PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 127, Number 5, Pages 1293–1300
S 0002-9939(99)04817-0
Article electronically published on January 27, 1999

# ON COVERING MULTIPLICITY

ZHI-WEI SUN

(Communicated by David E. Rohrlich)

ABSTRACT. Let $A = \{a_s + n_s\mathbb{Z}\}_{s=1}^k$ be a system of arithmetic sequences which forms an $m$-cover of $\mathbb{Z}$ (i.e. every integer belongs at least to $m$ members of $A$). In this paper we show the following surprising properties of $A$: (a) For each $J \subseteq \{1, \cdots, k\}$ there exist at least $m$ subsets $I$ of $\{1, \cdots, k\}$ with $I \neq J$ such that $\sum_{s\in I} 1/n_s - \sum_{s\in J} 1/n_s \in \mathbb{Z}$. (b) If $A$ forms a minimal $m$-cover of $\mathbb{Z}$, then for any $t = 1, \cdots, k$ there is an $\alpha_t \in [0, 1)$ such that for every $r = 0, 1, \cdots, n_t - 1$ there exists an $I \subseteq \{1, \cdots, k\} \setminus \{t\}$ for which $[\sum_{s\in I} 1/n_s] \geqslant m - 1$ and $\{\sum_{s\in I} 1/n_s\} = (\alpha_t + r)/n_t$.

## 1. Introduction

For integer $a$ and positive integer $n$ we call

$$a(n) = \{x \in \mathbb{Z} : \ x \equiv a \ (\mathrm{mod}\, n)\} = a + n\mathbb{Z}$$

an *arithmetic sequence* with *common difference* $n$ or a *residue class* with *modulus* $n$. For a finite system

$$A = \{a_s(n_s)\}_{s=1}^k \tag{1}$$

of such sets, we define its *covering multiplicity* by

$$m(A) = \inf_{x\in\mathbb{Z}} |S(x)| \tag{2}$$

where $S(x) = \{1 \leqslant s \leqslant k : \ x \equiv a_s \ (\mathrm{mod}\, n_s)\}$. It is easy to show that

$$\sum_{s=1}^k \frac{1}{n_s} \geqslant m(A), \tag{3}$$

and the equality holds if and only if (1) covers each integer exactly $m$ times for some $m = 1, 2, 3, \cdots$. (Cf. [S2], [S4].)

Let $m$ be a nonnegative integer. If system (1) has covering multiplicity at least $m$, then we call (1) an *$m$-cover* (of $\mathbb{Z}$). A *minimal $m$-cover* (of $\mathbb{Z}$) is an $m$-cover whose proper subsystems are not. If $|S(x)| = m$ for all $x \in \mathbb{Z}$, then we say that $A$ forms an *exact $m$-cover* (of $\mathbb{Z}$). Notice that an exact 1-cover is a partition of $\mathbb{Z}$ into (finitely many) periodic sets. The Chinese Remainder Theorem tells that the intersection of residue classes $a_1(n_1), \cdots, a_k(n_k)$ is empty if and only if two of them are disjoint. So, as a dual question, when (1) forms a 1-cover is fundamental and

---

Received by the editors August 13, 1997.
1991 *Mathematics Subject Classification.* Primary 11B25; Secondary 11A07, 11B75, 11D68.
Supported by the National Natural Science Foundation of the People's Republic of China and the Return-from-abroad Foundation of the Chinese Educational Committee.

important. In fact, 1-covers and exact $m$-covers (especially exact 1-covers) have been investigated for many years; also some famous conjectures remain open. (See R. K. Guy [G].)

Now we introduce some notation. As usual, if $m$ and $n$ are integers, then $(m,n)$ represents the greatest common divisor of $m$ and $n$. For a real number $x$, we set $\binom{x}{0}=1$ and let $\binom{x}{n}=\prod_{j=0}^{n-1}\frac{x-j}{n-j}$ for $n=1,2,3,\cdots$; also $[x]$ and $\{x\}$ denote the integral and the fractional parts of $x$ respectively.

In this paper we study the covering multiplicity of a general system of residue classes. Our main result is as follows.

**Theorem 1.** *Let* (1) *be a system of arithmetic sequences, and let* $J$ *be a subset of* $\{1,\cdots,k\}$. *Put* $J^-=\{1,\cdots,k\}\setminus J$.

