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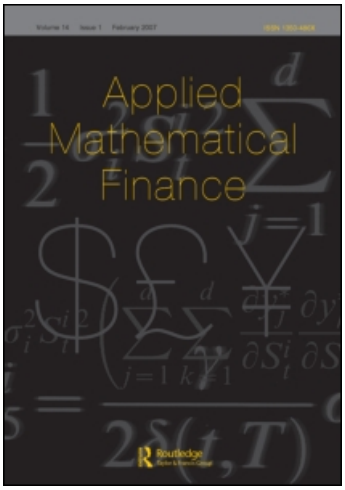

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On Cross-Currency Models with Stochastic Volatility and Correlated Interest Rates

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ABSTRACT *We construct multi-currency models with stochastic volatility (SV) and correlated stochastic interest rates with a full matrix of correlations. We first deal with a foreign exchange (FX) model of Heston-type, in which the domestic and foreign interest rates are generated by the short-rate process of Hull–White (Hull, J. and White, A. [1990] Pricing interest-rate derivative securities,* Review of Financial Studies*, 3, pp. 573–592). We then extend the framework by modelling the interest rate by an SV displaced-diffusion (DD) Libor Market Model (Andersen, L. B. G. and Andreasen, J. [2000] Volatility skews and extensions of the libor market model,* Applied Mathematics Finance*, 1[7], pp. 1–32), which can model an interest rate smile. We provide semi-closed form approximations which lead to efficient calibration of the multi-currency models. Finally, we add a correlated stock to the framework and discuss the construction, model calibration and pricing of equity–FX–interest rate hybrid pay-offs.*

KEY WORDS: foreign exchange (FX), stochastic volatility, Heston model, stochastic interest rates, interest rate smile, forward characteristic function, hybrids, efficient calibration

1. Introduction

Since the financial crisis, investors tend to look for products with a long-time horizon that are less sensitive to short-term market fluctuations. When pricing these *exotic contracts*, it is desirable to incorporate in a mathematical model the patterns present in the market that are relevant to the product.

Due to the existence of complex foreign exchange (FX) products, like the *Power-Reverse Dual-Currency* (Sippel and Ohkoshi, 2002), the *Equity-CMS Chameleon* or the *Equity-Linked Range Accrual TRAN swaps* (Caps, 2007), that all have a long lifetime and are sensitive to smiles or skews in the market, improved models with stochastic interest rates need to be developed.

The literature on modelling FX rates is rich and many stochastic models are available. An industrial standard is a model from Frey and Sommer (1996) and Sippel

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and Ohkoshi (2002), where log-normally distributed FX dynamics are assumed and Gaussian, one-factor, interest rates are used. This model gives analytic expressions for the prices of basic products for at-the-money options. Extensions on the interest rate side were presented in Schlögl (2002b) and Mikkelsen (2001), where the short-rate model was replaced by a Libor Market Model framework.

A Gaussian interest rate model was also used in Piterbarg (2006), in which a local volatility model was applied for generating the skews present in the FX market. In another paper, Kawai and Jäckel (2007), a displaced-diffusion (DD) model for FX was combined with the interest rate Libor Market Model.

Stochastic volatility (SV) FX models have also been investigated. For example, in Haastrecht and Pelsser (2011), the Schöbel–Zhu model was applied for pricing FX in combination with short-rate processes. This model leads to a semi-closed form for the characteristic function. However, for a normally distributed volatility process, it is difficult to outperform the Heston model with independent stochastic interest rates (Haastrecht and Pelsser, 2011).

Research on the Heston dynamics in combination with *correlated* interest rates has led to some interesting models. In Andreasen (2006) and Giese (2006), an indirectly imposed correlation structure between Gaussian short-rates and FX was presented. The model is intuitively appealing, but it may give rise to very large model parameters (Antonov *et al.*, 2008). An alternative model was presented in Antonov and Misirpashaev (2006) and Antonov *et al.* (2008), in which calibration formulas were developed by means of Markov projection techniques.

In this article we present an FX Heston-type model in which the interest rates are stochastic processes, correlated with the governing FX processes. We first discuss the Heston FX model with Gaussian interest rate (Hull–White model (Hull and White 1990)) short-rate processes. In this model a full matrix of correlations is used.

This model, denoted by *FX-HHW* here, is a generalization of our work in Grzelak and Oosterlee (2011), where we dealt with the problem of finding an affine approximation of the Heston equity model with a correlated stochastic interest rate. In this article, we apply this technique in the world of FX.

Secondly, we extend the framework by modelling the interest rates by a market model, that is, by the SV-DD Libor Market Model (Andersen and Andreasen, 2002; Piterbarg, 2005). In this hybrid model, called *FX-HLMM* here, we incorporate a non-zero correlation between the FX and the interest rates and between the rates from different currencies. Because it is not possible to obtain closed form formulas for the associated characteristic function, we use a linearization approximation, developed earlier, in Grzelak and Oosterlee (2010).

For both models we provide details on how to include a foreign stock in the multi-currency pricing framework.

Fast model evaluation is highly desirable for FX options in practice, especially during the calibration of the hybrid model. This is the main motivation for the generalization of the linearization techniques in Grzelak and Oosterlee (2010, 2011) to the world of FX. We will see that the resulting approximations can be used very well in the FX context.

This article is organized as follows: Section 2 discusses the extension of the Heston model by stochastic interest rates, described by short-rate processes. We provide details about some approximations in the model and then derive the related forward

characteristic function. We also discuss the model's accuracy and calibration results. Section 3 gives the details for the cross-currency model with interest rates driven by the market model and Section 4 concludes.

2. Multi-Currency Model with Short-Rate Interest Rates

Here, we derive the model for the spot FX, $\xi(t)$, expressed in units of domestic currency, per unit of a foreign currency.

We start the analysis with the specification of the underlying interest rate processes, $r_d(t)$ and $r_f(t)$. At this stage we assume that the interest rate dynamics are defined via *short-rate processes*, which under their spot measures, that is, $\mathbb{Q}$ − domestic and $\mathbb{Z}$ − foreign, are driven by the Hull–White (Hull and White, 1990) one-factor model:

$$\mathrm{d}r_d(t) = \lambda_d(\theta_d(t) - r_d(t))\mathrm{d}t + \eta_d \mathrm{d}W_d^{\mathbb{Q}}(t), \tag{1}$$

$$\mathrm{d}r_f(t) = \lambda_f(\theta_f(t) - r_f(t))\mathrm{d}t + \eta_f \mathrm{d}W_f^{\mathbb{Z}}(t), \tag{2}$$

where $W_d^{\mathbb{Q}}(t)$ and $W_f^{\mathbb{Z}}(t)$ are Brownian motions under $\mathbb{Q}$ and $\mathbb{Z}$, respectively. Parameters λ_d, λ_f determine the speed of mean reversion to the time-dependent term structure functions $\theta_d(t)$, $\theta_f(t)$, and parameters η_d, η_f are the volatility coefficients.

These processes, under the appropriate measures, are linear in their state variables, so that for a given maturity T $(0 < t < T)$, the zero-coupon bonds (ZCBs) are known to be of the following form:

$$\begin{aligned} P_d(t,T) &= \exp(A_d(t,T) + B_d(t,T)r_d(t)), \\ P_f(t,T) &= \exp(A_f(t,T) + B_f(t,T)r_f(t)), \end{aligned} \tag{3}$$

with $A_d(t,T)$, $A_f(t,T)$ and $B_d(t,T)$, $B_f(t,T)$ are analytically known quantities (see, e.g. Brigo and Mercurio, 2007). In the model the money market accounts are given by

$$\mathrm{d}M_d(t) = r_d(t)M_d(t)\mathrm{d}t, \quad \text{and} \quad \mathrm{d}M_f(t) = r_f(t)M_f(t)\mathrm{d}t. \tag{4}$$

Using the Heath–Jarrow–Morton arbitrage-free argument (Heath *et al.*, 1992), the dynamics for the ZCBs, under their own measures generated by the money savings accounts, are known and given by the following result:

Result 2.1 (ZCB dynamics under the risk-free measure.) The risk-free dynamics of the ZCBs, $P_d(t,T)$ and $P_f(t,T)$, with maturity T are given by

$$\frac{\mathrm{d}P_d(t,T)}{P_d(t,T)} = r_d(t)\mathrm{d}t - \left(\int_t^T \Gamma_d(t,s)\mathrm{d}s\right)\mathrm{d}W_d^{\mathbb{Q}}(t), \tag{5}$$

$$\frac{\mathrm{d}P_f(t,T)}{P_f(t,T)} = r_f(t)\mathrm{d}t - \left(\int_t^T \Gamma_f(t,s)\mathrm{d}s\right)\mathrm{d}W_f^{\mathbb{Z}}(t), \tag{6}$$

where $\Gamma_d(t,T)$, $\Gamma_f(t,T)$ are the volatility functions of the instantaneous forward rates $f_d(t,T), f_f(t,T)$, respectively, that are given by

$$\mathrm{d}f_d(t,T) = \Gamma_d(t,T)\int_t^T \Gamma_d(t,s)\mathrm{d}s + \Gamma_d(t,T)\mathrm{d}W_d^{\mathbb{Q}}(t), \tag{7}$$

$$\mathrm{d}f_f(t,T) = \Gamma_f(t,T)\int_t^T \Gamma_f(t,s)\mathrm{d}s + \Gamma_f(t,T)\mathrm{d}W_f^{\mathbb{Z}}(t). \tag{8}$$

Proof. For the proof see Musiela and Rutkowski (1997).

The spot-rates at time t are defined by $r_d(t) \equiv f_d(t,t)$, $r_f(t) \equiv f_f(t,t)$.

By means of the volatility structures, $\Gamma_d(t,T)$, $\Gamma_f(t,T)$, one can define a number of short-rate processes. In our framework the volatility functions are chosen to be $\Gamma_d(t,T) = \eta_d \exp(-\lambda_d(T-t))$ and $\Gamma_f(t,T) = \eta_f \exp\left(-\lambda_f(T-t)\right)$. The Hull–White short-rate processes, $r_d(t)$ and $r_f(t)$ as in Equations (1) and (2), are then obtained and the term structures, $\theta_d(t)$, $\theta_f(t)$, expressed in terms of instantaneous forward rates, are also known. The choice of specific volatility determines the dynamics of the ZCBs:

$$\begin{aligned}\frac{\mathrm{d}P_d(t,T)}{P_d(t,T)} &= r_d(t)\mathrm{d}t + \eta_d B_d(t,T)\mathrm{d}W_d^{\mathbb{Q}}(t),\\ \frac{\mathrm{d}P_f(t,T)}{rP_f(t,T)} &= r_f(t)\mathrm{d}t + \eta_f B_f(t,T)\mathrm{d}W_f^{\mathbb{Z}}(t),\end{aligned} \tag{9}$$

with $B_d(t,T)$ and $B_f(t,T)$ as in Equation (3), given by

$$B_d(t,T) = \frac{1}{\lambda_d}\left(\mathrm{e}^{-\lambda_d(T-t)} - 1\right), \quad B_f(t,T) = \frac{1}{\lambda_f}\left(\mathrm{e}^{-\lambda_f(T-t)} - 1\right). \tag{10}$$

For a detailed discussion on short-rate processes, we refer to the analysis of Musiela and Rutkowski (1997). In the next subsection we define the FX hybrid model.

2.1. The Model with Correlated Gaussian Interest Rates

The FX-HHW model, with all processes defined under the domestic risk-neutral measure, $\mathbb{Q}$, is of the following form:

$$\begin{aligned}\tfrac{\mathrm{d}\xi(t)}{\xi(t)} &= \left(r_d(t) - r_f(t)\right)\mathrm{d}t + \sqrt{v(t)}\mathrm{d}W_\xi^{\mathbb{Q}}(t), \quad \xi(0) > 0,\\ \mathrm{d}v(t) &= \kappa(\bar{v} - v(t))\mathrm{d}t + \gamma\sqrt{v(t)}\mathrm{d}W_v^{\mathbb{Q}}(t), \quad v(0) > 0,\\ \mathrm{d}r_d(t) &= \lambda_d(\theta_d(t) - r_d(t))\mathrm{d}t + \eta_d\mathrm{d}W_d^{\mathbb{Q}}(t), \quad r_d(0) > 0,\\ \mathrm{d}r_f(t) &= \left(\lambda_f(\theta_f(t) - r_f(t)) - \eta_f\rho_{\xi,f}\sqrt{v(t)}\right)\mathrm{d}t + \eta_f\mathrm{d}W_f^{\mathbb{Q}}(t), \quad r_f(0) > 0.\end{aligned} \tag{11}$$

Here, the parameters κ, λ_d and λ_f determine the speed of mean reversion of the latter three processes and their long-term mean is given by $\bar{v}$, $\theta_d(t)$ and $\theta_f(t)$, respectively. The volatility coefficients for the processes $r_d(t)$ and $r_f(t)$ are given by η_d and η_f and the volatility-of-volatility parameter for process $v(t)$ is γ.

In the model we assume a full matrix of correlations between the Brownian motions $\mathbf{W}(t) = [W_\xi^{\mathbb{Q}}(t), W_v^{\mathbb{Q}}(t), W_d^{\mathbb{Q}}(t), W_f^{\mathbb{Q}}(t)]^T$:

$$\mathrm{d}\mathbf{W}(t)(\mathrm{d}\mathbf{W}(t))^T = \begin{pmatrix} 1 & \rho_{\xi,v} & \rho_{\xi,d} & \rho_{\xi,f} \\ \rho_{\xi,v} & 1 & \rho_{v,d} & \rho_{v,f} \\ \rho_{\xi,d} & \rho_{v,d} & 1 & \rho_{d,f} \\ \rho_{\xi,f} & \rho_{v,f} & \rho_{d,f} & 1 \end{pmatrix} \mathrm{d}t. \tag{12}$$

Under the domestic-spot measure the drift in the short-rate process, $r_f(t)$, gives rise to an additional term, $-\eta_f \rho_{\xi,f} \sqrt{v(t)}$. This term ensures the existence of martingales, under the domestic-spot measure, for the following prices (for more discussion, see Shreve (2004)):

$$\chi_1(t) := \xi(t) \frac{M_f(t)}{M_d(t)} \quad \text{and} \quad \chi_2(t) := \xi(t) \frac{P_f(t,T)}{M_d(t)},$$

where $P_f(t,T)$ is the price foreign ZCB (Equation (9)), and the money savings accounts $M_d(t)$ and $M_f(t)$ are from Equation (4).

