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ON COUPLING AND WEAK CONVERGENCE TO STATIONARITY

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This paper studies coupling methods for proving convergence in distribution of (typically Markovian) stochastic processes in continuous time to their stationary distribution. The paper contains: (a) a simple lemma on ε-coupling; (b) conditions for Markov processes to couple in compact sets; (c) new variants of the coupling proof of the renewal theorem; (d) a convergence result for stochastically monotone Markov processes in an ordered Polish space; and (e) a case study of a queue with superposed renewal input. In a companion paper with Foss, similar discussion is given for many-server queues in continuous time.

1. Introduction. Throughout this paper, we consider two stochastic processes $\{Z_t\}, \{Z'_t\}$ with the same metric state space E (equipped with the Borel-σ-algebra $\mathscr{E}$). Typically, E is a nice subset of Euclidean space $\mathbb{R}^p$, both processes are Markovian with the same transition function and $\{Z'_t\}$ is stationary with invariant probability distribution π (though such structure need not be assumed from the outset). The problem is then to obtain weak convergence to the stationary distribution, $Z_t \to_{\mathscr{D}} \pi$, $t \to \infty$.

Standard model classes suitable for dealing with this problem are Markov chains in discrete time or Markov jump processes in continuous time with a discrete state space, Harris-recurrent Markov chains in discrete time and regenerative processes in discrete time. Here even $Z_t \to \pi$ in total variation under conditions which often are very easy to verify in concrete cases. Total variation convergence also holds for regenerative processes in continuous time provided the cycle length distribution is spread out ([1], Chapter VI.2), whereas in the general nonlattice case weak convergence comes out equally easily. In fact, regenerative processes cover the bulk of standard problems occurring in areas like queuing theory and storage processes (in some cases, it is then important to allow the weakened assumption that the process after a regeneration point is independent only of the preceding cycle lengths, not necessarily of the process itself; cf. [1], Chapter V, [3], [23] and [24]).

Nevertheless, there are some cases well motivated from applications where regenerative processes do not appear sufficient to get weak convergence under minimal assumptions, for example, continuous-time Markov processes with a general state space [4], one-dependent regenerative processes [24], synchronous processes [11], many-server queues in continuous time [2] and Markov storage processes [14, 13, 10]. What has been obtained here is typically the existence of

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a stationary distribution π (e.g., by Palm theory [8, 22], or monotonicity arguments [18]) and the convergence of Cesaro averages,

$$\frac{1}{T}\int_0^T f(Z_t)\,dt \to_{\text{a.s.}} \pi(f) = \int_E f(z)\pi(dz), \tag{1.1}$$

whereas weak convergence seems a much harder problem, at least without some nonsingularity or spread-out conditions (in fact, even uniqueness of π may present a problem but follows of course once weak convergence has been established).

The present paper studies some techniques for dealing with this problem, all of which exploit coupling in some way or another (see Lindvall [17] for a survey of this set of ideas). We start in Section 2 by a simple general lemma stating that ε-coupling is sufficient for weak convergence. In Section 3 some general discussion of continuous-time Markov processes is given. What comes out is essentially that we can realize $\{Z_t\}, \{Z_t'\}$ so as to visit some large compact set K at the same time infinitely often (i.o.). In special models, one can then frequently show without pain that there is a uniformly positive small probability of ε-coupling following each visit to $K \times K$ and thereby obtain weak convergence by combining with a geometric trials argument and the lemma of Section 2. This approach is then pursued in Section 4 to give a new and short variant of Lindvall's proof of the renewal theorem [15]. Also, for didactic purposes, a further variant is given which is slightly longer but is totally elementary by avoiding both the general Markov process machinery of Section 3 and the Hewitt–Savage 0–1 law employed in [15]. In Section 5 we show by coupling the processes in an ordered way that stochastically monotone Markov chains in an ordered Polish space converge weakly to their stationary distribution provided a very mild irreducibility condition is satisfied. In Section 6 we study an example, queues with superimposed renewal input [24], which requires a combination of various ideas of the paper. Finally, the companion paper [2] with Foss obtains weak convergence in many-server queues in continuous time by related methods.

2. A general lemma on ε-coupling.

The concept of ε-coupling seems to originate from Lindvall [15] who used the idea of coupling renewal point processes to obtain a proof of the renewal theorem (note that in such cases one can only obtain an exact coupling with spread-out conditions; cf. [16]). A minor variant is given in [1], Chapter VI.2, where instead the forward recurrence time processes are coupled. The following lemma gives a general formulation of the concept and shows that this suffices for weak convergence.