(i) *For any* $m_1,\cdots,m_k\in\mathbb{Z}$ *we have*

$$\left|\left\{I\subseteq\{1,\cdots,k\}:\ I\neq J\ \&\ \left\{\sum_{s\in I}\frac{m_s}{n_s}\right\}=\left\{\sum_{s\in J}\frac{m_s}{n_s}\right\}\right\}\right|\geqslant m(A). \tag{4}$$

(ii) *Suppose* $\emptyset\neq J\subseteq S(x)$ *for some* $x\in\mathbb{Z}$ *with* $|S(x)|=m(A)$. *For each* $s\in J^-$ *let* $m_s$ *be a positive integer prime to* $n_s$. *Then there exists an* $\alpha\in[0,1)$ *such that*

$$\begin{aligned}&\left\{\left\{\sum_{s\in I}\frac{m_s}{n_s}\right\}:\ I\subseteq J^-,\ \left[\sum_{s\in I}\frac{m_s}{n_s}\right]\geqslant m(A)-|J|\right\}\\ &\supseteq\left\{\frac{a}{N(J)}:\ 0\leqslant a<N(J),\ \{a\}=\alpha\right\},\end{aligned} \tag{5}$$

*where* $N(J)$ *denotes the least common multiple of those* $n_s$ *with* $s\in J$.

In view of Theorem 1, an $m$-cover $A=\{a_s(n_s)\}_{s=1}^k$ possesses the following properties:

(a) For each $J\subseteq\{1,\cdots,k\}$, there exist at least $m$ subsets $I$ of $\{1,\cdots,k\}$ with $I\neq J$ such that $\sum_{s\in I}1/n_s-\sum_{s\in J}1/n_s\in\mathbb{Z}$.

(b) If $A$ forms a minimal $m$-cover of $\mathbb{Z}$, then for any $t=1,\cdots,k$ there is an $\alpha_t\in[0,1)$ such that, for every $r=0,1,\cdots,n_t-1$, there exists an $I\subseteq\{1,\cdots,k\}\setminus\{t\}$ for which $[\sum_{s\in I}1/n_s]\geqslant m-1$ and $\{\sum_{s\in I}1/n_s\}=(\alpha_t+r)/n_t$.

Part (i) of Theorem 1 can be strengthened in the case $J=\emptyset$. By Theorem 1, if (1) forms a 1-cover, then $\sum_{s\in I}1/n_s\in\mathbb{Z}$ for some nonempty subset $I$ of $\{1,\cdots,k\}$, which is the main result of M. Z. Zhang [Z] obtained by means of the Riemann zeta function. For an exact $m$-cover (1), the author proved in [S1] that for each $n=0,1,\cdots,m$ there exist at least $\binom{m}{n}$ subsets $I$ of $\{1,\cdots,k\}$ with $\sum_{s\in I}1/n_s=n$. When (1) is an $m$-cover and $m_1,\cdots,m_k$ are positive integers, it was shown in [S3] that there are at least $m$ positive integers in the form $\sum_{s\in I}m_s/n_s$ where $I\subseteq\{1,\cdots,k\}$; we even conjecture that there exist nonempty subsets $I_1,\cdots,I_m$ of $\{1,\cdots,k\}$ for which $I_1\subset\cdots\subset I_m$ and $\sum_{s\in I_t}m_s/n_s\in\mathbb{Z}$ for all $t=1,\cdots,m$.

The first part of Theorem 1 yields

**Corollary 1.** *Let* (1) *be an* $m$*-cover of* $\mathbb{Z}$ *and* $m_1,\cdots,m_k$ *any integers. Then*

$$\left|\left\{\left\{\sum_{s\in I}\frac{m_s}{n_s}\right\}:\ I\subseteq\{1,\cdots,k\}\right\}\right|\leqslant\frac{2^k}{m+1}. \tag{6}$$

*Proof.* By part (i) of Theorem 1, for any $J\subseteq\{1,\cdots,k\}$ there are at least $m+1$ subsets $I$ of $\{1,\cdots,k\}$ with $\{\sum_{s\in I}m_s/n_s\}=\{\sum_{s\in J}m_s/n_s\}$. Since $\{1,\cdots,k\}$ has exactly $2^k$ subsets, Corollary 1 follows immediately. $\square$

*Remark 1.* A conjecture of P. Erdös proved by R. B. Crittenden and C. L. Vanden Eynden [CV] states that (1) forms a 1-cover of $\mathbb{Z}$ if it covers $1,\cdots,2^k$. In [S2], [S3] the author showed that (1) forms an $m$-cover of $\mathbb{Z}$ if there exist $W$ consecutive integers each of which lies in at least $m$ members of (1), where $W$ is the least integer equal to the left hand side of (6) for some integers $m_1,\cdots,m_k$ prime to $n_1,\cdots,n_k$ respectively.

