

Research Article

On Cubic KU-Ideals of KU-Algebras

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We introduce the notion of cubic KU-ideals of KU-algebras and several results are presented in this regard. The image, preimage, and cartesian product of cubic KU-ideals of KU-algebras are defined.

1. Introduction

BCK-algebras form an important class of logical algebras introduced by Iséki and were extensively investigated by several researchers. The class of all BCK-algebras is quasivariety. Iséki and Tanaka introduced two classes of abstract algebras, BCK-algebras and BCI-algebras [1–3]. In connection with this problem, Komori [4] introduced a notion of BCC-algebras.

Prabpayak and Leerawat [5] introduced a new algebraic structure which is called KU-algebra. They gave the concept of homomorphisms of KU-algebras and investigated some related properties in [5, 6].

Zadeh [7] introduced the notion of fuzzy sets. At present this concept has been applied to many mathematical branches, such as groups, functional analysis, probability theory, and topology. Mostafa et al. [8] introduced the notion of fuzzy KU-ideals of KU-algebras and then they investigated several basic properties which are related to fuzzy KU-ideals, also see [9]. Abdullah et al. [10, 11] introduced the concept of direct product of intuitionistic fuzzy sets in BCK-algebras.

Jun et al. [12] introduced the notion of cubic subalgebras/ideals in BCK/BCI-algebras, and then they investigated several properties. They discussed the relationship between a cubic subalgebra and a cubic ideal. Also, they provided characterizations of a cubic subalgebra/ideal and considered a method to make a new cubic subalgebra from an old one, also see [13–17].

In this paper, we introduce the notion of cubic KU-ideals of KU-algebras and then we study the homomorphic image and inverse image of cubic KU-ideals.

2. Preliminaries

In this section we will recall some concepts related to KU-algebra and cubic sets.

Definition 1 (see [5]). By a KU-algebra we mean an algebra $(X, *, 0)$ of type $(2, 0)$ with a single binary operation $*$ that satisfies the following identities: for any $x, y, z \in X$,

$$(ku1) (x * y) * [(y * z) * (x * z)] = 0,$$

$$(ku2) x * 0 = 0,$$

$$(ku3) 0 * x = x,$$

$$(ku4) x * y = 0 = y * x \text{ implies } x = y.$$

In what follows, let $(X, *, 0)$ denote a KU-algebra unless otherwise specified. For brevity we also call X a KU-algebra. In X we can define a binary relation \leq by: $x \leq y$ if and only if $y * x = 0$.

Definition 2 (see [5]). $(X, *, 0)$ is a KU-algebra if and only if it satisfies

$$(ku5) (y * z) * (x * z) \leq (x * y),$$

$$(ku6) 0 \leq x,$$

TABLE 1

.	0	1	2	3	4
0	0	1	2	3	4
1	0	0	2	3	4
2	0	1	0	3	3
3	0	0	2	0	2
4	0	0	0	0	0

TABLE 2

.	0	1	2	3	4	5
0	0	1	2	3	4	5
1	0	0	2	2	4	5
2	0	0	0	1	4	5
3	0	0	0	0	4	5
4	0	0	0	1	0	5
5	0	0	0	0	0	0

(ku7) $x \leq y$, $y \leq x$ implies $x = y$,

(ku8) $x \leq y$ if and only if $y * x = 0$.

Definition 3 (see [8]). In a KU-algebra, the following identities are true:

- (1) $z * z = 0$,
- (2) $z * (x * z) = 0$,
- (3) $x \leq y$ imply $y * z \leq x * z$,
- (4) $z * (y * x) = y * (z * x)$, for all $x, y, z \in X$,
- (5) $y * [(y * x) * x] = 0$.

Example 4 (see [8]). Let $X = \{0, 1, 2, 3, 4\}$ in which $*$ is defined by Table 1.

It is easy to see that X is a KU-algebra.

Definition 5 (see [6]). A subset S of a KU-algebra X is called subalgebra of X if $x * y \in S$, whenever $x, y \in S$.

Definition 6 (see [6]). A nonempty subset A of a KU-algebra X is called a KU-ideal of X if it satisfies the following conditions:

- (1) $0 \in A$,
- (2) $x * (y * z) \in A$, $y \in A$ implies $x * z \in A$, for all $x, y, z \in X$.

Example 7. Let $X = \{0, 1, 2, 3, 4, 5\}$ in which $*$ is defined by Table 2.