LEMMA 2.1. *Let $\{Z_t\}_{0 \le t < \infty}, \{Z_t'\}_{-\infty < t < \infty}$ be right-continuous stochastic processes with the same metric state space E, and assume that:*

(a) *$\{Z_t'\}$ is strictly stationary with invariant distribution π,*

(b) *for each $\varepsilon > 0$ one can find versions of $\{Z_t\}, \{Z_t'\}$ defined on a common probability space and a.s. finite random times $D = D_\varepsilon$, $T = T_\varepsilon$ such that*

$$|D| \le \varepsilon, \qquad Z_t = Z'_{t+D}, \qquad t \ge T. \tag{2.1}$$

Then $Z_t \to_{\mathscr{D}} \pi$, $t \to \infty$.

PROOF. We must show $\mathbb{E} f(Z_t) \to \pi(f)$ [$= \mathbb{E} f(Z'_u)$, $u \geq 0$] when f is continuous and bounded, say $\|f\| \leq 1$. Define

$$M_{\varepsilon,t} = \sup_{|s|\leq\varepsilon} |f(Z'_{s+t}) - f(Z'_t)|, \qquad M_\varepsilon = M_{\varepsilon,0} = \sup_{|s|\leq\varepsilon} |f(Z'_s) - f(Z'_0)|.$$

It follows by stationarity and Tonelli's theorem that the probability that $\{Z'_t\}$ has a discontinuity at $t = 0$ is 0 ([1], pages 304 and 305; nothing else than right continuity is required for this). Thus $M_\varepsilon \downarrow 0$, so that also $\mathbb{E} M_\varepsilon \downarrow 0$ by monotone (or dominated) convergence. Furthermore,

$$\begin{aligned}|\mathbb{E} f(Z_t) - \pi(f)| &= |\mathbb{E} f(Z_t) - \mathbb{E} f(Z'_t)| \\ &\leq |\mathbb{E}[f(Z_t) - f(Z'_t); t < T]| + |\mathbb{E}[f(Z'_{t+D}) - f(Z'_t); t \geq T]| \\ &\leq 2\mathbb{P}(T > t) + \mathbb{E} M_{\varepsilon,t} = 2\mathbb{P}(T > t) + \mathbb{E} M_\varepsilon.\end{aligned}$$

Letting first $t \to \infty$ and then $\varepsilon \downarrow 0$ completes the proof. □

REMARK 2.1. It is not difficult to see that the condition $Z_t = Z'_{t+D}$, $t \geq T$, can be weakened to $d(Z_t, Z'_{t+D}) \leq \varepsilon$, $t \geq T$, where d is the distance function on E (in the proof of Lemma 2.1, one then takes f uniformly continuous). At present, we do not know of examples where this added generality is useful. Another extension of Lemma 2.1 is implicit in the proof of Proposition 6.1 below.

Though our main purpose for studying ε-coupling is to prove $Z_t \to_{\mathscr{D}} \pi$, one may note that the approach applies also to certain other types of problems. For example, we have the following result covering also aspects of null recurrent and transient behavior [part (b) was inspired by some suggestions by Dr. Peter Glynn].

PROPOSITION 2.1. *Assume that* $\{Z_t\}$ *is Markovian and that for some* $\varepsilon > 0$ (2.1) *holds for any two versions* $\{Z_t\}, \{Z'_t\}$ *with different initial distributions. Then*:

(a) *The Cesaro average* $(1/t)\int_0^t f(Z_v)\,dv$ *has a* $\mathbb{P}_x$-*a.s. limit either for all* $x \in E$ *or for no* x. *If so, the limit is independent of* x.

(b) *The shift-invariant* σ-*field* $\mathscr{I}$ *is* $\mathbb{P}_\mu$-*trivial for any initial distribution* μ.

PROOF. Part (a) follows by straightforward estimates, taking $Z_0 = x$, $Z'_0 = y$. For part (b), let Z be $\mathscr{I}$-measurable and bounded, and define $f(x) = \mathbb{E}_x Z$. By shift invariance, $\mathbb{E}(Z|\mathscr{F}_t) = f(Z_t)$, and therefore by a standard result on convergence of conditional expectations, $f(Z_t)$ converges $\mathbb{P}_x$-a.s. to a random variable $Z(x)$ which has the $\mathbb{P}_x$-distribution of Z. Therefore also Cesaro averages converge, and thus

$$f(x) = \mathbb{E} Z(x) = \mathbb{E} \lim_{t\to\infty} \frac{\int_0^t f(Z_v)\,dv}{t}.$$

Using part (a), this is the same as

$$\mathbb{E} \lim_{t \to \infty} \frac{\int_0^t f(Z'_v)\,dv}{t},$$

which by the same argument equals $f(y)$. Hence $f(x)$ is independent of x, say $f(x) = c$, and $f(Z_t) \to Z(x)$ $\mathbb{P}_x$-a.s. now implies that $Z = c$ $\mathbb{P}_x$-a.s. Since x is arbitrary, the proof is complete. □

3. Harris-recurrent Markov processes in continuous time. Throughout this section, $\{Z_t\}$ is a continuous-time Markov process with a Polish state space E. We use standard notation like $\mathbb{P}_x$ to indicate $Z_0 = x$, and so on, and define $\tau(A) = \inf\{t: Z_t \in A\}$.

