$f \in \Gamma(A)$ which has an extension $\bar{f}$ not belonging to $\Gamma(B)$, or even $\operatorname{cor} B$ for some superalgebra $B \supset A$. To see this take as B the sup-norm disc algebra of all continuous functions on the unit disc of the complex plane, holomorphic in its interior and let $A = \{x \in B\colon x(0) = x(1)\}$. The maximal ideal space of A is the closed unit disc with identified 0 and 1 and the Šilov boundary of A is the unit circle (with 1 identified with 0). So the functional $f(x) = x(0) = x(1)$ is in $\Gamma(A)$ and it has two extensions onto B: $f_1(x) = x(1)$ and $f_0(x) = x(0)$ such that $f_1 \in \Gamma(B)$ but $f_0 \notin \operatorname{cor} B$.

The following purely algebraic result can support the conjecture that $l(A)$ coincides with the family of all non-removable closed ideals of A. Let R and P be arbitrary rings with unit elements. P is an extension of R if there is an isomorphic imbedding of R into P sending the unit of R into unit of P. Call an ideal I of R non-removable if in any extension P of R the ideal I is contained in a proper ideal of P. A subset S of R consists of joint divisors of zero if for any finite subset $\{x_1, \ldots, x_n\} \subset R$ there is a non zero element $y \in R$ such that $x_i y = 0$ for $i = 1, 2, \ldots, n$.

PROPOSITION 4. *An ideal I of a commutative ring R is a non-removable ideal if and only if it consists of joint divisors of zero.*

The proof can be obtained from a reasoning in [2].

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On cosine operator functions and one-parameter groups of operators

by

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Dedicated to Professor Antoni Zygmund

Abstract. If A is a complex number then

$$(*)\qquad \exp\left(t\begin{pmatrix}0 & 1\\ A & 0\end{pmatrix}\right) = \begin{pmatrix}\cos(-A)^{\frac{1}{2}}t & \int\limits_0^t \cos(-A)^{\frac{1}{2}}\tau\, d\tau \\ \frac{d}{dt}\cos(-A)^{\frac{1}{2}}t & \cos(-A)^{\frac{1}{2}}t\end{pmatrix}, \qquad -\infty < t < \infty.$$

The paper gives a generalization of this formula to the case, when A is an unbounded linear operator in a Banach space.

1. Preliminaries.

1.1. If E and F are Banach spaces over the same, real or complex, field of scalars then $\mathscr{L}(E; F)$ denotes the space of all linear bounded operators from E to F. Let $\mathscr{L}_s(E; F)$ denote $\mathscr{L}(E; F)$ equipped with the topology of pointwise convergence (called also the strong topology). An $\mathscr{L}(E; F)$-valued function of a real variable is called *strongly continuous*, or *strongly continuously differentiable*, if it is continuous or continuously differentiable, when regarded as a mapping from $(-\infty, \infty)$ to $\mathscr{L}_s(E; F)$. For instance, by an application of the Banach–Steinhaus theorem, it follows that a function $K\colon (-\infty, \infty) \to \mathscr{L}(E; F)$ is strongly continuously differentiable on $(-\infty, \infty)$ if and only if for any fixed $x \in E$ the F-valued function $t \to K(t)x$ is continuously differentiable on $(-\infty, \infty)$ in the sense of the norm in F.

1.2. Let E be a Banach space. A strongly continuous mapping $G\colon (-\infty, \infty) \to \mathscr{L}(E; F)$ is called a *one-parameter strongly continuous group of operators* if $G(0) = 1$ and

$$G(t)G(s) = G(t+s) \quad \text{for every } s, t \in (-\infty, \infty).$$

The infinitesimal generator of the one parameter group G is the linear operator A from E to E, with the domain $D(A)$ defined by the conditions

$$D(A) = \left\{x\colon\ x \in E,\ \lim_{t\to 0} \frac{1}{t}\big(G(t)x - x\big) \text{ exists}\right\},$$

$$Ax = \lim_{t\to 0} \frac{1}{t}\big(G(t)x - x\big) \quad \text{for } x \in D(A),$$

where the limit is taken in the sense of the norm in E.

It is known (see e.g. [3], chapter IX) that if G is a strongly continuous one-parameter group of bounded linear operators in a Banach space E and if A is the infinitesimal generator of G, then

(1.2.1) there are constants $M \geqslant 1$ and $k \geqslant 0$ such that

$$\|G(t)\| \leqslant Me^{k|t|} \quad \text{for every } t \in (-\infty, \infty);$$

(1.2.2) for every $n = 1, 2, \ldots$ the domain $D(A^n)$ of A^n is dense in E and A^n is a closed operator from E to E;

(1.2.3) $G(t)D(A) = D(A)$ for every $t \in (-\infty, \infty)$ and, for every fixed $x \in D(A)$, the E-valued function $t \to G(t)x$ is continuously differentiable on $(-\infty, \infty)$ in the sense of the norm in E and

$$\frac{dG(t)x}{dt} = AG(t)x = G(t)Ax, \quad t \in (-\infty, \infty).$$

1.3. Let E be a Banach space. A mapping $\mathscr{C}\colon (-\infty, \infty) \to \mathscr{L}(E; E)$ is called cosine operator function if it satisfies the d'Alembert functional equation

$$\mathscr{C}(t+s) + \mathscr{C}(t-s) = 2\mathscr{C}(t)\mathscr{C}(s)$$

for $s, t \in (-\infty, \infty)$, and if, moreover, $\mathscr{C}(0) = 1$. As it is easy to see, any cosine operator function is a pair function on $(-\infty, \infty)$, its range being a commutative family of operators.

The theory of $\mathscr{L}(E; E)$-valued strongly continuous cosine functions was developped by M. Sova [2].

