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# On coverings of algebraic varieties

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Let $U$ and $V$ be algebraic varieties, and $f: U\rightarrow V$ a Galois covering of degree $n$, defined over a field $k$; let $A$ and $A_0$ be Albanese varieties attached to $U$ and $V$ respectively. Then, in the preceding paper [3], we have proved, among several other results, the following two statements:

1) Suppose that $V$ is embedded in some projective space. Let $C$ be a generic hyperplane section curve on $V$ over $k$ and $W=f^{-1}(C)$ the inverse image of $C$ on $U$; let $J$ and $J_0$ be Jacobian varieties attached to (the normalization of) $W$ and $C$ respectively. Then the curve $W$ generates $A$ and we have the inequality

$$(*)\qquad \dim J-\dim A\geqq\dim J_0-\dim A_0\,.$$

2) Suppose that $U$ and $V$ are complete and non-singular. Then, under the assumption that the degree $n$ is prime to the characteristic of the universal domain, the equality $\dim\mathfrak{D}_0(U)=\dim\mathfrak{D}_0(A)$ implies the equality $\dim\mathfrak{D}_0(V)=\dim\mathfrak{D}_0(A_0)$.[1)]

In the present paper, we shall generalize these results to an arbitrary (i. e. not necessarily Galois) covering $f: U\rightarrow V$. Moreover, the result 2) will be replaced by a better one, i. e. the inequality

$$(**)\qquad \dim\mathfrak{D}_0(U)-\dim\mathfrak{D}_0(A)\geqq\dim\mathfrak{D}_0(V)-\dim\mathfrak{D}_0(A_0)\,.$$

Here we note that the numbers on the both sides of (*) and (**) are non-negative (cf. Lang [4] and Igusa [1]) and that the assumption on the degree $n$ in (**) is essential as easily seen in Igusa [2]. It seems to be worth noting that the inequalities (*) and (**) may be rewritten in the following forms:

$$(*)'\qquad \dim J-\dim J_0\geqq\dim A-\dim A_0\,.$$

$$(**)'\qquad \dim\mathfrak{D}_0(U)-\dim\mathfrak{D}_0(V)\geqq\dim\mathfrak{D}_0(A)-\dim\mathfrak{D}_0(A_0)\,.$$

The numbers on the both sides of (*)′ and (**)′ are also non-negative. As in [3], using the formula of Hurwitz on the genera of curves, we can deduce from (*)′ an estimation of the irregularity of the covering variety $U$ of $V$. In addition to these two inequalities, we shall prove, for this arbitrary covering $f: U\rightarrow V$, some analogous results to the main theorems in [3].

1) For a complete, non-singular variety $W$, we donote by $\mathfrak{D}_0(W)$ the space of the linear differential forms of the first kind on $W$.

## 1. Preliminaries.

Let $f: U \to V$ be a covering of degree $n$, defined over an algebraically closed field $k$. Then the function field $k(U)$ of $U$ over $k$ may be considered as a separable extension over $k(V)$ of degree $n$. Let $K^*$ be the smallest Galois extension over $k(V)$ containing $k(U)$, which is clearly a regular extension over $k$. We denote by $G$ and $H$ the Galois groups of $K^*/k(V)$ and $K^*/k(U)$ respectively. Now let $U^*$ be the normalization of $V$ in $K^*$. Then we have the Galois coverings

$$f^*: U^* \to V \quad \text{and} \quad f': U^* \to U,$$

defined over $k$, and we have

$$f^* = f \circ f'. \tag{1}$$

We denote also by the same letters $G$ and $H$ the Galois groups of these coverings respectively, which consist of everywhere biregular, birational transformations $T_\sigma^*$ of $U^*$ into itself defined over $k$ (cf. [**3**]). We set

$$n' = [U^* : U] = (H:1),$$
$$n^* = n'n = [U^* : V] = (G:1),$$

and decompose $G$ into the cosets of $H$ as follows:

$$G = \sum_{i=1}^{n} HT_{\rho_i}^*.$$

Now we list here some results in [**3**], which we shall need in the following arguments, without proof. Let $A^*$ be an Albanese variety attached to $U^*$ and $\alpha^*$ a canonical mapping of $U^*$ into $A^*$, both defined over $k$, such that there exists a simple point $p^*$ on $U^*$ with $\alpha^*(p^*) = 0$. Then each element $T_\sigma^*$ of $G$ determines an automorphism $\eta_\sigma^*$ of $A^*$ and a constant point $a_\sigma^*$ of $A^*$, both rational over $k$, such that

$$\alpha^* \circ T_\sigma^*(u^*) = \eta_\sigma^* \circ \alpha^*(u^*) + a_\sigma^*, \tag{2}$$

where $u^*$ is a generic point of $U^*$ over $k$. The mapping $T_\sigma^* \to \eta_\sigma^*$ is a group homomorphism.

The main theorem in [**3**] asserts that there exist Albanese varieties $A$ and $A_0$ attached to $U$ and $V$ respectively, defined over $k$, which are quotient abelian varieties of $A^*$ and have the following properties: Let $\mu'$ and $\mu^*$ be the canonical separable homomorphisms of $A^*$ onto $A$ and $A_0$ respectively. Then canonical mappings $\alpha$ and $\alpha_0$ of $U$ and $V$ into $A$ and $A_0$ may be taken to satisfy the relations

$$\left.\begin{aligned} \alpha \circ f' &= \mu' \circ \alpha^* \\ \alpha_0 \circ f^* &= \mu^* \circ \alpha^* \end{aligned}\right\} \text{ on } U^*, \tag{3}$$

respectively. We set $C_{\sigma}^{*}=(\eta_{\sigma}^{*}-\delta_{A^{*}})(A^{*})$[2] and let $C^{*}$ be the abelian subvariety of $A^{*}$, generated by all $C_{\sigma}^{*}$ for all $T_{\sigma}^{*}$ in $G$. Then the kernel $C_{G}^{*}$ of $\mu^{*}$ is the algebraic subgroup of $A^{*}$ defined over $k$, which is the union of $C^{*}$ and all its translations by $a_{\sigma}^{*}$ for all $T_{\sigma}^{*}$ in $G$. The kernel $C_{H}^{*}$ of $\mu'$ is defined for $H$, in a simillar way as $C_{G}^{*}$ for $G$. Since $C_{G}^{*}$ contains $C_{H}^{*}$ and $\mu'$ is canonical, there exists a homomorphism $\mu$ of $A$ onto $A_{0}$, defined over $k$, such that we have

$$\mu^{*}=\mu\circ\mu' \qquad \text{on } A^{*}. \tag{4}$$

Since $\mu^{*}$ is separable and $\mu'$ is surjective, the homomorphism $\mu$ is also separable. Moreover, by (1), (3) and (4), we have

$$\alpha_{0}\circ f\circ f'=\mu\circ\mu'\circ\alpha^{*}=\mu\circ\alpha\circ f'$$

and so, as $f'$ is surjective, we have

$$\alpha_{0}\circ f=\mu\circ\alpha \qquad \text{on } U. \tag{5}$$

Then it is easily verified that the abelian variety $A_{0}=A^{*}/C_{G}^{*}$ is also the quotient abelian variety of $A$ with respect to the algebraic subgroup $\mu'(C_{G}^{*})$ and the homomorphism $\mu$ defined in (4) is the canonical separable homomorphism of $A$ onto $A_{0}$ (cf. Rosenlicht [**5**]). Moreover, we have seen that a canonical mapping $\alpha_{0}$ of $V$ into $A_{0}$ may be taken to satisfy (5).

The following formulas will be used in the next section.

$$(6)\qquad \begin{aligned} &\mu^{*}\circ\eta_{\sigma_{i}}^{*}=\mu^{*} \text{ on } A^{*}, \quad \mu^{*}(a_{\sigma}^{*})=0 \text{ for all } T_{\sigma}^{*} \text{ in } G.\\ &\mu'\circ\eta_{\tau}^{*}=\mu' \text{ on } A^{*}, \quad \mu'(a_{\tau}^{*})=0 \text{ for all } T_{\tau}^{*} \text{ in } H.\end{aligned}$$

$$\eta_{\sigma_{1}}^{*}(a_{\sigma_{2}}^{*})=a_{\sigma_{1}\sigma_{2}}^{*}-a_{\sigma_{1}}^{*} \text{ for all } T_{\sigma_{1}}^{*},\ T_{\sigma_{2}}^{*} \text{ in } G. \tag{7}$$

## 2. The endomorphism $\rho$.

First we prove the existence of an endomorphism of $A$, which plays an important role in the proof of the inequality (**).