No entirely satisfying analogue of the theory of discrete-time Harris-recurrent Markov chains seems to have been developed in this setting, and in fact not even the concept of Harris recurrence in continuous time is unambiguous. The definition employed in [4] is:

ASSUMPTION 3.1. There exists a nonzero positive measure v on $(E, \mathscr{E})$ such that $\int_0^\infty I(Z_t \in A)\,dt = \infty$ $\mathbb{P}_x$-a.s. for all $x \in E$ whenever $v(A) > 0$.

This is a close parallel to the classical definition of Harris recurrence in discrete time and allows us to conclude the existence of a unique invariant measure π and convergence of Cesaro averages (see Corollary 3.1 below). However, simple conditions for the convergence to π in the positive recurrent case (where we may take $\|\pi\| = 1$) have not been established and also verification of Assumption 3.1 does not always seem straightforward in concrete cases. To overcome these problems, the author used in [1] an ad hoc definition ensuring total variation convergence but this definition is also more restrictive; an approach of a somewhat similar spirit can be found in Niemi and Nummelin [19]. It was noted by Kaspi and Mandelbaum [13] that using the following assumption instead may be more convenient in many cases:

ASSUMPTION 3.2. There exists a nonzero positive measure λ on $(E, \mathscr{E})$ such that $\tau(A) < \infty$ $\mathbb{P}_x$-a.s. for all $x \in E$ whenever $\lambda(A) > 0$.

(For example, Assumption 3.2 holds whenever $\{Z_t\}$ has an embedded discrete-time Harris chain, whereas this is not the case for the definitions in [1] and [19]). In fact, the following result was shown in [13].

THEOREM 3.1. *Asumption* 3.1 *is equivalent to Assumption* 3.2.

From [4], one then gets the following corollary.

COROLLARY 3.1. *If either Assumption* 3.1 *or* 3.2 *hold*, *then* $\{Z_t\}$ *has an invariant measure* π *which is unique up to a constant. If furthermore* $\|\pi\| = 1$, *then Cesaro averages converge $\mathbb{P}_x$-a.s. as in* (1.1) *for any x and any π-integrable f.*

REMARK 3.1. In Assumption 3.2, we have omitted certain technicalities needed to ensure measurability of hitting times. As is well known (e.g., [7]), these difficulties play no essential role when, as here, E is Polish. However, in the typical case where Assumption 3.2 is verified using an embedded discrete-time Harris chain, such discussion is not necessary at all.

For the purposes of coupling, our main application of these results will be to employ convergence of Cesaro averages to establish the following proposition.

PROPOSITION 3.1. *Let $\{Z_t\}$ be a Harris-recurrent Markov process (in the sense of either of Assumption* 3.1 *or* 3.2) *with invariant probability distribution π and let $\{Z_t'\}$ be a stationary version defined on the same probability space (but not necessarily independent of $\{Z_t\}$). If K is a set with $\pi(K) > 1/2$, then $\limsup_{t\to\infty} I((Z_t, Z_t') \in K \times K) = 1$ $\mathbb{P}_x$-a.s. for all x.*

PROOF. Define $G_T = \{t \le T\colon\ Z_t \in K\}$, $G'_T = \{t \le T\colon\ Z_t' \in K\}$. Then, with $|\cdot|$ denoting Lebesgue measure, we have

$$|G_T \cap G'_T| = |G_T| + |G'_T| - |G_T \cup G'_T| \ge |G_T| + |G'_T| - T.$$

Since $|G_T|/T \to \pi(K) > 1/2$ $\mathbb{P}_x$-a.s. according to Corollary 3.1 (and similarly for G'_T), it follows that

$$0 < \liminf_{T\to\infty} \frac{|G_T \cap G'_T|}{T} = \liminf_{T\to\infty} \frac{|\{t \le T\colon (Z_t, Z') \in K \times K\}|}{T},$$

from which the desired conclusion follows. □

Though simple to prove, Proposition 3.1 seems potentially quite useful. In particular, for locally compact state spaces, one can always find a compact K with $\pi(K) > 1/2$. In applications (e.g., the next section or [2]), one can then frequently show without pain that there is a uniformly positive small probability of ε-coupling following each visit to $K \times K$ and thereby obtain weak convergence by combining with a geometric trials argument and Lemma 2.1.