It should be remarked, that in [2] a cosine operator function is defined only on $[0, \infty)$. However, as Sova proved, the range of any strongly continuous cosine operator function defined on $[0, \infty)$ is a commutative family of operators, and from this it follows easily, that the pair extension onto $(-\infty, \infty)$ of such a cosine function satisfies the d'Alembert's equation on whole $(-\infty, \infty)$. According to [2], the infinitesimal generator of an $\mathscr{L}(E; E)$-valued cosine function $\mathscr{C}$ is the linear operator A from E to E, with the domain $D(A)$, defined by the conditions

$$D(A) = \left\{x\colon\ x \in E,\ \lim_{t\to 0} \frac{2}{t^2}\big(\mathscr{C}(t)x - x\big) \text{ exists}\right\},$$

$$Ax = \lim_{t\to 0} \frac{2}{t^2}\big(\mathscr{C}(t)x - x\big) \quad \text{for } x \in D(A),$$

the limit taken in the sense of the norm in E.

As proved by Sova [2], if E is a Banach space and if $\mathscr{C}$ is an $\mathscr{L}(E; E)$-valued strongly continuous cosine function with the infinitesimal generator A, then

(1.3.1.) the domain $D(A)$ of A is dense in E and A is a closed operator from E to E;

(1.3.2.) $\mathscr{C}(t)D(A) \subset D(A)$ for every $t \in (-\infty, \infty)$ and, for every fixed $x \in D(A)$, the E-valued function $t \to \mathscr{C}(t)x$ is twice continuously differentiable on $(-\infty, \infty)$ in the sense of the norm in E and

$$\frac{d^2\mathscr{C}(t)x}{dt^2} = A\mathscr{C}(t)x = \mathscr{C}(t)Ax, \qquad -\infty < t < \infty.$$

The following lemma will be also useful in our further reasonings.

(1.3.3.) Lemma. *Let E be a Banach space, let $\mathscr{C}$ be an $\mathscr{L}(E; E)$-valued function strongly continuous on $(-\infty, \infty)$ and let A be a closed linear operator from E to E with a domain $D(A)$ dense in E. If*

(a) *$\mathscr{C}(t)D(A) \subset D(A)$ for every $t \in (-\infty, \infty)$*

and

(b) *for any fixed $x \in D(A)$ the E-valued function $t \to \mathscr{C}(t)x$ is twice strongly continuously differentiable on $(-\infty, \infty)$ and we have*

$$\frac{d^2\mathscr{C}(t)x}{dt} = A\mathscr{C}(t)x = \mathscr{C}(t)Ax \quad \text{for } t \in (-\infty, \infty),$$

$$\mathscr{C}(0)x = x \quad \text{and} \quad \left.\frac{d\mathscr{C}(t)x}{dt}\right|_{t=0} = 0,$$

then $\mathscr{C}(t)$ is a cosine function and A is its infinitesimal generator.

Proof. In order to prove that $\mathscr{C}$ satisfies the d'Alembert functional equation it is sufficient to show that for any fixed $x_0 \in D(A)$ and any fixed s the E-valued function

$$x(t) = \mathscr{C}(t+s)x_0 + \mathscr{C}(s-t)x_0 - 2\mathscr{C}(s)\mathscr{C}(t)x_0$$

vanishes identically on $(-\infty, \infty)$. As it is easy to see, the function $t \to x(t)$ has the following properties:

(α) $x(t) \in D(A)$ for every $t \in (-\infty, \infty)$,

(β) $t \to x(t)$ is twice continuously differentiable on $(-\infty, \infty)$ in the sense of the norm in E and $x''(t) = Ax(t)$ for $t \in (-\infty, \infty)$,

(γ) $x(0) = 0$,

(δ) $x'(0) = \mathscr{C}'(s)x - \mathscr{C}'(s)x - 2\mathscr{C}(s)\mathscr{C}'(0)x = 0$.

If we define a norm on $D(A)$ by

$$\|x\|_{D(A)} = \|x\|_E + \|Ax\|_E$$

then, by the closedness of A, $D(A)$ with the norm $\| \; \|_{D(A)}$ is a Banach space. Moreover, by (α) and (β), $t \to x(t)$ is a $D(A)$-valued function continuous on $(-\infty, \infty)$ in the sense of the norm $\| \; \|_{D(A)}$ and, by (b), $t \to \mathscr{C}(t)|_{D(A)}$ is an $\mathscr{L}(D(A); E)$-valued function twice strongly continuously differentiable on $(-\infty, \infty)$. For any $t \in (-\infty, \infty)$ let $\mathscr{C}'(t) \in \mathscr{L}(D(A); E)$ and $\mathscr{C}''(t) \in \mathscr{L}(D(A); E)$ denote the corresponding derivatives at the point t. It follows that for any fixed $t \in (-\infty, \infty)$ the E-valued function $\tau \to y(\tau) = \mathscr{C}(t-\tau)x(\tau)$ is continuously differentiable on $(-\infty, \infty)$ in the sense of the norm in E and that

$$y'(\tau) = \mathscr{C}(t-\tau)x'(t) - \mathscr{C}'(t-\tau)x(\tau), \qquad -\infty < \tau < \infty.$$

Since $y(t) = \mathscr{C}(0)x(t) = x(t)$ by (b), and $y(0) = \mathscr{C}(t)x(0) = 0$ by (γ), we have $x(t) = \int_0^t y'(\tau)\,d\tau$, i.e.

$$x(t) = \int_0^t \{\mathscr{C}(t-\tau)x'(\tau) - \mathscr{C}'(t-\tau)x(\tau)\}\,d\tau, \qquad -\infty < t < \infty.$$

By (δ) and (β) we have $x'(\tau) = \int_0^\tau Ax(\varrho)\,d\varrho$. By (b) we have $\mathscr{C}'(t-\tau)x(\tau) = \int_0^{t-\tau} \mathscr{C}(\sigma)Ax(\tau)\,d\sigma$. Therefore

$$x(t) = \int_0^t \Big[\int_0^\tau \mathscr{C}(t-\tau)Ax(\varrho)\,d\varrho\Big]\,d\tau - \int_0^t \Big[\int_0^{t-\tau} \mathscr{C}(\sigma)Ax(\tau)\,d\sigma\Big]\,d\tau = 0,$$

by the Fubini theorem.