LEMMA. *There exists an endomorphism $\rho$ of $A$, defined over $k$, such that we have*

$$\rho\circ\mu'=\mu'\circ\sum_{i=1}^{n}\eta_{\rho_{i}}^{*} \qquad \textit{on } A^{*}. \tag{8}$$

PROOF. Since $\mu'$ is the canonical homomorphism, we have only to prove that the kernel of the homomorphism $\mu'\circ\sum_{i=1}^{n}\eta_{\rho_{i}}^{*}$ of $A^{*}$ into $A$ contains the kernel $C_{H}^{*}$ of $\mu'$. First we fix an element $T_{\tau}^{*}$ in $H$. Then, for $i=1,\cdots,n$, each element $T_{\rho_{i}}^{*}\circ T_{\tau}^{*}$ belongs to one and only one coset $HT_{\rho_{j}}^{*}$. Clearly the mapping $i\rightarrow j=s(i)$ defines a permutation of the set $\{1,\cdots,n\}$. Hence we can write

2) For an abelian variety $B$, we denote by $\delta_{B}$ the identity automorphism of $B$.

$$\mu' \circ \sum_{i=1}^{n} \eta^*_{\rho_i} \circ (\eta^*_\tau - \delta_{A^*}) = \mu' \circ \sum_i (\eta^*_{\rho_i \tau} - \eta^*_{\rho_i})$$

$$= \mu' \circ \sum_i (\eta^*_{\tau_i} \eta^*_{\rho_{s(i)}} - \eta^*_{\rho_i})$$

with some $T^*_{\tau_i}$ in $H$ and so, by (6),

$$= \mu' \circ \sum_i (\eta^*_{\rho_{s(i)}} - \eta^*_{\rho_i}) = 0 ,$$

i. e. we have

$$(\mu' \circ \sum_{i=1}^{n} \eta^*_{\rho_i})(\eta^*_\tau - \delta_{A^*})(A^*) = 0 .$$

On the other hand, by (7), we have

$$(\mu' \circ \sum_{i=1}^{n} \eta^*_{\rho_i})(a^*_\tau) = \mu'(\sum_i (a^*_{\rho_i \tau} - a^*_{\rho_i})) = \mu'(\sum_i (a^*_{\tau_i \rho_{s(i)}} - a^*_{\rho_i})$$

with some $T^*_{\tau_i}$ in $H$. Then, also by (7) and (6), we have

$$\mu'(a^*_{\tau_i \rho_{s(i)}}) = \mu'(a^*_{\tau_i} + \eta^*_{\tau_i}(a^*_{\rho_{s(i)}})) = \mu'(a^*_{\rho_{s(i)}})$$

and so

$$(\mu' \circ \sum_{i=1}^{n} \eta^*_{\rho_i})(a^*_\tau) = \mu'(\sum_i (a^*_{\rho_{s(i)}} - a^*_{\rho_i})) = 0 .$$

Therefore we have $(\mu' \circ \sum_{i=1}^{n} \eta^*_{\rho_i})(C^*_H) = 0$.

The endomorphism $\rho$ satisfies the relation

$$\mu \circ \rho = n\mu \qquad \text{on } A . \tag{9}$$

In fact, by (8), (4) and (6), we have

$$\mu \circ \rho \circ \mu' = \mu \circ \mu' \circ \sum_{i=1}^{n} \eta^*_{\rho_i} = \mu^* \circ \sum_i \eta^*_{\rho_i} = n\mu^* = n\mu \circ \mu'.$$

Then, as $\mu'$ is surjective, we have (9).