4. The renewal theorem. We consider a renewal process with interarrival distribution B and let Z_t be the forward recurrence time at time t (the time until next renewal). We assume that B is nonlattice with finite mean μ and let π denote the stationary distribution for $\{Z_t\}$, that is, the distribution with density $(1 - B(x))/\mu$ with respect to Lebesgue measure. The renewal theorem in this setting then states that $Z_t \to_{\mathscr{D}} \pi$ no matter the initial condition $Z_0 = x$ (from this other versions like the key renewal theorem or Blackwell's renewal theorem then follow easily; cf. [1], Chapter IV.4). We let $\{Z_t'\}$ be an independent version of $\{Z_t\}$ with possibly different initial conditions (e.g., a stationary process). Furthermore, $R(x, y)$ denotes the event of a renewal in the interval (x, y) [or, equivalently, of $Z_{t-} = 0$ for some $t \in (x, y)$] and similarly, $R'(x, y)$ refers to $\{Z_t'\}$.

The following lemma is standard in all proofs of the renewal theorem we know of. It is here that the nonlattice property is used in an essential way. Usually, only the zero-delayed case is stated in textbooks like [1], Chapter IV.5, but the general case follows immediately from this.

LEMMA 4.1. *For any* $\varepsilon > 0$ *there is a* $T < \infty$ *such that for any* $x \geq 0$, $\mathbb{P}_x R(t, t + \varepsilon/2) > 0$ *when* $t \geq T + x$.

From this we get the following lemma.

LEMMA 4.2. *For any* k *and* $\varepsilon > 0$ *there is an* $S < \infty$ *such that*

$$\delta = \inf_{x,y \leq k} \mathbb{P}_{x,y}(R(S - \varepsilon/2, S + \varepsilon/2) \cap R'(S - \varepsilon/2, S + \varepsilon/2)) > 0.$$

PROOF. Let $x_0, \dots, x_N$ be the points in $[0, k]$ of the form $i \cdot \varepsilon/2$ and choose $S > k + T$ with T as in Lemma 4.1. A sample path inspection shows that

$$\mathbb{P}_{x,y}(R(S - \varepsilon/2, S + \varepsilon/2) \cap R'(S - \varepsilon/2, S + \varepsilon/2))$$
$$\geq \mathbb{P}_{x_i, x_j}(R(S, S + \varepsilon/2) \cap R'(S, S + \varepsilon/2)),$$

when $x_{i-1} \leq x \leq x_i$, $x_{j-1} \leq y \leq x_j$. Thus

$$\delta \geq \min_{i=1,\dots,N} \left[\mathbb{P}_{x_i} R(S, S + \varepsilon/2)\right]^2,$$

which is positive by Lemma 4.1. □

LEMMA 4.3. *Let* $K = (0, k]$. *Then w.p.*1: *For all large enough* k, $\{(Z_t, Z_t')\}$ *visits* $K \times K$ *i.o.*

PROOF. Considering $\{Z_t\}$ just after jumps we get a sequence of i.i.d. random variables with distribution B. Thus obviously Assumption 3.2 holds with $\lambda = B$ so that Proposition 3.1 applies whenever k is chosen so large that $\pi(K) > 1/2$. □

We can now easily give the following proof.

PROOF OF THE RENEWAL THEOREM $Z_t \to_{\mathscr{D}} \pi$. Let $\{Z_t'\}$ be stationary. According to Lemma 4.3, w.p.1 a sequence $\{\sigma(k)\}$ of stopping times exists with $(Z_{\sigma(k)}, Z'_{\sigma(k)}) \in K \times K$ and $\sigma(k + 1) - \sigma(k) > S + \varepsilon/2$, and using a geometric trials argument and Lemma 4.2 it follows that

$$\omega = \inf\{t: R(t - \varepsilon/2, t + \varepsilon/2) \cap R'(t - \varepsilon/2, t + \varepsilon/2)\}$$

is finite a.s. After ω, we can in an obvious way modify the processes such that the interarrival intervals of $\{Z_t\}$ and $\{Z_t'\}$ coincide, and obtain in this way an ε-coupling. Lemma 1.1 completes the proof. □

It has sometimes been argued that, even though the proof of [15] is probabilistic, it is not elementary due to the fact that the Hewitt–Savage 0–1 law is used in an essential way. The same type of objection applies of course to the present proof since the background material from Section 3 on Markov processes needed for Lemma 4.3 is not elementary either. For didactic purposes, it is therefore of some interest to give the following proof.