To see that the processes $\chi_1(t)$ and $\chi_2(t)$ are martingales, one can apply the Itô product rule, which gives

$$\frac{\mathrm{d}\chi_1(t)}{\chi_1(t)} = \sqrt{v(t)}\mathrm{d}W_\xi^{\mathbb{Q}}(t), \tag{13}$$

$$\frac{\mathrm{d}\chi_2(t)}{\chi_2(t)} = \sqrt{v(t)}\mathrm{d}W_\xi^{\mathbb{Q}}(t) + \eta_f B_f(t,T)\mathrm{d}W_f^{\mathbb{Q}}(t). \tag{14}$$

The change of dynamics of the underlying processes, from the foreign-spot to the domestic-spot measure, also influences the dynamics for the associated bonds, which, under the domestic risk-neutral measure, $\mathbb{Q}$, with the money savings account considered as a numéraire, have the following representations:

$$\frac{\mathrm{d}P_d(t,T)}{P_d(t,T)} = r_d(t)\mathrm{d}t + \eta_d B_d(t,T)\mathrm{d}W_d^{\mathbb{Q}}(t), \tag{15}$$

$$\frac{\mathrm{d}P_f(t,T)}{P_f(t,T)} = \left(r_f(t) - \rho_{\xi,f}\eta_f B_f(t,T)\sqrt{v(t)}\right)\mathrm{d}t + \eta_f B_f(t,T)\mathrm{d}W_f^{\mathbb{Q}}(t), \tag{16}$$

with $B_d(t,T)$ and $B_f(t,T)$ as in Equation (10).

2.2. Pricing of FX Options

In order to perform efficient calibration of the model, we need to be able to price basic options on the FX rate, $V(t, \mathbf{X}(t))$, for a given state vector, $\mathbf{X}(t) = [\xi(t), v(t), r_d(t), r_f(t)]^T$:

$$V(t, \mathbf{X}(t)) = \mathbb{E}^{\mathbb{Q}}\left(\frac{M_d(t)}{M_d(T)} \max(\xi(T) - K, 0) \Big| \mathcal{F}(t)\right),$$

with

$$M_d(t) = \exp\left(\int_0^t r_d(s) \mathrm{d}s\right).$$

Now, we consider a forward price, $\Pi(t)$, such that

$$\mathbb{E}^{\mathbb{Q}}\left(\frac{\max(\xi(T) - K, 0)}{M_d(T)} \Big| \mathcal{F}(t)\right) = \frac{V(t, X(t))}{M_d(t)} =: \Pi(t).$$

By Itô's lemma we have

$$\mathrm{d}\Pi(t) = \frac{1}{M_d(t)} \mathrm{d}V(t) - r_d(t) \frac{V(t)}{M_d(t)} \mathrm{d}t, \tag{17}$$

with $V(t) := V(t, \mathbf{X}(t))$. We know that $\Pi(t)$ must be a martingale, that is, $\mathbb{E}(\mathrm{d}\Pi(t)) = 0$. Including this in Equation (17) gives the following *Fokker–Planck forward equation* for V:

$$\begin{aligned} r_d V = {} & \frac{1}{2}\eta_f^2 \frac{\partial^2 V}{\partial r_f^2} + \rho_{d,f}\eta_d\eta_f \frac{\partial^2 V}{\partial r_d \partial r_f} + \frac{1}{2}\eta_d^2 \frac{\partial^2 V}{\partial r_d^2} + \rho_{v,f}\gamma\eta_f\sqrt{v} \frac{\partial^2 V}{\partial v \partial r_f} \\ & + \rho_{v,d}\gamma\eta_d\sqrt{v} \frac{\partial^2 V}{\partial v \partial r_d} + \frac{1}{2}\gamma^2 v \frac{\partial^2 V}{\partial v^2} + \rho_{\xi,f}\eta_f\xi\sqrt{v} \frac{\partial^2 V}{\partial \xi \partial r_f} + \rho_{x,d}\eta_d\xi\sqrt{v} \frac{\partial^2 V}{\partial \xi \partial r_d} \\ & + \rho_{\xi,v}\gamma\xi v \frac{\partial^2 V}{\partial \xi \partial v} + \frac{1}{2}\xi^2 v \frac{\partial^2 V}{\partial \xi^2} + \left(\lambda_f(\theta_f(t) - r_f) - \rho_{\xi,f}\eta_f\sqrt{v}\right) \frac{\partial V}{\partial r_f} \\ & + \lambda_d(\theta_d(t) - r_d) \frac{\partial V}{\partial r_d} + \kappa(\bar{v} - v) \frac{\partial V}{\partial v} + (r_d - r_f)\xi \frac{\partial V}{\partial \xi} + \frac{\partial V}{\partial t}. \end{aligned}$$

This 4D partial differential equation (PDE) contains non-affine terms, like square-roots and products. It is therefore difficult to solve it analytically and a numerical PDE discretization, like finite differences, needs to be employed. Finding a numerical solution for this PDE is therefore rather expensive and not easily applicable for model calibration. In Subsection 2.2.1, we propose an *approximation* of the model, which is useful for calibration.

2.2.1 The FX Model under the Forward Domestic Measure. To reduce the complexity of the pricing problem, we move from the spot measure, generated by the money savings account in the domestic market, $M_d(t)$, to the *forward FX measure* where the numéraire is the domestic ZCB, $P_d(t,T)$. As indicated in Musiela and Rutkowski (1997) and Piterbarg (2006), the forward is given by

$$\mathrm{FX}^T(t) = \xi(t)\frac{P_f(t,T)}{P_d(t,T)}, \tag{18}$$

where $\mathrm{FX}^T(t)$ represents the forward exchange rate under the T-forward measure, and $\xi(t)$ stands for FX rate under the domestic-spot measure. The superscript should *not* be confused with the transpose notation used at other places in the text.

By switching from the domestic risk-neutral measure, $\mathbb{Q}$, to the domestic T-forward measure, $\mathbb{Q}^T$, the discounting will be decoupled from taking the expectation, that is,

$$\Pi(t) = P_d(t,T)\mathbb{E}^T\left(\max\left(\mathrm{FX}^T(T) - K, 0\right)|\mathcal{F}(t)\right). \tag{19}$$

In order to determine the dynamics for $\mathrm{FX}^T(t)$ in Equation (18), we apply Itô's formula:

$$\begin{aligned}
\mathrm{d}\,\mathrm{FX}^T(t) = {} & \frac{P_f(t,T)}{P_d(t,T)}\mathrm{d}\xi(t) + \frac{\xi(t)}{P_d(t,T)}\mathrm{d}P_f(t,T) - \xi(t)\frac{P_f(t,T)}{P_d^2(t,T)}\mathrm{d}P_d(t,T) \\
& + \xi(t)\frac{P_f(t,T)}{P_d^3(t,T)}\left(\mathrm{d}P_d(t,T)\right)^2 + \frac{1}{P_d(t,T)}(\mathrm{d}\xi(t)\mathrm{d}P_f(t,T)) \\
& - \frac{P_f(t,T)}{P_f^2(t,T)}(\mathrm{d}P_d(t,T)\mathrm{d}\xi(t)) - \frac{\xi(t)}{P_d^2(t,T)}\mathrm{d}P_d(t,T)\mathrm{d}P_f(t,T).
\end{aligned} \tag{20}$$

After substitution of SDEs (11), (15) and (16) into (20), we arrive at the following FX forward dynamics:

$$\begin{aligned}
\frac{\mathrm{d}\,\mathrm{FX}^T(t)}{\mathrm{FX}^T(t)} = {} & \eta_d B_d(t,T)\left(\eta_d B_d(t,T) - \rho_{\xi,d}\sqrt{v(t)} - \rho_{d,f}\eta_f B_f(t,T)\right)\mathrm{d}t \\
& + \sqrt{v(t)}\mathrm{d}W_\xi^{\mathbb{Q}}(t) - \eta_d B_d(t,T)\mathrm{d}W_d^{\mathbb{Q}}(t) + \eta_f B_f(t,T)\mathrm{d}W_f^{\mathbb{Q}}(t).
\end{aligned} \tag{21}$$

Since $\mathrm{FX}^T(t)$ is a martingale under the T-forward domestic measure, that is, $P_d(t,T)\mathbb{E}^T(\mathrm{FX}^T(T)|\mathcal{F}(t)) = P_d(t,T)\mathrm{FX}^T(t) =: P_f(t,T)\xi(t)$, the appropriate Brownian motions under the T-forward domestic measure, $\mathrm{d}W_\xi^T(t)$, $\mathrm{d}W_v^T(t)$, $\mathrm{d}W_d^T(t)$ and $\mathrm{d}W_f^T(t)$, need to be determined.

A change of measure from domestic-spot to domestic T-forward measure requires a change of numéraire from money savings account, $M_d(t)$, to ZCB $P_d(t,T)$. In the model we incorporate a full matrix of correlations, which implies that all processes

will change their dynamics by changing the measure from spot to forward. Lemma 2.2 provides the model dynamics under the domestic T-forward measure, $\mathbb{Q}^T$.

Lemma 2.2. (The FX-HHW model dynamics under the $\mathbb{Q}^T$ measure.) Under the T-forward domestic measure, the model in Equation (11) is governed by the following dynamics:

$$\frac{\mathrm{d}\mathrm{FX}^T(t)}{\mathrm{FX}^T(t)} = \sqrt{v(t)}\mathrm{d}W_\xi^T(t) - \eta_d B_d(t,T)\mathrm{d}W_d^T(t) + \eta_f B_f(t,T)\mathrm{d}W_f^T(t), \tag{22}$$

where

$$\mathrm{d}v(t) = \Big(\kappa(\bar{v} - v(t)) + \gamma\rho_{v,d}\eta_d B_d(t,T)\sqrt{v(t)}\Big)\,\mathrm{d}t + \gamma\sqrt{v(t)}\mathrm{d}W_v^T(t), \tag{23}$$

$$\mathrm{d}r_d(t) = \big(\lambda_d(\theta_d(t) - r_d(t)) + \eta_d^2 B_d(t,T)\big)\,\mathrm{d}t + \eta_d\mathrm{d}W_d^T(t), \tag{24}$$

$$\mathrm{d}r_f(t) = \Big(\lambda_f(\theta_f(t) - r_f(t)) - \eta_f\rho_{\xi,f}\sqrt{v(t)} + \eta_d\eta_f\rho_{d,f}B_d(t,T)\Big)\,\mathrm{d}t + \eta_f\mathrm{d}W_f^T(t), \tag{25}$$

with a full matrix of correlations given in Equation (12), and with $B_d(t,T)$, $B_f(t,T)$ given by Equation (10).

The proof can be found in Appendix A.

From the system in Lemma 2.2 we see that after moving from the domestic-spot $\mathbb{Q}$ measure to the domestic T-forward $\mathbb{Q}^T$ measure, the forward exchange rate $\mathrm{FX}^T(t)$ does not depend explicitly on the short-rate processes $r_d(t)$ or $r_f(t)$. It does not contain a drift term and only depends on $\mathrm{d}W_d^T(t)$, $\mathrm{d}W_f^T(t)$ (see Equation (22)).

Remark 1. Since the sum of three correlated, normally distributed random variables, $Q = X + Y + Z$, remains normal with the mean equal to the sum of the individual means and the variance equal to

$$v_Q^2 = v_X^2 + v_Y^2 + v_Z^2 + 2\rho_{X,Y}v_Xv_Y + 2\rho_{X,Z}v_Xv_Z + 2\rho_{Y,Z}v_Yv_Z,$$

we can represent the forward Equation (22) as

$$\frac{\mathrm{d}\mathrm{FX}^T}{\mathrm{FX}^T} = \Big(v + \eta_d^2B_d^2 + \eta_f^2B_f^2 - 2\rho_{\xi,d}\eta_dB_d\sqrt{v} + 2\rho_{\xi,f}\eta_fB_f\sqrt{v} - 2\rho_{d,f}\eta_d\eta_fB_dB_f\Big)^{\frac{1}{2}}\mathrm{d}W_F^T. \tag{26}$$

Although the representation in Equation (26) reduces the number of Brownian motions in the dynamics for the FX, one still needs to find the appropriate cross-terms, like $\mathrm{d}W_F^T(t)\mathrm{d}W_v^T(t)$, in order to obtain the covariance terms. For clarity we therefore prefer to stay with the standard notation.

Remark 2. The dynamics of the forwards, $\mathrm{FX}^T(t)$ in Equation (22) or in Equation (26), do not depend explicitly on the interest rate processes, $r_d(t)$ and $r_f(t)$, and are completely described by the appropriate diffusion coefficients. This suggests that the short-rate variables will not enter the pricing PDE. Note that this is only the case for models in which the diffusion coefficient for the interest rate does not depend on the level of the interest rate.

In Section 2.3, we derive the corresponding pricing PDE and provide model approximations.

2.3. Approximations and the Forward Characteristic Function

As the dynamics of the forward FX, $\mathrm{FX}^T(t)$, under the domestic forward measure involve only the interest rate diffusions $\mathrm{d}W_d^T(t)$ and $\mathrm{d}W_f^T(t)$, a significant reduction of the pricing problem is achieved.

In order to find the forward ChF we take, as usual, the log-transform of the forward rate $\mathrm{FX}^T(t)$, that is, $x^T(t) := \log \mathrm{FX}^T(t)$, for which we obtain the following dynamics:

$$\mathrm{d}x^T(t) = (\zeta(t, \sqrt{v(t)}) - \frac{1}{2}v(t))\mathrm{d}t + \sqrt{v(t)}\mathrm{d}W_\xi^T(t) - \eta_d B_d \mathrm{d}W_d^T(t) + \eta_f B_f \mathrm{d}W_f^T(t), \quad (27)$$

with the variance process, $v(t)$, given by

$$\mathrm{d}v(t) = \left(\kappa(\bar{v} - v(t)) + \gamma\rho_{v,d}\eta_d B_d\sqrt{v(t)}\right)\mathrm{d}t + \gamma\sqrt{v(t)}\mathrm{d}W_v^T(t), \quad (28)$$

where we used the notation $B_d := B_d(t, T)$ and $B_f := B_f(t, T)$, and

$$\zeta(t, \sqrt{v(t)}) = \left(\rho_{x,d}\eta_d B_d - \rho_{x,f}\eta_f B_f\right)\sqrt{v(t)} + \rho_{d,f}\eta_d\eta_f B_d B_f - \frac{1}{2}\left(\eta_d^2 B_d^2 + \eta_f^2 B_f^2\right).$$

By applying the Feynman–Kac theorem, we can obtain the characteristic function of the forward FX rate dynamics. The forward characteristic function

$$\phi^T := \phi^T(u, \mathbf{X}(t), t, T) = \mathbb{E}^T\left(\mathrm{e}^{iux^T(T)}\middle|\mathcal{F}(t)\right),$$

with final condition, $\phi^T(u, \mathbf{X}(T), T, T) = \mathrm{e}^{iux^T(T)}$, is the solution of the following Kolmogorov backward PDE:

$$-\frac{\partial\phi^T}{\partial t} = \left(\kappa(\bar{v} - v) + \rho_{v,d}\gamma\eta_d\sqrt{v}B_d(t, T)\right)\frac{\partial\phi^T}{\partial v} + \left(\frac{1}{2}v - \zeta(t, \sqrt{v})\right)\left(\frac{\partial^2\phi^T}{\partial x^2} - \frac{\partial\phi^T}{\partial x}\right)$$

$$+ \left(\rho_{x,v}\gamma v - \rho_{v,d}\gamma\eta_d\sqrt{v}B_d(t, T) + \rho_{v,f}\gamma\eta_f\sqrt{v}B_f(t, T)\right)\frac{\partial^2\phi^T}{\partial x\partial v} + \frac{1}{2}\gamma^2 v\frac{\partial^2\phi^T}{\partial v^2}.$$

This PDE contains, however, non-affine $\sqrt{v}$-terms so that it is non-trivial to find the solution. Recently, in Grzelak and Oosterlee (2011), we have proposed two methods for linearization of these non-affine[1] square roots of the square root process (Cox *et al.,*

1985). The first method is to project the non-affine square root terms on their first moments. This is also the approach followed here.[2]

The approximation of the non-affine terms in the corresponding PDE is then done as follows: We assume

$$\sqrt{v(t)} \approx \mathbb{E}\left(\sqrt{v(t)}\right) =: \varphi(t), \tag{29}$$

with the expectation of the square root of $v(t)$ given by

$$\mathbb{E}\left(\sqrt{v(t)}\right) = \sqrt{2c(t)}\mathrm{e}^{\frac{-\omega(t)}{2}} \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{\varepsilon(t)}{2}\right)^k \frac{\Gamma\left(\frac{1+\ell}{2}+k\right)}{\Gamma(\frac{\ell}{2}+k)}, \tag{30}$$

and

$$c(t) = \frac{1}{4\kappa}\gamma^2(1-\mathrm{e}^{-\kappa t}), \qquad \ell = \frac{4\kappa\bar{v}}{\gamma^2}, \qquad \varepsilon(t) = \frac{4\kappa v(0)\mathrm{e}^{-\kappa t}}{\gamma^2(1-\mathrm{e}^{-\kappa t})}. \tag{31}$$

$\Gamma(k)$ is the gamma function defined by

$$\Gamma(k) = \int_0^{\infty} t^{k-1}\mathrm{e}^{-t}\mathrm{d}t.$$

Although the expectation in Equation (30) is a closed form expression, its evaluation is rather expensive. One may prefer to use a proxy, for example,

$$\mathbb{E}(\sqrt{v(t)}) \approx \beta_1 + \beta_2\mathrm{e}^{-\beta_3 t}, \tag{32}$$

in which the constant coefficients β_1, β_2 and β_3 can be determined by asymptotic equality with Equation (30) (see Grzelak and Oosterlee (2011) for details).

