.7).

(c)$\Rightarrow$(a) If $K{:_M}L = M$, then $K \wedge^M L = M$.

(d)$\Rightarrow$(c) Assume M to be a self-cogenerator. Let K, L be proper fully invariant submodules of M and assume

$$(M)\ \pi_K \diamond \mathrm{Hom}_R(M/K, M) \diamond \pi_L \diamond \mathrm{Hom}_R(M/L, M) = 0.$$

Since M is a self-cogenerator, we obtain by Lemma 1.2,

$$(M)\ \pi_K \diamond \mathrm{Hom}_R(M/K, M) \subset \mathrm{Ker}\ \pi_L \diamond \mathrm{Hom}_R(M/L, M) = L,$$

and $M = \mathrm{Tr}(M, M) = \mathrm{Tr}(M/K, M) \subseteq L$, a contradiction. $\square$

3.14. Proposition. *Let M be a self-cogenerator and $S = \mathrm{End}_R(M)$. If S is prime, then M is fully coprime.*

Proof. Let K, L be proper fully invariant submodules of M and $M = K :_M L$. Then $(M)\ \pi_K \diamond \mathrm{Hom}_R(M/K, M) \diamond \pi_L \diamond \mathrm{Hom}_R(M/L, M) = 0$. Since S is prime, $\pi_K \diamond \mathrm{Hom}_R(M/K, M) = 0$ or $\pi_L \diamond \mathrm{Hom}_R(M/L, M) = 0$. Hence $K = M$ or $L = M$ since M is a self-cogenerator. □

Let I, J be ideals in $\mathrm{End}_R(M)$ and put Ker $I = K$, Ker $J = L$. Then

$$I \subseteq \mathrm{Hom}_R(M/K, M), \quad J \subseteq \mathrm{Hom}_R(M/L, M). \tag{1}$$

For the converse of Proposition 3.14 the equalities in (1) are of interest.

3.15. Proposition. *Let M be a self-cogenerator and $S = \mathrm{End}_R(M)$.*

(i) *If M is self-injective and fully coprime, then S is prime and M is coprime as a right S-module.*

(ii) *If M is coprime as a right S-module, then M is fully coprime.*

Proof. (i) Let M be a fully coprime module and I, J finitely generated right ideals in S with $IJ = 0$. Put $K = \mathrm{Ker}\, I$ and $L = \mathrm{Ker}\, J$. Then by Lemma 1.2, $\mathrm{Hom}(M/K, M) = I$ and $\mathrm{Hom}(M/L, M) = J$ and $K :_M L = M$. Then $M = K$ or $M = L$, thus $I = 0$ or $J = 0$. Hence the ring S is prime. Since ${}_RM$ is fi-coretractable, M is coprime as a right S-module.

(ii) We assume that M_S is coprime, hence S is prime. Then the assertion follows from Proposition 3.14. □

3.16. Corollary. *If M is a self-injective self-cogenerator and $S = \mathrm{End}_R(M)$, then the following assertions are equivalent:*

(a) *M is fully coprime;*

(b) *M is coprime as a right S-module;*

(c) *S is a prime ring.*

Proof. The equivalence holds by the Propositions 3.14 and 3.15. □

3.17. Proposition. *Let M be a fully coprime module with* $\mathrm{Rad}(M) \neq M$. *Then:*

(i) *M is generated by a module that is cogenerated by a simple module;*

(ii) *for any projective module P in $\sigma[M]$,* $\mathrm{Rad}(P) = 0$.

(iii) $\overline{R} := R/\mathrm{Ann}_R(M)$ *is a left primitive ring.*

Proof. (i) By assumption there is a maximal submodule K in M. Consider the fully invariant submodule $\mathrm{Rej}(M, M/K) \subset K \neq M$. By assumption M is generated by $M/\mathrm{Rej}(M, M/K)$ which is cogenerated by the simple module M/K.

(ii) By (i), P is subgenerated by $M/\mathrm{Rej}(M, M/K)$. Hence P is subgenerated by a product Q of copies of M/K, and $P \subset Q^{(\Lambda)}$, for some index Λ (see [19, 18.4]). Thus P is M/K-cogenerated and $\mathrm{Rad}(P) = 0$.

(iii) By Proposition 2.13, since M fully coprime implies that M is coprime. □

4 Strongly prime and coprime modules

A module M is called *strongly prime* (in Beidar-Wisbauer [3]) if for any non-zero fully invariant submodule $K \subseteq M$, $M \in \sigma[K]$. The regular module ${}_RR$ is strongly prime if and only if R is a *left strongly prime* ring in the sense of Handelman-Lawrence [9]. See [20, 13.3, 13.6] for details.

4.1. Proposition. *Let M be a strongly prime module. Then*

(i) *every M-cogenerated module in $\sigma[M]$ is strongly prime;*

(ii) *every projective module in $\sigma[M]$ is strongly prime;*

(iii) *for every finitely generated projective module P in $\sigma[M]$, $\mathrm{End}_R(P)$ is strongly prime.*

Proof. (i) It is easy to see that M^Λ is strongly prime for any index set Λ. Let U be a non-zero (fully invariant) submodule of an M-cogenerated module N. Then $N \subset M^\Lambda \in \sigma[U]$.

(ii) Consider a projective module P in $\sigma[M]$. Then $P \simeq X \subseteq M^{(\Lambda)}$ for some Λ and hence P is strongly prime.

(iii) Let $J \subset \mathrm{End}_R(P)$ be a proper finitely generated left ideal. Since P is strongly prime (by ii), PJ is a subgenerator in $\sigma[P]$ and hence $P \subset (PJ)^k$, $k \in \mathbb{N}$. Since P is finitely generated projective, $\mathrm{Hom}_R(P, PJ) = J$ (see [19, 18.4]) and

$$\mathrm{End}_R(P) \subset \mathrm{Hom}_R(P, (PJ)^k) \simeq \mathrm{Hom}_R(P, (PJ))^k = J^k$$

showing that $\mathrm{End}_R(P)$ is (left) strongly prime. □

4.2. Proposition. *Let M be a strongly prime module with* $\mathrm{Soc}(M) \neq 0$. *Then*

(i) *M is homogeneous semisimple;*

(ii) $\overline{R} := R/\mathrm{Ann}_R(M)$ *is primitive.*

Proof. (i) Let K be a simple submodule of M. Then $\mathrm{Tr}(K, M)$ is a fully invariant submodule and hence $M \in \sigma[\mathrm{Tr}(K, M)]$. Thus M is K-generated, i.e., homogeneous semisimple.