So we proved that $\mathscr{C}$ is a cosine function and now it follows at once from (b) that if A_0 denotes the infinitesimal generator of $\mathscr{C}$ then $A \subset A_0$, i.e. $D(A) \subset D(A_0)$ and $Ax = A_0x$ for $x \in D(A)$. In order to prowe that $A = A_0$ it is sufficient to apply Lemma 2 from Section 3 of the author's paper [1].

However, the reasoning in [1] is complicated by the fact that the general case of a locally convex sequentially complete space is treated

there. Therefore we shall repeat the argumentation in a simplified version for a Banach space. We have to prove that $D(A_0) \subset D(A)$.

If $x \in D(A)$ then, by (b) and by the closedness of A, we have $\mathscr{C}(t)x - x = \int_0^t\int_0^\tau A\mathscr{C}(\sigma)x\,d\sigma\,d\tau = A\int_0^t\int_0^\tau \mathscr{C}(\sigma)x\,d\sigma\,d\tau$. Again by the closedness of A and by the fact that $D(A)$ is dense in E it follows that

$$(*) \qquad \int_0^t\int_0^\tau \mathscr{C}(\sigma)x\,d\sigma\,d\tau \in D(A) \quad \text{and} \quad A\int_0^t\int_0^\tau \mathscr{C}(\sigma)x\,d\sigma\,d\tau = \mathscr{C}(t)x - x$$

for every $x \in E$ and $t \in (-\infty, \infty)$. If $x \in D(A_0)$ then $\lim_{t\to 0} \frac{2}{t^2}(\mathscr{C}(t)x - x) = A_0x$ and $\lim_{t\to 0} \frac{2}{t^2}\int_0^t\int_0^\tau \mathscr{C}(\sigma)x\,d\sigma\,d\tau = x$, so that, by (*) and by the closedness of A, $x \in D(A)$ and $Ax = A_0x$. Consequently $D(A_0) \subset D(A)$ and the proof is complete.

1.4. In the sequel we shall consider one-parameter groups of operators belonging to $\mathscr{L}(E_1 \times E_0; E_1 \times E_0)$, where E_0 and E_1 are Banach spaces. It will be convenient to write elements of $E_1 \times E_0$ in the form of columns $\begin{pmatrix} x \\ y \end{pmatrix}$, where $x \in E_1$, $y \in E_0$, and to represent any operator $B \in \mathscr{L}(E_1 \times E_0; E_1 \times E_0)$ as a matrix

$$B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix},$$

whose elements are operators $B_{ik} \in \mathscr{L}(E_{2-k}; E_{2-i})$ defined by the condition that, for every column $\begin{pmatrix} x \\ y \end{pmatrix} \in E_1 \times E_0$,

$$B\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} B_{11}x + B_{12}y \\ B_{21}x + B_{22}y \end{pmatrix},$$

according to the common rule of multiplication of matrices. A similar matricial representation may be used for any linear operator A from $E_1 \times E_0$ into $E_1 \times E_0$, having domain of the form $D(A) = D_1 \times D_0$, where D_i is a linear subset of E_i.

2. Generalization of the formula (*).

THEOREM. *Let E_0 be a Banach space and let A be a linear operator from E_0 to E_0 with domain $D(A)$. If E_1 is a Banach space such that $D(A) \subset E_1 \subset E_0$ and that the Banach space topology of E_1 is not weaker than the topology induced in E_1 by E_0, and if the operator $\mathscr{A}$ from $E_1 \times E_0$ into $E_1 \times E_0$, with domain $D(\mathscr{A})$, defined by the conditions*

$$(2.1) \qquad D(\mathscr{A}) = D(A) \times E_1, \qquad \mathscr{A} = \begin{pmatrix} 0 & 1 \\ A & 0 \end{pmatrix},$$

is the infinitesimal generator of a strongly continuous one-parameter group $\mathscr{G}\colon(-\infty,\infty)\to\mathscr{L}(E_1\times E_0;E_1\times E_0)$, *then* A *is the infinitesimal generator of a strongly continuous* $\mathscr{L}(E_0;E_0)$*-valued cosine function* $\mathscr{C}(t)$,

(2.2) $E_1=\{x\colon x\in E_0$, *the* E_0*-valued function* $t\to\mathscr{C}(t)x$ *is continuously differentiable on* $(-\infty,\infty)$ *in the sense of the norm in* $E_0\}$,

and

$$\mathscr{G}(t)\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}\mathscr{C}(t)x+\int\limits_0^t\mathscr{C}(\tau)y\,d\tau\\ \dfrac{d\mathscr{C}(t)x}{dt}+\mathscr{C}(t)y\end{pmatrix},\quad\begin{pmatrix}x\\ y\end{pmatrix}\in E_1\times E_0,\ -\infty<t<\infty. \tag{2.3}$$

On the other hand, if A *is the infinitesimal generator of a strongly continuous* $\mathscr{L}(E_0;E_0)$*-valued cosine function* $\mathscr{C}(t)$ *and if* E_1 *is defined by* (2.2), *then* E_1 *under the norm*

$$\|x\|_{E_1}=\|x\|_{E_0}+\sup_{0\leqslant t\leqslant 1}\left\|\frac{d\mathscr{C}(t)x}{dt}\right\|_{E_0},\qquad x\in E_1, \tag{2.4}$$

becomes a Banach space and the formula (2.3) *defines a strongly continuous one-parameter group* $\mathscr{G}\colon(-\infty,\infty)\to\mathscr{L}(E_1\times E_0;E_1\times E_0)$, *whose infinitesimal generator is the operator* $\mathscr{A}$ *defined by the conditions* (2.1).