ELEMENTARY PROOF OF LEMMA 4.3. Arguing as in the proof of Proposition 3.1, it is sufficient to show that $\int_0^T I(Z_t \le k)\,dt/T \to \pi(0,k]$ for any k. Let $x = Z_0$, let $Y_1, Y_2, \dots$ be the interarrival times and let N_T be the number of renewals before T. Then

$$(4.1)\qquad \sum_{i=1}^{N_T-1} (Y_i \wedge k) \le \int_0^T I(Z_t \le k)\,dt \le (x \wedge k) + \sum_{i=1}^{N_T} (Y_i \wedge k).$$

Here $\mathbb{E}(Y \wedge k) = \mu\pi(0,k]$. Also we can bound $Y_{N_T} \wedge k$ by k, whereas by the elementary renewal theorem, $N_T/T \to 1/\mu$ a.s. Hence by the strong law of large numbers,

$$\frac{1}{T}\int_0^T I(Z_t \le k)\,dt = \frac{1}{T}\sum_{i=1}^{N_T} (Y_i \wedge k) + o(1) = \frac{N_T}{T}\mathbb{E}(Y \wedge k) + o(1)$$
$$= \pi(0,k] + o(1) \quad \text{a.s.} \qquad □$$

For a further coupling proof of the renewal theorem, see [25].

5. Stochastically monotone Markov processes. We now assume that E is a Polish space equipped with a partial ordering $\le$ (see, e.g., [12]). Of course, a main example is an interval on the real line (open, closed or half-open), but our discussion applies with minor changes to this more general setting, at least when imposing some further regularity conditions. We write $\uparrow x = \{y\colon y \ge x\}$, $\downarrow x = \{y\colon y \le x\}$, $[y,x] = \uparrow y \cap \downarrow x$, and shall need the following assumptions.

ASSUMPTION 5.1. There exists a sequence $\{x_n\}$ such that $E = \bigcup \uparrow x_n = \bigcup \downarrow x_n$.

ASSUMPTION 5.2. E has the monotone coupling property.

By this we mean that if $\{Z_t'\}$ is stationary and we can find versions $\{Z_t^{(+)}\}, \{Z_t^{(-)}\}$ of $\{Z_t\}$ and random times T_+, T_- such that

$$(5.1)\qquad Z_t^{(+)} \ge Z_t' \ \text{ for } t \ge T_+, \qquad Z_t^{(-)} \le Z_t' \ \text{ for } t \ge T_-,$$

then $Z_t \to_{\mathscr{D}} \pi$, where π is the stationary distribution. A sufficient condition is the following [recall that a class $\mathscr{H}$ of functions is *convergence determining* if $\mathbb{E} f(Z_t) \to \pi(f)$ for all $f \in \mathscr{H}$ implies $Z_t \to_{\mathscr{D}} \pi$].

PROPOSITION 5.1. *If the class of nondecreasing bounded functions on E is convergence determining, then E has the monotone coupling property.*

PROOF. Let f be bounded and nondecreasing and choose T_- as above. Then

$$\begin{aligned}\limsup_{t\to\infty} \mathbb{E} f(Z_t) &= \limsup_{t\to\infty} \mathbb{E} f\big(Z_t^{(-)}\big)\\ &\le \limsup_{t\to\infty} \big\{\mathbb{E}[f(Z_t'); t \ge T_-] + \mathbb{E}\big[f\big(Z_t^{(-)}\big); t < T_-\big]\big\}\\ &= \limsup_{t\to\infty} \big\{\pi(f) + \mathbb{E}\big[f\big(Z_t^{(-)}\big) - f(Z_t'); t < T_-\big]\big\}\\ &= \pi(f) + 0.\end{aligned}$$

By symmetry, $\liminf \mathbb{E} f(Z_t) \ge \pi(f)$ so that $\mathbb{E} f(Z_t) \to \pi(f)$. □

Obviously, Proposition 5.1 applies to subspaces of R^n. Other instances of the monotone coupling property can be derived from results of [20] and cover, for example, continuous posets in the Lawson topology (cf. [9]) and hence their Polish subspaces.

The process $\{Z_t\}$ will in this section be strong Markov with respect to some (arbitrary) filtration, have right-continuous paths in the continuous time case and satisfy the following assumptions.

ASSUMPTION 5.3. $\{Z_t\}$ is stochastically monotone.

[Recall that $\{Z_t\}$ is stochastically monotone if $x < y$ implies $\mathbb{P}_x(Z_t \ge k) \le \mathbb{P}_y(Z_t \ge k)$ for all k and all t.]