Clearly $(X, *, 0)$ is a KU-algebra. It is easy to show that $A = \{0, 1\}$ and $B = \{0, 1, 2, 3, 4\}$ are KU-ideals of X .

Definition 8 (see [6]). Let $(X, *, 0)$ and $(X', *, 0')$ be KU-algebras. A homomorphism is a map $f : X \rightarrow X'$ satisfying $f(x * y) = f(x) *' f(y)$, for all $x, y \in X$.

Theorem 9 (see [6]). Let f be a homomorphism of a KU-algebra X into a KU-algebra X' , then

- (1) if 0 is the identity in X , then $f(0)$ is the identity in X' ;
- (2) if S is a KU-subalgebra of X , then $f(S)$ is a KU-subalgebra of X' ;
- (3) if I is a KU-ideal of X , then $f(I)$ is a KU-ideal in X' ;
- (4) if S is a KU-subalgebra of $f(X')$, then $f^{-1}(S)$ is a KU-subalgebra algebra of X ;
- (5) if B is a KU-ideal in $f(X')$, then $f^{-1}(B)$ is a KU-ideal in X ;
- (6) if f is a homomorphism from KU-algebra X to a KU-algebra X' then f is one to one if and only if $\ker f = \{0\}$.

Proposition 10 (see [8]). Suppose $f : X \rightarrow X'$ is a homomorphism of KU-algebras, then:

- (1) $f(0) = 0'$,
- (2) if $x \leq y$ implies $f(x) \leq f(y)$.

Proposition 11 (see [8]). Let $(X, *, 0)$ and $(X', *, 0')$ be KU-algebras and let $f : X \rightarrow X'$ be a homomorphism, then $\ker f$ is a KU-ideal of X .

Now we will recall the concept of interval-valued fuzzy sets.

An interval number is $\tilde{a} = [a^-, a^+]$, where $0 \leq a^- \leq a^+ \leq 1$. Let $D[0, 1]$ denote the family of all closed subintervals of $[0, 1]$, that is,

$$D[0, 1] = \{\tilde{a} = [a^-, a^+] : a^- \leq a^+, \text{ for } a^-, a^+ \in I\}. \quad (1)$$

We define the operations “ \succeq ,” “ \leq ,” “ $=$,” “rmin,” and “rmax” in case of two elements in $D[0, 1]$. We consider two elements $\tilde{a} = [a^-, a^+]$ and $\tilde{b} = [b^-, b^+]$ in $D[0, 1]$. Then

- (1) $\tilde{a} \succeq \tilde{b}$ if and only if $a^- \geq b^-$ and $a^+ \geq b^+$,
- (2) $\tilde{a} \leq \tilde{b}$ if and only if $a^- \leq b^-$ and $a^+ \leq b^+$,
- (3) $\tilde{a} = \tilde{b}$ if and only if $a^- = b^-$ and $a^+ = b^+$,
- (4) $\text{rmin}\{\tilde{a}, \tilde{b}\} = [\min\{a^-, b^-\}, \min\{a^+, b^+\}]$,
- (5) $\text{rmax}\{\tilde{a}, \tilde{b}\} = [\max\{a^-, b^-\}, \max\{a^+, b^+\}]$.

It is obvious that $(D[0, 1], \leq, \vee, \wedge)$ is a complete lattice with $\tilde{0} = [0, 0]$ as its least element and $\tilde{1} = [1, 1]$ as its greatest element. Let $\tilde{a}_i \in D[0, 1]$ where $i \in \Lambda$. We define

$$\text{rinf}_{i \in \Lambda} \tilde{a}_i = \left[\inf_{i \in \Lambda} a_i^-, \inf_{i \in \Lambda} a_i^+ \right], \quad \text{rsup}_{i \in \Lambda} \tilde{a}_i = \left[\sup_{i \in \Lambda} a_i^-, \sup_{i \in \Lambda} a_i^+ \right]. \quad (2)$$

An interval-valued fuzzy set (briefly, IVF-set) $\tilde{\mu}_A$ on X is defined as

$$\tilde{\mu}_A = \{\langle x, [\mu_A^-(x), \mu_A^+(x)] \rangle : x \in X\}, \quad (3)$$

where $\mu_A^-(x) \leq \mu_A^+(x)$, for all $x \in X$. Then the ordinary fuzzy sets $\mu_A^- : X \rightarrow [0, 1]$ and $\mu_A^+ : X \rightarrow [0, 1]$ are called a

TABLE 3

.	0	1	2	3	4
0	0	1	2	3	4
1	0	0	0	0	1
2	0	3	0	3	4
3	0	1	2	0	1
4	0	0	0	0	0

lower fuzzy set and an upper fuzzy set of $\tilde{\mu}$, respectively. Let $\tilde{\mu}_A(x) = [\mu_A^-(x), \mu_A^+(x)]$, then

$$A = \{\langle x, \tilde{\mu}_A(x) \rangle : x \in X\}, \quad (4)$$

where $\tilde{\mu}_A : X \rightarrow D[0, 1]$.

