

RESEARCH NOTE

On covariances of eigenvalues and eigenvectors of second-rank symmetric tensors

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SUMMARY

The applications of eigentheory to many branches of mathematical physics (e.g., rotational dynamics, continuum mechanics) is an unquestionable fact. This work expands the conventional methodology by introducing equations to compute the covariance matrices of eigenvalues and eigenvectors of second-rank 3-D symmetric tensors in terms of their six distinct elements error estimates. New analytical expressions derived herein are general and should be of interest to anyone concerned with the accuracy of the computed orientation of principal axes and their associated principal quantities (e.g., moments of inertia, stress, strain).

Key words: eigenvalues, eigenvectors, principal strain, principal stress.

INTRODUCTION

In mechanics, dynamics, statistics, etc. symmetric 3-D second-rank tensors play important roles. They are used to describe the mechanical properties of rigid or deformable bodies in terms of inertia, stress or strain. In addition, the statistical properties of the 3-D position of points on those bodies represented by covariance matrices, or its geometric interpretation in terms of 'error ellipsoids' are often required.

The equations to transform, under a known rotation, the components of these types of tensors between two arbitrary Cartesian coordinate frames are well known. Among all possible coordinate frames, emphasis is generally placed upon the one defining the principal axes.

Through the use of eigenvalues and eigenvectors 3×3 symmetric tensors consisting of six distinct elements can be diagonalized by choosing an appropriate reference frame. The original tensor will transform into a tensor of diagonal form (three eigenvalues), representing e.g., principal quantities (moments of inertia, strain, stress), and a rotation (eigenvector) matrix, specifying the rotation of the original arbitrary reference frame into the coordinate frame in which the mechanical or statistical properties become 'principal' or 'uncorrelated'.

In this note formulae to assess the precision of the components of the final rotated tensor as a function of the precision estimates of the components of the original given tensor are first established. Then, novel analytical expressions to compute the covariance matrix of the eigenvalues and eigenvectors in terms of the same initial variables are developed. This in turn provides the accuracies of important physical parameters such as magnitude and orientation of principal moments of inertia, strain, stress, etc.

The literature currently available discusses thoroughly the computation and properties of eigenvalues and eigenvectors of 3-D symmetric tensors but systematically ignores their accuracies. This paper supplements many textbooks in linear algebra and statistics by answering the question of how reliable the computed eigenvalues and eigenvectors are as implied from the known estimates of accuracy associated with the original undiagonalized tensor components.

COVARIANCE MATRIX OF ROTATED SYMMETRIC TENSOR COMPONENTS

Restricting the present discussion to the 3-D space of our ordinary experience, assume that a transformation of the type

$$[\varepsilon'] = R[\varepsilon]R^T \quad (1)$$

is established. The matrices $[\varepsilon]$ and $[\varepsilon']$ are 3-D second-rank symmetric tensors (e.g., inertia, stress, strain, etc.) whose components are transformed from one Cartesian coordinate frame (x, y, z) (e.g., global, platform) to another (x', y', z') (e.g., local, inertial). As usual, the 3×3 matrix R denotes the proper (pure) rotation (i.e., $R^{-1} = R'$ and $|R| = +1$) of the transformation between the two frames to which $[\varepsilon]$ and $[\varepsilon']$ refer (i.e., $\{x'\} = R\{x\}$). For simplicity the same origin and scale have been assumed for the two coordinate frames.

Moreover, suppose that the 6×6 covariance matrix (referred to by some authors as the variance-covariance matrix) of the six distinct elements of the tensor $[\varepsilon]$ is known. This implies that correlations between the six distinct elements ε_{ij} of $[\varepsilon]$ are also known.

The first problem addressed here is that of calculating the covariance matrix of the six distinct elements of the transformed tensor $[\varepsilon']$. Although this can be accomplished by strict application of the so-called 'covariance law', the procedure will require the computation of the corresponding Jacobian matrix after each individual element ε'_{ij} in equation (1) is expressed as an explicit function of the original ε_{ij} elements.