3. Proof of the part "from $\mathscr{C}$ to $\mathscr{G}$". Everywhere throughout this section it is assumed that E_0 is a Banach space, $\mathscr{C}$ is a strongly continuous $\mathscr{L}(E_0;E_0)$-valued cosine function and A is the infinitesimal generator of $\mathscr{C}$. We define the linear subset E_1 of E_0 by (2.2) and define the norm $\|\ \|_{E_1}$ on E_1 by (2.4).

LEMMA 3.1. E_1 *under the norm* $\|\ \|_{E_1}$ *is a Banach space. We have* $\mathscr{C}(t)|_{E_1}\in\mathscr{L}(E_1;E_1)$, $\int\limits_0^t\mathscr{C}(\tau)d\tau\in\mathscr{L}(E_0;E_1)$ *and* $\dfrac{d}{dt}\mathscr{C}(t)\Big|_{E_1}\in\mathscr{L}(E_1;E_0)$ *for every* $t\in(-\infty,\infty)$, *where the integral is the Riemann integral of a strongly continuous* $\mathscr{L}(E_0;E_0)$*-valued function and the derivative is taken in the sense of the strong topology in* $\mathscr{L}(E_1;E_0)$. *Moreover, the mappings*

(a) $$(-\infty,\infty)\ni t\to\mathscr{C}(t)|_{E_1}\in\mathscr{L}(E_1;E_1),$$

(b) $$(-\infty,\infty)\ni t\to\int\limits_0^t\mathscr{C}(\tau)d\tau\in\mathscr{L}(E_0;E_1),$$

(c) $$(-\infty,\infty)\ni t\to\frac{d}{dt}\mathscr{C}(t)\Big|_{E_1}\in\mathscr{L}(E_1;E_0)$$

are strongly continuous.

Proof. It follows from the d'Alembert functional equation, that if the function $t\to\mathscr{C}(t)x$ is continuously differentiable on $[0,1]$ in the sense of the norm in E_0, then $x\in E_1$. Therefore, by the theorem on term by term

differentation, E_1 is a complete space under the norm $\|\ \|_{E_1}$. If $x\in E_1$ and $s\in(-\infty,\infty)$ are fixed then the E_0-valued function $t\to\mathscr{C}(t)\mathscr{C}(s)x=\mathscr{C}(s)\mathscr{C}(t)x$ is continuously differentiable on $(-\infty,\infty)$ in the sense of the norm in E_0, so that $\mathscr{C}(s)E_1\subset E_1$. Since, for $x\in E_1$,

$$\begin{aligned}\|\mathscr{C}(t)x\|_{E_1}&=\|\mathscr{C}(t)x\|_{E_0}+\sup_{0\leqslant s\leqslant 1}\left\|\frac{d}{ds}\big(\mathscr{C}(s)\mathscr{C}(t)x\big)\right\|_{E_0}\\&=\|\mathscr{C}(t)x\|_{E_0}+\sup_{0\leqslant s\leqslant 1}\left\|\mathscr{C}(t)\frac{d\mathscr{C}(s)x}{ds}\right\|_{E_0}\\&\leqslant\|\mathscr{C}(t)\|_{\mathscr{L}(E_0;E_0)}\|x\|_{E_1},\end{aligned}$$

we see that $\mathscr{C}(t)|_{E_1}\in\mathscr{L}(E_1;E_1)$. Moreover, if $x\in E_1$, then, by the d'Alembert equation,

$$\|\mathscr{C}(t+h)x-\mathscr{C}(t)x\|_{E_1}=\|\mathscr{C}(t+h)x-\mathscr{C}(t)x\|_{E_0}$$
$$+\frac{1}{2}\sup_{0\leqslant s\leqslant 1}\left\|\frac{d\mathscr{C}(t+h+s)x}{dt}-\frac{d\mathscr{C}(t+h-s)x}{dt}-\frac{d\mathscr{C}(t+s)x}{dt}+\frac{d\mathscr{C}(t-s)x}{dt}\right\|_{E_0},$$

whence it follows that the mapping (a) is strongly continuous. This implies that if $x\in E_1$ then also $\int\limits_0^t\mathscr{C}(\tau)x\,d\tau\in E_1$ and

$$\frac{d}{ds}\Big(\mathscr{C}(s)\int\limits_0^t\mathscr{C}(\tau)x\,d\tau\Big)=\frac{d}{ds}\int\limits_0^t\mathscr{C}(s)\mathscr{C}(\tau)x\,d\tau=\frac{1}{2}\frac{d}{ds}\int\limits_0^t\{\mathscr{C}(\tau+s)x+\mathscr{C}(\tau-s)x\}d\tau$$
$$=\frac{1}{2}\int\limits_0^t\left\{\frac{d\mathscr{C}(\tau+s)x}{d\tau}-\frac{d\mathscr{C}(\tau-s)x}{d\tau}\right\}d\tau=\frac{1}{2}\mathscr{C}(t+s)x-\frac{1}{2}\mathscr{C}(t-s)x.$$