(ii) It is a consequence of Proposition 2.8, since M is strongly prime implies M is prime. □

For example, modules with descending chain condition for cyclic (finitely generated) submodules have non-zero socle. Thus strongly prime modules with this property are homogeneous semisimple.

We call a module M *strongly coprime* if for any proper fully invariant submodule $K \subset M$, $M \in \sigma[M/K]$.

4.3. Proposition. *If M is a strongly coprime module and K is a proper fully invariant submodule of M, then M/K is strongly coprime.*

Proof. Let K be a fully invariant submodule of M. We take any proper fully invariant submodule U/K of M/K, where $K \subset U \subset M$. By Lemma 1.10, U is fully invariant. Thus $M \in \sigma[M/U]$ and $M/K \in \sigma[M/U] = \sigma[(M/K)/(U/K)]$. □

4.4. Proposition. *Let M be a strongly coprime and semilocal module. Then M is homogeneous semisimple.*

Proof. $\mathrm{Rad}(M)$ is a fully invariant submodule of M, hence M is subgenerated by $M/\mathrm{Rad}(M)$ which is semisimple. Thus M is semisimple and hence every module in $\sigma[M]$ is injective. Now a proof similar to the proof of Lemma 3.9 shows that M is homogeneous semisimple. □

4.5. Proposition. *Let M be a module with* $\mathrm{Rad}(M) \neq M$*. If M is strongly coprime, then*

(i) *M is subgenerated by a product of copies of some simple module;*

(ii) *For any projective module P in $\sigma[M]$,* $\mathrm{Rad}(P) = 0$*;*

(iii) $\overline{R} := R/\mathrm{Ann}_R(M)$ *is primitive.*

Proof. (i) By assumption there is a maximal submodule K in M. Consider the fully invariant submodule $\mathrm{Rej}(M, M/K) \subset K \neq M$. By assumption M is $M/\mathrm{Rej}(M, M/K)$-subgenerated, where $M/\mathrm{Rej}(M, M/K)$ is cogenerated by the simple module M/K (product in $\sigma[M]$). Thus M is $(M/K)^{\Lambda}$-subgenerated for some index set Λ.

(ii) Since M is strongly coprime and $\mathrm{Rad}(M)$ is a fully invariant submodule of M, $M \in \sigma[M/\mathrm{Rad}(M)]$. By definition of the radical, $M/\mathrm{Rad}(M)$ is cogenerated by simple modules (see 14.5 of [19]). The projectivity of P implies that $P \subset (M/\mathrm{Rad}(M))^{(\Lambda)}$ for some index set Λ, and hence $\mathrm{Rad}(P) = 0$.

(iii) By Proposition 2.13, since M strongly coprime implies that M is coprime. □

M is called *duprime* if for any fully-invariant submodule K of M, $M \in \sigma[K]$ or $M \in \sigma[M/K]$ (see [17]). By definition it is clear that any strongly coprime module is duprime. The converse is true for self-injective modules.

4.6. Proposition. *If M is a self-injective R-module, then:*

(i) *M is duprime if and only if it is strongly coprime.*

(ii) *The following are equivalent :*

  (a) *M is fully coprime;*

  (b) *M is strongly coprime;*

  (c) *M is duprime.*

Proof. (i) Let K be a proper fully invariant submodule of M. By assumption, $M \in \sigma[K]$ or $M \in \sigma[M/K]$. If $M \in \sigma[K]$ then M is K-generated, since M is self-injective. But $\mathrm{Tr}(K, M) = K\mathrm{End}_R(M) = K \subset M$, hence M is not K-generated, i.e., $M \notin \sigma[K]$. Thus $M \in \sigma[M/K]$.

(ii) (a)⇔(b) It is clear that if M/K generates M then $M \in \sigma[M/K]$. Now let M/K be a subgenerator of M. Since M is injective, it is M/K-generated. This is based on the fact that any injective module is generated by subgenerators in $\sigma[M]$.

(ii) (b)⇒(c) holds by definition, and (c)⇒(b) by (i). □

5 Comodules and modules

Under weak conditions the category of comodules of an R-coalgebra C, R a commutative ring, can be considered as a module category over the dual algebra $C^* = \mathrm{Hom}_R(C,R)$. Hence the (co-)primeness conditions developed so far can be readily applied to comodules. We will do this in the next sections and relate our results to previously known observations about (co-)primeness of the coalgebra C itself (as in Xu et al. [21], Nekooei-Torkzadeh [12] and Jara et al. [11]) or of C-comodules (in Rodrigues [15] and Ferrero-Rodrigues [8]). We begin with a short outline of basic facts about coalgebras and comodules referring to Brzeziński-Wisbauer [5] for details.

From now R will denote a commutative ring with unit. An R-*coalgebra* is an R-module C with R-linear maps $\Delta : C \to C \otimes_R C$ and $\varepsilon : C \to R$ called (*coassociative*) *coproduct* and *counit* respectively, with the properties

$$(I_C \otimes \Delta) \circ \Delta = (\Delta \otimes I_C) \circ \Delta \quad \text{and} \quad (I_C \otimes \varepsilon) \circ \Delta = I_C = (\varepsilon \otimes I_C) \circ \Delta.$$

For $c \in C$ we use the *Sweedler* notation $\Delta(c) = \sum c_{\underline{1}} \otimes c_{\underline{2}}$.

The dual module $C^* = \mathrm{Hom}_R(C,R)$ is an R-algebra, called the *dual algebra* of C, by the convolution product: for any $f,g \in \mathrm{Hom}_R(C,A)$ put

$$f * g = \mu \circ (f \otimes g) \circ \Delta.$$

A *right* C*-comodule* is an R-module M with an R-linear map $\varrho^M : M \to M \otimes_R C$ called a *right* C*-coaction*, with the properties

$$(I_M \otimes \Delta) \circ \varrho^M = (\varrho^M \otimes I_C) \circ \varrho^M \text{ and } (I_M \otimes \varepsilon) \circ \varrho^M = I_M.$$

Denote by $\mathrm{Hom}^C(M,N)$ the set of C-comodule morphisms from M to N. The class of right comodules over C together with the comodule morphisms form an additive category which is denoted by $\mathbf{M}^C$.