Projection of the non-affine terms on their first moments allows us to derive the corresponding forward characteristic function, ϕ^T, which is then of the following form:

$$\phi^T(u, X(t), t, T) = \exp(A(u,\tau) + B(u,\tau)x^T(t) + C(u,\tau)v(t)),$$

where $\tau = T - t$, and the functions $A(\tau) := A(u,\tau)$, $B(\tau) := B(u,\tau)$ and $C(\tau) := C(u,\tau)$ are given by

$$\begin{aligned}
B'(\tau) &= 0,\\
C'(\tau) &= -\kappa C(\tau) + \frac{B^2(\tau) - B(\tau)}{2} + \rho_{x,v}\gamma B(\tau)C(\tau) + \frac{\gamma^2 C^2(\tau)}{2},\\
A'(\tau) &= \kappa\bar{v}C(\tau) + \rho_{v,d}\gamma\eta_d\varphi(\tau)B_d(\tau)C(\tau) - \zeta(\tau,\varphi(\tau))\left(B^2(\tau) - B(\tau)\right)\\
&\quad + \left(-\rho_{v,d}\eta_d\gamma\varphi(\tau)B_d(\tau) + \rho_{v,f}\gamma\eta_f\varphi(\tau)B_f(\tau)\right)B(\tau)C(\tau),
\end{aligned}$$

with $\varphi(t) = \mathbb{E}(\sqrt{v(t)})$ and $B_i(\tau) = \lambda_i^{-1}\left(\mathrm{e}^{-\lambda_i\tau} - 1\right)$ for $i = \{d, f\}$. The initial conditions are $B(0) = iu$, $C(0) = 0$ and $A(0) = 0$.

With $B(\tau) = iu$, the complex-valued function $C(\tau)$ is of the Heston-type (Heston, 1993), and its solution reads

$$C(\tau) = \frac{1 - \mathrm{e}^{-d\tau}}{\gamma^2(1 - g\mathrm{e}^{-d\tau})}\left(\kappa - \rho_{x,v}\gamma iu - d\right), \tag{33}$$

with $d = \sqrt{(\rho_{x,v}\gamma iu - \kappa)^2 - \gamma^2 iu(iu-1)}$, $g = \frac{\kappa - \gamma\rho_{x,v}iu - d}{\kappa - \gamma\rho_{x,v}iu + d}$.

The parameters κ, γ, $\rho_{x,v}$ are given in Equation (11).

Function $A(\tau)$ is given by

$$A(\tau) = \int_0^\tau (\kappa\bar{v} + \rho_{v,d}\gamma\eta_d\varphi(s)B_d(s) - \rho_{v,d}\eta_d\gamma\varphi(s)B_d(s)iu$$

$$+\rho_{v,f}\gamma\eta_f\varphi(s)B_f(s)iu)C(s)\mathrm{d}s + (u^2 + iu)\int_0^\tau \zeta(s, \varphi(s))\mathrm{d}s, \tag{34}$$

with $C(s)$ in Equation (33). It is most convenient to solve $A(\tau)$ *numerically* with, for example, Simpson's quadrature rule. With correlations $\rho_{v,d}$, $\rho_{v,f}$ equal to zero, a closed form expression for $A(\tau)$ would be available (Grzelak and Oosterlee, 2011).

We denote the *approximation*, by means of linearization, of the full-scale FX-HHW model by FX-HHW1. It is clear that efficient pricing with Fourier-based methods can be done with FX-HHW1, and not with FX-HHW.

By the projection of $\sqrt{v(t)}$ on its first moment in Equation (29), the corresponding PDE is affine in its coefficients, and reads

$$-\frac{\partial\phi^T}{\partial t} = (\kappa(\bar{v} - v) + \Psi_1)\frac{\partial\phi^T}{\partial v} + \left(\frac{1}{2}v - \zeta(t, \varphi(t))\right)\left(\frac{\partial^2\phi^T}{\partial x^2} - \frac{\partial\phi^T}{\partial x}\right)$$

$$+\left(\rho_{x,v}\gamma v - \Psi_2\right)\frac{\partial^2\phi^T}{\partial x\partial v} + \frac{1}{2}\gamma^2 v\frac{\partial^2\phi^T}{\partial v^2}, \tag{35}$$

with

$$\phi^T(u, X(T), T, T) = \mathbb{E}^T\left(\mathrm{e}^{iux^T(T)}\middle|\mathcal{F}(T)\right) = \mathrm{e}^{iux^T(T)},$$

and $\zeta(t, \varphi(t)) = \Psi_3 + \rho_{d,f}\eta_d\eta_f B_d(t, T)B_f(t, T) - \frac{1}{2}\left(\eta_d^2 B_d^2(t, T) + \eta_f^2 B_f^2(t, T)\right)$.

The three terms, Ψ_1, Ψ_2 and Ψ_3, in the PDE (Equation (35)) contain the function $\varphi(t)$:

$$\Psi_1 := \rho_{v,d}\gamma\eta_d B_d(t, T)\varphi(t),$$

$$\Psi_2 := \left(\rho_{v,d}\gamma\eta_d B_d(t, T) - \rho_{v,f}\gamma\eta_f B_f(t, T)\right)\varphi(t),$$

$$\Psi_3 := \left(\rho_{x,d}\eta_d B_d(t,T) - \rho_{x,f}\eta_f B_f(t,T)\right)\varphi(t).$$

When solving the pricing PDE for $t \to T$, the terms $B_d(t,T)$ and $B_f(t,T)$ tend to zero, and all terms that contain the approximation vanish. The case $t \to 0$ is furthermore trivial, since $\sqrt{v(t)} \overset{t\to 0}{\to} \mathbb{E}(\sqrt{v(0)})$.

Under the T-forward domestic FX measure, the projection of the non-affine terms on their first moments is expected to provide high accuracy. In Section 2.5 we perform a numerical experiment to validate this.

It is worth mentioning that also an alternative approximation for the non-affine terms $\sqrt{v(t)}$ is available (see Grzelak and Oosterlee, 2011). This alternative approach guarantees that the first two moments are exact. In this article we stay, however, with the first representation.

2.4. Pricing a Foreign Stock in the FX-HHW Model

Here, we focus our attention on pricing a foreign stock, $S_f(t)$, in a domestic market. With this extension we can in principle price equity–FX–interest rate hybrid products.

With an equity smile/skew present in the market, we assume that $S_f(t)$ is given by the Heston SV model:

$$\begin{aligned} \frac{\mathrm{d}S_f(t)}{S_f(t)} &= r_f(t)\mathrm{d}t + \sqrt{\omega(t)}\mathrm{d}W^{\mathbb{Z}}_{S_f}(t), \\ \mathrm{d}\omega(t) &= \kappa_f(\bar{\omega} - \omega(t))\mathrm{d}t + \gamma_f\sqrt{\omega(t)}\mathrm{d}W^{\mathbb{Z}}_{\omega}(t), \\ \mathrm{d}r_f(t) &= \lambda_f(\theta_f(t) - r_f(t))\mathrm{d}t + \eta_f\mathrm{d}W^{\mathbb{Z}}_{f}(t), \end{aligned} \tag{36}$$

where $\mathbb{Z}$ indicates the foreign-spot measure and the model parameters, κ_f, γ_f, λ_f, $\theta_f(t)$ and η_f, are as before.

Before deriving the stock dynamics in domestic currency, the model has to be calibrated in the foreign market to plain vanilla options. This can be efficiently done with the help of a fast pricing formula.

With the foreign short-rate process, $r_f(t)$, established in Equation (11), we need to determine the drifts for $S_f(t)$ and its variance process, $\omega(t)$, under the domestic-spot measure. The foreign stock, $S_f(t)$, can be expressed in domestic currency by multiplication with the FX, $\xi(t)$, and by discounting with the domestic money savings account, $M_d(t)$. Such a stock is a tradable asset, so the price $\xi(t)S_f(t)/M_d(t)$ (with $\xi(t)$ in Equation (11), $S_f(t)$ from Equation (36) and the domestic money-saving account $M_d(t)$ in Equation (4)) needs to be a martingale.

By applying Itô's lemma to $\xi(t)S_f(t)/M_d(t)$, we find

$$\frac{\mathrm{d}\left(\xi(t)\frac{S_f(t)}{M_d(t)}\right)}{\xi(t)\frac{S_f(t)}{M_d(t)}} = \rho_{\xi,S_f}\sqrt{v(t)}\sqrt{\omega(t)}\mathrm{d}t + \sqrt{v(t)}\mathrm{d}W^{\mathbb{Q}}_{\xi}(t) + \sqrt{\omega(t)}\mathrm{d}W^{\mathbb{Z}}_{S_f}, \tag{37}$$

where $\mathbb{Q}$ and $\mathbb{Z}$ indicate the domestic-spot and foreign-spot measures, respectively. To make process $\xi(t)S_f(t)/M_d(t)$ a martingale we set

$$\mathrm{d}W^{\mathbb{Z}}_{S_f}(t) = \mathrm{d}W^{\mathbb{Q}}_{S_f} - \rho_{\xi,S_f}\sqrt{v(t)}\mathrm{d}t,$$

where $v(t)$ is the variance process of FX defined in Equation (11).

Under the change of measure, from foreign- to domestic-spot, $S_f(t)$ has the following dynamics:

$$\begin{aligned}\frac{\mathrm{d}S_f(t)}{S_f(t)} &= r_f(t)\mathrm{d}t + \sqrt{\omega(t)}\mathrm{d}W^{\mathbb{Z}}_{S_f}(t)\\ &= \left(r_f(t) - \rho_{\xi,S_f}\sqrt{v(t)}\sqrt{\omega(t)}\right)\mathrm{d}t + \sqrt{\omega(t)}\mathrm{d}W^{\mathbb{Q}}_{S_f}(t). \end{aligned} \tag{38}$$

The variance process is correlated with the stock and by the Cholesky decomposition we find

$$\begin{aligned}\mathrm{d}\omega(t) &= \kappa_f(\bar{\omega} - \omega(t))\mathrm{d}t + \gamma_f\sqrt{\omega(t)}\left(\rho_{S_f,\omega}\mathrm{d}\widetilde{W}^{\mathbb{Z}}_{S_f}(t) + \sqrt{1-\rho^2_{S_f,\omega}}\mathrm{d}\widetilde{W}^{\mathbb{Z}}_{\omega}(t)\right)\\ &= \left(\kappa_f(\bar{\omega} - \omega(t)) - \rho_{S_f,\omega}\rho_{S_f,\xi}\gamma_f\sqrt{\omega(t)}\sqrt{v(t)}\right)\mathrm{d}t + \gamma_f\sqrt{\omega(t)}\mathrm{d}W^{\mathbb{Q}}_{\omega}(t). \end{aligned} \tag{39}$$

$S_f(t)$ in Equation (38) and $\omega(t)$ in Equation (39) are governed by several non-affine terms. Assuming that the foreign stock, $S_f(t)$, is already calibrated to market data, we only need to simulate the foreign stock dynamics in the domestic market. Monte Carlo simulation of the foreign stock under domestic measure can be done as, for example, presented in Andersen (2007). The outstanding property of Andersen's QE Monte Carlo scheme is that the Heston model can be accurately simulated when the Feller condition is satisfied as well as when this condition is violated.

2.5. Numerical Experiment for the FX-HHW Model

In this section we check the errors resulting from the various approximations of the FX-HHW1 model. We use the set-up from Piterbarg (2006), which means that the interest rate curves are modelled by ZCBs defined by $P_d(t=0,T) = \exp(-0.02T)$ and $P_f(t=0,T) = \exp(-0.05T)$. Furthermore,

$$\eta_d = 0.7\%, \quad \eta_f = 1.2\%, \quad \lambda_d = 1\%, \quad \lambda_f = 5\%.$$

We choose[3]

$$\kappa = 0.5, \quad \gamma = 0.3, \quad \bar{v} = 0.1, \quad v(0) = 0.1. \tag{40}$$

The correlation structure, defined in Equation (12), is given by

$$\begin{pmatrix} 1 & \rho_{\xi,v} & \rho_{\xi,d} & \rho_{\xi,f} \\ \rho_{\xi,v} & 1 & \rho_{v,d} & \rho_{v,f} \\ \rho_{\xi,d} & \rho_{v,d} & 1 & \rho_{d,f} \\ \rho_{\xi,f} & \rho_{v,f} & \rho_{d,f} & 1 \end{pmatrix} = \begin{pmatrix} 100\% & -40\% & -15\% & -15\% \\ -40\% & 100\% & 30\% & 30\% \\ -15\% & 30\% & 100\% & 25\% \\ -15\% & 30\% & 25\% & 100\% \end{pmatrix}. \tag{41}$$

The initial spot FX rate (Dollar, \$, per Euro, €) is set to 1.35. For the FX-HHW model, we compute a number of FX option prices with many expiries and strikes, using two different pricing methods.

The first method is the plain Monte Carlo method, with 50.000 paths and $20T_i$ steps, for the full-scale FX-HHW model, without any approximations.