Now we prove that the abelian subvariety $\rho(A)$ of $A$ is isogenous to $A_0$, an Albanese variety attached to $V$. We have, by (6), $n'\mu' = \mu' \circ \sum_H \eta^*_\tau$[3] and so, by (8),

$$n'\rho \circ \mu' = n'\mu' \circ \sum_{i=1}^{n} \eta^*_{\rho_i} = \mu' \circ \sum_H \eta^*_\tau \circ \sum_i \eta^*_{\rho_i} = \mu' \circ \sum_G \eta^*_\sigma .$$

Since the intersection $C^*_G \cap (\sum_G \eta^*_\sigma)(A^*)$ is a finite subgroup of $A^*$ (cf. [**3**]) and the kernel $C^*_H$ of $\mu'$ is contained in $C^*_G$, $\mu'$ induces a homomorphism of $(\sum_G \eta^*_\sigma)(A^*)$ onto $n'\rho(\mu'(A^*))$ with a finite kernel. As we have $\mu'(A^*) = A$, $\rho(A)$ is isogenous to $(\sum_G \eta^*_\sigma)(A^*)$, which is isogenous to $A_0$ (cf. Th. 2 of [**3**]).

3) The signs $\sum_H$ and $\sum_G$ mean the sums ranged over all the elements of $H$ and $G$ respectively.

Next we assume that the degree $n$ is prime to the characteristic of the universal domain. Let $a$ be any point of the intersection $\rho(A)\cap(\rho-n\delta_A)(A)$. Then we have $a=\rho(a')=(\rho-n\delta_A)(a'')$ with some $a', a''$ in $A$. Operating $\mu$ on this relation, we have, by (9), $\mu(na')=n\mu(a')=0$, i.e. $na'$ belongs to the kernel of $\mu$. So $na=\rho(na')$ belongs to $(\rho\circ\mu')(C_G^*)$, which is also written as $(\mu'\circ\sum_{i=1}^{n}\eta_{\rho_i}^*)$ $(C_G^*)$ by (8). However, by the similar argument as in the proof of Lemma, we can show that $(\mu'\circ\sum_i\eta_{\rho_i}^*)(C_G^*)=0$, because we have not used there the fact that $T_\tau^*$ is in $H$. Hence we have $na=0$, i.e. $\rho(A)\cap(\rho-n\delta_A)(A)$ is a finite subgroup of $A$. Since, clearly, $\rho(A)$ and $(\rho-n\delta_A)(A)$ generate $A$, $A$ is isogenous to the direct product $\rho(A)\times(\rho-n\delta_A)(A)$. Let $x$ be a generic point of $A$ over $k$. Then the mapping $\varphi(x)=\rho(x)\times(\rho-n\delta_A)(x)$ defines an isogeny of $A$ onto $\rho(A)\times(\rho-n\delta_A)(A)$ and, conversely, the mapping $\varphi'(\rho(x)\times(\rho-n\delta_A)(x))=\rho(x)-(\rho-n\delta_A)(x)=nx$ defines also an isogeny of $\rho(A)\times(\rho-n\delta_A)(A)$ onto $A$. Since we have $\varphi'\circ\varphi=n\delta_A$ and $n$ is assumed to be prime to the characteristic of the universal domain, $\varphi$ and $\varphi'$ are separable. Let $\tilde{\mu}$ be the canonical separable homomorphism of $\rho(A)\times(\rho-n\delta_A)(A)$ onto $\rho(A)$ with the kernel $0\times(\rho-n\delta_A)(A)$ (cf. Rosenlicht [**5**]). Then, as we have $(\rho\circ\mu')(C_G^*)=0$ as stated above, $\varphi(\mu'(C_G^*))$ is contained in $0\times(\rho-n\delta_A)(A)$ and so we have $(\tilde{\mu}\circ\varphi)(\mu'(C_G^*))=0$. Since $\mu$ is canonical, there exists an isogeny $\psi$ of $A_0$ onto $\rho(A)$ such that $\tilde{\mu}\circ\varphi=\psi\circ\mu$. Since $\tilde{\mu}$ and $\varphi$ are separable and $\mu$ is surjective, $\psi$ is also separable. Conversely we have, by (9), $(\mu\circ\varphi')(0\times(\rho-n\delta_A)(A))=\mu((\rho-n\delta_A)(A))=0$. Hence, by the similar arguments, we can prove the existence of a separable isogeny of $\rho(A)$ onto $A_0$.