Now let $x\in E_0$. It follows from 1.3.1 and 1.3.2 that E_1 is dense in E_0 and so there is a sequence x_n, $n=1,2,\ldots$, of elements of E_1, such that $\lim\limits_{n\to\infty}\|x_n-x\|_{E_0}=0$. If, for a fixed t, we put $y_n=\int\limits_0^t\mathscr{C}(\tau)x_n\,d\tau$, then

$$\lim_{n\to\infty}\sup_{0\leqslant s\leqslant 1}\left\|\mathscr{C}(s)y_n-\mathscr{C}(s)\int\limits_0^t\mathscr{C}(\tau)x\,d\tau\right\|_{E_0}=0$$

and

$$\frac{d\mathscr{C}(s)y_n}{ds}=\frac{1}{2}\mathscr{C}(t+s)x_n-\frac{1}{2}\mathscr{C}(t-s)x_n,$$

so that $\lim_{n\to\infty}\sup_{0\leqslant s\leqslant 1}\left\|\frac{d\mathscr{C}(s)y_n}{ds}-\frac{1}{2}\mathscr{C}(t+s)x+\frac{1}{2}\mathscr{C}(t-s)\right\|_{E_0}=0$. Hence, by the theorem on term by term differentation,

(3.1)

$$\int_0^t \mathscr{C}(\tau)x\,d\tau\in E_1 \quad\text{and}\quad \frac{d}{ds}\Big(\mathscr{C}(s)\int_0^t\mathscr{C}(\tau)x\,d\tau\Big)=\frac{1}{2}\mathscr{C}(t+s)x-\frac{1}{2}\mathscr{C}(t-s)x$$

for every $x\in E_0$ and $s,t\in(-\infty,\infty)$. From (3.1) it follows immediately that $\int_0^t\mathscr{C}(\tau)d\tau\in\mathscr{L}(E_0;E_1)$ and that the mapping (b) is strongly continuous. The statements that $\frac{d}{dt}\mathscr{C}(t)|_{E_1}\in\mathscr{L}(E_1;E_0)$ and that the mapping (c) is strongly continuous are trivial consequences of the definitions of E_1 and $\|\;\|_{E_1}$.

LEMMA 3.2. *The formula* (2.3) *defines a one-parameter strongly continuous group* $(-\infty,\infty)\ni t\to\mathscr{G}(t)\in\mathscr{L}(E_1\times E_0;E_1\times E_0)$, *whose infinitesimal generator is the operator* $\mathscr{A}$ *defined by* (2.1).

Proof. It follows from Lemma 3.1 that $\mathscr{G}(t)\in\mathscr{L}(E_1\times E_0;E_1\times E_0)$ for every $t\in(-\infty,\infty)$, and that the mapping $(-\infty,\infty)\ni t\to\mathscr{G}(t)\in\mathscr{L}(E_1\times E_0;E_1\times E_0)$ is strongly continuous. For any $\binom{x}{y}\in E_1\times E_0$ and $t,s\in(-\infty,\infty)$ we have

$$\mathscr{G}(t)\mathscr{G}(s)\binom{x}{y}$$
$$=\begin{pmatrix}\mathscr{C}(t)\mathscr{C}(s)x+\int_0^t\mathscr{C}(\tau)d\tau\frac{d\mathscr{C}(s)x}{ds}+\mathscr{C}(t)\int_0^s\mathscr{C}(\sigma)y\,d\sigma+\int_0^t\mathscr{C}(\tau)d\tau\mathscr{C}(s)y\\ \frac{d}{dt}\big(\mathscr{C}(t)\mathscr{C}(s)x\big)+\mathscr{C}(t)\frac{d\mathscr{C}(s)x}{ds}+\frac{d}{dt}\Big(\mathscr{C}(t)\int_0^s\mathscr{C}(\sigma)y\,d\sigma\Big)+\mathscr{C}(t)\mathscr{C}(s)y\end{pmatrix}$$

and so, in order to prove that $\mathscr{G}$ is a group, we have to show that the following equalities hold for any $s,t\in(-\infty,\infty)$, $x\in E_1$ and $y\in E_0$,

$$1^\circ\quad \mathscr{C}(t)\mathscr{C}(s)x+\int_0^t\mathscr{C}(\tau)d\tau\frac{d\mathscr{C}(s)x}{ds}=\mathscr{C}(t+s)x,$$

$$2^\circ\quad \mathscr{C}(t)\int_0^s\mathscr{C}(\sigma)y\,d\sigma+\int_0^t\mathscr{C}(\tau)d\tau\mathscr{C}(s)y=\int_0^{t+s}\mathscr{C}(\tau)y\,d\tau,$$

$$3^\circ\quad \frac{d}{dt}\big(\mathscr{C}(t)\mathscr{C}(s)x\big)+\mathscr{C}(t)\frac{d\mathscr{C}(s)x}{ds}=\frac{d\mathscr{C}(t+s)x}{dt},$$

$$4^\circ\quad \frac{d}{dt}\Big(\mathscr{C}(t)\int_0^s\mathscr{C}(\sigma)y\,d\sigma\Big)+\mathscr{C}(t)\mathscr{C}(s)y=\mathscr{C}(t+s)y.$$