Jun et al. [12] introduced the concept of cubic sets defined on a nonempty set X as objects having the form:

$$\Xi = \{\langle x, \tilde{\mu}_\Xi(x), \lambda_\Xi(x) \rangle : x \in X\}, \quad (5)$$

which is briefly denoted by $\Xi = \langle \tilde{\mu}_\Xi, \lambda_\Xi \rangle$, where the functions $\tilde{\mu}_\Xi : X \rightarrow D[0, 1]$ and $\lambda_\Xi : X \rightarrow [0, 1]$.

Denote $\mathcal{C}(X)$ by family of all cubic sets in X .

3. Cubic KU-Ideals of KU-Algebras

In this section, we will introduce a new notion called cubic KU-ideal of KU-algebras and study several properties of it.

Definition 12. Let X be a KU-algebra. A cubic set $\Xi = \langle \tilde{\mu}_\Xi, \lambda_\Xi \rangle$ in X is called cubic KU-subalgebra of X if it satisfies the following conditions:

- (S1) $\tilde{\mu}_\Xi(0) \succeq \tilde{\mu}_\Xi(x)$ and $\lambda_\Xi(0) \leq \lambda_\Xi(x)$,
- (S2) $\tilde{\mu}_\Xi(x) \succeq \text{rmin}\{\tilde{\mu}_\Xi(x * y), \tilde{\mu}_\Xi(y)\}$ and $\lambda_\Xi(x) \leq \max\{\lambda_\Xi(x * y), \lambda_\Xi(y)\}$, for all $x, y \in X$.

Definition 13. Let X be a KU-algebra. A cubic set $\Xi = \langle \tilde{\mu}_\Xi, \lambda_\Xi \rangle$ in X is called cubic KU-ideal of X if it satisfies the following conditions:

- (A1) $\tilde{\mu}_\Xi(0) \succeq \tilde{\mu}_\Xi(x)$ and $\lambda_\Xi(0) \leq \lambda_\Xi(x)$,
- (A2) $\tilde{\mu}_\Xi(x * z) \succeq \text{rmin}\{\tilde{\mu}_\Xi(x * (y * z)), \tilde{\mu}_\Xi(y)\}$ and $\lambda_\Xi(x * z) \leq \max\{\lambda_\Xi(x * (y * z)), \lambda_\Xi(y)\}$, for all $x, y, z \in X$.

Example 14. Let $X = \{0, 1, 2, 3, 4\}$ in which $*$ is defined by Table 3.

Clearly $(X, *, 0)$ is a KU-algebra. Define a cubic set $\Xi = \langle \tilde{\mu}_\Xi, \lambda_\Xi \rangle$ in X as follows:

$$\begin{aligned} \tilde{\mu}_\Xi(x) &= \begin{cases} [0.6, 0.7], & \text{if } x = 0, \\ [0.4, 0.5], & \text{if } x \in \{1, 3\}, \\ [0.1, 0.3], & \text{if } x \in \{2, 4\}, \end{cases} \\ \lambda_\Xi(x) &= \begin{cases} 0.1, & \text{if } x = 0, \\ 0.3, & \text{if } x \in \{1, 3\}, \\ 0.6, & \text{if } x \in \{2, 4\}. \end{cases} \end{aligned} \quad (6)$$

By routine calculations it can be seen that the cubic set $\Xi = \langle \tilde{\mu}_\Xi, \lambda_\Xi \rangle$ is a cubic KU-ideal of X .

Lemma 15. Let $\Xi = \langle \tilde{\mu}_\Xi, \lambda_\Xi \rangle$ be a cubic KU-ideal of KU-algebra X . If the inequality $x * y \leq z$ holds in X , then $\tilde{\mu}_\Xi(y) \succeq \text{rmin}\{\tilde{\mu}_\Xi(x), \tilde{\mu}_\Xi(z)\}$ and $\lambda_\Xi(y) \leq \max\{\lambda_\Xi(x), \lambda_\Xi(z)\}$.