For the second pricing method, we have used the ChF, based on the approximations in the FX-HHW1 model in Section 2.3. Efficient pricing of plain vanilla products is then done by means of the COS method (Fang and Oosterlee, 2008), based on a Fourier cosine series expansion of the probability density function, which is recovered by the ChF with 500 Fourier cosine terms.

We also define the experiments as in Piterbarg (2006), with expiries given by $T_1, \ldots, T_{10}$, and the strikes are computed by the formula

$$K_n(T_i) = \mathrm{FX}^{T_i}(0) \exp\left(0.1\delta_n\sqrt{T_i}\right), \tag{42}$$

with

$$\delta_n = \{-1.5,\ -1.0,\ -0.5,\ 0,\ 0.5,\ 1.0,\ 1.5\},$$

and $\mathrm{FX}^{T_i}(0)$ as in Equation (18) with $\xi(0) = 1.35$. This formula for the strikes is convenient, since for $n = 4$, strikes $K_4(T_i)$ with $i = 1, \ldots, 10$ are equal to the forward FX rates for time T_i. The strikes and maturities are presented in Table B1 in Appendix B.

The option prices resulting from both models are expressed in terms of the implied Black volatilities. The differences between the volatilities are tabulated in Table 1. The approximation FX-HHW1 appears to be highly accurate for the parameters considered.

We report a maximum error of about 0.1% volatility for at-the-money options with a maturity of 30 years and less than 0.07% for the other options.

Given that the exchange rate is defined in units of domestic currency per unit of foreign currency, it is also plausible to assume positive correlation between the exchange

Table 1. Differences, in implied volatilities, between the FX-HHW and FX-HHW1 models.

T_i	$K_1(T_i)$ (%)	$K_2(T_i)$ (%)	$K_3(T_i)$ (%)	$K_4(T_i)$ (%)	$K_5(T_i)$ (%)	$K_6(T_i)$ (%)	$K_7(T_i)$ (%)
6m	−0.03	−0.02	0.00	0.02	0.03	0.04	0.05
1y	−0.01	−0.01	−0.01	−0.01	−0.01	−0.01	−0.01
3y	0.05	0.04	0.02	−0.01	−0.03	−0.06	−0.09
5y	0.06	0.04	0.02	0.00	−0.03	−0.07	−0.10
7y	0.08	0.06	0.04	0.03	0.01	−0.01	−0.03
10y	−0.02	−0.03	−0.03	−0.05	−0.07	−0.09	−0.12
15y	−0.12	−0.10	−0.09	−0.09	−0.09	−0.09	−0.10
20y	0.09	0.09	0.09	0.08	0.08	0.07	0.06
25y	−0.15	−0.11	−0.08	−0.06	−0.05	−0.04	−0.04
30y	0.10	0.11	0.12	0.12	0.12	0.12	0.12

Notes: The corresponding FX option prices and the standard deviations are tabulated in Table B5. Strike $K_4(T_i)$ is the at-the-money strike.

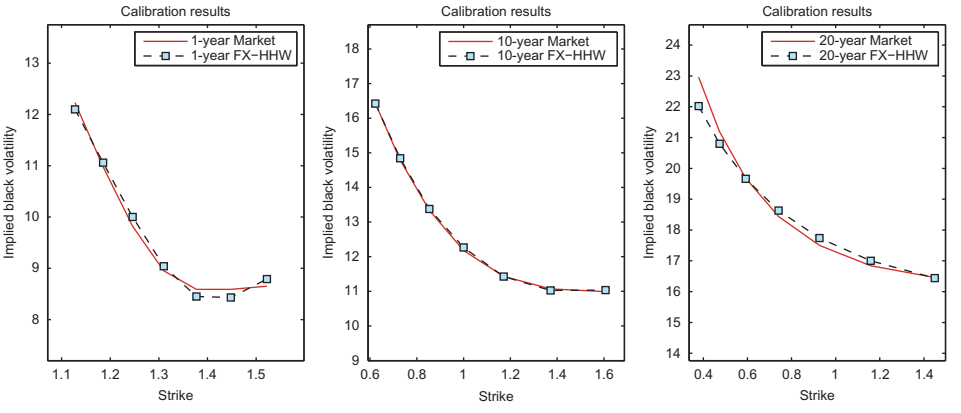

Figure 1. Comparison of implied volatilities from the market and the FX-HHW1 model for FX European call options for maturities of 1, 10 and 20 years. The strikes are provided in Table B1 in Appendix B ($\xi(0) = 1.35$).

rate and foreign interest rates. Similar results in terms of accuracy are obtained when the correlations $\rho_{\xi,d}$ and $\rho_{\xi,f}$ are positive (see Table B3 in Appendix B).

In Subsection 2.5.1, the calibration results to FX market data are presented.

2.5.1 Calibration to Market Data. We discuss the calibration of the FX-HHW model to FX market data. In the simulation the reference market-implied volatilities are taken from Piterbarg (2006) and are presented in Table B2 in Appendix B. In the calibration routine the approximate model FX-HHW1 was applied. The correlation structure is as in Equation (41). In Figure 1 some of the calibration results are presented.

Our experiments show that the model can be well calibrated to the market data. For long maturities and for deep-in-the-money options, some discrepancy is present. This is however typical when dealing with the Heston model (not related to our approximation), since the skew/smile pattern in FX does not flatten for long maturities. This was sometimes improved by adding jumps to the model (Bates' model). In Appendix B in Table B4 the detailed calibration results are tabulated.

Short-rate interest rate models can typically provide a satisfactory fit to at-the-money interest rate products. They are therefore not used for pricing derivatives that are sensitive to the interest rate skew. This is a drawback of the short-rate interest rate models. In Section 3, an extension of the framework, so that interest rate smiles and skews can be modelled as well, is presented.

3. Multi-Currency Model with Interest Rate Smile

In this section, we discuss a second extension of the multi-currency model, in which an interest rate smile is incorporated. This hybrid model models two types of smiles, the smile for the FX rate and the smiles in the domestic and foreign fixed income markets. We abbreviate the model by FX-HLMM. It is especially interesting for FX products

that are exposed to interest rate smiles. A description of such FX hybrid products can be found in the handbook by Hunter (Hunter and Picot 2005).

A first attempt to model the FX by SV and interest rates driven by a market model was proposed in Takahashi and Takehara (2008), assuming independence between log-normal-Libor rates and FX. In our approach we define a model with *non-zero correlation* between FX and interest rate processes.

As in the previous sections, the SV FX is of the Heston type, which under domestic risk-neutral measure, $\mathbb{Q}$, follows the following dynamics:

$$\begin{array}{rcl} \frac{\mathrm{d}\xi(t)}{\xi(t)} &=& (\ldots)\mathrm{d}t \quad + \quad \sqrt{v(t)}\mathrm{d}W_{\xi}^{\mathbb{Q}}(t), \qquad S(0)>0, \\ \mathrm{d}v(t) &=& \kappa(\bar{v}-v(t))\mathrm{d}t \quad + \quad \gamma\sqrt{v(t)}\mathrm{d}W_{v}^{\mathbb{Q}}(t), \qquad v(0)>0, \end{array} \tag{43}$$

with the parameters as in Equation (11). Since we consider the model under the forward measure, the drift in the first SDE does not need to be specified (the dynamics of domestic-forward FX $\xi(t)P_f(t,T)/P_d(t,T)$ do not contain a drift term).

In the model we assume that the domestic and foreign currencies are independently calibrated to interest rate products available in their own markets. For simplicity, we also assume that the tenor structure for both currencies is the same, that is, $\mathcal{T}_d \equiv \mathcal{T}_f = \{T_0, T_1, \ldots, T_N \equiv T\}$ and $\tau_k = T_k - T_{k-1}$ for $k = 1, \ldots, N$. For $t < T_{k-1}$ we define the forward Libor rates $L_{d,k}(t) := L_d(t, T_{k-1}, T_k)$ and $L_{f,k}(t) := L_f(t, T_{k-1}, T_k)$ as

$$L_{d,k}(t) := \frac{1}{\tau_k}\left(\frac{P_d(t,T_{k-1})}{P_d(t,T_k)}-1\right), \quad L_{f,k}(t) := \frac{1}{\tau_k}\left(\frac{P_f(t,T_{k-1})}{P_f(t,T_k)}-1\right). \tag{44}$$

For each currency we choose the SV-DD Libor Market Model from Andersen and Andreasen (2002) for the interest rates, under the T-forward measure generated by the numéraires $P_d(t,T)$ and $P_f(t,T)$, given by

$$\begin{array}{rcl} \mathrm{d}L_{d,k}(t) &=& v_{d,k}\phi_{d,k}(t)\sqrt{v_d(t)}\left(\mu_d(t)\sqrt{v_d(t)}\mathrm{d}t + \mathrm{d}W_k^{d,T}(t)\right), \\ \mathrm{d}v_d(t) &=& \lambda_d(v_d(0)-v_d(t))\mathrm{d}t + \eta_d\sqrt{v_d(t)}\mathrm{d}W_v^{d,T}(t), \end{array} \tag{45}$$

and

$$\begin{array}{rcl} \mathrm{d}L_{f,k}(t) &=& v_{f,k}\phi_{f,k}(t)\sqrt{v_f(t)}\left(\mu_f(t)\sqrt{v_f(t)}\mathrm{d}t + \mathrm{d}\widehat{W}_k^{f,T}(t)\right), \\ \mathrm{d}v_f(t) &=& \lambda_f(v_f(0)-v_f(t))\mathrm{d}t + \eta_f\sqrt{v_f(t)}\mathrm{d}\widehat{W}_v^{f,T}(t), \end{array} \tag{46}$$

with

$$\mu_d(t) = -\sum_{j=k+1}^{N}\frac{\tau_j\phi_{d,j}(t)v_{d,j}}{1+\tau_j L_{d,j}(t)}\rho_{k,j}^{d}, \qquad \mu_f(t) = -\sum_{j=k+1}^{N}\frac{\tau_j\phi_{f,j}(t)v_{f,j}}{1+\tau_j L_{f,j}(t)}\rho_{k,j}^{f}, \tag{47}$$

where

$$\phi_{d,k} = \beta_{d,k} L_{d,k}(t) + (1 - \beta_{d,k}) L_{d,k}(0),$$

$$\phi_{f,k} = \beta_{f,k} L_{f,k}(t) + (1 - \beta_{f,k}) L_{f,k}(0).$$

The Brownian motion, $\mathrm{d}W_k^{d,T}$, corresponds to the kth domestic Libor rate, $L_{d,k}(t)$, under the T-forward domestic measure; and the Brownian motion, $\mathrm{d}\widehat{W}_k^{f,T}$, relates to the kth foreign market Libor rate, $L_{f,k}(t)$, under the terminal foreign measure T.

In the model, $v_{d,k}(t)$ and $v_{f,k}(t)$ determine the level of the interest rate volatility smile; the parameters $\beta_{d,k}(t)$ and $\beta_{f,k}(t)$ control the slope of the volatility smile; and λ_d, λ_f determine the speed of mean-reversion for the variance and influence the speed at which the interest rate volatility smile flattens as the swaption expiry increases (Piterbarg, 2005). Parameters η_d, η_f determine the curvature of the interest rate smile.

The following correlation structure is imposed[4], between

$$\begin{array}{lrcl}
\text{FX and its variance process, } v(t): & \mathrm{d}W_\xi^T(t)\mathrm{d}W_v^T(t) & = & \rho_{\xi,v}\mathrm{d}t, \\
\text{FX and domestic Libors, } L_{d,j}(t): & \mathrm{d}W_\xi^T(t)\mathrm{d}W_j^{d,T}(t) & = & \rho_{\xi,j}^d\mathrm{d}t, \\
\text{FX and foreign Libors, } L_{f,j}(t): & \mathrm{d}W_\xi^T(t)\mathrm{d}\widehat{W}_j^{f,T}(t) & = & \rho_{\xi,j}^f\mathrm{d}t, \\
\text{Libors in domestic market:} & \mathrm{d}W_k^{d,T}(t)\mathrm{d}W_j^{d,T}(t) & = & \rho_{k,j}^d\mathrm{d}t, \\
\text{Libors in foreign market:} & \mathrm{d}\widehat{W}_k^{f,T}(t)\mathrm{d}\widehat{W}_j^{f,T}(t) & = & \rho_{k,j}^f\mathrm{d}t, \\
\text{Libors in domestic and foreign markets:} & \mathrm{d}W_k^{d,T}(t)\mathrm{d}\widehat{W}_j^{f,T}(t) & = & \rho_{k,j}^{d,f}\mathrm{d}t.
\end{array} \tag{48}$$

We prescribe a zero correlation between the remaining processes, that is, between

Libors and their variance process,

$$\mathrm{d}W_k^{d,T}(t)\mathrm{d}W_v^{d,T}(t) = 0, \quad \mathrm{d}\widehat{W}_k^{f,T}(t)\mathrm{d}\widehat{W}_v^{f,T}(t) = 0,$$

Libors and the FX variance process,

$$\mathrm{d}W_k^{d,T}(t)\mathrm{d}W_v^T(t) = 0, \quad \mathrm{d}\widehat{W}_k^{f,T}(t)\mathrm{d}W_v^T(t) = 0,$$

all variance processes,

$$\mathrm{d}W_v^T(t)\mathrm{d}W_v^{d,T}(t) = 0, \quad \mathrm{d}W_v^T(t)\mathrm{d}\widehat{W}_v^{f,T}(t) = 0, \quad \mathrm{d}W_v^{d,T}(t)\mathrm{d}\widehat{W}_v^{f,T}(t) = 0,$$

FX and the Libor variance processes,

$$\mathrm{d}W_\xi^T(t)\mathrm{d}W_v^{d,T}(t) = 0, \quad \mathrm{d}W_\xi^T(t)\mathrm{d}\widehat{W}_v^{f,T}(t) = 0.$$

The correlation structure is graphically displayed in Figure 2.

Throughout this article we assume that the SV-DD model in Equations (45) and (46) is already in the *effective* parameter framework as developed in Piterbarg (2005). This means that approximate time-homogeneous parameters are used instead of the time-dependent parameters, that is, $\beta_k(t) \equiv \beta_k$ and $v_k(t) \equiv v_k$.

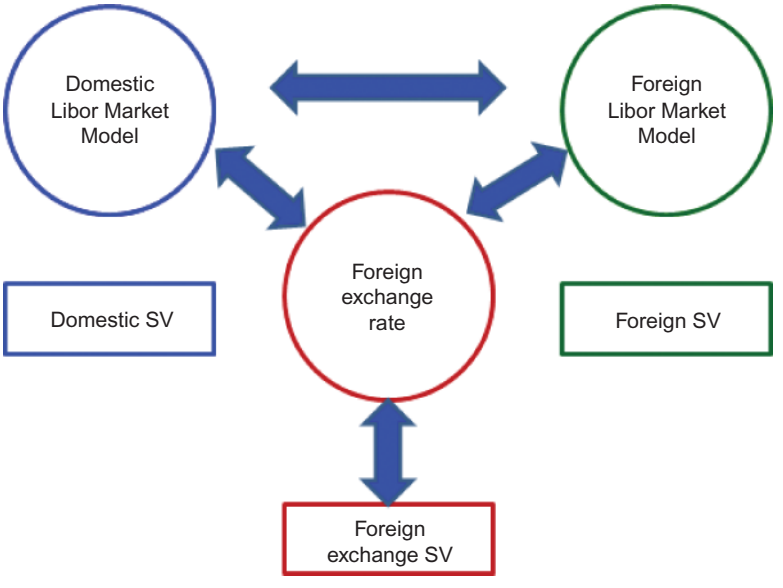

Figure 2. The correlation structure for the FX-HLMM model. Arrows indicate non-zero correlations. SV is Stochastic Volatility.

