ANNALES
POLONICI MATHEMATICI
XX (1968)

On cosine and sine functional equations

by PL. KANNAPPAN (Annamalai Nagar)

Introduction. In this paper, the functional equations

$$f(xy)+f(xy^{-1}) = 2f(x)f(y) \tag{A}$$

and

$$f(xy)f(xy^{-1}) = f(x)^2-f(y)^2, \tag{C}$$

where f is a complex-valued function on an arbitrary group G, are considered. Let g be a homomorphism of G into the multiplicative group of the complex numbers, K. Then, it is evident that

$$f(x) = \frac{g(x)+g^*(x)}{2} \tag{B}$$

and

$$f(x) = \frac{g(x)-g^*(x)}{2} \tag{D}$$

are solutions of (A) and (C) respectively, where $g^*(x) = g(x)^{-1}$.

1. First we shall consider the equation (A) and prove the following

THEOREM 1. *Let I be any arbitrary index set. Well order I. Let $\{G_a\}$, ($a \in I$) be a family of groups such that for every a in I, each solution of* (A) *on G_a is of the form* (B) *and $G_a \subsetneq G_\beta$, $a<\beta$, $a, \beta \in I$. Then every solution of* (A) *on $G = \bigcup G_a$, a in I, is of the form* (B), *provided $f(xyz) = f(xzy)$, for all x, y, z in G.*

Proof. Here we quote the following results, Lemma 1 and Lemma 2 (which are Lemma 1 and Theorem 3 of [2]) without proof, which will be used in the sequel.

LEMMA 1. *Let G be any group. Let f be a function on G with the properties that* (1) *f satisfies* (A) *on G,* (2) *$f(x)$ assumes the values ± 1 only on G and* (3) *$f(xyz) = f(xzy)$, for all x, y, z in G. Then f is of the form* (B).

LEMMA 2. *Let*

$$\frac{g_1(x)+g_1^*(x)}{2} = \frac{g_2(x)+g_2^*(x)}{2},$$

for all x in G, where g_1 and g_2 are homomorphisms of G into K. Then either $g_2 = g_1$ or $g_2^* = g_1$.

The proof of the theorem is based on transfinite induction. Let $G = \bigcup G_\alpha$, $\alpha \in I$. Let f: $G \to K$ be a solution of (A). Now we will prove that f is of the form (B). Let $f_\alpha = f|G_\alpha$ (f restricted to G_α) assume the values ± 1, for every α in I. Then evidently $f = \pm 1$ on G and the proof follows by Lemma 1. Without loss of generality we can assume that

$$f|G_\alpha = f_\alpha \neq \pm 1 \, , \quad \text{for each } \alpha \text{ in } I \, . \tag{1.1}$$

Otherwise there is a least index $\beta \in I$ such that $f_\beta \neq \pm 1$. Then $G = G_\beta \cup \cup G_{\beta+1} \cup \ldots$ Then we can consider the index set $J = \{\beta, \beta+1, \ldots\}$ and we have $f_\gamma \neq \pm 1$, $\gamma \in J$. Let $\alpha < \beta$, $\alpha, \beta \in I$. By hypothesis, there are homomorphisms g_γ: $G_\gamma \to K$ ($\gamma = \alpha, \beta$) such that

$$f_\gamma = \frac{g_\gamma + g_\gamma^*}{2} \, . \tag{1.2}$$

Since $\alpha < \beta$ by Lemma 2, it follows that either g_β or g_β^* is the extension of g_α. If both are extensions of g_α, we would have $g_\alpha = \pm 1$ and so $f_\alpha = \pm 1$, a contradiction to (1.1). We call the unique extension of g_α to G_β as g_β. Now, we shall apply the transfinite induction. Let $\alpha \in I$ be such that the above result is true for the initial segment $s(\alpha)$, that is, for every β and $\gamma < \alpha$ (we may take $\beta < \gamma < \alpha$), g_β: $G_\beta \to K$ a homomorphism satisfying (B) on G_β has a unique extension g_γ to G_γ, also a homomorphism on G_γ into K, satisfying (1.2) on G_γ. Now, we will prove that there is a unique extension to G_α; that is, there is a g_α on G_α such that g_α is the extension of g_β, $\beta < \alpha$. Let $H = \bigcup G_\beta$, $\beta < \alpha$. Let us define g: $H \to K$, as follows:

$$g(x) = g_\gamma(x) \, , \quad x \in G_\gamma \backslash G_\beta \, , \quad \beta < \gamma < \alpha \, . \tag{1.3}$$