Proof. Assume that the inequality $x * y \leq z$ holds in X , then $z * (x * y) = 0$ and by (A2) $\tilde{\mu}_\Xi(z * y) \succeq \text{rmin}\{\tilde{\mu}_\Xi(z * (x * y)), \tilde{\mu}_\Xi(x)\}$, if we put $z = 0$ then

$$\begin{aligned} \tilde{\mu}_\Xi(0 * y) &= \tilde{\mu}_\Xi(y) \succeq \text{rmin}\{\tilde{\mu}_\Xi(0 * (x * y)), \tilde{\mu}_\Xi(x)\} \\ &= \text{rmin}\{\tilde{\mu}_\Xi(x * y), \tilde{\mu}_\Xi(x)\} \end{aligned} \quad (i)$$

but

$$\begin{aligned} \tilde{\mu}_\Xi(x * y) &\succeq \text{rmin}\{\tilde{\mu}_\Xi(x * (z * y)), \tilde{\mu}_\Xi(z)\} \\ &= \text{rmin}\{\tilde{\mu}_\Xi(z * (x * y)), \tilde{\mu}_\Xi(z)\} \\ &= \text{rmin}\{\tilde{\mu}_\Xi(0), \mu(z)\} = \tilde{\mu}_\Xi(z). \end{aligned} \quad (ii)$$

From (i) and (ii), we get $\tilde{\mu}_\Xi(y) \succeq \text{rmin}\{\tilde{\mu}_\Xi(x), \tilde{\mu}_\Xi(z)\}$. Similarly we can show that $\lambda_\Xi(y) \leq \max\{\lambda_\Xi(x), \lambda_\Xi(z)\}$. This completes the proof. \square

Lemma 16. If $\Xi = \langle \tilde{\mu}_\Xi, \lambda_\Xi \rangle$ is a cubic KU-ideal of KU-algebra X and if $x \leq y$ then $\tilde{\mu}_\Xi(x) \succeq \tilde{\mu}_\Xi(y)$ and $\lambda_\Xi(x) \leq \lambda_\Xi(y)$.

Proof. If $x \leq y$ then $y * x = 0$. This together with $0 * x = x$ and $\tilde{\mu}_\Xi(0) \succeq \tilde{\mu}_\Xi(y)$ also $\lambda_\Xi(0) \leq \lambda_\Xi(y)$, we get

$$\begin{aligned} \tilde{\mu}_\Xi(0 * x) &= \tilde{\mu}_\Xi(x) \succeq \text{rmin}\{\tilde{\mu}_\Xi(0 * (y * x)), \tilde{\mu}_\Xi(y)\} \\ &= \text{rmin}\{\tilde{\mu}_\Xi(0 * 0), \tilde{\mu}_\Xi(y)\} \\ &= \text{rmin}\{\tilde{\mu}_\Xi(0), \tilde{\mu}_\Xi(y)\} = \tilde{\mu}_\Xi(y). \end{aligned} \quad (7)$$

Also

$$\begin{aligned} \lambda_\Xi(0 * x) &= \lambda_\Xi(x) \leq \max\{\lambda_\Xi(0 * (y * x)), \lambda_\Xi(y)\} \\ &= \max\{\lambda_\Xi(0 * 0), \lambda_\Xi(y)\} \\ &= \max\{\lambda_\Xi(0), \lambda_\Xi(y)\} = \lambda_\Xi(y). \end{aligned} \quad (8)$$

This completes the proof. \square

Let $\Xi = \langle \tilde{\mu}_\Xi, \lambda_\Xi \rangle$ and $\mathcal{F} = \langle \tilde{\mu}_\mathcal{F}, \lambda_\mathcal{F} \rangle$ be two cubic sets in a KU-algebra X , then

$$\begin{aligned} \Xi \cap \mathcal{F} &= \{\langle x, \text{rmin}\{\tilde{\mu}_\Xi(x), \tilde{\mu}_\mathcal{F}(x)\}, \max\{\lambda_\Xi(x), \lambda_\mathcal{F}(x)\} \rangle \\ &\quad : x \in X\} \\ &= \{\langle x, \tilde{\mu}_\Xi(x) \cap \tilde{\mu}_\mathcal{F}(x), \lambda_\Xi(x) \cup \lambda_\mathcal{F}(x) \rangle : x \in X\}. \end{aligned} \quad (9)$$

Proposition 17. Let $\{\Xi_i\}_{i \in I}$ be a family of cubic KU-ideals of a KU-algebra X , then $\bigcap_{i \in I} \Xi_i$ is a cubic KU-ideal of X .