The approach suggested herein also uses the covariance law but is more general in scope; it only uses matrix algebra concepts and avoids the computation of any partial derivatives. Still more important, the methodology introduced can be later applied to solve the more complex problem of obtaining the covariance matrix of the eigenvalues and eigenvectors of $[\varepsilon]$ as a function of the covariance matrix of the elements ε_{ij} .

Denote the columns of $[\varepsilon]$ respectively by $\{\varepsilon_1\}$, $\{\varepsilon_2\}$, and $\{\varepsilon_3\}$, that is

$$[\varepsilon] = [\{\varepsilon_1\} : \{\varepsilon_2\} : \{\varepsilon_3\}] = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} \end{bmatrix} \tag{2}$$

where $\varepsilon_{ij} = \varepsilon_{ji}$ when $i \neq j$.

Then, by definition the *vec* of the matrix $[\varepsilon]$ can be written (Rogers 1980; Graham 1981)

$$\text{vec}[\varepsilon] = \begin{Bmatrix} \{\varepsilon_1\} \\ \{\varepsilon_2\} \\ \{\varepsilon_3\} \end{Bmatrix}. \tag{3}$$

It can be proved (see appendix A), that applying the *vec* operator to both sides of equation (1) we get

$$\text{vec}[\varepsilon'] = [T] \text{vec}[\varepsilon] \tag{4}$$

where

$$[T] = \begin{bmatrix} R & \otimes & R \\ 9 \times 9 & 3 \times 3 & 3 \times 3 \end{bmatrix} \tag{5}$$

and the symbol \otimes denotes the ordinary Kronecker product [to give rigorous historical credit, better referred as 'Zehfuss product' according to Henderson, Pukelsheim & Searle (1983)]. An example of an application of this product in geodetic literature may be found in e.g. Schaffrin (1985). Recall that given two general conformable matrices $[A]$ and $[B]$ we may write the Zehfuss product as

$$[A] \otimes [B] = \llbracket a_{ij}[B] \rrbracket = \begin{bmatrix} a_{11}[B] & \cdots & a_{1n}[B] \\ \vdots & & \vdots \\ a_{m1}[B] & \cdots & a_{mn}[B] \end{bmatrix} \tag{6}$$

where a_{ij} as usual represent the individual elements of the matrix $[A]$.

Rearranging the elements of $\text{vec}[\varepsilon]$ in a more convenient way by ordering first the diagonal elements of $[\varepsilon]$ and explicitly partitioning its duplicated symmetric elements, we have

$$\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{12} \\ \varepsilon_{31} \\ \varepsilon_{23} \\ \dots \\ \varepsilon_{21} \\ \varepsilon_{13} \\ \varepsilon_{32} \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{21} \\ \varepsilon_{31} \\ \varepsilon_{12} \\ \varepsilon_{22} \\ \varepsilon_{32} \\ \varepsilon_{13} \\ \varepsilon_{23} \\ \varepsilon_{33} \end{Bmatrix} \tag{7}$$

Introducing the notation

$$[D]_{6 \times 9} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \tag{8}$$

equation (7) can be rewritten

$$v[\varepsilon] = \begin{bmatrix} \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{bmatrix} \text{vec}[\varepsilon] = [E]_{9 \times 9} \text{vec}[\varepsilon] \tag{9}$$

where now $v[\varepsilon]$ defines a new type of vec operator that arranges in vector form the elements of the symmetric matrix $[\varepsilon]$ starting along the diagonal and followed by the elements corresponding to the row location of positive values in 3×3 skew-symmetric (antisymmetric) matrices (see equation 22). It is left to the reader to prove that $[E]$ is an orthogonal matrix (i.e., $[E]^{-1} = [E]'$).

Also notice that from (7) and (8) we may write

$$v_d[\varepsilon] = [D]_{6 \times 9} \text{vec}[\varepsilon] \tag{10}$$

where the subscript d stands for ‘distinct’, indicating that only the distinct elements of the matrix $[\varepsilon]$ are included in $v_d[\varepsilon]$, starting with the ‘diagonal’ elements, namely

$$\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{12} \\ \varepsilon_{31} \\ \varepsilon_{23} \end{Bmatrix}_{6 \times 1} = [D]_{6 \times 9} \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{21} \\ \varepsilon_{31} \\ \varepsilon_{12} \\ \varepsilon_{22} \\ \varepsilon_{32} \\ \varepsilon_{13} \\ \varepsilon_{23} \\ \varepsilon_{33} \end{Bmatrix}_{9 \times 1} \tag{11}$$

This definition of v_d is not introduced in the standard literature treating the subject (see Henderson & Searle 1981), but it is very convenient when symmetric matrices are involved and it will be used in the discussion that follows.