With this correlation structure, we derive the dynamics for the forward FX, given by

$$\mathrm{FX}^T(t) = \xi(t)\frac{P_f(t,T)}{P_d(t,T)}, \tag{49}$$

(see also Equation (18)) with $\xi(t)$ the spot exchange rate and $P_d(t,T)$ and $P_f(t,T)$ ZCB. Note that the bonds are not yet specified.

When deriving the dynamics for Equation (49), we need expressions for the ZCB, $P_d(t,T)$ and $P_f(t,T)$. With Equation (44), the following expression for the final bond can be obtained:

$$\frac{1}{P_i(t,T)} = \frac{1}{P_i(t,T_{m(t)})}\prod_{j=m(t)+1}^{N}\left(1+\tau_j L_{i,j}(t)\right), \quad \text{for } i=\{d,f\}, \tag{50}$$

with $T = T_N$ and $m(t) = \min(k : t \leq T_k)$ (empty products in Equation (50) are defined to be equal to 1). The bond $P_i(t,T_N)$ in Equation (50) is fully determined by the Libor rates $L_{i,k}(t)$, $k = 1,\ldots,N$ and the bond $P_i(t,T_{m(t)})$. Whereas the Libors $L_{i,k}(t)$ are defined by Equation (45) and (46), the bond $P_i(t,T_{m(t)})$ is not yet well defined in the current framework.

To define continuous time dynamics for a ZCB, *interpolation techniques* are available (see, e.g. Beveridge and Joshi, 2009; Davis and Mataix-Pastor, 2009; Piterbarg, 2004; Schlögl, 2002a). We consider here the linear interpolation scheme, proposed in Schlögl (2002a), which reads

$$\frac{1}{P_i(t,T_{m(t)})} = 1 + (T_{m(t)} - t)L_{i,m(t)}(T_{m(t)-1}), \text{for } T_{m(t)-1} < t < T_{m(t)}. \tag{51}$$

In our previous work (Grzelak and Oosterlee, 2010), this basic interpolation technique was very satisfactory for the calibration. By combining Equation (51) with Equation (50), we find for the domestic and foreign bonds:

$$\frac{1}{P_d(t,T)} = \left(1 + (T_{m(t)} - t)L_{d,m(t)}(T_{m(t)-1})\right) \prod_{j=m(t)+1}^{N} \left(1 + \tau_j L_d(t, T_{j-1}, T_j)\right),$$

$$\frac{1}{P_f(t,T)} = \left(1 + (T_{m(t)} - t)L_{f,m(t)}(T_{m(t)-1})\right) \prod_{j=m(t)+1}^{N} \left(1 + \tau_j L_f(t, T_{j-1}, T_j)\right).$$

When deriving the dynamics for $\mathrm{FX}^T(t)$ in Equation (49) we will not encounter any $\mathrm{d}t$-terms (as $\mathrm{FX}^T(t)$ has to be a martingale under the numéraire $P_d(t,T)$).

For each ZCB, $P_d(t,T)$ or $P_f(t,T)$, the dynamics are determined under the appropriate T-forward measures (for $P_d(t,T)$ the domestic T-forward measure, and for $P_f(t,T)$ the foreign T-forward measure). The dynamics for the ZCB, driven by the Libor dynamics in Equations (45) and (46), are given by

$$\frac{\mathrm{d}P_d(t,T)}{P_d(t,T)} = (\ldots)\mathrm{d}t - \sqrt{v_d(t)} \sum_{j=m(t)+1}^{N} \frac{\tau_j v_{d,j} \phi_{d,j}(t)}{1 + \tau_j L_{d,j}(t)} \mathrm{d}W_j^{d,T}(t), \tag{52}$$

$$\frac{\mathrm{d}P_f(t,T)}{P_f(t,T)} = (\ldots)\mathrm{d}t - \sqrt{v_f(t)} \sum_{j=m(t)+1}^{N} \frac{\tau_j v_{f,j} \phi_{f,j}(t)}{1 + \tau_j L_{f,j}(t)} \mathrm{d}\widehat{W}_j^{f,T}(t), \tag{53}$$

and the coefficients were defined in Equations (45) and (46).

By changing the numéraire from $P_f(t,T)$ to $P_d(t,T)$ for the foreign bond, only the drift terms will change. Since $\mathrm{FX}^T(t)$ in Equation (49) is a martingale under the $P_d(t,T)$ measure, it is *not* necessary to determine the appropriate drift correction.

By taking Equation (20) for the general dynamics of Equation (49) and neglecting all the $\mathrm{d}t$-terms, we get

$$\frac{\mathrm{d}\,\mathrm{FX}^T(t)}{\mathrm{FX}^T(t)} = \sqrt{v(t)}\mathrm{d}W_\xi^T(t) + \sqrt{v_d(t)} \sum_{j=m(t)+1}^{N} \frac{\tau_j v_{d,j} \phi_{d,j}(t)}{1 + \tau_j L_{d,j}(t)} \mathrm{d}W_j^{d,T}(t)$$

$$- \sqrt{v_f(t)} \sum_{j=m(t)+1}^{N} \frac{\tau_j v_{f,j} \phi_{f,j}(t)}{1 + \tau_j L_{f,j}(t)} \mathrm{d}W_j^{f,T}(t). \tag{54}$$

Note that the *hat* in $\widehat{W}$ disappeared from the Brownian motion $\mathrm{d}W_j^{f,T}(t)$ in Equation (54), which is an indication for the change of measure from the foreign to the domestic measure for the foreign Libors.

Since the SV process, $v(t)$, for FX is independent of the domestic and foreign Libors, $L_{d,k}(t)$ and $L_{f,k}(t)$, the dynamics under the $P_d(t,T)$-measure do not change[5] and are given by

$$\mathrm{d}v(t) = \kappa(\bar{v} - v(t))\mathrm{d}t + \gamma\sqrt{v(t)}\mathrm{d}W_v^T(t). \tag{55}$$

The model given in Equation (54) with the stochastic variance in Equation (55) and the correlations between the main underlying processes is not affine. In Section 3.1 we discuss a linearization.

3.1. Linearization and Forward Characteristic Function

The model in Equation (54) is not of the affine form, as it contains terms like $\phi_{i,j}(t)/(1+\tau_{i,j}L_{i,j}(t))$ with $\phi_{i,j} = \beta_{i,j}L_{i,j}(t) + (1-\beta_{i,j})L_{i,j}(0)$ for $i = \{d,f\}$. In order to derive a characteristic function, we *freeze* the Libor rates, which is standard practice (see, e.g. Glasserman and Zhao, 1999; Hull and White, 2000; Jäckel and Rebonato, 2000), that is,

$$\begin{aligned} L_{d,j}(t) \approx L_{d,j}(0) \quad &\Rightarrow \quad \phi_{d,j} \equiv L_{d,j}(0), \\ L_{f,j}(t) \approx L_{f,j}(0) \quad &\Rightarrow \quad \phi_{f,j} \equiv L_{f,j}(0). \end{aligned} \tag{56}$$

This approximation gives the following $\mathrm{FX}^T(t)$-dynamics:

$$\begin{cases} \dfrac{\mathrm{d}\,\mathrm{FX}^T(t)}{\mathrm{FX}^T(t)} &\approx \sqrt{v(t)}\mathrm{d}W_\xi^T(t) + \sqrt{v_d(t)}\displaystyle\sum_{j\in\mathcal{A}}\psi_{d,j}\mathrm{d}W_j^{d,T}(t) - \sqrt{v_f(t)}\sum_{j\in\mathcal{A}}\psi_{f,j}\mathrm{d}W_j^{f,T}(t), \\ \mathrm{d}v(t) &= \kappa(\bar{v} - v(t))\mathrm{d}t + \gamma\sqrt{v(t)}\mathrm{d}W_v^T(t), \\ \mathrm{d}v_i(t) &= \lambda_i(v_i(0) - v_i(t))\mathrm{d}t + \eta_i\sqrt{v_i(t)}\mathrm{d}W_v^{i,T}(t), \end{cases}$$

with $i = \{d,f\}$, $\mathcal{A} = \{m(t)+1,\ldots,N\}$, the correlations are given in Equation (48) and

$$\psi_{d,j} := \frac{\tau_j v_{d,j} L_{d,j}(0)}{1+\tau_j L_{d,j}(0)}, \quad \psi_{f,j} := \frac{\tau_j v_{f,j} L_{f,j}(0)}{1+\tau_j L_{f,j}(0)}. \tag{57}$$

We derive the dynamics for the log-transform of $\mathrm{FX}^T(t)$, $x^T(t) = \log \mathrm{FX}^T(t)$, for which we need to calculate the square of the diffusion coefficients.[6]

With the notation,

$$a := \sqrt{v(t)}\mathrm{d}W_{\xi}^{T}(t), \quad b := \sqrt{v_d(t)}\sum_{j\in\mathcal{A}}\psi_{d,j}\mathrm{d}W_j^{d,T}(t), \quad c := \sqrt{v_f(t)}\sum_{j\in\mathcal{A}}\psi_{f,j}\mathrm{d}W_j^{f,T}(t), \tag{58}$$

we find for the square diffusion coefficient $(a+b-c)^2 = a^2+b^2+c^2+2ab-2ac-2bc$. So, the dynamics for the log-forward, $x^T(t) = \log \mathrm{FX}^T(t)$, can be expressed as

$$\begin{aligned}\mathrm{d}x^T(t) \approx &-\frac{1}{2}(a+b-c)^2 + \sqrt{v(t)}\mathrm{d}W_{\xi}^{T}(t) + \sqrt{v_d(t)}\sum_{\mathcal{A}}\psi_{d,j}\mathrm{d}W_j^{d,T}(t)\\ &-\sqrt{v_f(t)}\sum_{\mathcal{A}}\psi_{f,j}\mathrm{d}W_j^{f,T}(t),\end{aligned} \tag{59}$$

with the coefficients a, b and c given in Equation (58). Since

$$\left((\sum_{j=1}^{N}x_j)^2\right) = \sum_{j=1}^{N}x_j^2 + \sum_{\substack{i,j=1,\ldots,N\\ i\neq j}}x_ix_j, \quad \text{for } N>0,$$

we find

$$a^2 = v(t)\mathrm{d}t,$$

$$b^2 = v_d(t)\left(\sum_{j\in\mathcal{A}}\psi_{d,j}^2 + \sum_{\substack{i,j\in\mathcal{A}\\ i\neq j}}\psi_{d,i}\psi_{d,j}\rho_{i,j}^{d}\right)\mathrm{d}t =: v_d(t)A_d(t)\mathrm{d}t, \tag{60}$$

$$c^2 = v_f(t)\left(\sum_{j\in\mathcal{A}}\psi_{f,j}^2 + \sum_{\substack{i,j\in\mathcal{A}\\ i\neq j}}\psi_{f,i}\psi_{f,j}\rho_{i,j}^{f}\right)\mathrm{d}t =: v_f(t)A_f(t)\mathrm{d}t, \tag{61}$$

$$ab = \sqrt{v(t)}\sqrt{v_d(t)}\sum_{j\in\mathcal{A}}\psi_{d,j}\rho_{j,x}^{d}\mathrm{d}t,$$

$$ac = \sqrt{v(t)}\sqrt{v_f(t)}\sum_{j\in\mathcal{A}}\psi_{f,j}\rho_{j,x}^{f}\mathrm{d}t,$$

$$bc = \sqrt{v_d(t)}\sqrt{v_f(t)}\sum_{j\in\mathcal{A}}\psi_{d,j}\sum_{k\in\mathcal{A}}\psi_{f,k}\rho_{j,k}^{d,f}\mathrm{d}t,$$

with $\rho^d_{j,x}$, $\rho^f_{j,x}$ the correlation between the FX and jth domestic and foreign Libor, respectively. The correlation between the kth domestic and jth foreign Libor is $\rho^{d,f}_{k,j}$.

By setting $f(t,\sqrt{v(t)},\sqrt{v_d(t)},\sqrt{v_f(t)}) := (2ab-2ac-2bc)/\mathrm{d}t$, we can express the dynamics for $\mathrm{d}x^T(t)$ in Equation (59) by

$$\begin{aligned}\mathrm{d}x^T(t) \approx &-\frac{1}{2}\left(v(t)+A_d(t)v_d(t)+A_f(t)v_f(t)+f(t,\sqrt{v(t)},\sqrt{v_d(t)},\sqrt{v_f(t)})\right)\mathrm{d}t\\ &+\sqrt{v(t)}\mathrm{d}W^T_\xi(t)+\sqrt{v_d(t)}\sum_{\mathcal{A}}\psi_{d,j}\mathrm{d}W^{d,T}_j(t)-\sqrt{v_f(t)}\sum_{\mathcal{A}}\psi_{f,j}\mathrm{d}W^{f,T}_j(t).\end{aligned}$$

The coefficients $\psi_{d,j}$, $\psi_{f,j}$, A_d and A_f in Equations (57), (60) and (61) are deterministic and piecewise constant.

In order to make the model affine, we *linearize* the non-affine terms in the drift in $f(t,\sqrt{v(t)},\sqrt{v_d(t)},\sqrt{v_f(t)})$ by a projection on the first moments, that is,

$$f(t,\sqrt{v(t)},\sqrt{v_d(t)},\sqrt{v_f(t)}) \approx f(t,\mathbb{E}(\sqrt{v(t)}),\mathbb{E}(\sqrt{v_d(t)}),\mathbb{E}(\sqrt{v_f(t)})) =: f(t). \tag{62}$$

The variance processes $v(t)$, $v_d(t)$ and $v_f(t)$ are independent CIR-type processes (Cox *et al.*, 1985), so the expectation of their products equals the product of the expectations. Function $f(t)$ can be determined with the help of the formula in Equation (30).

