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ON CR-SUBMANIFOLDS OF THE SIX-DIMENSIONAL SPHERE

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ABSTRACT. We consider proper CR-submanifolds of the six-dimensional sphere S^6. We prove that S^6 does not admit compact proper CR-submanifolds with non-negative sectional curvature and integrable holomorphic distribution.

KEY WORDS AND PHRASES. CR-submanifolds, Kaehler manifold, nearly Kaehler manifold, the six-dimensional sphere, almost complex structures.
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1. INTRODUCTION. The study of CR-submanifolds of a Kaehler manifold was initiated by Bejancu [1]. This study generalizes both the complex submanifolds as well as the totally real submanifolds. For this reason, it has become the subject of interest to many mathematicians [3]. Of all the Euclidian spheres, only S^2 and S^6 admit the almost complex structure of which S^2 is complex and S^6 is not. It is known that S^6 is an almost hermitian manifold which is nearly Kaehler but not Kaehler, that is, the almost complex structure is not parallel with respect to the Riemannian connection on S^6 [4]. CR-submanifolds of S^6 have been studied by several mathematicians. For instance Sekigawa [7] proved that S^6 does not contain any CR-product submanifold. Gray [5] has shown that S^6 does not admit a 4-dimensional complex submanifold.

In this paper, we consider compact proper CR-submanifolds of S^6. We obtain the following:

THEOREM. S^6 does not admit any compact proper CR-submanifold with non-negative sectional curvature and integrable holomorphic distribution.

2. PRELIMINARIES. Let C be the set of all purely imaginary Cayley numbers. C can be viewed as a 7-dimensional linear subspace $\mathbb{R}^7$ of $\mathbb{R}^8$. Consider the unit hypersurface which is centered at the origin:

$$S^6(1) = \{x \in C: < x, x > \ = 1\}$$

The tangent space T_xS^6 of S^6 at a point x may be identified with the affine subspace of C which is orthogonal to x. $A(1,1)$ tensor field J on S^6 is defined by

$$J_x U = X \times U$$

where the above product is defined as in [4] for $x \varepsilon S^6$ and $U \varepsilon T_x S^6$. The tensor field J determines an almost complex structure (i.e., $J^2 = -id$) on S^6. If $\overline{\nabla}$ is the Riemannian connection on S^6, then $(\overline{\nabla}_X J)X = 0$ for any $X \varepsilon \mathfrak{X}(S^6)$, i.e., S^6 is nearly Kaehler.

$A(2p+q)$-dimensional submanifold M of S^6 is called a CR-submanifold if there exists a pair of orthogonal complementary distribution D and $\overset{\perp}{D}$ such that $JD = D$ and $J\overset{\perp}{D}\varepsilon\nu$ where ν is the normal bundle of M. The distributions D and $\overset{\perp}{D}$ are called the holomorphic distribution and the totally real distribution respectively with $dim D = 2p$ and $dim D^{\perp} = q$. The normal bundle ν splits as $\nu = J\overset{\perp}{D} \oplus \mu$ where μ is invariant sub-bundle of ν under J. The CR-submanifold is said to be proper if neither $D = \{0\}$ nor $\overset{\perp}{D} = \{0\}$. A proper CR-submanifold M of S^6 is said to be a CR-product submanifold if it is locally the Riemannian product of a holomorphic submanifold and a totally real submanifold of S^6. It is known that there does not exist any CR-product submanifolds in S^6 [7].

Let ∇ be the Riemannian connection on (M,g) where g is the induced metric. Then the curvature tensor R of (M,g) of type $(1,3)$ is defined by

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, \qquad X,Y,Z\varepsilon\mathfrak{X}(M)$$

The sectional curvature $K(X,Y)$ of the plane section determined by $\{X,Y\}$ is defined by

$$K(X,Y) = R(X,Y,Y,X)\{\|X\|^2\|Y\|^2 - g(X,Y)^2\}^{-1} \text{ where } R(X,Y,Z,W) = g(R(X,Y)Z,W).$$

The Ricci tensor of (M,g) is defined by

$$Ric(X,Y) = \sum_{i=1}^{n} R(e_i, X, Y, e_i), \qquad X,Y\varepsilon\mathfrak{X}(M)$$

where $\{e_1, \cdots, e_n\}$ is a local orthonormal frame field on M. On a compact Riemannian manifold the following integral formula holds for any $X\varepsilon\mathfrak{X}(M)$ (cf [8]).

$$\int_M \{Ric(X,X) + \|\nabla X\|^2 - \frac{1}{2}\|d\eta\|^2 - (div\ X)^2\}dv = 0,$$

where η is a 1-form dual to X, i.e., $g(X,Z) = \eta(Z)$, for

$$Z\varepsilon\mathfrak{X}(M) \text{ and } \|\nabla X\|^2 = \sum_{i=1}^{n} g(\nabla_{e_i} X, \nabla_{e_i} X).$$

Let h be the second fundamental form. M is said to be totally geodesic if $h \equiv 0$ and M is said to be totally umbilical if $h(X,Y) = g(X,Y)H$ where H is the mean curvature tensor defined by $H = \frac{1}{n}$ trace h.