Similarly to (10) it is possible to write, using (4):

$$v_d[\varepsilon'] = [D] \text{vec}[\varepsilon'] = [D][T] \text{vec}[\varepsilon] = [D][T][D]' v_d[\varepsilon]. \tag{12}$$

Therefore, finally the relationship between the v_d ’s of the original tensor $[\varepsilon]$ and the transformed one $[\varepsilon']$ reduces to

$$v_d[\varepsilon'] = [V]_{6 \times 6} v_d[\varepsilon]_{6 \times 1} \tag{13}$$

where

$$[V] = [D][T][D]' \tag{14}$$

and $[T]$ and $[D]$ were previously defined respectively by equations (5) and (8). Equation (13) gives explicitly the relationship of the distinct elements in $[\varepsilon']$ as a function of the distinct elements in $[\varepsilon]$.

Now the standard covariance law [also called ‘propagation of error’ method, e.g. Hamilton (1964)] can be invoked to obtain the covariance matrix of the distinct elements in $[\varepsilon']$ when the covariance matrix of the elements in $[\varepsilon]$ is known, thus

$$\Sigma_{v_d[\varepsilon']} = [V] \Sigma_{v_d[\varepsilon]} [V]'. \tag{15}$$

If we are also interested in the covariance matrix of the components of the vector $\{\omega\}$ rotating the frame (x, y, z) into the (x', y', z') , expressed in the new coordinate frame (x', y', z') , the matrix $[V]$ must be bordered with R as follows:

$$[P] = \begin{bmatrix} [V] & 0 \\ 6 \times 6 & \\ 0 & R \\ & 3 \times 3 \end{bmatrix} \quad (16)$$

and finally

$$\Sigma_{(v_d[\varepsilon'], \{\omega'\})} = [P] \Sigma_{(v_d[\varepsilon], \{\omega\})} [P]^t \quad (17)$$

COVARIANCE MATRIX OF EIGENVALUES AND ROTATIONS

Assume that the eigenvalues and eigenvectors of the symmetric matrix $[\varepsilon]$ (or equivalently $[\varepsilon']$) are known. Thus, we may write

$$[\varepsilon] = S^t[\lambda]S \quad (18)$$

where $[\lambda]$ is the diagonal matrix of eigenvalues

$$[\lambda] = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \quad (19)$$

and S is the rotation matrix whose rows are the three orthogonal eigenvectors of $[\varepsilon]$ corresponding to the λ_i eigenvalues. S rotates the original (x, y, z) axes to which $[\varepsilon]$ refers, into the principal axes (x^p, y^p, z^p) to which the diagonal principal tensor $\text{diag}[\varepsilon]$ refers (i.e., $\{x^p\} = S\{x\}$).

Differentiating equation (18) we get

$$d[\varepsilon] = dS^t[\lambda]S + S^t d[\lambda]S + S^t[\lambda] dS \quad (20)$$

but it is known that

$$dS^t = [\Omega]S^t \quad \text{and} \quad dS = -S[\Omega] \quad (21)$$

where the following short notation, also applied later, was used:

$$[\Omega] = \begin{bmatrix} 0 & \Omega_3 & -\Omega_2 \\ -\Omega_3 & 0 & \Omega_1 \\ \Omega_2 & -\Omega_1 & 0 \end{bmatrix}. \quad (22)$$

$[\Omega]$ is the antisymmetric matrix associated with the rotation vector $\{\Omega\} = \{\Omega_1, \Omega_2, \Omega_3\}'$ whose components are given along the (x, y, z) coordinate frame. The signs in equation (22) are consistent with positive counterclockwise rotations of axes as viewed toward the origin of the coordinate frame.