Proof. Let $\{\Xi_i\}_{i \in I}$ be a family of cubic KU-ideals of a KU-algebra X , then for any $x, y, z \in X$,

$$\begin{aligned} & (\bigcap \tilde{\mu}_{\Xi_i})(0) \\ &= \text{rinf}(\tilde{\mu}_{\Xi_i}(0)) \succeq \text{rinf}(\tilde{\mu}_{\Xi_i}(x)) \\ &= (\bigcap \tilde{\mu}_{\Xi_i})(x), \\ & (\bigcap \tilde{\mu}_{\Xi_i})(x * z) \\ &= \text{rinf}(\tilde{\mu}_{\Xi_i}(x * z)) \\ &\geq \text{rinf}(\text{rmin}\{\tilde{\mu}_{\Xi_i}(x * (y * z)), \tilde{\mu}_{\Xi_i}(y)\}) \\ &= \text{rmin}\{\text{rinf}(\tilde{\mu}_{\Xi_i}(x * (y * z))), \text{rinf}(\tilde{\mu}_{\Xi_i}(y))\} \\ &= \text{rmin}\{(\bigcap \tilde{\mu}_{\Xi_i})(x * (y * z)), (\bigcap \tilde{\mu}_{\Xi_i})(y)\}. \end{aligned} \tag{10}$$

Also

$$\begin{aligned} & (\bigcup \lambda_{\Xi_i})(0) \\ &= \sup(\lambda_{\Xi_i}(0)) \leq \sup(\lambda_{\Xi_i}(x)) \\ &= (\bigcup \lambda_{\Xi_i})(x), \\ & (\bigcup \lambda_{\Xi_i})(x * z) \\ &= \sup(\lambda_{\Xi_i}(x * z)) \\ &\leq \sup(\max\{\lambda_{\Xi_i}(x * (y * z)), \lambda_{\Xi_i}(y)\}) \\ &= \max\{\sup(\lambda_{\Xi_i}(x * (y * z))), \sup(\lambda_{\Xi_i}(y))\} \\ &= \max\{(\bigcup \lambda_{\Xi_i})(x * (y * z)), (\bigcup \lambda_{\Xi_i})(y)\}. \end{aligned} \tag{11}$$

This completes the proof. \square

For any $\tilde{t} \in D[0, 1]$ and $s \in [0, 1]$, let $\Xi = \langle \tilde{\mu}_{\Xi}, \lambda_{\Xi} \rangle$ be a cubic set in a KU-algebra X , then the set

$$U(\Xi; \tilde{t}, s) = \{x \in X : \tilde{\mu}_{\Xi}(x) \succeq \tilde{t}, \lambda_{\Xi}(x) \leq s\} \tag{12}$$

is called the cubic level set of $\Xi = \langle \tilde{\mu}_{\Xi}, \lambda_{\Xi} \rangle$.

Theorem 18. Let $\Xi = \langle \tilde{\mu}_{\Xi}, \lambda_{\Xi} \rangle$ be a cubic subset in X then $\Xi = \langle \tilde{\mu}_{\Xi}, \lambda_{\Xi} \rangle$ is a cubic KU-ideal of X if and only if for all $\tilde{t} \in D[0, 1]$ and $s \in [0, 1]$, the set $U(\Xi; \tilde{t}, s)$ is either empty or a KU-ideal of X .

Proof. Assume that $\Xi = \langle \tilde{\mu}_{\Xi}, \lambda_{\Xi} \rangle$ is a cubic KU-ideal of X , let $\tilde{t} \in D[0, 1]$ and $s \in [0, 1]$, be such that $U(\Xi; \tilde{t}, s) \neq \emptyset$, and let $x, y \in X$ be such that $x \in U(\Xi; \tilde{t}, s)$, then $\tilde{\mu}_{\Xi}(x) \succeq \tilde{t}$ and $\lambda_{\Xi}(x) \leq s$. By (A2) we get

$$\begin{aligned} \tilde{\mu}_{\Xi}(y * 0) &= \tilde{\mu}_{\Xi}(0) \succeq \text{rmin}\{\tilde{\mu}_{\Xi}(y * (x * 0)), \tilde{\mu}_{\Xi}(x)\} \\ &= \text{rmin}\{\tilde{\mu}_{\Xi}(y * 0), \tilde{\mu}_{\Xi}(x)\} \\ &= \text{rmin}\{\tilde{\mu}_{\Xi}(0), \tilde{\mu}_{\Xi}(x)\} = \tilde{t}, \end{aligned} \tag{13}$$

also

$$\begin{aligned} \lambda_{\Xi}(y * 0) &= \lambda_{\Xi}(0) \leq \max\{\lambda_{\Xi}(y * (x * 0)), \lambda_{\Xi}(x)\} \\ &= \max\{\lambda_{\Xi}(y * 0), \lambda_{\Xi}(x)\} \\ &= \max\{\lambda_{\Xi}(0), \lambda_{\Xi}(x)\} = s. \end{aligned} \tag{14}$$