Similar to the classical Hom-tensor relations (e.g. [19]), there is a functorial isomorphism for $M \in \mathbf{M}^C$ and $X \in {}_R\mathbf{M}$ (see [5, 3.10]),

$$\phi : \mathrm{Hom}^C(M, X \otimes_R C) \to \mathrm{Hom}_R(M,X), \quad f \mapsto (I_X \otimes \varepsilon) \circ f,$$

with inverse map $h \mapsto (h \otimes I_X) \circ \rho^M$. For $X = R$ and $M = C$, ϕ yields $\mathrm{End}^C(C) \simeq C^*$.

Any $M \in \mathbf{M}^C$ is a (unital) left C^*-module by

$$\rightharpoonup : C^* \otimes_R M \to M, \quad f \otimes m \mapsto (I_M \otimes f) \circ \varrho^M(m),$$

and any morphism $h : M \to N$ in $\mathbf{M}^C$ is a left C^*-module morphism, i.e.,

$$\mathrm{Hom}^C(M,N) \subset {}_{C^*}\mathrm{Hom}(M,N).$$

C is a subgenerator in $\mathbf{M}^C$, that is all C-comodules are subgenerated by C as C-comodules and C^*-modules. Thus we have a faithful functor from $\mathbf{M}^C$ to ${}_{C^*}\mathbf{M}$, where the latter denotes the category of left C^*-modules.

C *satisfies the* α*-condition* if the following map is injective for every $N \in \mathbf{M}_R$,

$$\alpha_N : N \otimes_R C \to \mathrm{Hom}_R(C^*, N), \quad n \otimes c \mapsto [f \mapsto f(c)n].$$

$\mathbf{M}^C$ is a full subcategory of ${}_{C^*}\mathbf{M}$ if and only if C satisfies the α-condition, and then $\mathbf{M}^C$ is isomorphic to $\sigma[{}_{C^*}C]$ (see [5, Proposition 4.3]). This provides the possibility to apply the results of primeness and coprimeness in categories of type $\sigma[M]$ to the category $\mathbf{M}^C$. In particular, $\mathbf{M}^C$ is a Grothendieck category.

Throughout C will be an R-coalgebra which satisfies the α-condition.

Let $f : M \to N$ be a comodule morphism and $U \subset M$ and $L \subset N$ be subcomodules. Then $(U)f \subset N$ and $(L)f^{-1} \subset M$ are again subcomodules ($=C^*$-submodules) and Lemma 1.10 yields:

5.1. Lemma. *Let M, N be C-comodules, $f \in \mathrm{Hom}^C(M, N)$ an epimorphism and assume the coalgebra C satisfies the α-condition.*

(i) *If $\mathrm{Ker}\, f$ is a fully invariant subcomodule of M and L is a fully-invariant subcomodule of N, then $(L)f^{-1}$ is a fully-invariant subcomodule of M.*

(ii) *If M is self-projective and U is a fully invariant subcomodule of M, then $(U)f$ is a fully-invariant subcomodule of N.*

5.2. Orthogonal sets. For any submodule $A \subset C$ and any subset $I \subset C^*$, put

$$\begin{aligned} A^{\perp C^*} &:= \{f \in C^* |\ (A)f = 0\} = \pi_A \diamond \mathrm{Hom}_R(C/A, R) \subset C^*, \\ I^{\perp C} &:= \bigcap \{\mathrm{Ker}\, f |\ f \in I\} \subset C, \end{aligned}$$

where $\pi_A : C \to C/A$ is the canonical projection.

For any R-submodule A of C, $A \subset (A^{\perp C^*})^{\perp C}$ always holds. To get the equality, we need special properties (see [19, 28.1]).

5.3. Lemma. *If every R-factor module of C is R-cogenerated, then for any R-submodule A of C, $\mathrm{Ker}\,\mathrm{Hom}_R(C/A, R) = A$. Thus $A = (A^{\perp C^*})^{\perp C}$.*

Further properties of the annihilator and the kernel are given in [5, 6.2 and 6.3].

For subsets $A \subseteq C$ and $I \subseteq C^*$ we also have the annihilators

$$\mathrm{Ann}_{C^*}(A) := \{f \in C^* \mid f \rightharpoonup A = 0\}, \quad \mathrm{Ann}_C(I) := \{c \in C \mid I \rightharpoonup c = 0\},$$

with the following relationships:

5.4. Lemma. *Let C be an R-coalgebra and $A \subset C$.*

(i) *If A is an R-submodule of C, then $\mathrm{Ann}_{C^*}(A) \subset A^{\perp C^*}$.*

(ii) *If A is a left and right subcomodule (a left and right C^*-submodule) of C, then $A^{\perp C^*} = \mathrm{Ann}_{C^*}(A)$.*

Proof. (i) Let $g \in \mathrm{Ann}_{C^*}(A)$. Then $g \rightharpoonup a = 0$ for any $a \in A$ and $(I_C \otimes g) \circ \Delta(a) = 0$. Applying ε we obtain $0 = \varepsilon((I_C \otimes g) \circ \Delta(a)) = g(a)$. Thus $\mathrm{Ann}_{C^*}(A) \subset A^{\perp C^*}$.

(ii) It is sufficient to prove that $A^{\perp C^*} \subset \mathrm{Ann}_{C^*}(A)$. Take any $f \in A^{\perp C^*}$, that is $f(a) = 0$ for any $a \in A$. Then

$$f \rightharpoonup a = \sum (I_C \otimes_R f) \circ \Delta(a) = \sum a_{\underline{1}} f(a_{\underline{2}}) = 0,$$

since $\Delta(a) \in C \otimes_R A$. Thus $A^{\perp C^*} \subset \mathrm{Ann}_{C^*}(A)$. □

Notice that for any subcoalgebra A of C holds $A^{\perp C^*} = \mathrm{Ann}_{C^*}(A)$, since A is a left and right subcomodule of C.