Substituting (21) into (20) we have

$$d[\varepsilon] = [\Omega]S^t[\lambda]S + S^t d[\lambda]S - S^t[\lambda]S[\Omega]. \quad (23)$$

Making use of (18), the differential corrections to the elements of $[\varepsilon]$ can be expressed as

$$d[\varepsilon] = [\Omega][\varepsilon] - [\varepsilon][\Omega] + S^t d[\lambda]S. \quad (24)$$

Pre- and post-multiplying (24) by S and S^t , respectively,

$$S d[\varepsilon]S^t = S[\Omega][\varepsilon]S^t - S[\varepsilon][\Omega]S^t + d[\lambda] \quad (25)$$

and using (18), noting that

$$S[\Omega]S^t = [\Omega^p] \quad \text{where the vector } \{\Omega^p\} = S\{\Omega\} \quad (26)$$

we arrive at

$$S d[\varepsilon]S^t = d[\mu] \quad (27)$$

where

$$d[\mu] = [\Omega^p][\lambda] - [\lambda][\Omega^p] + d[\lambda]. \quad (28)$$

Notice that $d[\mu]$ is symmetric and equal to

$$d[\mu] = \begin{bmatrix} d\lambda_1 & (\lambda_2 - \lambda_1)\Omega_3^p & (\lambda_1 - \lambda_3)\Omega_2^p \\ & d\lambda_2 & (\lambda_3 - \lambda_2)\Omega_1^p \\ \text{sym} & & d\lambda_3 \end{bmatrix}. \quad (29)$$

Applying the *vec* operator to (27) making use of equation (A4) from the appendix, we have

$$[S \otimes S] \text{vec}[d[\varepsilon]] = \text{vec}[d[\mu]] \quad (30)$$

but using (29)

$$\text{vec}[d[\mu]] = [G]\{\beta\} \quad (31)$$

where

$$[G] = \begin{bmatrix} 1 & 0 & 0 & : & 0 & 0 & 0 \\ 0 & 0 & 0 & : & 0 & 0 & \lambda_2 - \lambda_1 \\ 0 & 0 & 0 & : & 0 & \lambda_1 - \lambda_3 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & : & 0 & 0 & \lambda_2 - \lambda_1 \\ 0 & 1 & 0 & : & 0 & 0 & 0 \\ 0 & 0 & 0 & : & \lambda_3 - \lambda_2 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & : & 0 & \lambda_1 - \lambda_3 & 0 \\ 0 & 0 & 0 & : & \lambda_3 - \lambda_2 & 0 & 0 \\ 0 & 0 & 1 & : & 0 & 0 & 0 \end{bmatrix} \quad (32)$$

and

$$\{\beta\}_{6 \times 1} = \begin{Bmatrix} d\lambda_1 \\ d\lambda_2 \\ d\lambda_3 \\ \Omega_1^p \\ \Omega_2^p \\ \Omega_3^p \end{Bmatrix} = \begin{Bmatrix} \{d\lambda\} \\ \{\Omega^p\} \end{Bmatrix}. \quad (33)$$

Recalling equation (11) we may write the correction vector of the distinct elements of $[\varepsilon]$ as

$$\{d\varepsilon\}_{6 \times 1} = v_d[d[\varepsilon]] = \begin{Bmatrix} d\varepsilon_{11} \\ d\varepsilon_{22} \\ d\varepsilon_{33} \\ d\varepsilon_{12} \\ d\varepsilon_{31} \\ d\varepsilon_{23} \end{Bmatrix} = [D]_{6 \times 9} \text{vec}[d[\varepsilon]]_{9 \times 1} \quad (34)$$

where $[D]$ was given previously by equation (8).

From (30) it follows that

$$\text{vec}[d[\varepsilon]]_{9 \times 1} = [S' \otimes S']_{9 \times 9} \text{vec}[d[\mu]]_{9 \times 1} \quad (35)$$

thus, using (34)

$$\{d\varepsilon\} = [D][S' \otimes S'] \text{vec}[d[\mu]]. \quad (36)$$

When we want to include the variances and covariances of the components of the rotation vector $\{\Omega\}$, the value of $[D]$ above should be replaced by

$$[Q] = \begin{bmatrix} [1] & 0 \\ 0 & S \end{bmatrix}_{6 \times 9} [D]_{6 \times 9} \quad (37)$$

where $[1]$ is the 3×3 unit matrix. Thus, finally $\{d\varepsilon\}$ has the form

$$\{d\varepsilon\} = [F]\{\beta\} \quad (38)$$

where

$$[F] = \begin{bmatrix} [Q] & [S]' \otimes S' \\ [S] \otimes S' & [G] \end{bmatrix} \quad (39)$$