Now, g is well defined and further g is a homomorphism on H. In fact, let $x, y \in H$. Then there are $\beta, \gamma < \alpha$ such that $x \in G_\beta \backslash G_\mu$, $\mu < \beta < \alpha$ and $y \in G_\gamma \backslash G_\nu$, $\nu < \gamma < \alpha$. Let $\beta \leqslant \gamma$. Then $xy \in G_\gamma \backslash G_\nu$, $\nu < \gamma < \alpha$. By using the hypothesis and (1.3), we obtain

$$g(xy) = g_\gamma(xy) = g_\gamma(x) g_\gamma(y) = g_\beta(x) g_\gamma(y) = g(x) g(y) \, .$$

Therefore, g is a homomorphism on H. Further

$$f|H = \frac{g + g^*}{2} \, . \tag{1.4}$$

Now consider G_α. By hypothesis, there is a g_α: $G_\alpha \to K$, a homomorphism such that

$$f_\alpha = \frac{g_\alpha + g_\alpha^*}{2} \, . \tag{1.5}$$

So,

$$f|H = \frac{g_a+g_a^*}{2}\Big|H = \frac{g+g^*}{2}.$$

Hence, by Lemma 2, either g_a or g_a^* restricted to H gives g. Otherwise, we would have, $g_\beta = \pm 1$, for every $\beta < a$, a contradiction to (1.1). We shall call the unique extension of g to G_a as g_a. Then g_a is the required unique extension to G_a, satisfying (1.5). Thus we have g_a on G_a is such that $g_a|G_\beta = g_\beta$, $\beta < a$, $a \in I$. Now, let us define h: $G \to K$, as follows:

$$h(x) = g_a(x), \quad \text{where} \quad x \in G_a \setminus G_\beta, \ a < \beta, \ a, \beta \in I. \tag{1.6}$$

Then h is well defined. Further, proceeding as above we can show that h is a homomorphism on G such that $f = (h+h^*)/2$. This completes the proof of the theorem.

From Theorem 1, we can easily deduce the following two corollaries using the following Lemma 3 (which is Theorem (2.5) of [1]).

LEMMA 3. *Let G be any cyclic group finite or infinite. Then every complex valued function on G which is also a solution of* (A) *has the form* (B).

COROLLARY 1. *Consider the additive group of rationals Q. Let $G_n = \{k/n!: k \in Z\}$, where Z denotes the set of integers and n any positive integer greater than or equal to* 1. *Then evidently $Q = \bigcup G_n$, $n \geqslant 1$. G_n is a cyclic group for every n and $G_n \subsetneq G_m$, for $n < m$. Also it follows by Lemma* 3 *that every solution of* (A) *on G_n, for every n, has the form* (B). *Hence it follows from Theorem* 1 *that every solution of* (A) *on Q has the form* (B).

COROLLARY 2. *Consider the multiplicative group $Z(p^\infty)$, p a prime. Then*

$$Z(p^\infty) = \bigcup Z(p^n), \quad \textit{for} \quad n = 0, 1, 2, \ldots$$

where $Z(p^n) = \{\exp(2k\pi i/p^n): k \in Z\}$, Z being the set of integers. $Z(p^n)$ is a cyclic group for every n and $Z(p^n) \subsetneq Z(p^m)$, for $n < m$. Therefore by Lemma 3, *it follows that every solution of* (A) *on $Z(p^n)$ for every n, has the form* (B). *Hence, again using Theorem* 1, *we see that every solution of* (A) *on $Z(p^\infty)$ has the form* (B).

2. Now we take up the equation (C) and prove the following

THEOREM 2. *Let G be a cyclic group. Let f be a complex-valued function on G satisfying* (C) *with the properties that* (1) *f is not identically zero and* (2) *f is of the form* (D). *Then we assert that there is one and only one homomorphism g of G into K satisfying* (D).

Proof. Suppose that there are two homomorphisms g_i: $G \to K$ $(i = 1, 2)$ such that

$$f(x) = \frac{g_i(x) - g_i^*(x)}{2}, \tag{2.1}$$

for all x in G.