5.5. Lemma. *Let C be an R-coalgebra and $J \subset C^*$.*

(i) *If J is an R-submodule of C^*, then $\mathrm{Ann}_C(J) \subset J^{\perp C}$.*

(ii) *If J is a two-sided ideal of C^*, then $J^{\perp C} = \mathrm{Ann}_C(J)$.*

Proof. (i) Let $c \in \mathrm{Ann}_C(J)$. Then $h \rightharpoonup c = 0$ for any $h \in J$ and $(I_C \otimes h) \circ \Delta(c) = 0$. Applying ε we obtain $0 = \varepsilon((I_C \otimes h) \circ \Delta(c)) = h(c)$. Thus $\mathrm{Ann}_C(J) \subset J^{\perp C}$.

(ii) By [5, 6.3], $J^{\perp C}$ is a (C^*, C^*)-subbimodule of C. It is sufficient to prove that $J^{\perp C} \subset \mathrm{Ann}_C(J)$. Take any $c \in J^{\perp C}$, that is $l(c) = 0$ for any $l \in J$. Then

$$l \rightharpoonup c = \sum (I_C \otimes_R l) \circ \Delta(c) = \sum c_{\underline{1}} l(c_{\underline{2}}) = 0,$$

since $\Delta(c) \in C \otimes_R J^{\perp C}$. Thus $J^{\perp C} \subset \mathrm{Ann}_C(J)$. □

5.6. Lemma. *For any proper (C^*, C^*)-subbimodule A of C, if C/A is R-cogenerated, then $\mathrm{Ann}_{C^*}(A) \neq 0$ and equivalently C satisfies condition $(**)$ as a C^*-module. Thus if R is a cogenerator in ${}_R\mathbf{M}$, then $\mathrm{Ann}_{C^*}(A) \neq 0$.*

Proof. Since C/A is R-cogenerated, there is some $\tilde{f} : C/A \to R$ such that $0 \neq f := \pi_A \diamond \tilde{f} : C \to R$. Thus $\mathrm{Ann}_{C^*}(A) \neq 0$. □

6 Prime and coprime comodules

We call a right C-comodule M *prime* if M is prime as a left C^*-module.

6.1. Prime comodules. *For a right C-comodule M, the following are equivalent:*

(a) *M is a prime comodule;*

(b) *$\mathrm{Ann}_{C^*}(K) = \mathrm{Ann}_{C^*}(M)$ for any subcomodule K of M;*

(c) *$\overline{C^*} := C^*/\mathrm{Ann}_{C^*}(M)$ is cogenerated by K for any subcomodule K of M;*

(d) *$\overline{C^*}$ is cogenerated by K for any fully invariant subcomodule K of M.*

If these conditions hold, then:

(i) *$\overline{C^*}$ is a prime algebra and a right C-comodule that is finitely generated as an R-module.*

(ii) *$R/\mathrm{Ann}_R(\overline{C^*})$ is a prime ring.*

Proof. From 2.1 we have the equivalences stated.

(i) Consider a cyclic $\overline{C^*}$-submodule of M, say $U := \overline{C^*} \rightharpoonup m$, for some $m \in M$. According to the Finiteness Theorem 2 ([5],4.12), U is a finitely generated R-module with generators say $\{u_1, \ldots, u_k\}$. Define a mapping

$$\varphi : \overline{C^*} \to U^k, \quad f \mapsto (f \rightharpoonup u_1, \ldots, f \rightharpoonup u_k) = f \rightharpoonup (u_1, \ldots, u_k) \in U^k.$$

If $f \rightharpoonup (u_1, \ldots, u_k) = 0$, then $f \rightharpoonup u_i = 0$ and $f \rightharpoonup (r_1 u_1 + \ldots + r_k u_k) = 0$ for any $r_i \in R$, $i = 1, 2, \ldots, k$. Thus $f \rightharpoonup U = 0$, i.e., $f \in \mathrm{Ann}_{\overline{C^*}}(U) = 0$, since M is prime. It follows that the map φ is a monomorphism and $\overline{C^*}$ is a right C-comodule. Moreover, it is cyclic as a left C^*–module and, by the Finiteness Theorem 2, this implies that it is finitely generated as an R-module.

(ii) Since R is commutative and $\overline{C^*}$ is prime, $R/\mathrm{Ann}_R(\overline{C^*})$ is a prime ring. □

6.2. Proposition. *Let M be a comodule with* $\mathrm{Soc}(M) \neq 0$. *If M is prime, then $\overline{C^*}$ is a simple artinian algebra and finitely generated as an R-module. Thus M is a homogeneous semisimple comodule.*

Proof. Consider a simple $\overline{C^*}$-submodule of M, say $V := \overline{C^*} \rightharpoonup m$, for some $m \in M$. Then by 6.1 part (i), $\overline{C^*}$ is a direct summand of the homogeneous semisimple C^*-module V^k, i.e., $\overline{C^*}$ is a simple artinian algebra. If M is a faithful C^*-module, then C^* is a simple artinian algebra and finitely generated as an R-module. □

Proposition 3.2 of [8] is a corollary of our Proposition 6.2.

6.3. Corollary. *If R is a perfect ring and M is a prime comodule over the R-coalgebra C, then $C^*/\mathrm{Ann}_{C^*}(M)$ is a simple artinian algebra.*

Proof. If R is a perfect ring, then M satisfies the descending chain condition for finitely generated R-submodules and hence for finitely generated C-subcomodules (see [5, 4.16]). Thus $\mathrm{Soc}(M) \neq 0$ and Proposition 6.2 applies. □

6.4. Prime coalgebras. *For a coalgebra C, the following are equivalent :*

(a) *C is prime as a right C-comodule.*

(b) $\mathrm{Ann}_{C^*}(A) = 0$ *for any non-zero right subcomodule A of C.*

(c) *C^* is cogenerated by A for any non-zero right subcomodule A of C.*

(d) *C^* is cogenerated by A for any non-zero (C^*, C^*)-subbimodule A of C.*

If these conditions hold, then:

(i) *C is a finitely generated R-module and* $\mathbf{M}^C = {}_{C^*}\mathbf{M}$.