ASSUMPTION 5.4. $\{Z_t\}$ has an invariant distribution π such that $\pi(\uparrow k) > 0$, $\pi(\downarrow k) > 0$ for some $k \in E$.

ASSUMPTION 5.5. For each $h, l, x, y, \in E$ there is a t such that $\mathbb{P}_h(Z_t \ge x)$ and $\mathbb{P}_l(Z_t \le y)$ are both strictly positive.

Note that Assumption 5.5 is essentially a very weak irreducibility condition which seems easy to verify in concrete cases. It holds for example if the following assumption is satisfied:

ASSUMPTION 5.5′. For each $h, k \in E$, $\mathbb{P}_h(Z_t > k)$ and $\mathbb{P}_h(Z_t < k)$ are both strictly positive for all large enough t.

In fact, Assumption 5.5′ is a necessary condition for weak convergence, say in $\mathbb{R}^n$, with $\operatorname{supp}(\pi) = E$, and we shall see that nothing more is needed.

THEOREM 5.1. *If Assumptions* 5.1–5.5 *hold, then* $Z_t \to_{\mathscr{D}} \pi$.

The main step of the proof is the following lemma.

LEMMA 5.1. *For any two independent versions* $\{Z_t\}, \{Z_t'\}$ *of the process and any initial values* $x, y \in E$,

$$T_- = \inf\{t \geq 0: Z_t \leq Z_t'\} < \infty \quad \mathbb{P}_{x,y} - a.s.$$

PROOF. Let $A = \downarrow k \times \uparrow k$ with k as in Assumption 5.4. Then according to Assumptions 5.4 and 5.1, we can choose h, l such that $\tilde{A} = [h, k] \times [k, l]$ has positive $\pi \otimes \pi$ measure. Let

$$I_j = I((Z_t, Z_t') \in A \text{ for some } t \in [j, j+1)), \qquad N(A) = \sum_{j=0}^{\infty} I_j,$$

and let $N(\tilde{A})$ be defined similarly. It is then more than sufficient to show $\mathbb{P}_{x,y}(N(A) = \infty) = 1$.

We use an indirect argument, so assume $\mathbb{P}_{x_0,y_0}(N(A) = \infty) < 1$ for some x_0, y_0. Then also $\delta = \mathbb{P}_{x_1,y_1}(N(A) = 0) > 0$ for some x_1, y_1 so that by monotonicity,

$$\mathbb{P}_{x,y}(N(A) = 0) \geq \delta \quad \text{when } x \geq x_1, y \leq y_1. \tag{5.2}$$

According to Assumption 5.5, there exists an integer S such that

$$\eta = \mathbb{P}_{h,l}(Z_t \geq x_1,\, Z_t' \leq y_1 \text{ for some } t \leq S) > 0. \tag{5.3}$$

Thus by monotonicity,

$$\mathbb{P}_{x,y}(Z_t \geq x_1,\, Z_t' \leq y_1 \text{ for some } t \leq S) \geq \eta \tag{5.4}$$

for $x, y \in \tilde{A}$. We shall show that this implies

$$\mathbb{P}_{x,y}\big(N(\tilde{A}) < \infty\big) = 1 \tag{5.5}$$

for all $x, y \in E$; from (5.5) it follows that $\mathbb{P}_{x,y}((Z_t, Z_t') \in \tilde{A}) \to 0$, so that the desired contradiction is obtained from

$$\begin{aligned} 0 < (\pi \otimes \pi)(\tilde{A}) &= \mathbb{P}_{\pi,\pi}\big((Z_t, Z_t') \in \tilde{A}\big) \\ &= \iint \mathbb{P}_{x,y}\big((Z_t, Z_t') \in \tilde{A}\big)\pi(dx)\pi(dy) \to 0. \end{aligned}$$

To show (5.5), let

$$\sigma_1 = \inf\big\{t \geq 0: (Z_t, Z_t') \in \tilde{A}\big\}, \qquad \sigma_{k+1} = \inf\big\{t \geq \lfloor \sigma_k \rfloor + S: (Z_t, Z_t') \in \tilde{A}\big\}$$

($\lfloor\cdot\rfloor$ = integer part). Then

$$\{N(\tilde{A}) = \infty\} = \left\{\sum_0^\infty I(\sigma_k < \infty)\right\} = \bigcup_{n=1}^\infty \bigcap_{k=n}^\infty \{\sigma_k < \infty\}.$$