It is easy to see, 3° and 4° follow from 1° and 2° by differentiation. If $x\in E_1$, then

$$\int_0^t\mathscr{C}(\tau)\frac{d\mathscr{C}(s)x}{ds}d\tau=\int_0^t\frac{d}{ds}\big(\mathscr{C}(\tau)\mathscr{C}(s)x\big)d\tau=\frac{1}{2}\frac{d}{ds}\int_0^t\big(\mathscr{C}(\tau+s)x+\mathscr{C}(s-\tau)x\big)d\tau$$
$$=\frac{1}{2}\int_0^t\frac{d}{d\tau}\big(\mathscr{C}(\tau+s)x-\mathscr{C}(s-\tau)x\big)d\tau=\frac{1}{2}\mathscr{C}(t+s)x-\frac{1}{2}\mathscr{C}(s-t)x$$
$$=\mathscr{C}(t+s)x-\mathscr{C}(t)\mathscr{C}(s)x$$

and so 1° is proved. Recalling that $\mathscr{C}(t)$ is a pair function of t we have

$$\mathscr{C}(t)\int_0^s\mathscr{C}(\sigma)d\sigma+\int_0^t\mathscr{C}(\tau)d\tau\mathscr{C}(s)=\frac{1}{2}\int_0^s[\mathscr{C}(t+\sigma)+\mathscr{C}(t-\sigma)]d\sigma+\frac{1}{2}\int_0^t[\mathscr{C}(\tau-s)+$$
$$+\mathscr{C}(\tau+s)]d\tau=\frac{1}{2}\Big[\int_t^{t+s}+\int_{t-s}^{t}+\int_{-s}^{t-s}+\int_{-t-s}^{-s}\Big]\mathscr{C}(\tau)d\tau=\frac{1}{2}\int_{-t-s}^{t+s}\mathscr{C}(\tau)d\tau=\int_0^{t+s}\mathscr{C}(\tau)d\tau$$

and so 2° is proved. Therefore $\mathscr{G}$ is a one-parameter group. Let $\mathscr{A}_0$ be its infinitesimal generator. If $x\in D(A)$ then, by 1.3.2 and by Lemma 3.1, $t\to\int_0^t\mathscr{C}(\tau)Ax\,d\tau=\frac{d\mathscr{C}(t)x}{dt}$ is an E_1-valued function continuous on $(-\infty,\infty)$ in the sense of the norm $\|\;\|_{E_1}$. Since $\mathscr{C}(t)$ is a pair function of t, it follows that $\lim_{t\to0}\frac{1}{t}\big(\mathscr{C}(t)x-x\big)=0$ in the sense of the norm $\|\;\|_{E_1}$. Moreover

$$\lim_{t\to0}\frac{1}{t}\frac{d\mathscr{C}(t)x}{dt}=\lim_{t\to0}\frac{1}{t}\int_0^t\mathscr{C}(\tau)Ax\,d\tau=Ax$$

in the sense of the norm $\|\;\|_{E_0}$. If $y\in E_1$ then, by Lemma 3.1, $t\to\mathscr{C}(t)y$ is an E_1-valued function continuous on $(-\infty,\infty)$ in the sense of the norm $\|\;\|_{E_1}$, so that

$$\lim_{t\to0}\frac{1}{t}\int_0^t\mathscr{C}(\tau)y\,d\tau=\mathscr{C}(0)y=y$$

in the sense of the norm $\|\;\|_{E_1}$. Moreover, if $y\in E_1$ then, since $\mathscr{C}(t)$ is pair,

$$\lim_{t\to0}\frac{1}{t}\big(\mathscr{C}(t)y-y\big)=0$$

in the sense of the norm $\|\;\|_{E_0}$. It follows, that if $\binom{x}{y}\in D(\mathscr{A})=D(A)\times E_1$ then

$$\lim_{t\to0}\frac{1}{t}\Big(\mathscr{G}(t)\binom{x}{y}-\binom{x}{y}\Big)=\binom{y}{Ax}=\mathscr{A}\binom{x}{y}$$

in the sense of the norm in $E_1\times E_0$. This means that $\mathscr{A}\subset\mathscr{A}_0$.

On the other hand, if $\begin{pmatrix} x \\ y \end{pmatrix} \in D(\mathscr{A}_0)$ then, by 1.2.3, the $(E_1 \times E_0)$-valued function

$$t \to \mathscr{G}(t)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \mathscr{C}(t)x + \int_0^t \mathscr{C}(\tau)y\,d\tau \\ \dfrac{d\mathscr{C}(t)x}{dt} + \mathscr{C}(t)y \end{pmatrix}$$

is continuously differentiable on $(-\infty, \infty)$ in the sense of the norm in $E_1 \times E_0$. It follows that $\frac{d\mathscr{C}(t)x}{dt} + \mathscr{C}(t)y \in E_1$ for every $t \in (-\infty, \infty)$ and so $y = \left(\frac{d\mathscr{C}(t)x}{dt} + \mathscr{C}(t)y\right)\Big|_{t=0} \in E_1$. Therefore $t \to \frac{d\mathscr{C}(t)x}{dt} = \left(\frac{d\mathscr{C}(t)x}{dt} + \mathscr{C}(t)y\right) - \mathscr{C}(t)y$ is an E_0-valued function continuously differentiable on $(-\infty, \infty)$ in the sense of the norm $\| \; \|_{E_0}$, and so $x \in D(A)$. Hence $D(\mathscr{A}_0) \subset D(A) \times E_1 = D(\mathscr{A})$. Since we already know, that $\mathscr{A} \subset \mathscr{A}_0$, it follows that $\mathscr{A} = \mathscr{A}_0$ and the proof is complete.

4. Proof of the part "from $\mathscr{G}$ to $\mathscr{C}$". Everywhere throughout this section it is assumed that E_1 and E_0 are Banach spaces such that $E_1 \subset E_0$ and the Banach space topology of E_1 is not weaker then the topology induced in E_1 by E_0.

Moreover, it is assumed that A is a linear operator from E_1 to E_0 and that the operator $\mathscr{A}$ defined by (2.1) is the infinitesimal generator of a strongly continuous one parameter group $\mathscr{G}$: $(-\infty, \infty) \to \mathscr{L}(E_1 \times E_0; E_1 \times E_0)$. We have to prove that A is the infinitesimal generator of a strongly continuous $\mathscr{L}(E_0; E_0)$-valued cosine function $\mathscr{C}(t)$ and that (2.2) and (2.3) hold.