Thus, $0 \in U(\Xi; \tilde{t}, s)$. Now letting $x * (y * z) \in U(\Xi; \tilde{t}, s)$, $y \in U(\Xi; \tilde{t}, s)$, it implies that

$$\tilde{\mu}_{\Xi}(x * z) \succeq \text{rmin}\{\tilde{\mu}_{\Xi}(x * (y * z)), \tilde{\mu}_{\Xi}(y)\} = \tilde{t}, \tag{15}$$

also

$$\lambda_{\Xi}(x * z) \leq \max\{\lambda_{\Xi}(x * (y * z)), \lambda_{\Xi}(y)\} = s. \tag{16}$$

So that $x * z \in U(\Xi; \tilde{t}, s)$. Hence, $U(\Xi; \tilde{t}, s)$ is a KU-ideal of X .

Conversely, suppose that $U(\Xi; \tilde{t}, s)$ is a KU-ideal of X and let $x, y, z \in X$ be such that

$$\begin{aligned} \tilde{\mu}_{\Xi}(x * z) &< \text{rmin}\{\tilde{\mu}_{\Xi}(x * (y * z)), \tilde{\mu}_{\Xi}(y)\}, \\ \lambda_{\Xi}(x * z) &> \max\{\lambda_{\Xi}(x * (y * z)), \lambda_{\Xi}(y)\}, \end{aligned} \tag{17}$$

taking

$$\tilde{\beta}_1 = \frac{1}{2} \{\tilde{\mu}_{\Xi}(x * z) + \text{rmin}\{\tilde{\mu}_{\Xi}(x * (y * z)), \tilde{\mu}_{\Xi}(y)\}\}, \tag{18}$$

and taking

$$\beta_2 = \frac{1}{2} \{\lambda_{\Xi}(x * z) + \max\{\lambda_{\Xi}(x * (y * z)), \lambda_{\Xi}(y)\}\}, \tag{19}$$

we have $\tilde{\beta}_1 \in D[0, 1]$ and $\beta_2 \in [0, 1]$, and

$$\begin{aligned} \tilde{\mu}_{\Xi}(x * z) &< \tilde{\beta}_1 < \text{rmin}\{\tilde{\mu}_{\Xi}(x * (y * z)), \tilde{\mu}_{\Xi}(y)\}, \\ \lambda_{\Xi}(x * z) &> \beta_2 > \max\{\lambda_{\Xi}(x * (y * z)), \lambda_{\Xi}(y)\}. \end{aligned} \tag{20}$$

It follows that $x * (y * z) \in U(\Xi; \tilde{\beta}_1, \beta_2)$ and $x * z \notin U(\Xi; \tilde{\beta}_1, \beta_2)$. This is a contradiction and therefore $\Xi = \langle \tilde{\mu}_{\Xi}, \lambda_{\Xi} \rangle$ is cubic KU-ideal of X . \square

Proposition 19. If $\Xi = \langle \tilde{\mu}_{\Xi}, \lambda_{\Xi} \rangle$ is a cubic KU-ideal of KU-algebra X , then

$$\tilde{\mu}_{\Xi}(x * (x * y)) \succeq \tilde{\mu}_{\Xi}(y), \quad \lambda_{\Xi}(x * (x * y)) \leq \lambda_{\Xi}(y). \tag{21}$$