(ii) *Every C-cogenerated comodule is prime.*

(iii) *Every projective comodule P is prime.*

(iv) *C^* is a prime algebra and finitely generated as R-module.*

Proof. Considering C as a right C-comodule we get the equivalences and (iv) from 6.1. (i) By 6.1, $C^* \in \mathbf{M}^C$ and hence C is finitely generated as R-module. Now apply [5, 4.7]. (ii)-(iii) Transfer the situation of Proposition 2.5 into $\mathbf{M}^C$. □

6.5. Proposition. *Let C be a coalgebra that is prime as right C-comodule with* $\mathrm{Soc}(C) \neq 0$. *Then*

(i) *C^* is a simple algebra and finitely generated as R-module.*

(ii) *If C is cocommutative, then C^* is a field.*

Proof. (i) By Proposition 6.2. (ii) By Corollary 2.8. □

If C is prime as right C-comodule, then for any fully invariant subbicomodule A of C, $\mathrm{Ann}_{C^*}(A) = A^{\perp C^*} = 0$ (see Lemma 5.4).

6.6. Example. Let R be a commutative ring. Consider the free R-module $T := R^n$, where $n \in \mathbb{N}$, T is a finitely generated and projective R-module. $C := T^* \otimes_R T$ is a coalgebra (see [7]) and moreover, as R-module,

$$C = (R^n)^* \otimes_R R^n \simeq \mathrm{End}_R(R^n_R),$$

that is the matrix coalgebra of all $n \times n$ matrices over R, which we denote as $\mathbb{M}^n(R)$. Let $\{e_{ij}\}_{1 \leq i,j \leq n}$ be the canonical basis of $\mathbb{M}^n(R)$. Then the coproduct and counit of C are

$$\begin{aligned} \Delta : \mathbb{M}^n(R) &\rightarrow \mathbb{M}^n(R) \otimes_R \mathbb{M}^n(R), \ e_{ij} \mapsto \sum_{k=1}^{n} e_{ik} \otimes_R e_{kj}. \\ \varepsilon : \mathbb{M}^n(R) &\rightarrow R, \ \ e_{ij} \mapsto \delta_{ij}. \end{aligned}$$

For the dual algebra of C, there are (anti-)algebra morphisms

$$C^* \simeq \mathrm{End}_R((R^n)^*_R) \simeq \mathrm{End}_R((R^*)^n_R) \simeq \mathbb{M}_n(\mathrm{End}_R(R^*_R)) \simeq \mathbb{M}_n(R),$$

the matrix ring of all $n \times n$ matrices over R. Thus if R is prime then C^* is a prime algebra. The fully invariant subcomodules of $C = \mathbb{M}^n(R)$ are (C^*, C^*)-subbimodules, that is the two-sided ideals of $\mathbb{M}_n(R)$, and hence are of the form $\mathbb{M}_n(I)$, where I is an ideal of R. Since $\mathrm{Ann}_{\mathbb{M}_n(R)}(\mathbb{M}_n(R)/\mathbb{M}_n(I)) \neq 0$, C satisfies $(*fi)$, and thus C is prime as C^*-module (by Proposition 2.2 (ii)).

We call a right C-comodule M *coprime* if M is coprime as a C^*-module. Here 2.9 and Lemma 2.10 read as follows.

6.7. Coprime comodules. *Let M be a right C-comodule with $S = \mathrm{End}^C(M)$.*

(i) *The following assertions are equivalent :*
  - (a) *M is a coprime comodule;*
  - (b) $\mathrm{Ann}_{C^*}(M/K) = \mathrm{Ann}_{C^*}(M)$ *for any proper subcomodule K of M;*
  - (c) $C^*/\mathrm{Ann}_{C^*}(M)$ *is cogenerated by M/K for any proper subcomodule K of M;*
  - (d) $C^*/\mathrm{Ann}_{C^*}(M)$ *is cogenerated by M/K for any proper fully invariant subcomodule K of M.*

(ii) *If M is coprime, then $C^*/\mathrm{Ann}_{C^*}(M)$ is prime.*

(iii) *If $C^*/\mathrm{Ann}_{C^*}(M)$ is prime and for any proper (fully invariant) subcomodule K of M holds* $\mathrm{Ann}_{C^*}(K) \neq \mathrm{Ann}_{C^*}(M)$*, then M is coprime.*

6.8. Proposition. *Let M be a right C-comodule. If M is coprime and* $\mathrm{Rad}(M) \neq M$*, then $\overline{C^*} := C^*/\mathrm{Ann}_{C^*}(M)$ is a simple algebra and finitely generated as R-module.*

Proof. By Proposition 2.13, $\overline{C^*} := C^*/\mathrm{Ann}_{C^*}(M)$ is a primitive algebra. Then the proof is similar to 6.2. □

6.9. Coprime coalgebras. *Let C be a coalgebra.*

(1) *The following are equivalent :*

   (a) *C is coprime as a right C-comodule;*

   (b) $\mathrm{Ann}_{C^*}(C/A) = 0$ *for any proper right subcomodule A of C;*

   (c) *C^* is cogenerated by C/A for any proper right subcomodule A of C;*

   (d) *C^* is cogenerated by C/A for any proper (C^*, C^*) subbimodule A of C.*

(2) *If the conditions* (a)-(d) *hold, then*

   (i) *C^* is prime;*

   (ii) *if C is cocommutative, then C^* is an integral domain;*

   (iii) *for any proper fully-invariant (C^*, C^*)-subbimodule A of C, C/A is coprime as C^*-module;*

   (iv) $\mathrm{Rad}(C) \neq C$ *implies that C^* is a simple algebra and finitely generated as R-module;*

   (v) *if C is cocommutative with* $\mathrm{Rad}(C) \neq C$, *then C^* is a field.*

(3) *If C^* is prime and* $\mathrm{Ann}_{C^*}(A) \neq 0$ *for any proper (C^*, C^*)-subbimodule A of C, then C is coprime as a right C-comodule.*

Proof. (1) is obtained by applying 6.7.

(2)(i) and (3) follow from 6.7; (ii) C^* is prime and commutative; (iii) and (iv) follow by Proposition 6.8, and (v) is a consequence of (iv). □

A coalgebra C with C^* prime may not be coprime if it does not satisfy the condition $(**)$, i.e., for any proper (C^*, C^*)-subbimodule A of C holds $\mathrm{Ann}_{C^*}(A) \neq 0$. We can see this in the following example.