Then, together with the result in **1**, we have the following

THEOREM 1. *Let the notations be as explained above. Then the quotient abelian variety $A_0=A/\mu'(C_G^*)$ is an Albanese variety attached to $V$ and a canonical mapping $\alpha_0$ of $V$ into $A_0$ may be taken to satisfy the relation: $\alpha_0\circ f=\mu\circ\alpha$, where $\mu$ is the canonical homomorphism of $A$ onto $A_0$. On the other hand, $\rho(A)$ is isogenous to $A_0$, where $\rho$ is the endomorphism of $A$ defined in* (8). *Moreover, if the degree $n$ is prime to the characteristic of the universal domain, then there exist separable isogenies between $\rho(A)$ and $A_0$.*

### 3. The inequality (*).

In this section, we suppose that $V$ is embedded in some projective space. Let $C$ be a generic hyperplane section curve on $V$ over $k$; let $W=f^{-1}(C)$ and $W^*=f^{*-1}(C)$ be the inverse images of $C$ on $U$ and $U^*$ respectively, which are irreducible curves. The curves $C$, $W$ and $W^*$ are defined over a regular extension $K$ of $k$; let $\bar{K}$ be the algebraic closure of $K$. Let $W'$ and $W^{*\prime}$ be complete, non-singular curves, which are birationally equivalent to $W$ and $W^*$ over $\bar{K}$ respectively. Then, in a natural way, we can define the Galois coverings

$$g^*: W^{*\prime} \to C \quad \text{and} \quad g': W^{*\prime} \to W',$$

defined over $\bar{K}$ and with the Galois groups isomorphic to $G$ and $H$ respectively (cf. [**3**]).

Let $J^*$ be a Jacobian variety attached to $W^{*\prime}$. Then, by Lang [**4**] as seen in [**3**], $W^{*\prime}$ generates $A^*$ and so there exists a homomorphism $\lambda^*$ of $J^*$ onto $A^*$. For each element $T^*_\sigma$ in $G$, there correspond the automorphisms $\xi^*_\sigma$ and $\eta^*_\sigma$ of $J^*$ and $A^*$, respectively, by the relations of type (2). These automorphisms satisfy the following relations:

$$\lambda^* \circ \xi^*_\sigma = \eta^*_\sigma \circ \lambda^* \qquad \text{on } J^*. \tag{10}$$

$$(\sum_H \xi^*_\tau)(J^*) \sim J, \quad (\sum_G \xi^*_\sigma)(J^*) \sim J_0,$$
$$(\sum_H \eta^*_\tau)(A^*) \sim A, \quad (\sum_G \eta^*_\sigma)(A^*) \sim A_0, \tag{11}$$
[4]

where $J$ and $J_0$ are Jacobian varieties attached to $W'$ and $C$ respectively (cf. [**3**]). Then, by (10), $\lambda^*$ induces, in a natural way, the homomorphisms $\lambda$ of $(\sum_H \xi^*_\tau)(J^*)$ onto $(\sum_H \eta^*_\tau)(A^*)$ and $\lambda_0$ of $(\sum_G \xi^*_\sigma)(J^*)$ onto $(\sum_G \eta^*_\sigma)(A^*)$. Since we have

$$\sum_G \xi^*_\sigma = (\sum_H \xi^*_\tau)(\sum_{i=1}^n \xi^*_{\rho_i}),$$

$(\sum_G \xi^*_\sigma)(J^*)$ is contained in $(\sum_H \xi^*_\tau)(J^*)$ and so the kernel of $\lambda_0$ is contained in that of $\lambda$. On the other hand, as $\lambda$ and $\lambda_0$ are surjective, the dimensions of the kernels of $\lambda$ and $\lambda_0$ are equal to $\dim J - \dim A$ and $\dim J_0 - \dim A_0$, by (11), respectively. Hence we have the following

THEOREM 2. *Let the notations be as explained above. Then we have the inequality*

$$\dim J - \dim J_0 \geqq \dim A - \dim A_0.$$

Let $Z$ be a $k$-closed algebraic subset of $V$, containing all the points on $V$ which ramify in the covering $f: U \to V$. Then, since $W'$ is unramified over every point of $C - C \cap Z$ (cf. [**3**]), we have easily the following corollary by Theorem 2 and the formula of Hurwitz.