As for part (ii) of Theorem 1 we should mention the following result obtained by the author ([S4]) recently: Let (1) be an exact $m$-cover of $\mathbb{Z}$ , and $J$ a nonempty subset of $\{1,\cdots,k\}$ with $(n_s,n_t)\mid a_s-a_t$ for all $s,t\in J$ (i.e. $\emptyset\neq J\subseteq S(x)$ for some $x\in\mathbb{Z}$). Then

$$\left|\left\{I\subseteq J^-:\ \left\{\sum_{s\in I}\frac{1}{n_s}\right\}=\frac{a}{N(J)}\right\}\right|\geqslant\frac{\prod_{s\in J}n_s}{N(J)}$$

for every $a=0,1,\cdots,N(J)-1$, and

$$\left|\left\{I\subseteq J^-:\ \sum_{s\in I}\frac{1}{n_s}=\frac{a}{N(J)}\right\}\right|\geqslant\binom{m-1}{[a/N(J)]}$$

for all $a=0,1,2,\cdots$ if $|J|=1$.

**Corollary 2.** *Let* (1) *be an $m$-cover of $\mathbb{Z}$ with $n_1\leqslant\cdots\leqslant n_{k-1}\leqslant n_k$. Suppose that $B=\{a_s(n_s)\}_{s=1}^{k-1}$ fails to be an $m$-cover of $\mathbb{Z}$. If $\sum_{s=1}^{k-1}1/n_s=m$, then $n_{k-1}=n_k>1$ and*

$$\left\{\sum_{s\in I}\frac{1}{n_s}:\ I\subseteq\{1,\cdots,k-1\}\right\}\supseteq\left\{\frac{r}{n_k}:\ r=0,1,\cdots,n_k-1\right\}. \tag{7}$$

*Proof.* Assume that $\sum_{s=1}^{k-1}1/n_s=m$. By part (ii) of Theorem 1 there exists an $\alpha\in[0,1)$ such that

$$\begin{aligned}&\left\{\left\{\sum_{s\in I}\frac{1}{n_s}\right\}:\ I\subseteq\{1,\cdots,k-1\},\ \left[\sum_{s\in I}\frac{1}{n_s}\right]\geqslant m-1\right\}\\&\qquad\supseteq\left\{\frac{a}{n_k}:\ 0\leqslant a<n_k,\ \{a\}=\alpha\right\}.\end{aligned}$$

Thus

$$\begin{aligned}&\left\{\sum_{s\in J}\frac{1}{n_s}:\ J\subseteq\{1,\cdots,k-1\},\ \sum_{s\in J}\frac{1}{n_s}\notin\mathbb{Z}\right\}\\
\supseteq&\left\{\sum_{s=1}^{k-1}\frac{1}{n_s}-\sum_{s\in I}\frac{1}{n_s}:\ I\subseteq\{1,\cdots,k-1\},\ m-1<\sum_{s\in I}\frac{1}{n_s}<m=\sum_{s=1}^{k-1}\frac{1}{n_s}\right\}\\
=&\left\{1-\left\{\sum_{s\in I}\frac{1}{n_s}\right\}:\ I\subseteq\{1,\cdots,k-1\},\ \left[\sum_{s\in I}\frac{1}{n_s}\right]\geqslant m-1\right\}\setminus\{1\}\\
\supseteq&\left\{1-\frac{a}{n_k}:\ 0\leqslant a<n_k,\ \{a\}=\alpha\right\}\setminus\{1\}=\left\{\frac{b}{n_k}:\ 0<b<n_k,\ \{b\}=\{-\alpha\}\right\}.\end{aligned}$$