$\begin{matrix} 6 \times 6 & 6 \times 9 & 9 \times 9 & 9 \times 6 \end{matrix}$

Therefore, the differential changes in the eigenvalues and angular rotation components as a function of the known differential values of the components of $[\epsilon]$ may be obtained by inverting equation (38). Although the inversion of the matrix $[F]$ can always be performed numerically, an analytical procedure will be described next involving the concept of generalized inverse and pseudoinverse matrix (e.g. Graybill 1983).

The matrix $[G]$ in (39) is not square, therefore, the conventional Caleyian inverse cannot be used. It is known from numerous references, as for example Golub & van Loan (1983), that because $[G]'[G]$ is a regular matrix (i.e., its determinant is not equal to zero), the pseudoinverse of $[G]$ is

$$[G]^+ = \{ \{ [G]'[G] \} \}^{-1} [G]' \quad (40)$$

where clearly

$$[G]^+[G] = [I] \quad (41)$$

and $[I]$ is a 6×6 unit matrix. However, notice that

$$[G][G]^+ \neq [I]. \quad (42)$$

This can be easily verified by substituting (32) in (41) and (42). Consequently, the value of $\{\beta\}$ can be determined from (38)

$$\{\beta\} = [F]^{-1} \{d\epsilon\} = \{ \{ [G]^+[S \otimes S][Q]' \} \} \{d\epsilon\}. \quad (43)$$

It can be proved by simple multiplication of matrices in (40) using the given value of $[G]$ in (32) that

$$[G]^+_{6 \times 9} = \begin{bmatrix} 1 & 0 & 0 & : & 0 & 0 & 0 & : & 0 & 0 & 0 \\ 0 & 0 & 0 & : & 0 & 1 & 0 & : & 0 & 0 & 0 \\ 0 & 0 & 0 & : & 0 & 0 & 0 & : & 0 & 0 & 1 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & : & 0 & 0 & \frac{1}{2(\lambda_3 - \lambda_2)} & : & 0 & \frac{1}{2(\lambda_3 - \lambda_2)} & 0 \\ 0 & 0 & \frac{1}{2(\lambda_1 - \lambda_3)} & : & 0 & 0 & 0 & : & \frac{1}{2(\lambda_1 - \lambda_3)} & 0 & 0 \\ 0 & \frac{1}{2(\lambda_2 - \lambda_1)} & 0 & : & \frac{1}{2(\lambda_2 - \lambda_1)} & 0 & 0 & : & 0 & 0 & 0 \end{bmatrix} \quad (44)$$

The Zehfuss product in equation (43) is easily computed analytically, or if preferred, general computer algorithms to manipulate tensor products are available in many software packages. The matrix $[Q]$ was given in equation (37). Consequently, equation (43) represents the functional relationship between the variation of the distinct elements of the symmetric tensor $[\epsilon]$ with respect to the eigenvalues and rotations of $[\epsilon]$.

Hence, the covariance matrix of $\{\beta\}$, (i.e., the vector of eigenvalues and rotations) as a function of the known covariance matrix of the distinct elements in $[\epsilon]$ may finally be written

$$\Sigma_{\{\beta\}} = [F]^{-1} \Sigma_{\{d\epsilon\}} [F]^{-t} \quad (45)$$

where the notation $\{ \{ [F]^{-1} \} \} = [F]^{-t}$ has been implemented. Still to be determined is the covariance matrix of the eigenvectors.