COROLLARY. *If the dimension of $Z$ is less than* $\dim V - 1$, *then we have the inequality*

$$\dim A \leqq \dim A_0 + (n-1)(\dim J_0 - 1).$$

Here we note that the dimension of $J_0$ does not depend on the choice of

4) The sign $\sim$ means the isogenous relation between abelian varieties.

the generic curve $C$ but depends only on $V$.

REMARK. By Theorem 2, there are the following two possibilities as for the relations between the numbers $\dim J - \dim A$ and $\dim J_0 - \dim A_0$:

(a) $$\dim J - \dim A = \dim J_0 - \dim A_0 .$$

(b) $$\dim J - \dim A > \dim J_0 - \dim A_0 .$$

We can give examples of the above two cases respectively.

The example of (a): Consider the case where $U$ and $V$ are algebraic curves. Or, consider an unramified covering of a normal algebraic surface of degree 3 in the projective space of dimension 3 (cf. §4 of [**3**]).

The example of (b): Let $X$ be a normal variety with the irregularity larger than 1. Let $s$ and $t$ be rational integers larger than 1 and let $U = X(s)(t)$ and $V = X(st)$ be the $t$-fold symmetric product of the $s$-fold symmetric product of $X$ and the $st$-fold symmetric product of $X$ respectively. Then, taking their normalizations, we have a covering $f: U \to V$ of degree larger than 1. Using the above notations, we have clearly $\dim A = \dim A_0$. On the other hand, the genus of $C$ is not less than the irregularity of $V$ and so it is larger than 1. Then, by the formula of Hurwitz, any covering curve of $C$ has the genus larger than that of $C$, i.e. we have $\dim J > \dim J_0$.

#### 4. The inequality (**).

In this section, we suppose that $U$ and $V$ are complete and non-singular. (But the non-singularity of $U^*$ is not necessary.) Let $\theta$ be an element of $\mathfrak{D}_0(A)$ such that there exists an element $\omega_0$ of $\mathfrak{D}_0(V)$ and $\delta\alpha(\theta) = \delta f(\omega_0)$. Then we have

(12) $$\delta\rho(\theta) = n\theta .$$

In fact, by (3), (1), (2) and the definition of $f^*$, we have

$$\begin{aligned}\delta\alpha^* \circ \delta\mu'(\theta) &= \delta f' \circ \delta\alpha(\theta) = \delta f' \circ \delta f(\omega_0) = \delta f^*(\omega_0) = \delta T^*_{\rho_i} \circ \delta f^*(\omega_0) \\ &= \delta T^*_{\rho_i} \circ \delta\alpha^* \circ \delta\mu'(\theta) = \delta\alpha^* \circ \delta\eta^*_{\rho_i} \circ \delta\mu'(\theta) .\end{aligned}$$

Here we used the fact that, as $\delta\mu'(\theta)$ is of the first kind on $A^*$, it is invariant by the translation of $a^*_{\rho_i}$. Since $\delta\alpha^*$ is injective by Igusa [**1**], we have

$$\delta\mu'(\theta) = \delta\eta^*_{\rho_i} \circ \delta\mu'(\theta)$$

and so, by (8),

$$n\delta\mu'(\theta) = \delta\Big(\sum_{i=1}^{n} \eta^*_{\rho_i}\Big) \circ \delta\mu'(\theta) = \delta\mu' \circ \delta\rho(\theta) .$$

Since $\mu'$ is separable and surjective, $\delta\mu'$ is injective (cf. Igusa [**1**]) and so we have (12).