The approximation in Equation (62) linearizes all non-affine terms in the corresponding PDE. As before, the forward characteristic function, $\phi^T := \phi^T(u,\mathbf{X}(t),t,T)$, is defined as the solution of the following backward PDE:

$$\begin{aligned}0 = &\frac{\partial\phi^T}{\partial t}+\frac{1}{2}\left(v+A_d(t)v_d+A_f(t)v_f+f(t)\right)\left(\frac{\partial^2\phi^T}{\partial x^2}-\frac{\partial\phi^T}{\partial x}\right)\\ &+\lambda_d(v_d(0)-v_d)\frac{\partial\phi^T}{\partial v_d}+\lambda_f(v_f(0)-v_f)\frac{\partial\phi^T}{\partial v_f}+\kappa(\bar{v}-v)\frac{\partial\phi^T}{\partial v}\\ &+\frac{1}{2}\eta_d^2 v_d\frac{\partial^2\phi^T}{\partial v_d^2}+\frac{1}{2}\eta_f^2 v_f\frac{\partial^2\phi^T}{\partial v_f^2}+\frac{1}{2}\gamma^2 v\frac{\partial^2\phi^T}{\partial v^2}+\rho_{x,v}\gamma v\frac{\partial^2\phi^T}{\partial x\partial v},\end{aligned} \tag{63}$$

with the final condition $\phi^T(u,\mathbf{X}(T),T)=\mathrm{e}^{iux^T(T)}$. Since all coefficients in this PDE are linear, the solution is of the following form:

$$\begin{aligned}\phi^T(u,\mathbf{X}(t),t,T) = &\exp(A(u,\tau)+B(u,\tau)x^T(t)+C(u,\tau)v(t)\\ &+D_d(u,\tau)v_d(t)+D_f(u,\tau)v_f(t)),\end{aligned} \tag{64}$$

with $\tau := T-t$. Substitution of Equation (64) in Equation (63) gives us the following system of ordinary differential equations (ODEs) for the functions $A(\tau) := A(u,\tau)$, $B(\tau) := B(u,\tau)$, $C(\tau) := C(u,\tau)$, $D_d(\tau) := D_d(u,\tau)$ and $D_f(\tau) := D_f(u,\tau)$:

$$
\begin{aligned}
A'(\tau) &= f(t)\frac{B^2(\tau)-B(\tau)}{2} + \lambda_d v_d(0) D_1(\tau) + \lambda_f v_f(0) D_2(\tau) + \kappa\bar{v}C(\tau),\\
B'(\tau) &= 0,\\
C'(\tau) &= \frac{B^2(\tau)-B(\tau)}{2} + \left(\rho_{x,v}\gamma B(\tau) - \kappa\right) C(\tau) + \frac{\gamma^2 C^2(\tau)}{2},\\
D_d'(\tau) &= A_d(t)\frac{B^2(\tau)-B(\tau)}{2} - \lambda_d D_d(\tau) + \frac{\eta_d^2 D_d^2(\tau)}{2},\\
D_f'(\tau) &= A_f(t)\frac{B^2(\tau)-B(\tau)}{2} - \lambda_f D_f(\tau) + \frac{\eta_f^2 D_f^2(\tau)}{2},
\end{aligned}
$$

with initial conditions $A(0) = 0$, $B(0) = iu$, $C(0) = 0$, $D_d(0) = 0$, $D_f(0) = 0$ with $A_d(t)$ and $A_f(t)$ from Equations (60) and (61), respectively, and $f(t)$ as in Equation (62).

With $B(\tau) = iu$, the solution for $C(\tau)$ is analogous to the solution for the ODE for the FX-HHW1 model in Equation (33). As the remaining ODEs involve the piecewise constant functions $A_d(t)$, $A_f(t)$, the solution must be determined iteratively, like for the pure Heston model with piecewise constant parameters in Andersen and Andreasen (2000). For a given grid $0 = \tau_0 < \tau_1 < \ldots < \tau_N = \tau$, the functions $D_d(u,\tau)$, $D_f(u,\tau)$ and $A(u,\tau)$ can be expressed as

$$
\begin{aligned}
D_d(u,\tau_j) &= D_d(u,\tau_{j-1}) + \chi_d(u,\tau_j),\\
D_f(u,\tau_j) &= D_f(u,\tau_{j-1}) + \chi_f(u,\tau_j),
\end{aligned}
$$

for $j = 1,\ldots,N$ and

$$
A(u,\tau_j) = A(u,\tau_{j-1}) + \chi_A(u,\tau_j) \; - \; \frac{1}{2}(u^2+u)\int_{\tau_{j-1}}^{\tau_j} f(s)\mathrm{d}s,
$$

with $f(s)$ in Equation (62) and analytically known functions $\chi_k(u,\tau_j)$, for $k = \{d,f\}$ and $\chi_A(u,\tau_j)$:

$$
\chi_k(u,\tau_j) := \frac{\left(\lambda_k - \delta_{k,j} - \eta_k^2 D_k(u,\tau_{j-1})\right)\left(1-\mathrm{e}^{-\delta_{k,j}s_j}\right)}{\left(\eta_k^2\left(1-\ell_{k,j}\mathrm{e}^{-\delta_{k,j}s_j}\right)\right)},
$$

and

$$
\begin{aligned}
\chi_A(u,\tau_j) &= \frac{\kappa\bar{v}}{\gamma^2}\left((\kappa - \rho_{x,v}\gamma iu - d_j)s_j - 2\log\left(\frac{1-g_j\mathrm{e}^{-d_j s_j}}{1-g_j}\right)\right)\\
&\quad + v_d(0)\frac{\lambda_d}{\eta_d^2}\left((\lambda_d - \delta_{d,j})s_j - 2\log\left(\frac{1-\ell_{d,j}\mathrm{e}^{-\delta_{d,j}s_j}}{1-\ell_{d,j}}\right)\right)\\
&\quad + v_f(0)\frac{\lambda_f}{\eta_f^2}\left((\lambda_f - \delta_{f,j})s_j - 2\log\left(\frac{1-\ell_{f,j}\mathrm{e}^{-\delta_{f,j}s_j}}{1-\ell_{f,j}}\right)\right),
\end{aligned}
$$

where

$$d_j = \sqrt{(\rho_{x,v}\gamma iu-\kappa)^2+\gamma^2(iu+u^2)}, \qquad g_j = \frac{(\kappa-\rho_{x,v}\gamma iu)-d_j-\gamma^2 C(u,\tau_{j-1})}{(\kappa-\rho_{x,v}\gamma iu)+d_j-\gamma^2 C(u,\tau_{j-1})},$$
$$\delta_{k,j} = \sqrt{\lambda_k^2+\eta_k^2 A_k(t)(u^2+iu)}, \qquad \ell_{k,j} = \frac{\lambda_k-\delta_{k,j}-\eta_k^2 D_k(u,\tau_{j-1})}{\lambda_k+\delta_{k,j}-\eta_k^2 D_k(u,\tau_{j-1})},$$

with $s_j=\tau_j-\tau_{j-1}, j=1,\ldots,N$, $A_d(t)$ and $A_f(t)$ are from Equations (60) and (61).

The resulting approximation of the full-scale FX-HLMM model is called FX-LMM1 here.

3.2. Foreign Stock in the FX-HLMM Framework

We also consider a foreign stock, $S_f(t)$, driven by the Heston SV model, with the interest rates driven by the market model. The stochastic processes of the stock model are assumed to be of the same form as the FX (with one, foreign, interest rate curve) with the dynamics, under the forward foreign measure, given by

$$\frac{\mathrm{d}S_f^T(t)}{S_f^T(t)} = \sqrt{\omega(t)}\mathrm{d}W_{S_f}^{f,T}(t)+\sqrt{v_f(t)}\sum_{j=m(t)+1}^{N}\frac{\tau_j v_{f,j}\phi_{f,j}(t)}{1+\tau_j L_{f,j}(t)}\mathrm{d}W_j^{f,T}(t),$$
$$\mathrm{d}\omega(t) = \kappa_f(\bar{\omega}-\omega(t))\mathrm{d}t+\gamma_f\sqrt{\omega(t)}\mathrm{d}W_\omega^{f,T}(t). \tag{65}$$

Variance process, $\omega(t)$, is correlated with forward stock $S^T(t)$.

We move to the domestic-forward measure. The forward stock, S_f^T, and forward FX rate, $\mathrm{FX}^T(t)$, are defined by

$$S_f^T(t)=\frac{S_f(t)}{P_f(t,T)}, \qquad \mathrm{FX}^T(t)=\xi(t)\frac{P_f(t,T)}{P_d(t,T)}. \tag{66}$$

The quantity,

$$S_f^T(t)\mathrm{FX}^T(t)=\frac{S_f(t)}{P_f(t,T)}\xi(t)\frac{P_f(t,T)}{P_d(t,T)}=\frac{S_f(t)}{P_d(t,T)}\xi(t), \tag{67}$$

is therefore a tradable asset. So, foreign stock exchanged by an FX rate and denominated in the domestic ZCB is a tradable quantity, which implies that $S_f^T(t)\mathrm{FX}^T(t)$ is a martingale. By Itô's lemma, one finds

$$\frac{\mathrm{d}\left(S_f^T(t)\mathrm{FX}^T(t)\right)}{S_f^T(t)\mathrm{FX}^T(t)}=\frac{\mathrm{d}\mathrm{FX}^T(t)}{\mathrm{FX}^T(t)}+\frac{\mathrm{d}S_f^T(t)}{S_f^T(t)}+\left(\frac{\mathrm{d}\mathrm{FX}^T(t)}{\mathrm{FX}^T(t)}\right)\left(\frac{\mathrm{d}S_f^T(t)}{S_f^T(t)}\right). \tag{68}$$

The two first terms at the RHS of Equation (68) do not contribute to the drift. The last term involves all $\mathrm{d}t$-terms, that, by a change of measure, will enter the drift of the variance process $\mathrm{d}\omega(t)$ in Equation (65).

3.3. Numerical Experiments with the FX-HLMM Model

We here focus on the FX-HLMM model covered in Section 3 and consider the errors generated by the various approximations that led to the model FX-HLMM1. We have performed basically two linearization steps to define FX-HLMM1: We have frozen the Libors at their initial values and projected the non-affine covariance terms on a deterministic function. We check, by a numerical experiment, the size of the errors of these approximations.

We have chosen the following interest rate curves $P_d(t=0,T)=\exp(-0.02T)$, $P_f(t=0,T)=\exp(-0.05T)$, and, as before, for the FX SV model we set:

$$\kappa = 0.5, \quad \gamma = 0.3, \quad \bar{v} = 0.1, \quad v(0) = 0.1. \tag{69}$$

In the simulation, we have chosen the following parameters for the domestic and foreign markets:

$$\beta_{d,k} = 95\%, \quad v_{d,k} = 15\%, \quad \lambda_d = 100\%, \quad \eta_d = 10\%, \tag{70}$$

$$\beta_{f,k} = 50\%, \quad v_{f,k} = 25\%, \quad \lambda_f = 70\%, \quad \eta_f = 20\%. \tag{71}$$

In the correlation matrix, a number of correlations need to be specified. For the correlations between the Libor rates in each market, we prescribe large positive values, as frequently observed in fixed income markets (see, e.g. Brigo and Mercurio, 2007), $\rho^d_{i,j} = 90\%$, $\rho^f_{i,j} = 70\%$, for $i,j = 1,\ldots,N$ $(i \neq j)$. In order to generate skew for FX, we prescribe a *negative* correlation between $\mathrm{FX}^T(t)$ and its SV process, $v(t)$, that is, $\rho_{\xi,v} = -40\%$. The correlation between the FX and the domestic Libors is set as $\rho^d_{\xi,k} = -15\%$, for $k = 1,\ldots,N$, and the correlation between FX and the foreign Libors is $\rho^f_{\xi,k} = -15\%$. The correlation between the domestic and foreign Libors is $\rho^{d,f}_{i,j} = 25\%$ for $i,j = 1,\ldots,N$ $(i \neq j)$. The following block correlation matrix results

$$\mathbf{C} = \begin{bmatrix} \mathbf{C}_{\mathrm{d}} & \mathbf{C}_{\mathrm{d,f}} & \mathbf{C}_{\xi,\mathrm{d}} \\ \mathbf{C}^{\mathrm{T}}_{\mathrm{d,f}} & \mathbf{C}_{\mathrm{f}} & \mathbf{C}_{\xi,\mathrm{f}} \\ \mathbf{C}^{\mathrm{T}}_{\xi,\mathrm{d}} & \mathbf{C}^{\mathrm{T}}_{\xi,\mathrm{f}} & 1 \end{bmatrix}, \tag{72}$$

with the domestic Libor correlations given by

$$\mathbf{C_d} = \begin{bmatrix} 1 & \rho^d_{1,2} & \cdots & \rho^d_{1,N} \\ \rho^d_{1,2} & 1 & \cdots & \rho^d_{2,N} \\ \vdots & \vdots & \ddots & \vdots \\ \rho^d_{1,N} & \rho^d_{2,N} & \cdots & 1 \end{bmatrix} = \begin{bmatrix} 1 & 90\% & \cdots & 90\% \\ 90\% & 1 & \cdots & 90\% \\ \vdots & \vdots & \ddots & \vdots \\ 90\% & 90\% & \cdots & 1 \end{bmatrix}_{N\times N}, \tag{73}$$

the foreign Libors correlations given by

$$\mathbf{C_f} = \begin{bmatrix} 1 & \rho^f_{1,2} & \dots & \rho^f_{1,N} \\ \rho^f_{1,2} & 1 & \dots & \rho^f_{2,N} \\ \vdots & \vdots & \ddots & \vdots \\ \rho^f_{1,N} & \rho^f_{2,N} & \dots & 1 \end{bmatrix} = \begin{bmatrix} 1 & 70\% & \dots & 70\% \\ 70\% & 1 & \dots & 70\% \\ \vdots & \vdots & \ddots & \vdots \\ 70\% & 70\% & \dots & 1 \end{bmatrix}_{N\times N}, \tag{74}$$

the correlation between Libors from the domestic and foreign markets given by

$$\mathbf{C_{d,f}} = \begin{bmatrix} 1 & \rho^{d,f}_{1,2} & \dots & \rho^{d,f}_{1,N} \\ \rho^{d,f}_{1,2} & 1 & \dots & \rho^{d,f}_{2,N} \\ \vdots & \vdots & \ddots & \vdots \\ \rho^{d,f}_{1,N} & \rho^{d,f}_{2,N} & \dots & 1 \end{bmatrix} = \begin{bmatrix} 1 & 25\% & \dots & 25\% \\ 25\% & 1 & \dots & 25\% \\ \vdots & \vdots & \ddots & \vdots \\ 25\% & 25\% & \dots & 1 \end{bmatrix}_{N\times N}, \tag{75}$$

and the vectors $\mathbf{C_{\xi,d}}$ and $\mathbf{C_{\xi,f}}$ as used in Piterbarg (2006) are given by

$$\mathbf{C_{\xi,d}} = \begin{bmatrix} \rho^d_{\xi,1} \\ \rho^d_{\xi,2} \\ \vdots \\ \rho^d_{\xi,N} \end{bmatrix} = \begin{bmatrix} -15\% \\ -15\% \\ \vdots \\ -15\% \end{bmatrix}_{N\times 1}, \mathbf{C_{\xi,f}} = \begin{bmatrix} \rho^f_{\xi,1} \\ \rho^f_{\xi,2} \\ \vdots \\ \rho^f_{\xi,N} \end{bmatrix} = \begin{bmatrix} -15\% \\ -15\% \\ \vdots \\ -15\% \end{bmatrix}_{N\times 1}. \tag{76}$$

Since in both markets the Libor rates are assumed to be independent of their variance processes, we can neglect these correlations here.