6.10. Example. Let H be a free R-module with basis $\{c_m \mid m \in \mathbb{N}\}$. Define the comultiplication $c_m \mapsto \sum_{i=0,m} c_i \otimes c_{m-i}$ and the counit by $\varepsilon(c_m) = \delta_{0,m}$. H is a coalgebra and the dual algebra H^* has multiplication for $f, g \in H^*$, $(f * g)(c_m) := \sum_{i=0,m} f(c_i)g(c_{m-i})$ and unit $u : R \to H^*$ where $u(\alpha)(c_m) = \alpha\delta_{0,m}$ for any $\alpha \in R$, $m \in \mathbb{N}$. There is an isomorphism (see Example 1.3.8 of [6]) $\Phi : H^* \to R[[X]]$, where $f \mapsto \sum_{m \geq 0} f(c_m)X^m$. Notice that the formal power series ring $R[[X]]$ is prime provided R is a prime ring. As a special case one may take $H = R[X]$ with comultiplication

$$\Delta : R[X] \to R[X] \otimes_R R[X], \quad X^m \mapsto \sum_{i=0,m} X^i \otimes X^{m-i},$$

and the counit is $\varepsilon(X^m) = \delta_{0,m}$. If R is a field, then $R[[X]]$ is a prime ring, and $R[X]$ is a coprime comodule by 6.9 part (3). For $R = \mathbb{Z}$, the primeness of $\mathbb{Z}[[X]]$ does not imply the coprimeness of $\mathbb{Z}[X]$, since for the subcomodule $n\mathbb{Z}[X]$, for $0 \neq n \in \mathbb{N}$, we have $\mathrm{Ann}_{\mathbb{Z}[X]^*}(\mathbb{Z}[X]/n\mathbb{Z}[X]) \neq 0$.

7 Fully prime and coprime comodules

We call a comodule M *fully prime* if for any non-zero fully invariant subcomodule K of M, M is K-cogenerated.

As in Section 3, a product of fully invariant subcomodules K, L of M is defined by $K *_M L := K\mathrm{Hom}^C(M, L)$.

7.1. Fully prime comodules. *The following are equivalent for a comodule M:*

(a) *M is a fully prime comodule;*

(b) $\mathrm{Rej}(M, K) = 0$ *for any non-zero fully invariant subcomodule $K \subset M$;*

(c) $K *_M L \neq 0$ *for any non-zero fully invariant subcomodules $K, L \subset M$;*

(d) $\mathrm{Rej}(-, M) = \mathrm{Rej}(-, K)$ *for any non-zero fully invariant subcomodule K of M, i.e., any M-cogenerated comodule is also K-cogenerated.*

If these conditions hold and $\mathrm{Soc}(M) \neq 0$*, then*

(i) *M is semisimple;*

(ii) $\overline{C^*} := C^*/\mathrm{Ann}_{C^*}(M)$ *is a simple artinian algebra and finitely generated as R-module.*

Proof. From 3.1 we get the equivalences stated. M is fully prime implies that it is prime. Now apply Proposition 6.2 to get (i) and (ii). □

A fully invariant subcomodule N of M is called *fully prime in M* if for any fully invariant subcomodules K, L of M,

$$K *_M L \subseteq N \text{ implies } K \subseteq N \text{ or } L \subseteq N.$$

The comodule M is fully prime if zero is a fully prime subcomodule.

Applying Proposition 18 of [13] and Lemma 5.1 yield

7.2. Proposition. *Let N be a proper fully-invariant subcomodule of M.*

(i) *If N is fully prime in M, then M/N is a fully prime comodule.*

(ii) *If M is self-projective and M/N is fully prime, then N is fully prime in M.*

For $M = C$ the assertions in 7.1 yield

7.3. Fully prime coalgebras. *The following are equivalent for a coalgebra C:*

(a) *C is fully prime as a right C-comodule;*

(b) $\mathrm{Rej}(C, A) = 0$ *for any non-zero (C^*, C^*)-subbimodule A of C;*

(c) $A *_C B \neq 0$ *for any non-zero (C^*, C^*)-subbimodules A, B of C;*

(d) $\mathrm{Rej}(-, C) = \mathrm{Rej}(-, A)$ *for any non-zero (C^*, C^*)-subbimodule A of C, i.e., any C-cogenerated coalgebra is also A-cogenerated.*

If these conditions hold and $\mathrm{Soc}(C) \neq 0$*, then*

(i) *C is a semisimple C^*-module;*

(ii) *C^* is a simple artinian algebra and finitely generated as R-module.*

We call a comodule M *fully coprime* if for any proper fully invariant subcomodule K of M, M is M/K-generated.

For fully invariant subcomodules $K, L \subset M$, we have the internal coproduct

$$K :_M L := \bigcap \{(L)f^{-1} \mid f \in \mathrm{End}^C(M), K \subseteq \mathrm{Ker} f\}.$$

7.4. Fully coprime comodules. *Let M be a C-comodule and $S = \mathrm{End}^C(M)$. The following are equivalent:*

(a) *M is a fully coprime comodule;*

(b) *If $K :_M L = M$, then $K = M$ or $L = M$, for any fully invariant subcomodules K, L of M;*

(c) *$K :_M L \neq M$ for any proper fully invariant subcomodules K, L of M;*

(d) *$\mathrm{Tr}(M/K, -) = \mathrm{Tr}(M, -)$ for any proper fully invariant subcomodules K of M, i.e. any M-generated comodule is also M/K-generated.*

If these conditions hold, then

(i) *M is coprime, fi-coretractable and indecomposable as (C^*, S)-bimodule;*

(ii) *if $\mathrm{Rad}(M) \neq M$, then $\overline{C^*} := C^*/\mathrm{Ann}_{C^*}(M)$ is a simple algebra and finitely generated as R-module. Moreover, M is homogeneous semisimple.*

Proof. By 3.7 we get the equivalences stated.

(i) follows by Proposition 3.11 and Lemma 3.9.