Observe that (7) follows if $\alpha=0$. Since $B$ doesn't form an $m$-cover of $\mathbb{Z}$, we cannot have $n_1=\cdots=n_{k-1}=1$ (otherwise $k-1=\sum_{s=1}^{k-1}1/n_s=m$). So $n_k\geqslant n_{k-1}>1$;

hence by the above for some nonempty $J \subseteq \{1, \cdots, k-1\}$ we have

$$\frac{1}{n_{k-1}} \leqslant \min_{s \in J} \frac{1}{n_s} \leqslant \sum_{s \in J} \frac{1}{n_s} = \frac{1-\alpha}{n_k} \leqslant \frac{1}{n_k} \leqslant \frac{1}{n_{k-1}}.$$

Therefore $n_k = n_{k-1}$ and $\alpha = 0$. We are done. □

*Remark 2.* Let (1) be an $m$-cover of $\mathbb{Z}$ with $n_1 \leqslant \cdots \leqslant n_{k-1} < n_k$. By part (iv) of Theorem I of [S3], $\sum_{s=1}^{k-1} 1/n_s \geqslant m$. In view of Corollary 2, if $\{a_s(n_s)\}_{s=1}^{k-1}$ fails to be an $m$-cover of $\mathbb{Z}$, then $\sum_{s=1}^{k-1} 1/n_s$ must be greater than $m$. This extends and improves a confirmed conjecture of Erdös which states that $\sum_{s=1}^{k} 1/n_s > 1$ for any 1-cover (1) with $1 < n_1 < \cdots < n_{k-1} < n_k$ (see [E] and [G]).

**Corollary 3.** *Let* (1) *be an $m$-cover of $\mathbb{Z}$, and $J$ a nonempty subset of* $\{1, \cdots, k\}$ *with* $|\{s \in J^- : \ x \in a_s(n_s)\}| = m - |J|$ *for some $x \in \mathbb{Z}$. Let $\varepsilon_s \in \{1, -1\}$ for those $s \in J^-$. Then*

$$\left| \left\{ \left\{ \sum_{s \in I} \frac{\varepsilon_s}{n_s} \right\} : \ I \subseteq J^- \right\} \right| \geqslant N(J). \tag{8}$$

*Proof.* This follows immediately from the second part of Theorem 1. □

*Remark 3.* With the help of a local-global result proved in [S2], in 1994 the author found Corollary 3 in the case $|J| = 1$ (see Section 3 of [S3]).

**Corollary 4.** *Let* (1) *be a minimal $m$-cover of $\mathbb{Z}$, and $m_1, \cdots, m_k$ any positive integers prime to $n_1, \cdots, n_k$ respectively. Then for every $t = 1, \cdots, k$ all the numbers $0, 1/n_t, \cdots, (n_t - 1)/n_t$ lie in the set*

$$\left\{ \left\{ \sum_{s \in I} \frac{m_s}{n_s} - \sum_{s \in J} \frac{m_s}{n_s} \right\} : \ I, J \subseteq \{1, \cdots, k\} \setminus \{t\} \ \& \ \sum_{s \in I} \frac{m_s}{n_s}, \sum_{s \in J} \frac{m_s}{n_s} \geqslant m-1 \right\}. \tag{9}$$

*Proof.* By part (ii) of Theorem 1 there is an $\alpha_t \in [0,1)$ such that

$$\left\{ \left\{ \sum_{s \in I} \frac{m_s}{n_s} \right\} : \ I \subseteq \{1, \cdots, k\} \setminus \{t\}, \ \left[ \sum_{s \in I} \frac{m_s}{n_s} \right] \geqslant m-1 \right\}$$

contains $S_t = \{a/n_t : 0 \leqslant a < n_t, \ \{a\} = \alpha_t\}$. As $r/n_t = (\alpha_t + r)/n_t - \alpha_t/n_t$ for each $r = 0, 1, \cdots, n_t - 1$, the desired result follows. □

*Remark 4.* In [S3] the author was able to prove Corollary 4 with $\sum_{s \in J} m_s/n_s \geqslant m-1$ in (9) replaced by $\sum_{s \in J} m_s/n_s \geqslant m-2$.

## 3. Proof of Theorem 1

Let's recall a key result given by the author in [S2].