Now we assume that the degree $n$ is prime to the characteristic of the universal domain. Then we have

$$\delta\mu(\mathfrak{D}_0(A_0)) = \delta\rho(\mathfrak{D}_0(A)) . \tag{13}$$

In fact, by (9) and the assumption on $n$, we have

$$\delta\mu(\mathfrak{D}_0(A_0)) = \delta\rho \circ \delta\mu(\mathfrak{D}_0(A_0)) \subset \delta\rho(\mathfrak{D}_0(A)) .$$

Moreover, as $\mu$ is separable and surjective, $\delta\mu$ is injective and so we have

$$\dim \delta\mu(\mathfrak{D}_0(A_0)) = \dim \mathfrak{D}_0(A_0) = \dim A_0 ,$$

which is equal to $\dim \rho(A)$ by Theorem 1. Now we consider the endomorphism $\rho$ of $A$ as a homomorphism $\rho'$ of $A$ onto another abelian varieiy $\rho(A)$. Denoting by $\iota$ the injection of $\rho(A)$ into $A$, we have $\iota \circ \rho' = \rho$ and so $\delta\rho = \delta\rho' \circ \delta\iota$. Then we have

$$\delta\rho(\mathfrak{D}_0(A)) = \delta\rho' \circ \delta\iota(\mathfrak{D}_0(A)) \subset \delta\rho'(\mathfrak{D}_0(\rho(A)))$$

and so

$$\dim \delta\rho(\mathfrak{D}_0(A)) \leqq \dim \delta\rho'(\mathfrak{D}_0(\rho(A))) \leqq \dim \mathfrak{D}_0(\rho(A)) = \dim \rho(A) = \dim A_0 .$$

Therefore, the linear space $\delta\rho(\mathfrak{D}_0(A))$ of dimension $\leqq \dim A_0$ contains the subspace $\delta\mu(\mathfrak{D}_0(A_0))$ of dimension $= \dim A_0$ and so we must have (13).

THEOREM 3. *We assume that the degree $n$ is prime to the characteristic of the universal domain. If, for an element $\omega_0$ in $\mathfrak{D}_0(V)$, $\delta f(\omega_0)$ belongs to $\delta\alpha(\mathfrak{D}_0(A))$, then there exists an element $\theta_0$ of $\mathfrak{D}_0(A_0)$ such that we have*

$$\omega_0 = \delta\alpha_0(\theta_0) .$$

PROOF. Let $\theta$ be an element of $\mathfrak{D}_0(A)$ such that $\delta f(\omega_0) = \delta\alpha(\theta)$. From the assumption on the degree $n$, $\frac{1}{n} \cdot \theta$ belongs to $\mathfrak{D}_0(A)$, and so, by (12), we have

$$\delta\rho\Big(\frac{1}{n} \cdot \theta\Big) = \frac{1}{n} \cdot \delta\rho(\theta) = \frac{1}{n} \cdot n\theta = \theta ,$$

i.e. $\theta$ is contained in $\delta\rho(\mathfrak{D}_0(A))$. Then, by (13), there exists an element $\theta_0$ of $\mathfrak{D}_0(A_0)$ such that we have $\delta\mu(\theta_0) = \theta$. Hence, by (5), we have

$$\delta f(\omega_0) = \delta\alpha(\theta) = \delta\alpha \circ \delta\mu(\theta_0) = \delta f \circ \delta\alpha_0(\theta_0) .$$

Since $f$ is separable and surjective, $\delta f$ is injective and so the statement of our theorem is proved.

Theorem 3 implies that, if an element $\omega_0$ of $\mathfrak{D}_0(V)$ does not belong to the subspace $\delta\alpha_0(\mathfrak{D}_0(A_0))$, then also $\delta f(\omega_0)$ does not belong to the subspace $\delta\alpha(\mathfrak{D}_0(A))$ of $\mathfrak{D}_0(U)$. Since $\delta f$, $\delta\alpha$ and $\delta\alpha_0$ are injective, we have the following

COROLLARY. *Under the same assumption on $n$ as in Theorem 3, there holds the inequality*

$$\dim \mathfrak{D}_0(U) - \dim \mathfrak{D}_0(A) \geqq \dim \mathfrak{D}_0(V) - \dim \mathfrak{D}_0(A_0) .$$

*Especially, if* $\dim \mathfrak{D}_0(U) = \dim \mathfrak{D}_0(A)$, *then we have the equality* $\dim \mathfrak{D}_0(V) = \dim \mathfrak{D}_0(A_0)$.

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