COVARIANCE MATRIX OF EIGENVECTORS

Recall that S is the rotation matrix whose rows are the eigenvectors of $[\epsilon]$, i.e.,

$$S = \begin{bmatrix} s_{11} & s_{12} & s_{13} \\ s_{21} & s_{22} & s_{23} \\ s_{31} & s_{32} & s_{33} \end{bmatrix} = \begin{bmatrix} \{s_1\}' \\ \{s_2\}' \\ \{s_3\}' \end{bmatrix}. \quad (46)$$

Using similar reasoning as above, we need to compute $d\{vec S\}$ in this particular case. However

$$d\{vec S\} = vec[dS] \quad (47)$$

and from (21) we know $dS = -S[\Omega]$. Thus, after using equation (A2) from the appendix, recalling that $[1]$ is the 3×3 unit

matrix, equation (47) takes the form

$$d\{vec S\} = -vec\{[S]\{\Omega\}\} = -\{[1] \otimes S\}vec\{\Omega\}. \tag{48}$$

The above expression can be written independently of \otimes as

$$d_{9 \times 1}\{vec S\} = -\left\{ \begin{matrix} [s_1] \\ [s_2] \\ [s_3] \end{matrix} \right\}_{9 \times 3} \{ \Omega \}_{3 \times 1} = -\{[S]\}\{\Omega\} \tag{49}$$

where $\{[S]\}$ may be considered an antisymmetric tensor of third-rank. Cartesian symmetric tensors higher than rank two can also be written in matrix form (e.g., Soler 1984, 1985).

We may write $\{[S]\}$ explicitly as

$$\{[S]\} = \left\{ \begin{matrix} \begin{bmatrix} 0 & s_{13} & -s_{12} \\ -s_{13} & 0 & s_{11} \\ s_{12} & -s_{11} & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & s_{23} & -s_{22} \\ -s_{23} & 0 & s_{21} \\ s_{22} & -s_{21} & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & s_{33} & -s_{32} \\ -s_{33} & 0 & s_{31} \\ s_{32} & -s_{31} & 0 \end{bmatrix} \end{matrix} \right\}. \tag{50}$$

Thus, applying the covariance law to (49) we get

$$\Sigma_{d\{vec S\}} = \{[S]\}\Sigma_{\{\Omega\}}\{[S]\}' \tag{51}$$

It is also possible to express equation (51) as a function of the variance-covariance matrix of the components of the rotation vector with respect to the principal axes.

Similarly to equation (48) we write

$$d\{vec S\} = -vec\{[S^p]\{\Omega^p\}\} = -\{[S^p] \otimes [1]\}vec\{\Omega^p\} = \{[S^p]\}\{\Omega^p\} \tag{52}$$

where $\{[S^p]\}$ is explicitly given by

$$\{[S^p]\} = \left\{ \begin{matrix} \begin{bmatrix} 0 & s_{31} & -s_{21} \\ -s_{31} & 0 & s_{11} \\ s_{21} & -s_{11} & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & s_{32} & -s_{22} \\ -s_{32} & 0 & s_{12} \\ s_{22} & -s_{12} & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & s_{33} & -s_{23} \\ -s_{33} & 0 & s_{13} \\ s_{23} & -s_{13} & 0 \end{bmatrix} \end{matrix} \right\}. \tag{53}$$

It may be noticed that the matrix $\{[S^p]\}$ is the same as $\{[S]\}$ but with the subscripts interchanged. Observe that the operation of interchanging subscripts is not equal to the transpose operation.

Thus, similar to equation (51) we have now

$$\Sigma_{d\{vec S\}} = \{[S^p]\}\Sigma_{\{\Omega^p\}}\{[S^p]\}' \tag{54}$$

Finally, it is possible to combine (45) and (54) into a single equation. First we have

$$\begin{Bmatrix} d\{vec S\} \\ \{d\lambda\} \end{Bmatrix}_{9 \times 1} = \begin{bmatrix} 0 & \{[S^p]\} \\ [1] & 0 \end{bmatrix}_{9 \times 3} \begin{Bmatrix} \{d\lambda\} \\ \{\Omega^p\} \end{Bmatrix}_{3 \times 1} = \begin{bmatrix} 0 & \{[S^p]\} \\ [1] & 0 \end{bmatrix}_{6 \times 6} [F]^{-1}_{6 \times 6} \{d\varepsilon\}_{6 \times 1} = [K]_{12 \times 6} \{d\varepsilon\}_{6 \times 1} \quad (55)$$

and finally

$$\Sigma_{(d\{vec S\}, \{d\lambda\})} = [K] \Sigma_{\{d\varepsilon\}} [K]^t \quad (56)$$

Equation (55) gives the differential changes in the eigenvector components and eigenvalues of a real 3×3 symmetric matrix $[\varepsilon]$ as a function of the differential changes of the elements ε_{ij} , $i \neq j$, of $[\varepsilon]$ (see equation 34).