3. PROOF OF THE THEOREM.

Since D is integrable, then the integral submanifold of the distribution D is a Kaehler manifold. Since M is proper then $dim D = 4$ is ruled out by a result of Gray [5] namely S^6 does not contain a 4-dimensional complex submanifold. Therefore $dim D = 2$. Since $\nu = J\overset{\perp}{D} \oplus \mu$ and M is a proper CR-submanifold of S^6 we have $dim\overset{\perp}{D} = 1$, i.e., M is 3-dimensional. Now let w be a 2-form on the integral submanifold of D and let η be its dual. Since the integral submanifold of D is Kaehler, w is harmonic (cf. [6]). Using Poincare duality theorem, its dual η is also harmonic, i.e., $d\eta = \delta\eta = 0$.

Now from the hypothesis of the theorem, we get $Ric(Z,Z) \geq 0$. Using the integral formula on this page and $Z\varepsilon\overset{\perp}{D}$ we have

$$\int_M \{Ric(Z,Z) - \frac{1}{2}\|d\eta\|^2 + \|\nabla Z\|^2 - (\delta\eta)^2\}dv = 0,$$

from which we get $\nabla_X \overset{\perp}{Z} = 0$ for all $X\varepsilon\mathfrak{X}(M)$ and $Z\varepsilon\overset{\perp}{D}$, i.e., the distribution $\overset{\perp}{D}$ is parallel. Also $g(Y,Z) = 0$ for all $Y\varepsilon D$ gives $\nabla_X Y = 0$ for all $X\varepsilon\mathfrak{X}(M)$ and $Y\varepsilon D$. This means that D is also parallel. D and $\overset{\perp}{D}$ being parallel implies that M is a CR-product, which is a contradiction to the fact that S^6 does not have any CR-product submanifold [7]. Therefore our theorem is proven.

COROLLARY 1. There does not exist a compact totally umbilical proper CR-submanifolds of S^6 with integrable distribution D.

PROOF. Since S^6 is of constant positive curvature, the curvature tensor $\overline{R}$ of S^6 is given by $\overline{R}(X,Y,Z,W) = c\{g(X,W)g(Y,Z) - g(Z,X)g(Y,W)\}$. Using this in Gauss equation

$$R(X,Y,Z,W) = \overline{R}(X,Y,Z,W) + g(h(X,W),h(Y,Z)) - g(h(Z,X),h(Y,W))$$

with the assumption that M is totally umbilical (i.e., $h(X,Y) = g(X,Y)H$) we get $R(X,Y,Y,X) = c + \| H \|^2 > 0 \cdot X, Y\varepsilon\mathfrak{X}(M)$. This implies that M is of positive sectional curvature. Then the corollary follows from the theorem.

COROLLARY 2. There does not exist a compact totally geodesic proper CR-submanifold of S^6 with integrable distribution D.

PROOF. Since M is totally geodesic in S^6, then it follows immediately from Gauss equation that M is of positive sectional curvature. Thus the corollary follows from the theorem.

REMARK. If $dimM = 3$, then Corollary 1 holds without the assumption that D is integrable. This is a result proved previously by Bashir [2].

REFERENCES

1. BEJANCU, A., CR-submanifolds of a Kaehler manifold, *Proc. Amer. Math. Soc.* **69** (1978), 135-142.

2. BASHIR, M., On the three-dimensional CR-submanifolds of the six-dimensional sphere, *Internat. J. Math. and Math. Sci.* 14 (1991), 675-678.

3. CHEN, B.Y., CR-submanifolds of a Kaehler manifold I and II, *J. Diff.* **16** (1981), 305-322 and 493-502.

4. FUKAMI, T. and ISHIHARA, S., Almost Hermitian structure on S^6, *Tohoku Math. J.* **7** (1955), 151-156.

5. GRAY, A., Almost complex submanifolds of six sphere, *Proc. Amer. Math. Soc.* **20** (1969), 277-279.

6. GOLDBERG, S., *Curvature and Homology*, Academic Press, New York, 1962.

7. SEKIGAWA, K., Some CR-submanifolds in a 6-dimensional sphere, *Tensor, N.S.* **41** (1984), 13-20.

8. YANO, K. and BOCHNER, S., Curvature and Betti numbers, *Ann. of Math. Stud.* **32**, 1953.