If the time parameter t is discrete, we have $(Z_{\sigma_k}, Z'_{\sigma_k}) \in \tilde{A}$ on the event $\{\sigma_k < \infty\}$, and using (5.2), (5.3), (5.4) and the strong Markov property yields

$$\mathbb{P}_{x,y}(\sigma_{k+1} < \infty | \sigma_k < \infty) \le 1 - \eta\delta < 1,$$

so that by a geometric trials argument $\sigma_k = \infty$ eventually and hence $N(\tilde{A}) < \infty$. In continuous time, the difficulty is that $(Z_{\sigma_k}, Z'_{\sigma_k}) \in \tilde{A}$ need not hold. However, by Meyer's section theorem ([26], Section 52, or [21], Section 6.5, here applied to a finite segment of the process), stopping times τ_k with

$$\tau_k \in [\sigma_k, \lfloor\sigma_k\rfloor + 1) \cup \{\infty\}, \qquad (Z_{\tau_k}, Z'_{\tau_k}) \in \tilde{A} \quad \text{on } \{\sigma_k < \infty\},$$

$$\mathbb{P}(\tau_k = \infty, \sigma_k < \infty) < 2^{-k}$$

exist. Arguing as above, $\mathbb{P}(\tau_k = \infty \text{ eventually}) = 1$. But, by the Borel–Cantelli lemma, σ_k and τ_k are either both finite or both infinite for all sufficiently large k. □

Proof of Theorem 5.1. Take $\{Z'_t\}$ stationary and note that it is well known that it is possible to couple two versions $\{Z_t\}, \{Z'_t\}$ of the process starting from x, resp. y, where $x \le y$, in such a way that $Z_t \le Z'_t$ for all t. Using the strong Markov property, it thus follows from Lemma 5.1 that we may modify $\{Z_t\}$ to a version $\{Z_t^{(-)}\}$ such that $Z_t^{(-)} \le Z'_t$ for $t \ge T_-$. Invoking a symmetry argument and the monotone coupling property completes the proof. □

For a result related to Theorem 5.1, see [5] where, however, the conditions are close to compactness (and the conclusions correspondingly stronger; e.g., the existence of π is automatic as well as exponential ergodicity). A typical class of models where Theorem 5.1 would appear relevant are storage processes. Here often the existence of π has been derived directly (by analytical arguments in [6] and [14] and by monotonicity in [10]), but weak convergence is harder to get at, in particular if the process does not hit points [14], and also the uniqueness of π is a problem in [10] (but follows of course from Theorem 5.1 by adding Assumption 5.5 to those of [10]).

6. An example: queues with superposed renewal input. Let $\{\tau_n(1)\}, \dots, \{\tau_n(c)\}$ be independent renewal processes with interarrival distributions $F_1, \dots, F_c$, arrival rates $\lambda_1 > 0, \dots, \lambda_c > 0$ ($\lambda_i^{-1} = \mathbb{E}[\tau_n(i) - \tau_{n-1}(i)]$) and forward recurrence times $B_1(t), \dots, B_c(t)$. Consider as in Sigman [24] a queue where a customer with service time $U_n(i)$ arrives at time $\tau_n(i)$ and let V_t be the work load at time t (the amount of time the server will have to work if no new customers arrive). All service times are assumed independent, i.i.d. for each i and independent of the arrival processes, but the queue discipline is not necessarily first in first out (FIFO): As in [24], any work-conserving discipline (meaning that the sample paths of the workload process $\{V_t\}$ is as for the FIFO case) will be allowed. We let $Z_t = (B_1(t), \dots, B_c(t), V_t)$. Assuming $\rho = \rho_1 + \cdots + \rho_c < 1$ where $\rho_i = \lambda_i \mathbb{E}U(i)$, the existence of a stationary distribution π can then easily be inferred using stationarity properties of renewal processes and Theorem 2.2.2 of [8]. We shall show the following result, which is a strengthening of [24] in that it does not require spread-out conditions.

PROPOSITION 6.1. *For the above queue with superposed renewal input, $Z_t \to_{\mathscr{D}} \pi$ provided all interarrival distributions F_i are nonlattice.*

PROOF. Consider a stationary version $\{Z'_t\}$ corresponding to $\tau'_n(i)$, $B'_i(t)$, V'_t and so on, and let $\varepsilon > 0$ be given. Then, since renewal processes can be ε-coupled, we can find random times D_i, T_1 such that $|D_i| \le \varepsilon$ and $B_t(i) = B'_{t+D_i}(i)$ for $t \ge T_1$, $i = 1, \dots, c$. Without loss of generality, assume that $T_1 = 0$ and that the service times have been coupled as well in the obvious way.