(ii) By Proposition 3.17 part (iii), $\overline{C^*}$ is a primitive algebra. Then the proof is similar to 6.2. □

Let M be a comodule and $N \subset M$ be a fully invariant subcomodule. We say that N is *fully coprime in M* if for any fully invariant subcomodules $K, L \subset M$, $N \subseteq K :_M L$ implies $N \subset K$ or $N \subset L$.

By 7.4, M is a fully coprime comodule if and only if M is fully coprime in M.

7.5. Proposition. *Let M be a self-cogenerator right comodule and $S = \mathrm{End}^C(M)$.*

(i) *If S is prime, then M is fully coprime.*

(ii) *If M is self-injective and fully coprime, then M is coprime as a right S-module and hence S is prime.*

(iii) *If M is coprime as a right S-module, then M is fully coprime.*

Proof. By Lemma 3.14 and Lemma 3.15. □

7.6. Fully coprime coalgebras. *The following are equivalent for a coalgebra C.*

(a) *C is fully coprime as a right C-comodule;*

(b) *If $A :_C B = C$, then $A = C$ or $B = C$, for any (C^*, C^*)-subbimodules A, B of C;*

(c) *$A :_C B \neq C$ for any proper (C^*, C^*)-subbimodules A, B of C;*

(d) *$\mathrm{Tr}(C/A, -) = \mathrm{Tr}(C, -)$ for any proper (C^*, C^*)-subbimodule A of C, i.e. any C-generated coalgebra is also C/A-generated.*

If these conditions hold, then

(i) *C is indecomposable as (C^*, C^*)-bimodule;*

(ii) *if $\mathrm{Rad}(C) \neq C$, then C^* is a simple algebra and finitely generated as R-module;*

(iii) *for any projective comodule P in $\mathbf{M}^C$, $\mathrm{Rad}(P) = 0$.*

Proof. Put $M = C$ and apply 7.4. For (iii) see Proposition 3.17 part (ii). □

8 Strongly prime and strongly coprime comodules

We call a comodule M *strongly prime* if it is strongly prime as C^*-module. This property extends to the self-injective hull and we have

8.1. Strongly prime comodules. *For a comodule M denote by $\widehat{M}$ its injective hull in $\mathbf{M}^C$. The following are equivalent:*

(a) *M is a strongly prime comodule;*

(b) *M is subgenerated by each of its non-zero subcomodules;*

(c) *$\widehat{M}$ is generated by each of its nonzero (fully-invariant) subcomodules;*

(d) *$\widehat{M}$ has no non-trivial fully invariant subcomodules.*

If these conditions hold and $\mathrm{Soc}M \neq 0$*, then*

(i) *M is homogeneous semisimple;*

(ii) *$\overline{C^*} := C^*/\mathrm{Ann}_{C^*}(M)$ is a simple algebra and finitely generated as R-module.*

Proof. Apply [20, 13.3] to get the equivalences stated. (i) and (ii) follow by Proposition 4.2 and "finitely generated" is obtained similarly to the proof of Proposition 6.2. □

As a corollary of 8.1 we obtain [8, Theorem 3.3 and Corollary 3.5], since R is a perfect ring implies $\mathrm{Soc}(M) \neq 0$. For $M = C$, 8.1 yields:

8.2. Strongly prime coalgebras. *For a coalgebra C with C-injective hull $\widehat{C}$, the following are equivalent:*

(a) *C is strongly prime as a right C-comodule;*

(b) *$\widehat{C}$ is generated by each of its nonzero subcomodules;*

(c) *$\widehat{C}$ has no non-trivial (C^*, C^*)-subbimodule.*

If these conditions hold and $\mathrm{Soc}(C) \neq 0$*, then*

(i) *C is homogeneous semisimple;*

(ii) *C^* is simple and finitely generated as an R-module;*

(iii) *every comodule is strongly prime;*

(iv) *for any finitely generated projective comodule P, $\mathrm{End}^C(P) \simeq \mathrm{End}_{C^*}(P)$ is strongly prime.*

Proof. (iii) and (iv) follow by Proposition 4.1. □

For any two fully invariant subcomodules $A, B \subset C$ we have from 3.12,

$$A \wedge^C B = \mathrm{Ker}\,\pi_A \diamond \mathrm{Hom}^C(C/A, M) \diamond \pi_B \diamond \mathrm{Hom}^C(C/B, C).$$

This can be written in the form known as *wedge product* in coalgebra theory.

8.3. Lemma. *For any (proper) subbicomodules $A, B \subset C$,*

$$\mathrm{Ker}\,\Delta \diamond (\pi_A \otimes \pi_B) \subseteq A \wedge^C B = (A^{\perp C^*} * B^{\perp C^*})^{\perp C}.$$

Equality holds provided R is a field.

Proof. We first show the identity. For this let

$$u \in \operatorname{Ker} \pi_A \diamond \operatorname{Hom}^C(C/A, C) \diamond \pi_B \diamond \operatorname{Hom}_C(C/B, C)$$

and take any $\tilde{f} \in (C/A)^*$, $\tilde{g} \in (C/B)^*$. Then there are
$\alpha := \Delta \diamond (\pi_A \diamond \tilde{f} \otimes I_C) \in \pi_A \diamond \operatorname{Hom}^C(C/A, C)$,
$\beta := \Delta \diamond (\pi_B \diamond \tilde{g} \otimes I_C) \in \pi_B \diamond \operatorname{Hom}^C(C/B, C)$, such that

$$(u)(\pi_A \diamond \tilde{f}) * (\pi_B \diamond \tilde{g}) = (u)(\alpha \diamond \varepsilon) * (\beta \diamond \varepsilon) = (u)(\alpha \diamond \beta) \diamond \varepsilon = 0.$$

Conversely, let $v \in ((C/A)^* * (C/B)^*)^{\perp C}$. Then $v \in \operatorname{Ann}_C((C/A)^* * (C/B)^*)$ (by Lemma 5.5) and we have

$$\begin{aligned} 0 &= (v)\Delta \diamond (I_C \otimes \pi_A \diamond \tilde{f} * \pi_B \diamond \tilde{g}) \\ &= (v)\Delta \diamond (I_C \otimes \alpha \diamond \varepsilon * \beta \diamond \varepsilon) \\ &= (v)\Delta \diamond (I_C \otimes \alpha \diamond \beta) \diamond (I_C \otimes \varepsilon) \\ &= (v)(\alpha \diamond \beta) \diamond \Delta \diamond (I_C \otimes \varepsilon) \;=\; (v)(\alpha \diamond \beta). \end{aligned}$$