**Proposition 1.** *Let $\mathcal{A} = \{\alpha_s + \beta_s \mathbb{Z}\}_{s=1}^k$ where $\alpha_1, \cdots, \alpha_k$ are real numbers and $\beta_1, \cdots, \beta_k$ are positive reals. Let $m$ be a positive integer. Then $\mathcal{A}$ forms an $m$-cover of $\mathbb{Z}$ (i.e.* $|\{1 \leqslant s \leqslant k : \ (x - \alpha_s)/\beta_s \in \mathbb{Z}\}| \geqslant m$ *for all $x \in \mathbb{Z}$) if and only if*

$$\sum_{\substack{I \subseteq \{1, \cdots, k\} \\ \{\sum_{s \in I} 1/\beta_s\} = \theta}} (-1)^{|I|} \binom{[\sum_{s \in I} 1/\beta_s]}{n} e^{2\pi i \sum_{s \in I} \alpha_s/\beta_s} = 0 \tag{10}$$

*holds for all $\theta \in [0,1)$ and $n = 0, 1, \cdots, m-1$.*

**Lemma 1.** *Let $k, m, n$ be positive integers with $k > m - n \geqslant 0$. Then* (1) *forms an $m$-cover of $\mathbb{Z}$ if and only if for each $I \subseteq \{1, \cdots, k\}$ with $|I| = m - n$ system $A_I = \{a_s(n_s)\}_{s \in I^-}$ forms an $n$-cover of $\mathbb{Z}$.*

*Proof.* If (1) is an $m$-cover of $\mathbb{Z}$ and $I$ is a subset of $\{1, \cdots, k\}$ with $|I| = m - n$, then for any integer $x$ we have

$$|\{s \in I^- :\ x \equiv a_s \ (\mathrm{mod}\, n_s)\}| \geqslant m - |I| = n;$$

therefore $A_I$ is an $n$-cover of $\mathbb{Z}$.

Now suppose that $A_I$ forms an $n$-cover of $\mathbb{Z}$ for all $I \subseteq \{1, \cdots, k\}$ with $|I| = m - n$. Let's show that $A = A_\emptyset$ forms an $m$-cover of $\mathbb{Z}$. Assume on the contrary that for some integer $x$ set $J = \{1 \leqslant s \leqslant k :\ x \equiv a_s \ (\mathrm{mod}\, n_s)\}$ has cardinality $l < m$. Choose a subset $I$ of $\{1, \cdots, k\}$ with cardinality $m - n$ such that either $I \subseteq J$ or $I \supseteq J$. Observe that $x$ belongs to less than $n$ members of $A_I$. This contradiction ends our proof. □

*Remark 5.* Apparently for (1) to be an $m$-cover of $\mathbb{Z}$ it is necessary that $k \geqslant m$.

*Proof of part (i) of Theorem 1.* It suffices to handle the case $m = m(A) > 0$.

At first we assume that $n_1, \cdots, n_k$ are all greater than one. Since $m \leqslant \sum_{s=1}^k 1/n_s \leqslant k/2$, either $J$ or $J^-$ has cardinality not less than $m$.

*Case 1.* $|J^-| \geqslant m$. Among $I \subseteq \{1, \cdots, k\}$ with $\{\sum_{s \in I} m_s/n_s\} = \{\sum_{s \in J} m_s/n_s\}$, we select a $J_0$ with the least cardinality. Apparently $|J_0^-| \geqslant |J^-| \geqslant m$. Let $I_0 = \{s_1, \cdots, s_{m-1}\}$ be a subset of $\{1, \cdots, k\}$ with $|I_0| = m - 1$ and $I_0 \cap J_0 = \emptyset$. By Lemma 1 and Remark 5, system $\{a_s(n_s)\}_{s \in I_0^-}$ forms a 1-cover of $\mathbb{Z}$ and hence so does $\mathcal{A}_0 = \{a_s + (n_s/m_s)\mathbb{Z}\}_{s \in I_0^-}$. As $J_0 \subseteq I_0^-$, by Proposition 1 or Theorem 2 of [S2] there is a $J_1 \subseteq I_0^-$ for which $J_1 \neq J_0$ and $\{\sum_{s \in J_1} m_s/n_s\} = \{\sum_{s \in J_0} m_s/n_s\}$. According to the choice of $J_0$ we must have $J_1 \not\subseteq J_0$. Choose $t_1 \in J_1 \setminus J_0$ and put $I_1 = \{t_1, s_2, \cdots, s_{m-1}\}$. Observe that $I_1 \cap J_0 = \emptyset$. Since $\mathcal{A}_1 = \{a_s + (n_s/m_s)\mathbb{Z}\}_{s \in I_1^-}$ forms a 1-cover of $\mathbb{Z}$, there exists a $J_2 \subseteq I_1^-$ with $J_2 \neq J_0$ such that $\{\sum_{s \in J_2} m_s/n_s\} = \{\sum_{s \in J_0} m_s/n_s\}$. Choose $t_2 \in J_2 \setminus J_0$ and put $I_2 = \{t_1, t_2, s_3, \cdots, s_{m-1}\}$. Then continue this procedure to find $J_3, t_3, I_3; \cdots; J_{m-1}, t_{m-1}, I_{m-1}; J_m, t_m$ in the same way. Apparently $J_1, J_2, \cdots, J_m$ are all different from $J_0$. If $1 \leqslant i < j \leqslant m$, then $t_i \in J_i \setminus J_j$ because $t_i \in I_{j-1}$ and $J_j \cap I_{j-1} = \emptyset$. So the $m+1$ subsets $J_0, J_1, J_2, \cdots, J_m$ of $\{1, \cdots, k\}$ are distinct; therefore