LEMMA 4.1. *The operator A is closed as an operator from E_0 to E_0 and its domain $D(A)$ is dense in E_0.*

Proof. We have

$$D(\mathscr{A}^2) = \{x\colon x \in D(A),\, Ax \in E_1\} \times D(A),$$

$$\mathscr{A}^2\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} Ax \\ Ay \end{pmatrix}, \quad \begin{pmatrix} x \\ y \end{pmatrix} \in D(\mathscr{A}^2),$$

and, according to 1.2.2, $\mathscr{A}^2$ is a closed operator from $E_1 \times E_0$ to $E_1 \times E_0$, with the domain $D(\mathscr{A}^2)$ dense in $E_1 \times E_0$.

LEMMA 4.2. *The operator A is the infinitesimal generator of an $\mathscr{L}(E_0; E_0)$-valued strongly continuous cosine function $\mathscr{C}$ such that $\mathscr{C}(t)E_1 \subset E_1$ for every $t \in (-\infty, \infty)$, and for any fixed $x \in E_1$ the E_1-valued function $t \to C(t)x$ is continuous on $(-\infty, \infty)$ in the sense of the norm in E_1 and is continuously differentiable on $(-\infty, \infty)$ in the sense of the norm in E_0. Moreover, the formula (2.3) holds.*

Proof. Represent $\mathscr{G}(t)$ as a matrix

$$\mathscr{G}(t) = \begin{pmatrix} G_{11}(t) & G_{12}(t) \\ G_{21}(t) & G_{22}(t) \end{pmatrix}.$$

Then any $t \to G_{ik}(t)$ is a strongly continuous $\mathscr{L}(E_{2-k}; E_{2-i})$-valued function and it follows from 1.2.3 that for any $\begin{pmatrix} x \\ y \end{pmatrix} \in D(\mathscr{A})$, i.e. for $x \in D(A)$ and $y \in E_1$, and for any $t \in (-\infty, \infty)$ we have

(4.1)
$$\begin{pmatrix} \dfrac{dG_{11}(t)x}{dt} + \dfrac{dG_{12}(t)y}{dt} \\ \dfrac{dG_{21}(t)x}{dt} + \dfrac{dG_{22}(t)y}{dt} \end{pmatrix} = \begin{pmatrix} G_{21}(t)x + G_{22}(t)y \\ AG_{11}(t)x + AG_{12}(t)y \end{pmatrix} = \begin{pmatrix} G_{12}(t)Ax + G_{11}(t)y \\ G_{22}(t)Ax + G_{21}(t)y \end{pmatrix},$$

where the derivatives in the first row are taken in the sense of the norm in E_1 and the derivatives in the second row are taken in the sense of the norm in E_0.

From these equalities and from the fact that $D(\mathscr{A})$ is dense in $E_1 \times E_0$ it is easy to see that for $\mathscr{C}(t) = G_{22}(t)$ all the continuity and differentiability properties stated in the lemma are valid and moreover the formula (2.2) holds. Therefore it remains only to prove that $\mathscr{C}(t)$ is an $\mathscr{L}(E_0; E_0)$-valued cosine function and that A is its infinitesimal generator.

To that end we shall apply Lemma 1.3.3.

Let operators $\pi_0 \in \mathscr{L}(E_1 \times E_0; E_0)$ and $J_0 \in \mathscr{L}(E_0; E_1 \times E_0)$ be defined by the formulae

$$\pi_0\begin{pmatrix} x \\ y \end{pmatrix} = y, \qquad J_0 y = \begin{pmatrix} 0 \\ y \end{pmatrix}.$$

Then $\mathscr{C}(t) = \pi_0\mathscr{G}(t)J_0$, $J_0 D(A) \subset D(\mathscr{A}^2)$, $\pi_0 D(\mathscr{A}^2) = D(A)$, $J_0 A = \mathscr{A}^2 J_0$, $\pi_0\mathscr{A}^2 = A\pi_0$, so that, by 1.2.2 and 1.2.3,

(i) $\quad \mathscr{C}(t)D(A) = \pi_0\mathscr{G}(t)J_0 D(A) \subset \pi_0\mathscr{G}(t)D(\mathscr{A}^2) = \pi_0 D(\mathscr{A}^2) = D(A)$

and

$$\text{(ii)} \quad \frac{d^2\mathscr{C}(t)x}{dt^2} = \pi_0\frac{d^2\mathscr{G}(t)J_0x}{dt^2} = \pi_0\mathscr{A}^2\mathscr{G}(t)J_0x = \pi_0\mathscr{G}(t)\mathscr{A}^2J_0x$$
$$= A\pi_0\mathscr{G}(t)J_0x = \pi_0\mathscr{G}(t)J_0Ax = A\mathscr{C}(t)x = \mathscr{C}(t)Ax$$

for any $t \in (-\infty, \infty)$ and $x \in D(A)$. We know from Lemma 4.1 that A is closed and $D(A)$ is dense in E_0. Therefore all the assumptions of Lemma 1.3.3 are satisfied and consequently $\mathscr{C}$ is a cosine function and A is its infinitesimal generator.