CONCLUDING REMARKS

Symmetric tensors of second-rank are widely used in mechanics, dynamics, geodesy and geophysics and other related scientific fields. Diagonalization of these tensors through transformations from arbitrary reference frames to appropriate principal frames is desirable and commonly practiced.

However, little attention is devoted in the published literature to the accuracy assessment of these principal tensors and their transformations. With this in mind, analytical expressions have been developed in this note to complement this didactic void. The approach uses contemporary mathematical concepts (e.g., *vec* operators and Zehfuss' products) often not found in traditional textbooks on geophysics.

Numerous problems ranging from the orientation of principal axes of inertia in analyses of Earth's rotation, to the evaluation of maximum and minimum stress and strain fields in active tectonic areas, require accuracy estimates presently not addressed by investigators. For instance, the resulting principal axes orientation may not be statistically significant merely because their final attached variances exceed certain tolerance limits.

With the advent of modern positioning technology knowledge of the accuracy of these various eigentransformations has become increasingly important in geodynamics, e.g., critical evaluation of strain in networks of points spanning the globe. This configuration of sites occupied permanently or intermittently by Satellite Laser Ranging, Very Long Baseline Interferometry or Global Positioning System equipment are continuously monitoring (relative) plate tectonic motions, intraplate deformation, Earth rotation, etc.

The nature of these techniques can sometimes lead to rank-deficient tensors, resulting in zero eigenvalues. Algorithms to transform rank-deficient tensors may sometimes fail due to near-rank-deficiency caused by computational inaccuracies such as those produced by round-off errors.

The formulae given here are able to assess the inaccuracies of these 3-D tensor transformations providing a better interpretation of the quality of the original observations. This is particularly important to differentiate between real deformations and computational deformations. These equations have grown increasingly valuable because of the intense recent interest in determining the accuracies of the eigenvalues and eigenvectors obtained from data processed using numerical matrix computation routines readily available in most software packages.

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APPENDIX A: THE *vec* OF THE PRODUCT OF THREE MATRICES

It is easily verified that

$$vec[B] = [I \otimes B]vec[I] = [B' \otimes I]vec[I] = [I \otimes I]vec[B] \tag{A1}$$

and from Graham (1981, p. 26) we have

$$vec[AB] = [I \otimes A]vec[B] = [B' \otimes I]vec[A]. \tag{A2}$$

Consequently

$$vec[ABC] = [I \otimes A]vec[BC] = [I \otimes A][C' \otimes I]vec[B] \tag{A3}$$

and finally

$$vec[ABC] = [C' \otimes A]vec[B]. \tag{A4}$$

Equation (A4) applied to (1) immediately gives (4).

APPENDIX B: COVARIANCE MATRICES OF PRINCIPAL DIRECTIONS IN A LOCAL GEODETIC FRAME

Assume for simplicity that the components of each orthogonal eigenvector $\{s_i\}$, $i = 1, 2, 3$ are denoted by $(\vartheta_1, \vartheta_2, \vartheta_3)_i$, and refer, for example, to a 'local geodetic' coordinate frame [i.e., the Cartesian axes of this frame point respectively to geodetic east (e), north (n) and zenith = up (u)]. Then it will be more useful to know not the covariance matrix of the eigenvector components but the covariance matrix of the directions of the three principal axes defined by these eigenvectors. These principal directions can be parametrized through two polar angles such as the vertical angle v and geodetic azimuth α (see Fig. B1). Denote the direction of the i ($i = 1, 2, 3$) eigenvector along the i principal axis by p_i , and the axes normal to each p_i by v_i and a_i , as shown in the figure, positive in the sense of positive v_i and α_i respectively.