The left inclusion is obvious. To show equality (see also [16]), let R be a field and let $d \in (A^{\perp C^*} * B^{\perp C^*})^{\perp C}$. For all $f \in A^{\perp C^*}$ and $g \in B^{\perp C^*}$, writing $f = \pi_A \diamond \tilde{f}$ and $g = \pi_B \diamond \tilde{g}$, where $\tilde{f} \in (C/A)^*$ and $\tilde{g} \in (C/B)^*$, we get

$$0 = (d)(f * g) = (d)\Delta \diamond (f \otimes g) = (d)\Delta \diamond (\pi_A \otimes \pi_B) \diamond (\tilde{f} \otimes \tilde{g}).$$

As known from Linear Algebra this implies $(d)\Delta \diamond (\pi_A \otimes \pi_B) = 0$ and therefore $d \in \operatorname{Ker} \Delta \diamond (\pi_A \otimes \pi_B)$. □

8.4. Corollary. *Let C be an R-coalgebra, R a perfect ring. Then the following assertions are equivalent:*

(a) *C is prime as a right C-comodule.*

(b) *C is strongly prime as a right C-comodule.*

(c) *C is homogeneous semisimple.*

(d) *C^* is a simple ring and a finitely generated R-module.*

We call a comodule M *strongly coprime* if it is strongly coprime as C^*-module.

8.5. Strongly coprime comodules. *Let M be a strongly coprime comodule.*

(1) *For any proper fully-invariant subcomodule K of M, M/K is strongly coprime.*

(2) *If $\operatorname{Rad}(M) \neq M$, then*

  (i) *M is homogeneous semisimple;*

  (ii) *$\overline{C^*} := C^*/\operatorname{Ann}_{C^*}(M)$ is a simple algebra and finitely generated as R-module.*

Proof. (1) By Proposition 4.3.

(2) By Proposition 4.5, $\overline{C^*}$ is a primitive algebra. Then similar to the proof of 6.2 we get that $\overline{C^*}$ is a simple algebra. □

Recall that a comodule M is *duprime* if for any fully invariant subcomodule $K \subset M$, $M \in \sigma[K]$ or $M \in \sigma[M/K]$. By definition it is obvious that any strongly coprime comodule is duprime. The converse is true for self-injective comodules.

As a corollary of 8.5 we get:

8.6. Strongly coprime coalgebras. *Let C be strongly coprime as a right C-comodule. If* $\mathrm{Rad}(C) \neq C$, *then*

(i) *C is homogeneous semisimple;*

(ii) *C^* is a simple algebra and finitely generated as R-module.*

Coprime coalgebras over a field k, defined by the wedge product, are considered for example, in Jara, Merino, Ruiz [11] and Nekooei-Torkzadeh [12]. Xu, Lu, Zhu [21] describe coalgebras with prime dual algebra. Some of their results are included in the following list.

For the moment let us call a coalgebra C *wedge coprime* if for any subcoalgebras A, B of C, $A \wedge^C B = C$ implies $A = C$ or $B = C$ (see [11], [12]).

Coalgebras C over a field R are injective cogenerators and for any subcoalgebras A, B (see 8.3),

$$\mathrm{Ker}\, \Delta \diamond (\pi_A \otimes \pi_B) = A :_C B = A \ \wedge^C \ B = (A^{\perp C^*} * B^{\perp C^*})^{\perp C}.$$

8.7. Coprime coalgebras over fields. *For a coalgebra C over a field R, the following are equivalent:*

(a) *C is a wedge coprime coalgebra;*

(b) *C is coprime as a left (right) C-comodule;*

(c) *C is fully coprime as a right C-comodule;*

(d) *C^* is a prime algebra.*

(e) *For any proper (C^*, C^*)-subbimodule A of C,* $\mathrm{Hom}^C(C, A) = 0$.

(f) *For any* $0 \neq f \in C^*$, $C = C \leftharpoonup (f * C^*)$.

(g) *For any ideal* $0 \neq I \subset C^*$, $C = C \leftharpoonup I$.

(h) *For any* $0 \neq f \in C^*$, $C = (C^* * f) \rightharpoonup C$.

(i) *For any ideal* $0 \neq I \subset C^*$, $C = I \rightharpoonup C$.

Proof. The coalgebra C over a field R is a self-cogenerator and hence fi-coretractable. By the isomorphism $\mathrm{End}^C(C) \simeq C^*$ we have $(C)\mathrm{End}^C(C) = C \leftharpoonup C^*$. Now apply Proposition 1.6 to get the equivalences (d)⇔(e)⇔(f)⇔(h).

(a)⇔(c) Over a field, the wedge product $\wedge^C$ and $:_C$ coincide.

(b)⇔(c)⇔(d) C is a self-injective self-cogenerator; now apply the Propositions 3.14, 3.15 and Corollary 3.16.

(d)⇔(h) is symmetric to (d)⇔(f). It also follows by [21, Theorem 3].

(d)⇔(i) is symmetric to (d)⇔(g). □

Recall that our definition of coprimeness is related to fully invariant subcomodules of C. To conclude we observe some properties related to not necessarily fully invariant subcomodules of the coalgebra C.

8.8. Proposition. *Let C be a coalgebra over a field R. The following assertions are equivalent:*

(a) *C^* has no zero-divisor;*

(b) *For any proper left C^*-submodule A of C, $\mathrm{Hom}^C(C, A) = 0$;*

(c) *For any $0 \neq f \in C^*$, $C \leftharpoonup f = C$.*

Proof. The coalgebra C over a field R is a self-cogenerator and hence coretractable. By the isomorphism $\mathrm{End}^C(C) \simeq C^*$ we have $(C)\mathrm{End}^C(C) = C \leftharpoonup C^*$. Now apply Proposition 1.5.

The implication (a)⇔(c) is also shown in [21, Corollary]. □

For explicit examples of (wedge) coprime coalgebras we refer to [12] and [11]. For an extension of these and similar investigations for *corings* we refer to Abuhlail [1].

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