Now we find the prices of plain vanilla options on FX in Equation (49). The simulation is performed in the same spirit as in Section 2.5 where the FX-HHW model was considered. In Table 2 we present the differences, in terms of the implied volatilities between the models FX-HLMM and FX-HLMM1. While the prices for the FX-HLMM were obtained by Monte Carlo simulation (20.000 paths and 20 intermediate

Table 2. Differences, in implied Black volatilities, between the FX-HLMM and FX-LMM1 models.

T_i	$K_1(T_i)$ (%)	$K_2(T_i)$ (%)	$K_3(T_i)$ (%)	$K_4(T_i)$ (%)	$K_5(T_i)$ (%)	$K_6(T_i)$ (%)	$K_7(T_i)$ (%)
2y	0.19	0.14	0.09	0.05	0.00	−0.05	−0.10
3y	0.29	0.25	0.21	0.16	0.11	0.06	0.02
5y	0.32	0.28	0.23	0.17	0.10	0.05	0.00
7y	0.30	0.28	0.25	0.21	0.18	0.14	0.10
10y	0.39	0.32	0.25	0.18	0.12	0.05	−0.03
15y	0.38	0.29	0.21	0.13	0.05	−0.04	−0.14
20y	0.02	−0.09	−0.18	−0.27	−0.34	−0.40	−0.44
25y	0.08	0.04	−0.14	−0.25	−0.34	−0.40	−0.46
30y	0.11	0.07	0.00	−0.09	−0.18	−0.21	−0.24

Notes: The corresponding strikes $K_1(T_i), \dots, K_7(T_i)$ are tabulated in Table B1. The prices and associated standard deviations are presented in Table B6.

points between the dates T_{i-1} and T_i for $i = 1, \ldots, N$), the prices for FX-HLMM1 were obtained by the Fourier-based COS method (Fang and Oosterlee, 2008) with 500 Fourier series terms.

The FX-HLMM1 model performs very well, as the maximum difference in terms of implied volatilities is about 0.2−0.5%.

3.3.1 Sensitivity to the Interest Rate Skew. Approximation FX-HLMM1 was based on freezing the Libor rates. By freezing the Libors, that is, $L_{d,k}(t) \equiv L_{d,k}(0)$ and $L_{f,k}(t) \equiv L_{f,k}(0)$, we have

$$\phi_{d,k}(t) = \beta_{d,k} L_{d,k}(t) + (1 - \beta_{d,k}) L_{d,k}(0) = L_{d,k}(0), \tag{77}$$

$$\phi_{f,k}(t) = \beta_{f,k} L_{f,k}(t) + (1 - \beta_{f,k}) L_{f,k}(0) = L_{f,k}(0). \tag{78}$$

In the SV-DD models for the Libor rates $L_{d,k}(t)$ and $L_{f,k}(t)$ for any k, the parameters $\beta_{d,k}$, $\beta_{f,k}$ control the slope of the interest rate volatility smiles. Freezing the Libors to $L_{d,k}(0)$ and $L_{f,k}(0)$ is equivalent to setting $\beta_{d,k} = 0$ and $\beta_{f,k} = 0$ in Equations (77) and (78) in the approximation FX-HLMM1.

By a Monte Carlo simulation, we obtain the FX implied volatilities from the full-scale FX-HLMM model for different values of β and by comparing them to those from FX-HLMM1 with $\beta = 0$ we check the influence of the parameters $\beta_{d,k}$ and $\beta_{f,k}$ on the FX. In Table 3, the implied volatilities for the FX European call options for

Table 3. Implied volatilities of the FX options from the FX-HLMM and FX-HLMM1 models, $T = 10$ and parameters were as in Section 3.3.

	FX-HLMM (Monte Carlo simulation)					FX-HLMM1 (Fourier)
	$\beta_f = 0.5$			$\beta_d = 0.5$		
Strike Equation (42)	$\beta_d = 0$(%)	$\beta_d = 0.5$(%)	$\beta_d = 1$(%)	$\beta_f = 0$(%)	$\beta_f = 1$(%)	$\beta_d = 0$ $\beta_f = 0$(%)
0.6224	31.98	31.91	31.98	31.99	31.96	31.56
	(0.20)	(0.17)	(0.17)	(0.15)	(0.18)	
0.7290	31.49	31.43	31.48	31.51	31.46	31.12
	(0.21)	(0.16)	(0.19)	(0.15)	(0.18)	
0.8538	31.02	30.96	31.01	31.04	30.97	30.69
	(0.21)	(0.17)	(0.20)	(0.15)	(0.18)	
1.0001	30.58	30.53	30.56	30.61	30.52	30.30
	(0.21)	(0.17)	(0.22)	(0.15)	(0.17)	
1.1714	30.16	30.11	30.15	30.20	30.08	29.93
	(0.20)	(0.17)	(0.24)	(0.15)	(0.16)	
1.3721	29.77	29.73	29.77	29.82	29.68	29.60
	(0.22)	(0.16)	(0.26)	(0.16)	(0.17)	
1.6071	29.41	29.38	29.43	29.48	29.31	29.30
	(0.24)	(0.17)	(0.28)	(0.17)	(0.18)	

Note: The numbers in parentheses correspond to the standard deviations (the experiment was performed 20 times with $20T$ time steps).

FX-HLMM and FX-HLMM1 are presented. The experiments are performed for different combinations of the interest rate skew parameters, β_d and β_f.

The experiment indicates that there is only a small impact of the different $\beta_{d,k}$ – and $\beta_{f,k}$ – values on the FX implied volatilities, implying that the approximate model, FX-HLMM1 with $\beta_{d,k} = \beta_{f,k} = 0$, is useful for the interest rate modelling, for the parameters studied. With $\beta_{d,k} \neq 0$ and $\beta_{f,k} \neq 0$, the implied volatilities obtained by the FX-HLMM model appear to be somewhat higher than those obtained by FX-HLMM1, a difference of approximately $0.1-0.15\%$, which is considered highly satisfactory.

4. Conclusion

In this article, we have presented two FX models with SV and correlated stochastic interest rates. Both FX models were based on the Heston FX model and differed with respect to the interest rate processes.

In the first model we considered a model in which the domestic and foreign interest rates were driven by single factor Hull–White short-rate processes. This model enables the pricing of FX–interest rate hybrid products that are not exposed to the smile in the fixed income markets.

For hybrid products sensitive to the interest rate skew, a second model was presented in which the interest rates were driven by the SV Libor Market Model. For both hybrid models we have developed approximate models for the pricing of European options on the FX. These pricing formulas form the basis for highly efficient model calibration strategies.

The approximate models are based on the linearization of the non-affine terms in the corresponding pricing PDE, in a very similar way as in our previous article (Grzelak and Oosterlee, 2011) on equity–interest rate options. The approximate models perform very well in the world of FX. These models can also be used to obtain an initial guess when the full-scale models are used.

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Notes

[1]According to Duffie *et al.* (2000) the n-dimensional system of SDEs,

$$\mathrm{d}\mathbf{X}(t) = \mu\left(\mathbf{X}(t)\right)\mathrm{d}t + v(\mathbf{X}(t))\mathrm{d}\mathbf{W}(t),$$

is of the affline form if

$$\mu\left(\mathbf{X}(t)\right) = a_0 + a_1\mathbf{X}(t), \text{for any}(a_0, a_1) \in \mathbb{R}^n \times \mathbb{R}^{n\times n},$$

$$v(\mathbf{X}(t))v(\mathbf{X}(t))^{\mathrm{T}} = (c_0)_{ij} + (c_1)_{ij}^{\mathrm{T}}\mathbf{X}(t), \text{ for arbitary}(c_0, c_1) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n\times n},$$

$$r(\mathbf{X}(t)) = (r_0) + (r_1)_1^{\mathrm{T}}\mathbf{X}(t), \text{ for } (r_0, r_1) \in \mathbb{R} \times \mathbb{R}^n,$$

for $i, j = 1,\ldots,n$, with $r(\mathbf{X}(t))$ being an interest rate component.

[2]Since the moments of the square root process under the T-forward measure are difficult to determine for $\sqrt{v(t)}$, we have set $\rho_{v,d} = 0$ or, in other words, the expectation is calculated under measure $\mathbb{Q}$.

[3]The model parameters do not satisfy the Feller condition, $\gamma^2 > 2\kappa\bar{v}$.

[4]As it is insightful to relate the covariance matrix with the necessary model approximations, the correlation structure is introduced here by means of instantaneous correlation of the scalar diffusions.

[5]In Grzelak and Oosterlee (2010), the proof for this statement is given when a single yield curve is considered.

[6]As in the standard Black–Scholes analysis for $\mathrm{d}S(t) = vS(t)\mathrm{d}W(t)$, the log-transform gives $\mathrm{d}\log S(t) = -\frac{1}{2}v^2\mathrm{d}t + v\mathrm{d}W(t)$.

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Appendix A: Proof of Lemma 2.2

Since the domestic short-rate process, $r_d(t)$, is driven by one source of uncertainty (only one Brownian motion $\mathrm{d}W_d^{\mathbb{Q}}(t)$), it is convenient to change the order of the state variables, from $\mathrm{d}\mathbf{X}(t) = [\mathrm{d}\mathrm{FX}^T(t)/\mathrm{FX}^T(t), \mathrm{d}v(t), \mathrm{d}r_d(t), \mathrm{d}r_f(t)]^T$ to $\mathrm{d}\mathbf{X}^*(t) = [\mathrm{d}r_d(t), \mathrm{d}r_f(t), \mathrm{d}v(t), \mathrm{d}\mathrm{FX}^T(t)/\mathrm{FX}^T(t)]^T$, and express the model in terms of the independent Brownian motions $\mathrm{d}\widetilde{\mathbf{W}}^{\mathbb{Q}}(t) = [\mathrm{d}\widetilde{W}_d(t), \mathrm{d}\widetilde{W}_f(t), \mathrm{d}\widetilde{W}_v(t), \mathrm{d}\widetilde{W}_\xi(t)]^T$, that is,

$$\begin{bmatrix} \mathrm{d}r_d \\ \mathrm{d}r_f \\ \mathrm{d}v \\ \mathrm{d}\mathrm{FX}^T/\mathrm{FX}^T \end{bmatrix} = \mu(\mathbf{X}^*)\mathrm{d}t + \begin{bmatrix} \eta_d & 0 & 0 & 0 \\ 0 & \eta_f & 0 & 0 \\ 0 & 0 & \gamma\sqrt{v} & 0 \\ -\eta_d B_d & \eta_f B_f & 0 & \sqrt{v} \end{bmatrix} \mathbf{H} \begin{bmatrix} \mathrm{d}\widetilde{W}_d^{\mathbb{Q}} \\ \mathrm{d}\widetilde{W}_f^{\mathbb{Q}} \\ \mathrm{d}\widetilde{W}_v^{\mathbb{Q}} \\ \mathrm{d}\widetilde{W}_\xi^{\mathbb{Q}} \end{bmatrix}, \quad \text{(A1)}$$

which, equivalently, can be written as

$$\mathrm{d}\mathbf{X}^*(t) = \mu(\mathbf{X}^*)\mathrm{d}t + \mathbf{A}\mathbf{H}\mathrm{d}\widetilde{\mathbf{W}}^{\mathbb{Q}}(t), \quad \text{(A2)}$$

where $\mu(\mathbf{X}^*)$ represents the drift for system $\mathrm{d}\mathbf{X}^*(t)$ and $\mathbf{H}$ is the Cholesky lower-triangular matrix of the following form:

$$\mathbf{H} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \mathrm{H}_{2,1} & \mathrm{H}_{2,2} & 0 & 0 \\ \mathrm{H}_{3,1} & \mathrm{H}_{3,2} & \mathrm{H}_{3,3} & 0 \\ \mathrm{H}_{4,1} & \mathrm{H}_{4,2} & \mathrm{H}_{4,3} & \mathrm{H}_{4,4} \end{bmatrix} \triangleq \begin{bmatrix} 1 & 0 & 0 & 0 \\ \rho_{f,d} & \mathrm{H}_{2,2} & 0 & 0 \\ \rho_{v,d} & \mathrm{H}_{3,2} & \mathrm{H}_{3,3} & 0 \\ \rho_{\xi,d} & \mathrm{H}_{4,2} & \mathrm{H}_{4,3} & \mathrm{H}_{4,4} \end{bmatrix}. \tag{A3}$$

The representation presented above seems to be favourable, since the short-rate process $r_d(t)$ can be considered independent of the other processes.

The matrix model representation in terms of orthogonal Brownian motions results in the following dynamics for the domestic short rate $r_d(t)$ under measure $\mathbb{Q}$:

$$\mathrm{d}r_d(t) = \lambda_d(\theta_d(t) - r_d(t))\mathrm{d}t + \zeta_1(t)\mathrm{d}\widetilde{\mathbf{W}}^{\mathbb{Q}}(t),$$

and for the domestic ZCB:

$$\frac{\mathrm{d}P_d(t,T)}{P_d(t,T)} = r_d(t)\mathrm{d}t + B_d(t,T)\zeta_1(t)\mathrm{d}\widetilde{\mathbf{W}}^{\mathbb{Q}}(t),$$

with $\zeta_k(t)$ being the kth row vector resulting from multiplying the matrices $\mathbf{A}$ and $\mathbf{H}$. Note that for the 1D Hull–White short-rate processes $\zeta_1(t) = [\eta_d, 0, 0, 0]$.