Proof. Taking $z = x * y$ in (A2) and using (ku2) and (A1), we get

$$\begin{aligned} \tilde{\mu}_\Xi(x * (x * y)) &\succeq \text{rmin}\{\tilde{\mu}_\Xi(x * (y * (x * y))), \tilde{\mu}_\Xi(y)\} \\ &= \text{rmin}\{\tilde{\mu}_\Xi(x * (x * (y * y))), \tilde{\mu}_\Xi(y)\} \\ &= \text{rmin}\{\tilde{\mu}_\Xi(x * (x * 0)), \tilde{\mu}_\Xi(y)\} \\ &= \text{rmin}\{\tilde{\mu}_\Xi(0), \tilde{\mu}_\Xi(y)\} = \tilde{\mu}_\Xi(y), \\ \lambda_\Xi(x * (x * y)) &\leq \max\{\lambda_\Xi(x * (y * (x * y))), \lambda_\Xi(y)\} \\ &= \max\{\lambda_\Xi(x * (x * (y * y))), \lambda_\Xi(y)\} \\ &= \max\{\lambda_\Xi(x * (x * 0)), \lambda_\Xi(y)\} \\ &= \max\{\lambda_\Xi(0), \lambda_\Xi(y)\} = \lambda_\Xi(y). \end{aligned} \quad (22)$$

Thus, we get $\tilde{\mu}_\Xi(x * (x * y)) \succeq \tilde{\mu}_\Xi(y)$ and $\lambda_\Xi(x * (x * y)) \leq \lambda_\Xi(y)$. This completes the proof. \square

4. Image and Preimage of Cubic KU-Ideals

In this section we will present some results on images and preimages of cubic KU-ideals in KU-algebras.

Definition 20. Let f be a mapping from a set X to a set Y . If $\Xi = \langle \tilde{\mu}_\Xi, \lambda_\Xi \rangle$ is a cubic subset of X , then the cubic subset $\beta = \langle \tilde{\mu}_\beta, \lambda_\beta \rangle$ of Y is defined by

$$\begin{aligned} f(\tilde{\mu}_\Xi)(y) &= \tilde{\mu}_\beta(y) = \begin{cases} \text{rsup}_{x \in f^{-1}(y)} \tilde{\mu}_\Xi(x), \\ \text{if } f^{-1}(y) = \{x \in X, f(x) = y\} \neq \emptyset, \\ 0, \quad \text{otherwise}, \end{cases} \\ f(\lambda_\Xi)(y) &= \lambda_\beta(y) = \begin{cases} \inf_{x \in f^{-1}(y)} \lambda_\Xi(x), \\ \text{if } f^{-1}(y) = \{x \in X, f(x) = y\} \neq \emptyset, \\ 1, \quad \text{otherwise}, \end{cases} \end{aligned} \quad (23)$$

is said to be the image of $\Xi = \langle \tilde{\mu}_\Xi, \lambda_\Xi \rangle$ under f .

Similarly if $\beta = \langle \tilde{\mu}_\beta, \lambda_\beta \rangle$ is a cubic subset of Y , then the cubic subset $\Xi = \beta \circ f$ in X (i.e., the cubic subset defined by $\tilde{\mu}_\Xi(x) = \tilde{\mu}_\beta(f(x))$ and $\lambda_\Xi(x) = \lambda_\beta(f(x))$ for all $x \in X$) is called the preimage of β under f .

Theorem 21. An onto homomorphic preimage of cubic KU-ideal is also cubic KU-ideal.

Proof. Let $f : X \rightarrow X'$ be an onto homomorphism of KU-algebras, $\beta = \langle \tilde{\mu}_\beta, \lambda_\beta \rangle$ a cubic KU-ideal of X' , and $\Xi = \langle \tilde{\mu}_\Xi, \lambda_\Xi \rangle$ the preimage of β under f . Then $\tilde{\mu}_\Xi(x) = \tilde{\mu}_\beta(f(x))$ and $\lambda_\Xi(x) = \lambda_\beta(f(x))$ for all $x \in X$. Let $x \in X$, then

$\langle \tilde{\mu}_\Xi, \lambda_\Xi \rangle$ the preimage of β under f , then $\tilde{\mu}_\Xi(x) = \tilde{\mu}_\beta(f(x))$ and $\lambda_\Xi(x) = \lambda_\beta(f(x))$ for all $x \in X$. Let $x \in X$, then

$$\begin{aligned} \tilde{\mu}_\Xi(0) &= \tilde{\mu}_\beta(f(0)) \succeq \tilde{\mu}_\beta(f(x)) = \tilde{\mu}_\Xi(x), \\ \lambda_\Xi(0) &= \lambda_\beta(f(0)) \leq \lambda_\beta(f(x)) = \lambda_\Xi(x). \end{aligned} \quad (24)$$