$$\begin{aligned} &\left|\left\{I \subseteq \{1, \cdots, k\} :\ I \neq J \ \&\ \sum_{s \in I} \frac{m_s}{n_s} - \sum_{s \in J} \frac{m_s}{n_s} \in \mathbb{Z}\right\}\right| \\ &\qquad \geqslant |\{J_i :\ 0 \leqslant i \leqslant m \ \&\ J_i \neq J\}| \geqslant m. \end{aligned}$$

*Case 2.* $|J| \geqslant m$, i.e. $|(J^-)^-| \geqslant m$. It follows from the above that

$$\left|\left\{I \subseteq \{1, \cdots, k\} :\ I \neq J^- \ \&\ \sum_{s \in I} \frac{m_s}{n_s} - \sum_{s \in J^-} \frac{m_s}{n_s} \in \mathbb{Z}\right\}\right| \geqslant m.$$

Thus

$$\begin{aligned}&\left|\left\{I'\subseteq\{1,\cdots,k\}:\ I'\neq J\ \&\ \sum_{s\in I'}\frac{m_s}{n_s}-\sum_{s\in J}\frac{m_s}{n_s}\in\mathbb{Z}\right\}\right|\\=&\left|\left\{I^-:\ I\subseteq\{1,\cdots,k\},\ I^-\neq J\ \&\ \sum_{s\in I^-}\frac{m_s}{n_s}-\sum_{s\in J}\frac{m_s}{n_s}\in\mathbb{Z}\right\}\right|\\=&\left|\left\{I\subseteq\{1,\cdots,k\}:\ I\neq J^-\ \&\ \sum_{s\in I}\frac{m_s}{n_s}-\sum_{s\in J^-}\frac{m_s}{n_s}\in\mathbb{Z}\right\}\right|\geqslant m.\end{aligned}$$

So far we have proven (4) in both cases.

Next let's consider the situation in which $K=\{1\leqslant s\leqslant k:\ n_s=1\}$ is nonempty. If $|K|<m$, then $\{a_s(n_s)\}_{s\in K^-}$ forms an $m-|K|$-cover of $\mathbb{Z}$ with all the moduli greater than one; hence by the above

$$\left|\left\{I\subseteq K^-:\ \sum_{s\in I}\frac{m_s}{n_s}-\sum_{s\in J\setminus K}\frac{m_s}{n_s}\in\mathbb{Z}\right\}\right|\geqslant m-|K|+1.$$