LEMMA 4.3. *The equality* (2.2) *is true.*

Proof. Let

$$\left\|\begin{pmatrix} x \\ y \end{pmatrix}\right\|_{E_1\times E_0} = \|x\|_{E_1}+\|y\|_{E_0}$$

and put

$$E_1^0 = \{x\colon x\in E_0,\ \text{the } E_0\text{-valued function } t\to \mathscr{C}(t)x \text{ is continuously differentiable on } (-\infty,\infty) \text{ in the sense of the norm in } E_0\}$$

and

$$\|x\|_{E_1^0} = \|x\|_{E_0}+\sup_{0\leqslant t\leqslant 1}\left\|\frac{d\mathscr{C}(t)x}{dt}\right\|_{E_0}$$

for every $x\in E_1^0$. According to Lemma 3.1, E_1^0 under the norm $\|\ \|_{E_1^0}$ is a Banach space. According to 1.2.1, there are constants $\lambda>0$ and $M\geqslant 1$ such that

$$\left\|\mathscr{G}(t)\begin{pmatrix} x \\ y \end{pmatrix}\right\|_{E_1\times E_0}\leqslant Me^{\lambda|t|}\left\|\begin{pmatrix} x \\ y \end{pmatrix}\right\|_{E_1\times E_0} \tag{4.2}$$

for every $\begin{pmatrix} x \\ y \end{pmatrix}\in E_1\times E_0$ and $t\in(-\infty,\infty)$. From 1.2.2 and from Lemma 3.2 it follows that $D(A)$ is dense in E_1^0 in the sense of the norm $\|\ \|_{E_1^0}$. From Lemma 4.2 it follows, that $E_1\subset E_1^0$. Since $D(A)\subset E_1$, it follows, that E_1 is dense in E_1^0 in the sense of the norm $\|\ \|_{E_1^0}$. Therefore the equality $E_1^0=E_1$ will follow, if we shall show, that there is a constant C, such that

$$\|x\|_{E_1}\leqslant C\|x\|_{E_1^0}\quad \text{for every } x\in E_1. \tag{4.3}$$

If $x\in E_1$, then, by Lemma 4.2, $\mathscr{C}(t)x$ is an E_1-valued function of t, continuous on $(-\infty,\infty)$ in the sense of the norm $\|\ \|_{E_1}$ and so, by the Lemma 4.2, by (2.3) and by (3.1), we have

$$\begin{aligned}\|x\|_{E_1} &\leqslant \tfrac{1}{2}\left\|\int_0^2\mathscr{C}(t)x\,dt\right\|_{E_1}+\frac{1}{2}\left\|\int_0^2\big(x-\mathscr{C}(t)x\big)\,dt\right\|_{E_1}\\ &\leqslant D\|x\|_{E_0}+\sup_{s,t\in[0,1]}\|\mathscr{C}(t+s)x-\mathscr{C}(t-s)x\|_{E_1}\\ &= D\|x\|_{E_0}+2\sup_{s,t\in[0,1]}\left\|\frac{d}{ds}\Big(\mathscr{C}(s)\int_0^t\mathscr{C}(\tau)x\,d\tau\Big)\right\|_{E_1}\end{aligned}$$

for every $x\in E_1$, where $D=\frac{1}{2}\left\|\int_0^2\mathscr{C}(t)\,dt\right\|_{\mathscr{L}(E_0;E_1)}$. Therefore, inequality (4.3) will be proved, if we shall show, that

$$\left\|\frac{d}{ds}\Big(\mathscr{C}(s)\int_0^t\mathscr{C}(\tau)x\,d\tau\Big)\right\|_{E_1}\leqslant Me^{\lambda|s|}\left\|\frac{d\mathscr{C}(t)x}{dt}\right\|_{E_0} \tag{4.4}$$

for every $x\in E_1$ and $s,t\in(-\infty,\infty)$.

If $x\in D(A)$, then $\begin{pmatrix} x \\ 0 \end{pmatrix}\in D(\mathscr{A})$, so that, by 1.2.3, $\mathscr{G}(t)\begin{pmatrix} x \\ 0 \end{pmatrix}\in D(\mathscr{A})$ and consequently $\frac{d\mathscr{C}(t)x}{dt}=\pi_0\mathscr{G}(t)\begin{pmatrix} x \\ 0 \end{pmatrix}\in E_1$. Let the operator $\pi_1\in\mathscr{L}(E_1\times E_0;E_1)$ be defined by the formula $\pi_1\begin{pmatrix} x \\ y \end{pmatrix}=x$. If $x\in D(A)$, then, by the Lemma 4.2, by (2.3) and 1.2.3, and by inequality (4.2), we have

$$\left\|\frac{d\mathscr{C}(s)x}{ds}\right\|_{E_1}=\left\|\pi_1\mathscr{G}(s)\mathscr{A}\begin{pmatrix} x \\ 0 \end{pmatrix}\right\|_{E_1}=\left\|\pi_1\mathscr{G}(s)\begin{pmatrix} 0 \\ Ax \end{pmatrix}\right\|_{E_1}\leqslant Me^{\lambda|s|}\|Ax\|_{E_0}.$$

If $x\in E_1$, then $\begin{pmatrix} 0 \\ x \end{pmatrix}\in D(\mathscr{A})$, so that also $\mathscr{G}(t)\begin{pmatrix} 0 \\ x \end{pmatrix}\in D(\mathscr{A})$ and hence $\int_0^t\mathscr{C}(\tau)x\,d\tau=\pi_1\mathscr{G}(t)\begin{pmatrix} 0 \\ x \end{pmatrix}\in D(A)$. It follows, that if $x\in E_1$, then

$$\left\|\frac{d}{ds}\Big(\mathscr{C}(s)\int_0^t\mathscr{C}(\tau)x\,d\tau\Big)\right\|_{E_1}\leqslant Me^{\lambda|s|}\left\|A\int_0^t\mathscr{C}(\tau)x\,d\tau\right\|_{E_0}.$$

But from equality (4.1) we have, that if $x\in E_1$, then $A\int_0^t\mathscr{C}(\tau)x\,d\tau=AG_{12}(t)x=G_{21}(t)x=\frac{d\mathscr{C}(t)x}{dt}$ and so, inequality (4.4) is proved. This completes the proof of Lemma 4.3 and, at the same time, the whole proof of the theorem from Section 2.

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