Now, we derive the Radon–Nikodým derivative (Geman *et al.*, 1996):

$$\Lambda_{\mathbb{Q}}^{T}(t) = \frac{\mathrm{d}\mathbb{Q}^T}{\mathrm{d}\mathbb{Q}} = \frac{P_d(t,T)}{P_d(0,T)M_d(t)}. \tag{A4}$$

By calculating the Itô derivative of Equation (A4) we get

$$\frac{\mathrm{d}\Lambda_{\mathbb{Q}}^{T}}{\Lambda_{\mathbb{Q}}^{T}} = B_d(t,T)\zeta_1(t)\mathrm{d}\widetilde{\mathbf{W}}^{\mathbb{Q}}(t), \tag{A5}$$

which implies that the Girsanov kernel for the transition from $\mathbb{Q}$ to $\mathbb{Q}^T$ is given by $B_d(t,T)\zeta_1(t)$, which is the T-bond volatility given by $\eta_d B_d(t,T)$, that is,

$$\Lambda_{\mathbb{Q}}^{T} = \exp\left(-\frac{1}{2}\int_0^T B_r^2(s,T)\zeta_1^2(s)\mathrm{d}s + \int_0^T B_r(s,T)\zeta_1(s)\mathrm{d}\widetilde{\mathbf{W}}^{\mathbb{Q}}(s)\right). \tag{A6}$$

So,

$$\mathrm{d}\widetilde{\mathbf{W}}^{T}(t) = -B_d(t,T)\zeta_1^T(t)\mathrm{d}t + \mathrm{d}\widetilde{\mathbf{W}}^{\mathbb{Q}}(t).$$

Since the vector $\zeta_1^T(t)$ is of scalar form, the Brownian motion under the T-forward measure is given by

$$\mathrm{d}\widetilde{\mathbf{W}}^{\mathbb{Q}}(t) = \left[\mathrm{d}\widetilde{W}_d^T(t) + \eta_d B_d(t,T)\mathrm{d}t, \mathrm{d}\widetilde{W}_f^T(t), \mathrm{d}\widetilde{W}_v^T(t), \mathrm{d}\widetilde{W}_\xi^T(t)\right]^T.$$

Now, from the vector representation in Equation (A2) we get that

$$\mathbf{H}\mathrm{d}\widetilde{\mathbf{W}}^{\mathbb{Q}} = \begin{bmatrix} \eta_d B_d & + & \mathrm{d}\widetilde{W}_d^T\mathrm{d}t \\ \rho_{d,f}\eta_d B_d\mathrm{d}t & + & \rho_{d,f}\mathrm{d}\widetilde{W}_d^T + \mathrm{H}_{2,2}\mathrm{d}\widetilde{W}_f^T \\ \rho_{v,d}\eta_d B_d\mathrm{d}t & + & \rho_{v,d}\mathrm{d}\widetilde{W}_d^T + \mathrm{H}_{3,2}\mathrm{d}\widetilde{W}_f^T + \mathrm{H}_{3,3}\mathrm{d}\widetilde{W}_\xi^T \\ \rho_{\xi,d}\eta_d B_d\mathrm{d}t & + & \rho_{\xi,d}\mathrm{d}\widetilde{W}_d^T + \mathrm{H}_{4,2}\mathrm{d}\widetilde{W}_f^T + \mathrm{H}_{4,3}\mathrm{d}\widetilde{W}_\xi^T + \mathrm{H}_{4,4}\mathrm{d}\widetilde{W}_v^T \end{bmatrix}. \tag{A7}$$

Returning to the dependent Brownian motions under the T-forward measure, gives us

$$\frac{\mathrm{d}\mathrm{FX}^T(t)}{\mathrm{FX}^T(t)} = \sqrt{v(t)}\mathrm{d}W_\xi^T(t) - \eta_d B_d(t,T)\mathrm{d}W_d^T(t) + \eta_f B_f(t,T)\mathrm{d}W_f^T(t),$$

$$\mathrm{d}v(t) = \Big(\kappa(\bar{v} - v(t)) + \gamma\rho_{v,d}\eta_d B_d(t,T)\sqrt{v(t)}\Big)\,\mathrm{d}t + \gamma\sqrt{v(t)}\mathrm{d}W_v^T(t),$$

$$\mathrm{d}r_d(t) = \big(\lambda_d(\theta_d(t) - r_d(t)) + \eta_d^2 B_d(t,T)\big)\,\mathrm{d}t + \eta_d\mathrm{d}W_d^T(t),$$

$$\mathrm{d}r_f(t) = \Big(\lambda_f(\theta_f(t) - r_f(t)) - \eta_f\rho_{\xi,f}\sqrt{v(t)} + \eta_d\eta_f\rho_{d,f}B_d(t,T)\Big)\,\mathrm{d}t + \eta_f\mathrm{d}W_f^T(t),$$

with full matrix of correlations given in Equation (12).

Appendix B: Tables

In this appendix we present tables with details for the numerical experiments.

Table B1. Expiries and strikes of FX options used in the FX-HHW model.

T_i	$K_1(T_i)$	$K_2(T_i)$	$K_3(T_i)$	$K_4(T_i)$	$K_5(T_i)$	$K_6(T_i)$	$K_7(T_i)$
6m	1.1961	1.2391	1.2837	1.3299	1.3778	1.4273	1.4787
1y	1.1276	1.1854	1.2462	1.3101	1.3773	1.4479	1.5221
3y	0.9515	1.0376	1.1315	1.2338	1.3454	1.4671	1.5999
5y	0.8309	0.9291	1.0390	1.1620	1.2994	1.4531	1.6250
7y	0.7358	0.8399	0.9587	1.0943	1.2491	1.4257	1.6274
10y	0.6224	0.7290	0.8538	1.0001	1.1714	1.3721	1.6071
15y	0.4815	0.5844	0.7093	0.8608	1.0447	1.2680	1.5389
20y	0.3788	0.4737	0.5924	0.7409	0.9265	1.1587	1.4491
25y	0.3012	0.3868	0.4966	0.6377	0.8188	1.0514	1.3500
30y	0.2414	0.3174	0.4174	0.5489	0.7218	0.9492	1.2482

Note: The strikes $K_n(T_i)$ were calculated as given in Equation (42) with $\xi(0) = 1.35$.

Table B2. Market-implied Black volatilities for FX options as given in Piterbarg (2006).

T_i	$K_1(T_i)$ (%)	$K_2(T_i)$ (%)	$K_3(T_i)$ (%)	$K_4(T_i)$ (%)	$K_5(T_i)$ (%)	$K_6(T_i)$ (%)	$K_7(T_i)$ (%)
6m	11.41	10.49	9.66	9.02	8.72	8.66	8.68
1y	12.23	10.98	9.82	8.95	8.59	8.59	8.65
3y	12.94	11.35	9.89	8.78	8.34	8.36	8.46
5y	13.44	11.84	10.38	9.27	8.76	8.71	8.83
7y	14.29	12.68	11.23	10.12	9.52	9.37	9.43
10y	16.43	14.79	13.34	12.18	11.43	11.07	10.99
15y	20.93	19.13	17.56	16.27	15.29	14.65	14.29
20y	22.96	21.19	19.68	18.44	17.50	16.84	16.46
25y	23.97	22.31	20.92	19.80	18.95	18.37	18.02
30y	25.09	23.48	22.17	21.13	20.35	19.81	19.48

Note: The strikes $K_n(T_i)$ were tabulated in Table B1.

Table B3. Differences, in implied volatilities, between the FX-HHW and FX-HHW1 models.

T_i	$K_1(T_i)$ (%)	$K_2(T_i)$ (%)	$K_3(T_i)$ (%)	$K_4(T_i)$ (%)	$K_5(T_i)$ (%)	$K_6(T_i)$ (%)	$K_7(T_i)$ (%)
6m	0.00	0.02	0.04	0.06	0.07	0.08	0.09
1y	−0.05	−0.03	−0.02	−0.01	0.00	0.01	0.01
3y	0.04	0.04	0.03	0.02	0.02	0.01	0.01
5y	0.10	0.09	0.07	0.05	0.03	0.00	−0.02
7y	0.05	0.04	0.03	0.02	0.01	0.00	−0.03
10y	0.14	0.13	0.12	0.11	0.10	0.08	0.07
15y	−0.01	0.01	0.03	0.03	0.02	0.02	0.00
20y	−0.08	−0.04	−0.02	0.00	0.02	0.02	0.02
25y	−0.21	−0.14	−0.08	−0.05	−0.02	−0.01	0.00
30y	−0.13	−0.07	−0.03	0.01	0.03	0.05	0.06

Note: The parameters were chosen as in Section 2.5 except for the correlations: $\rho_{\xi,d} = \rho_{\xi,f} = +15\%$.

Table B4. The calibration results for the FX-HHW model, in terms of the differences between the market (given in Table B2) and FX-HHW model-implied volatilities.

T_i	$K_1(T_i)$ (%)	$K_2(T_i)$ (%)	$K_3(T_i)$ (%)	$K_4(T_i)$ (%)	$K_5(T_i)$ (%)	$K_6(T_i)$ (%)	$K_7(T_i)$ (%)
6m	0.12	−0.12	−0.25	−0.23	−0.01	0.20	0.22
1y	0.13	−0.08	−0.18	−0.09	0.14	0.16	−0.14
3y	0.16	−0.07	−0.17	−0.08	0.18	0.22	−0.14
5y	0.11	−0.06	−0.12	−0.07	0.10	0.13	−0.14
7y	0.07	−0.03	−0.06	−0.03	0.06	0.10	−0.08
10y	0.04	−0.01	−0.01	−0.02	0.02	0.05	−0.02
15y	0.11	−0.05	−0.09	−0.04	0.03	0.09	−0.05
20y	0.94	0.39	0.02	−0.19	−0.24	−0.16	0.02
25y	1.43	0.59	−0.02	−0.43	−0.63	−0.64	−0.51
30y	1.65	0.70	0.00	−0.48	−0.74	−0.82	−0.74

Note: Strikes $K_n(T_i)$ are given in Table B1.

Table B5. Average FX call option prices obtained by the FX-HHW model with 20 Monte Carlo simulations, 50.000 paths and $20 \times T_i$ steps.

T_i	Method	$K_1(T_i)$	$K_2(T_i)$	$K_3(T_i)$	$K_4(T_i)$	$K_5(T_i)$	$K_6(T_i)$	$K_7(T_i)$
6m	*MC*	0.1907	0.1636	0.1382	0.1148	0.0935	0.0748	0.0585
	SD	0.0004	0.0004	0.0005	0.0004	0.0004	0.0004	0.0004
	COS	0.1908	0.1637	0.1382	0.1147	0.0934	0.0746	0.0583
1y	*MC*	0.2566	0.2209	0.1870	0.1553	0.1264	0.1008	0.0785
	SD	0.0007	0.0007	0.0007	0.0007	0.0007	0.0007	0.0007
	COS	0.2567	0.2210	0.1870	0.1554	0.1265	0.1008	0.0786
3y	*MC*	0.3768	0.3281	0.2805	0.2349	0.1923	0.1538	0.1200
	SD	0.0014	0.0015	0.0015	0.0015	0.0015	0.0015	0.0014
	COS	0.3765	0.3279	0.2804	0.2349	0.1926	0.1543	0.1207
5y	*MC*	0.4216	0.3709	0.3205	0.2713	0.2246	0.1816	0.1432
	SD	0.0021	0.0021	0.0021	0.0020	0.0020	0.0019	0.0018
	COS	0.4212	0.3706	0.3203	0.2713	0.2249	0.1822	0.1441
7y	*MC*	0.4368	0.3878	0.3383	0.2895	0.2426	0.1986	0.1587
	SD	0.0018	0.0018	0.0018	0.0018	0.0018	0.0017	0.0016
	COS	0.4362	0.3873	0.3380	0.2893	0.2425	0.1987	0.1590
10y	*MC*	0.4310	0.3871	0.3420	0.2967	0.2521	0.2096	0.1702
	SD	0.0033	0.0033	0.0033	0.0033	0.0033	0.0031	0.0030
	COS	0.4311	0.3873	0.3423	0.2971	0.2528	0.2106	0.1714
15y	*MC*	0.3894	0.3553	0.3195	0.2826	0.2455	0.2092	0.1744
	SD	0.0038	0.0037	0.0037	0.0036	0.0036	0.0036	0.0035
	COS	0.3900	0.3560	0.3202	0.2834	0.2463	0.2100	0.1754
20y	*MC*	0.3362	0.3109	0.2838	0.2553	0.2260	0.1966	0.1677
	SD	0.0037	0.0037	0.0037	0.0037	0.0037	0.0036	0.0036
	COS	0.3358	0.3104	0.2833	0.2548	0.2254	0.1960	0.1672
25*y*	*MC*	0.2809	0.2626	0.2425	0.2211	0.1987	0.1757	0.1526
	SD	0.0048	0.0048	0.0048	0.0048	0.0047	0.0046	0.0045
	COS	0.2814	0.2630	0.2429	0.2215	0.1990	0.1759	0.1529
30*y*	*MC*	0.2322	0.2191	0.2046	0.1888	0.1720	0.1545	0.1367
	SD	0.0050	0.0050	0.0050	0.0050	0.0049	0.0048	0.0048
	COS	0.2319	0.2188	0.2042	0.1883	0.1714	0.1539	0.1359

Notes: *MC* stands for Monte Carlo and *COS* for Fourier cosine expansion technique (Fang and Oosterlee, 2008) for the FX-HHW1 model with 500 expansion terms. The strikes $K_n(T_i)$ are tabulated in Table B1.

Table B6. Average FX call option prices obtained by the FX-HLMM model with 20 Monte Carlo simulations, 50.000 paths and $20 \times T_i$ steps.

T_i	Method	$K_1(T_i)$	$K_2(T_i)$	$K_3(T_i)$	$K_4(T_i)$	$K_5(T_i)$	$K_6(T_i)$	$K_7(T_i)$
2y	*MC*	0.3336	0.2889	0.2456	0.2046	0.1667	0.1327	0.1030
	SD	0.0008	0.0009	0.0010	0.0010	0.0011	0.0011	0.0012
	COS	0.3326	0.2880	0.2450	0.2043	0.1667	0.1330	0.1037
3y	*MC*	0.3786	0.3299	0.2823	0.2366	0.1939	0.1553	0.1213
	SD	0.0006	0.0007	0.0008	0.0009	0.0011	0.0012	0.0013
	COS	0.3768	0.3282	0.2808	0.2354	0.1931	0.1548	0.1212
5y	*MC*	0.4243	0.3738	0.3234	0.2743	0.2274	0.1843	0.1457
	SD	0.0012	0.0013	0.0014	0.0015	0.0016	0.0016	0.0016
	COS	0.4222	0.3717	0.3215	0.2727	0.2265	0.1838	0.1457
7y	*MC*	0.4399	0.3914	0.3424	0.2938	0.2470	0.2031	0.1631
	SD	0.0013	0.0014	0.0015	0.0016	0.0018	0.0019	0.0021
	COS	0.4379	0.3893	0.3402	0.2918	0.2453	0.2017	0.1621
10y	*MC*	0.4363	0.3928	0.3482	0.3031	0.2587	0.2162	0.1764
	SD	0.0012	0.0016	0.0019	0.0023	0.0026	0.0027	0.0028
	COS	0.4338	0.3905	0.3461	0.3014	0.2576	0.2157	0.1767
15y	*MC*	0.3964	0.3632	0.3280	0.2917	0.2550	0.2186	0.1834
	SD	0.0008	0.0010	0.0012	0.0014	0.0016	0.0019	0.0023
	COS	0.3944	0.3613	0.3265	0.2907	0.2545	0.2190	0.1848
20y	*MC*	0.3417	0.3171	0.2907	0.2629	0.2342	0.2052	0.1768
	SD	0.0010	0.0013	0.0015	0.0018	0.0021	0.0025	0.0030
	COS	0.3416	0.3176	0.2918	0.2647	0.2367	0.2085	0.1806
25y	*MC*	0.2886	0.2715	0.2525	0.2321	0.2107	0.1887	0.1664
	SD	0.0011	0.0014	0.0016	0.0019	0.0023	0.0027	0.0033
	COS	0.2883	0.2715	0.2532	0.2335	0.2127	0.1913	0.1697
30y	*MC*	0.2396	0.2281	0.2152	0.2011	0.1858	0.1699	0.1534
	SD	0.0012	0.0015	0.0018	0.0021	0.0024	0.0029	0.0035
	COS	0.2393	0.2279	0.2152	0.2014	0.1866	0.1710	0.1548

Notes: *MC* stands for Monte Carlo and *COS* for the Fourier cosine expansion technique (Fang and Oosterlee, 2008) for the FX-HLMM1 model with 500 expansion terms. The strikes $K_n(T_i)$ are tabulated in Table B1.