Therefore

$$\begin{aligned}&\left|\left\{I\subseteq\{1,\cdots,k\}:\ \sum_{s\in I}\frac{m_s}{n_s}-\sum_{s\in J}\frac{m_s}{n_s}\in\mathbb{Z}\right\}\right|\\=&\left|\left\{I\cup I':\ I\subseteq K,\ I'\subseteq K^-\ \&\ \sum_{s\in I'}\frac{m_s}{n_s}-\sum_{s\in J\setminus K}\frac{m_s}{n_s}\in\mathbb{Z}\right\}\right|\\\geqslant&\left|\left\{I\cup I':\ I\subseteq K,\ |I|\leqslant 1,\ I'\subseteq K^-\ \&\ \sum_{s\in I'}\frac{m_s}{n_s}-\sum_{s\in J\setminus K}\frac{m_s}{n_s}\in\mathbb{Z}\right\}\right|\\\geqslant&|K|+\left|\left\{I'\subseteq K^-:\ \sum_{s\in I'}\frac{m_s}{n_s}-\sum_{s\in J\setminus K}\frac{m_s}{n_s}\in\mathbb{Z}\right\}\right|\\\geqslant&|K|+\max\{m-|K|+1,1\}\geqslant m+1.\end{aligned}$$

This completes the proof. □

**Lemma 2.** *Let* (1) *be a system of arithmetic sequences, and $J$ a nonempty subset of $\{1,\cdots,k\}$ with $|J|\leqslant m(A)$ and $\bigcap_{s\in J}a_s(n_s)\neq\emptyset$. For each $s\in J^-$ let $m_s$ be a positive integer. Let $0\leqslant a<N(J)$ and*

$$C(a)=\sum_{\substack{I\subseteq J^-\\ \{\sum_{s\in I}\frac{m_s}{n_s}\}=\frac{a}{N(J)}}}(-1)^{|I|}\binom{[\sum_{s\in I}\frac{m_s}{n_s}]}{m(A)-|J|}e^{2\pi i\sum_{s\in I}\frac{m_s}{n_s}(a_s-a_J)}\tag{11}$$

*where $a_J$ is the unique integer in $\bigcap_{s\in J}a_s(n_s)$ with $0\leqslant a_J<N(J)$. Then $C(a)=C(\{a\})$.*

*Proof.* Apparently it suffices to show $C(a)=C(a-1)$ providing $a\geqslant 1$.

Let $m = m(A)$. Observe that the sequences $a_s + (n_s/m_s)\mathbb{Z}$ $(s \in J^-)$ together with $a_J + N(J)\mathbb{Z}$ form an $m - |J| + 1$-cover of $\mathbb{Z}$. In view of Proposition 1,

$$\sum_{\substack{I \subseteq J^- \\ \{\sum_{s\in I} \frac{m_s}{n_s}\} = \frac{a}{N(J)}}} (-1)^{|I|} \binom{[\sum_{s\in I} \frac{m_s}{n_s}]}{m-|J|} e^{2\pi i \sum_{s\in I} \frac{a_s m_s}{n_s}}$$

$$+ \sum_{\substack{I \subseteq J^- \\ \{\sum_{s\in I} \frac{m_s}{n_s} + \frac{1}{N(J)}\} = \frac{a}{N(J)}}} (-1)^{|I|+1} \binom{[\sum_{s\in I} \frac{m_s}{n_s} + \frac{1}{N(J)}]}{m-|J|} e^{2\pi i (\sum_{s\in I} \frac{a_s m_s}{n_s} + \frac{a_J}{N(J)})}$$

vanishes. So

$$\begin{aligned}
&e^{2\pi i a a_J/N(J)} C(a) \\
=& \sum_{\substack{I \subseteq J^- \\ \{\sum_{s\in I} \frac{m_s}{n_s}\} = \frac{a}{N(J)}}} (-1)^{|I|} \binom{[\sum_{s\in I} \frac{m_s}{n_s}]}{m-|J|} e^{2\pi i \sum_{s\in I} \frac{a_s m_s}{n_s}} \\
=& \sum_{\substack{I \subseteq J^- \\ \{\sum_{s\in I} \frac{m_s}{n_s}\} = \frac{a-1}{N(J)}}} (-1)^{|I|} \binom{[[\sum_{s\in I} \frac{m_s}{n_s}] + \frac{a}{N(J)}]}{m-|J|} e^{2\pi i (\sum_{s\in I} \frac{a_s m_s}{n_s} + \frac{a_J}{N(J)})} \\
=& e^{2\pi i a_J/N(J)} \sum_{\substack{I \subseteq J^- \\ \{\sum_{s\in I} \frac{m_s}{n_s}\} = \frac{a-1}{N(J)}}} (-1)^{|I|} \binom{[\sum_{s\in I} \frac{m_s}{n_s}]}{m-|J|} e^{2\pi i \sum_{s\in I} \frac{a_s m_s}{n_s}} \\
=& e^{2\pi i a_J/N(J)} e^{2\pi i (a-1) a_J/N(J)} C(a-1) = e^{2\pi i a a_J/N(J)} C(a-1).
\end{aligned}$$