The rotation matrix \mathbf{R} of the mapping $\mathbf{R}: (e, n, u) \rightarrow (v, a, p)_i$, can be computed from

$$\mathbf{R} = R_3(\pi)R_2(\frac{1}{2}\pi - v_i)R_3(\frac{1}{2}\pi - \alpha_i) = \begin{bmatrix} -\sin v \sin \alpha & -\sin v \cos \alpha & \cos v \\ \cos \alpha & -\sin \alpha & 0 \\ \cos v \sin \alpha & \cos v \cos \alpha & \sin v \end{bmatrix}_i \tag{B1}$$

From Fig. B1 it immediately follows that

$$\sin \alpha_i = \vartheta_1 / \sqrt{\vartheta_1^2 + \vartheta_2^2} \tag{B2a}$$

$$\cos \alpha_i = \vartheta_2 / \sqrt{\vartheta_1^2 + \vartheta_2^2} \tag{B2b}$$

$$\sin v_i = \vartheta_3 \tag{B2c}$$

$$\cos v_i = \sqrt{\vartheta_1^2 + \vartheta_2^2} \tag{B2d}$$

Restricting our computations to the covariance matrix of the two polar angles $(v, \alpha)_i$, we can write

$$\mathbf{R}_i = [1/\sqrt{\vartheta_1^2 + \vartheta_2^2}]_i \begin{bmatrix} -\vartheta_1 \vartheta_3 & -\vartheta_2 \vartheta_3 & \vartheta_1^2 + \vartheta_2^2 \\ \vartheta_2 & -\vartheta_1 & 0 \end{bmatrix}_i \tag{B3}$$

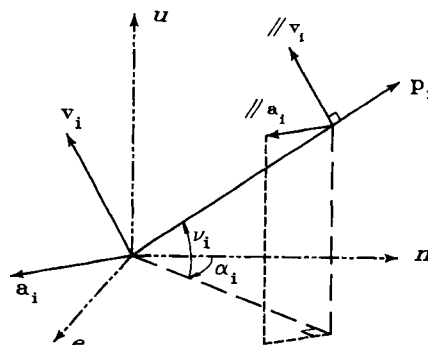


Figure B1. Principal axis p_i and local geodetic frame (e, n, u) .

and

$$\Sigma_{(v,a)_i} = \mathbf{R}_i \Sigma_{(\vartheta_1, \vartheta_2, \vartheta_3)_i} \mathbf{R}_i^t \quad (\text{B4})$$

Notice that the elements of the matrix $\Sigma_{(v,a)_i}$, that is, the variance and covariances of the angles v and α , are given in linear units squared. Recall that $(\vartheta_1, \vartheta_2, \vartheta_3)_i$ are the components of the vector $\{s_i\}$ along the Cartesian axes (e, n, u) . Thus, their covariance matrix has dimensions $[L]^2$. In other words, $\Sigma_{(v,a)_i}$ is a Cartesian tensor whose physical components are referred to the Cartesian frame $(v, a, p)_i$. If on the contrary angular (radian) units are desired, the following transformation (Jacobian) matrix should be used:

$$J_i = H_i^{-1} \mathbf{R}_i, \quad (\text{B5})$$

where H_i (also called Lamé's matrix) is given in this particular case by

$$H_i = \begin{bmatrix} 1 & 0 \\ 0 & \cos v \end{bmatrix}_i = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{\vartheta_1^2 + \vartheta_2^2} \end{bmatrix}_i. \quad (\text{B6})$$

Hence, finally

$$\Sigma_{(v,\alpha)_i} = J_i \Sigma_{(\vartheta_1, \vartheta_2, \vartheta_3)_i} J_i^t. \quad (\text{B7})$$

In conclusion,

$$\Sigma_{(v_1, \alpha_1, v_2, \alpha_2, v_3, \alpha_3)} = J \Sigma_{d\{vecS\}} J^t \quad (\text{B8})$$

where $\Sigma_{d\{vecS\}}$ was given implicitly in equation (56) and J is the matrix of J_i 's partitioned as follows:

$$J_{6 \times 9} = \begin{bmatrix} J_1 & 0 & 0 \\ 2 \times 3 & & \\ 0 & J_2 & 0 \\ & 2 \times 3 & \\ 0 & 0 & J_3 \\ & & 2 \times 3 \end{bmatrix}. \quad (\text{B9})$$