Now (2.1) gives that

$$g_1(x) - g_1^*(x) = g_2(x) - g_2^*(x) ,$$

for all x in G, equivalently,

$$g_1(x) - g_2(x) - g_1^*(x) + g_2^*(x) = 0 ,$$

hence

$$g_1(x) - g_2(x) + \frac{g_1(x) - g_2(x)}{g_1(x) g_2(x)} = 0 ,$$

which is the same as

$$[g_1(x) - g_2(x)][g_1(x) g_2(x) + 1] = 0 .$$

From the above we conclude that either for each x,

$$g_2(x) = g_1(x) \quad \text{or} \quad g_2(x) = -g_1^*(x) . \tag{2.2}$$

Let G be generated by a. From (2.2), we would have either

$$g_2(a) = g_1(a) \tag{2.3}$$

which in turn implies that g_2 is the same as g_1, since G is a cyclic group and there is nothing to prove, or

$$g_2(a) = -g_1^*(a) . \tag{2.4}$$

If for every n in Z, $g_2(a^n) = -g_1^*(a^n)$, then g_2 cannot be a homomorphism. So, this case cannot happen. The only other possibility is that with (2.4), for some n in Z, we have

$$g_2(a^n) = g_1(a^n) . \tag{2.5}$$

Let n_0 be the smallest positive integer satisfying (2.5). That is,

$$g_2(a^n) = \begin{cases} -g_1^*(a^n) , & n < n_0 , \\ g_1(a^n) , & n = n_0 . \end{cases} \tag{2.6}$$

Now let us consider $g_2(a^{n_0+1})$. From (2.2) we distinguish two cases according as $g_2(a^{n_0+1}) = g_1(a^{n_0+1})$ or $g_2(a^{n_0+1}) = -g_1^*(a^{n_0+1})$. First we consider the former case. Now,

$$g_2(a^{n_0+1}) = g_2(a^{n_0}) g_2(a) = g_1(a^{n_0})[-g_1^*(a)] = -g_1(a^{n_0-1}) .$$

Hence

$$g_1(a^{n_0+1}) = -g_1(a^{n_0-1}) ,$$

that is,

$$g_1(a^{n_0-1})[g_1(a^2) + 1] = 0 .$$

Now $g_1(a^{n_0-1}) = 0$ implies $g_1(a) = 0$, which cannot be, leaving the other possibility that $g_1(a^2) = g_1(a)^2 = -1$. So, $g_1(a) = \pm i$. Now $g_1(a) = i$ implies $-g_1^*(a) = -1/i = i$. Hence $g_2(a) = -g_1^*(a) = g_1(a)$, whence g_2 is identically equal to g_1. Similarly we can establish in the case where $g_2(a) = -i$, that g_2 is the same as g_1.

Finally, we are left with the remaining case, $g_2(a^{n_0+1}) = -g_1^*(a^{n_0+1})$. But, we have

$$g_2(a^{n_0+1}) = g_2(a^{n_0-1})g_2(a^2) = [-g_1^*(a^{n_0-1})][-g_1^*(a^2)] = g_1^*(a^{n_0+1}),$$

provided $n_0 \geqslant 3$.

So, we have, $g_1^*(a^{n_0+1}) = -g_1^*(a^{n_0+1})$. This means $g_1^*(a) = 0$, which is false. Hence this possibility cannot occur. If $n_0 = 2$, (2.6) becomes

$$g_2(a) = -g_1^*(a) \quad \text{and} \quad g_2(a^2) = g_1(a^2). \tag{2.7}$$

In case

$$g_2(a^3) = -g_1^*(a^3),$$

we have

$$g_2(a^3) = g_2(a^2)g_2(a) = g_1(a^2)[-g_1^*(a)] = -g_1(a) = -g_1^*(a^3).$$

So, $g_1(a^4) = 1$, giving $g_1(a) = \pm 1, \pm i$. $g_1(a) = \pm 1$, implies that f is a zero function, which contradicts the hypothesis (1). The equality $g_1(a) = \pm i$ as above shows that g_2 and g_1 coincide on G. Lastly we may have $g_2(a^3) = g_1(a^3)$. From (2.7), we have

$$g_2(a^3) = g_1(a^3) = g_1(a^2)g_1(a) = g_2(a^2)g_1(a).$$

Thus $g_2(a) = g_1(a)$. Hence g_2 and g_1 are one and the same on G. Thus we have established the uniqueness of the homomorphism g, when g satisfies (D). Hence the proof of the theorem is complete.

I wish to express my sincere thanks to Professor Edwin Hewitt for his guidance during the preparation of this paper.

References

[1] Pl. Kannappan, *The functional equation* $f(xy)+f(xy^{-1}) = 2f(x)f(y)$ *for Abelian groups*, Ph. D. thesis, University of Washington, Seattle 1964.

[2] — *The functional equation* $f(xy)+f(xy^{-1}) = 2f(x)f(y)$ *for groups*, Proc. Amer. Math. Soc. (to appear).

ANNAMALAI UNIVERSITY,
ANNAMALAI NAGAR, INDIA

Reçu par la Rédaction le 20. 9. 1966