Therefore $C(a) = C(a-1)$. We are done. □

*Remark 6.* If we replace $m(A) - |J|$ in (11) by a smaller nonnegative integer $n$, then the new $C(a)$ will equal zero by Proposition 1, because system $\{a_s + (n_s/m_s)\mathbb{Z}\}_{s\in J^-}$ forms an $m(A) - |J|$-cover of $\mathbb{Z}$.

*Proof of part (ii) of Theorem 1.* Since $|\{s \in J^- : x \in a_s(n_s)\}| = m(A) - |J|$ for some integer $x$ and $(m_s, n_s) = 1$ for all $s \in J^-$, system $\{a_s + (n_s/m_s)\mathbb{Z}\}_{s\in J^-}$ fails to form an $m(A) - |J| + 1$-cover of $\mathbb{Z}$ as well as $\{a_s(n_s)\}_{s\in J^-}$. As $x \in \bigcap_{s\in J} a_s(n_s)$, there is a unique integer $a_J$ with $0 \leqslant a_J < N(J)$ such that $a_J \equiv a_s \pmod{n_s}$ for all $s \in J$. By Proposition 1 and Remark 6 there exists a $\theta \in [0, 1)$ such that

$$C(N(J)\theta) e^{2\pi i a_J \theta} = \sum_{\substack{I \subseteq J^- \\ \{\sum_{s\in I} \frac{m_s}{n_s}\} = \theta}} (-1)^{|I|} \binom{[\sum_{s\in I} \frac{m_s}{n_s}]}{m(A)-|J|} e^{2\pi i \sum_{s\in I} \frac{a_s m_s}{n_s}} \neq 0.$$

Put $\alpha = \{N(J)\theta\}$. If $0 \leqslant a < N(J)$ and $\{a\} = \alpha$, then $a - N(J)\theta \in \mathbb{Z}$ and hence $C(a) = C(N(J)\theta) \neq 0$ by Lemma 2; therefore $\{\sum_{s\in I} \frac{m_s}{n_s}\} = \frac{a}{N(J)}$ for some $I \subseteq J^-$ with $[\sum_{s\in I} \frac{m_s}{n_s}] \geqslant m(A) - |J|$. This concludes the proof. □

## Acknowledgement

The author is indebted to the referee for his comments.

## References


[CV] R. B. Crittenden and C. L. Vanden Eynden, *Any $n$ arithmetic progressions covering the first $2^n$ integers cover all integers*, Proc. Amer. Math. Soc. **24** (1970), 475–481. MR **41:**3365

[E] P.Erdös, *Problems and results in number theory*, in: H. Halberstam and C. Holley, eds., Recent Progress in Analytic Number Theory, vol. 1, Academic Press, New York, 1981, pp. 1–13. MR **84j:**10001

[G] R. K. Guy, *Unsolved Problems in Number Theorey* (2nd, ed.), Springer-Verlag, New York, 1994, pp. 251–256. MR **96e:**11002

[S1] Z. W. Sun, *On exactly $m$ times covers*, Israel J. Math. **77** (1992), 345–348. MR **93k:**11007

[S2] Z. W. Sun, *Covering the integers by arithmetic sequences*, Acta Arith. **72** (1995), 109–129. MR **96k:**11013

[S3] Z. W. Sun, *Covering the integers by arithmetic sequences* II, Trans. Amer. Math. Soc. **348** (1996), 4279–4320. MR **97c:**11011

[S4] Z. W. Sun, *Exact $m$-covers and the linear form $\sum_{s=1}^{k} x_s/n_s$*, Acta Arith. **81** (1997), 175–198. CMP 97:14

[Z] M. Z. Zhang, *A note on covering systems of residue classes*, J. Sichuan Univ. (Nat. Sci. Ed.) **26** (1989), Special Issue, 185–188. MR **92c:**11003



Department of Mathematics, Nanjing University, Nanjing 210093, People's Republic of China

*E-mail address*: zwsun@netra.nju.edu.cn