Let $\{\tilde{Z}_t\}$ be defined in terms of the arrival times $\tilde{\tau}_n(i)$ defined by $\tilde{\tau}_n(i) = \tau'_n(i) + \varepsilon$ if $\tau'_n(i) \ge 0$, $= \tau'_n(i)$ otherwise, and the initial condition $\tilde{V}_0 = V'_0 + \varepsilon$. Note that this means that we shift the arrival times of $\{Z'_t\}$ in $[0, \infty)$ (only!) ε to the right, and hence $\tilde{Z}_t = Z'_{t-\varepsilon}$, $t \ge \varepsilon$. Consider a process $\{S_t\}$ which jumps the same amount as $\{V_t\}$ at each arrival, moves linearly downwards at a unit rate in between jumps and starts from $S_0 = 0$. Let $\{\tilde{S}_t\}$ be defined in a similar way but relative to the increments of $\{\tilde{Z}_t\}$. Then ([1], Chapter III.8)

$$V_t = \sup\{V_0 + S_t, S_t - S_s \colon s \le t\}, \tag{6.1}$$

$$\tilde{V}_t = \sup\left\{\tilde{V}_0 + \tilde{S}_t, \tilde{S}_t - \tilde{S}_s \colon s \le t\right\}. \tag{6.2}$$

Let $G_{t,\varepsilon}$ denote the event that $\{Z'_t\}$ has no jumps in $[t - 2\varepsilon, t + \varepsilon]$. If $G_{t,\varepsilon}$ occurs, then $\{Z_t\}$ and $\{\tilde{Z}_t\}$ have no jumps in $[t - \varepsilon, t]$, from which it follows that $S_t = \tilde{S}_t$. Also, since $\{\tilde{Z}_t\}$ has arrivals at later times than $\{Z_t\}$, we have $\tilde{S}_s \le S_s$ for all s. It is easy to check from $\rho < 1$ and the LLN that $S_t \to_{\text{a.s.}} -\infty$, $\tilde{S}_t \to_{\text{a.s.}} -\infty$ so that (6.2) implies $V_t = \sup\{S_t - S_s \colon s \le t\}$ for all large enough t and similarly for (6.2). Putting things together, we see that for some random time T we have $V_t \le \tilde{V}_t = V'_{t-\varepsilon}$ on $G_{t,\varepsilon} \cap \{T \le t\}$. Now let $\mathscr{H}$ be the class of functions on $\mathbb{R}^{c+1}$ of the form $f(v)g_1(b_1) \cdots g_c(b_c)$, where $f, g_1, \dots, g_c$ are $[0,1]$-valued functions such that f is nondecreasing and the g_i continuous. For a fixed but arbitrary function in $\mathscr{H}$, define

$$M_{t,\varepsilon} = \sup_{s_1, \dots, s_c \in [-\varepsilon, \varepsilon]} \left| \prod_{i=1}^{c} g_i\big(B'_{t+s_i}(i)\big) - \prod_{i=1}^{c} g_i\big(B'_t(i)\big) \right|.$$

By an easy extension of the proof of Lemma 2.1, $M_{s,\varepsilon} \to_{\mathscr{D}}$ as $\varepsilon \downarrow 0$, and hence (by boundedness) $\mathbb{E}M_{t,\varepsilon} \to 0$. Also obviously $\mathbb{P}G_{s,\varepsilon} \to 1$, and hence

$$\begin{aligned}\mathbb{E}\left[f(V_t)\prod_{i=1}^{c} g_i(B_t(i))\right] &\le \mathbb{E}\left[f(V'_{t-\varepsilon})\prod_{i=1}^{c} g_i(B_t(i))\right] + \mathbb{P}(T > t) + \mathbb{P}G^c_{t,\varepsilon}\\ &\le \mathbb{E}\left[f(V'_{t-\varepsilon})\prod_{i=1}^{c} g_i(B'_{t-\varepsilon}(i))\right] + \mathbb{E}M_{t-\varepsilon,2\varepsilon} + \mathbb{P}(T > t) + \mathbb{P}G^c_{t,\varepsilon}\\ &\to \mathbb{E}\left[f(V'_0)\prod_{i=1}^{c} g_i(B'_0(i))\right] + \mathbb{E}M_{-\varepsilon,2\varepsilon} + \mathbb{P}G^c_{0,\varepsilon},\end{aligned}$$

so that

$$\limsup_{t\to\infty} \mathbb{E}\left[f(V_t)\prod_{i=1}^{c} g_i(B_t(i))\right] \le \mathbb{E}\left[f(V'_0)\prod_{i=1}^{c} g_i(B'_0(i))\right].$$

In a similar manner, one obtains lim inf $\ge$, and the proof is complete since the class $\mathscr{H}$ is easily seen to be convergence determining on $\mathbb{R}^{c+1}$. □

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