Now let $x, y, z \in X$, then

$$\begin{aligned} \tilde{\mu}_\Xi(x * z) &= \tilde{\mu}_\beta(f(x * z)) = \tilde{\mu}_\beta(f(x) *' f(z)) \\ &\succeq \text{rmin}\{\tilde{\mu}_\beta(f(x) *' (f(y) *' f(z))), \tilde{\mu}_\beta(f(y))\} \\ &= \text{rmin}\{\tilde{\mu}_\beta(f(x * (y * z))), \tilde{\mu}_\beta(f(y))\} \\ &= \text{rmin}\{\tilde{\mu}_\Xi(x * (y * z)), \tilde{\mu}_\Xi(y)\}, \\ \lambda_\Xi(x * z) &= \lambda_\beta(f(x * z)) = \lambda_\beta(f(x) *' f(z)) \\ &\leq \max\{\lambda_\beta(f(x) *' (f(y) *' f(z))), \lambda_\beta(f(y))\} \\ &= \max\{\lambda_\beta(f(x * (y * z))), \lambda_\beta(f(y))\} \\ &= \max\{\lambda_\Xi(x * (y * z)), \lambda_\Xi(y)\}. \end{aligned} \quad (25)$$

This completes the proof. \square

Definition 22. A cubic subset $\Xi = \langle \tilde{\mu}_\Xi, \lambda_\Xi \rangle$ of X has sup and inf properties if for any subset T of X , there exist $t, s \in T$ such that $\tilde{\mu}_\Xi(t) = \text{rsup}_{t \in T} \tilde{\mu}_\Xi(t)$ and $\lambda_\Xi(s) = \inf_{t \in T} \lambda_\Xi(t)$.

Theorem 23. Let $f : X \rightarrow Y$ be a homomorphism between KU-algebras X and Y . For every cubic KU-ideal $\Xi = \langle \tilde{\mu}_\Xi, \lambda_\Xi \rangle$ in X , $f(\Xi)$ is cubic KU-ideal of Y .

Proof. By definition $\tilde{\mu}_\beta(y') = f(\tilde{\mu}_\Xi)(y') = \text{rsup}_{x \in f^{-1}(y')} \tilde{\mu}_\Xi(x)$ and $\lambda_\beta(y') = f(\lambda_\Xi)(y') = \inf_{x \in f^{-1}(y')} \lambda_\Xi(x)$ for all $y' \in Y$ and $\text{rsup} \emptyset = [0, 0]$ and $\inf \emptyset = 0$. We have to prove that $\tilde{\mu}_\beta(x' * z') \succeq \text{rmin}\{\tilde{\mu}_\beta(x' * (y' * z')), \tilde{\mu}_\beta(y')\}$ and $\lambda_\beta(x' * z') \leq \max\{\lambda_\beta(x' * (y' * z')), \lambda_\beta(y')\}$ for all $x', y', z' \in Y$. Let $f : X \rightarrow Y$ be an onto homomorphism of KU-algebras, $\Xi = \langle \tilde{\mu}_\Xi, \lambda_\Xi \rangle$ a cubic KU-ideal of X with sup and inf properties, and $\beta = \langle \tilde{\mu}_\beta, \lambda_\beta \rangle$ the image of $\Xi = \langle \tilde{\mu}_\Xi, \lambda_\Xi \rangle$ under f . Since $\Xi = \langle \tilde{\mu}_\Xi, \lambda_\Xi \rangle$ is cubic KU-ideal of X , we have

$$\tilde{\mu}_\Xi(0) \succeq \tilde{\mu}_\Xi(x), \quad \lambda_\Xi(0) \leq \lambda_\Xi(x) \quad \forall x \in X. \quad (26)$$

Note that $0 \in f^{-1}(0')$ where $0, 0'$ are the zero of X and Y , respectively. Thus,

$$\begin{aligned} \tilde{\mu}_\beta(0') &= \text{rsup}_{t \in f^{-1}(0')} \tilde{\mu}_\Xi(t) = \tilde{\mu}_\Xi(0) \succeq \tilde{\mu}_\Xi(x) \quad \forall x \in X, \\ \lambda_\beta(0') &= \inf_{t \in f^{-1}(0')} \lambda_\Xi(t) = \lambda_\Xi(0) \leq \lambda_\Xi(x) \quad \forall x \in X, \end{aligned} \quad (27)$$

which implies that $\tilde{\mu}_\beta(0') \succeq \text{rsup}_{t \in f^{-1}(x')} \tilde{\mu}_\Xi(t) = \tilde{\mu}_\beta(x')$ and $\lambda_\beta(0') \leq \inf_{t \in f^{-1}(x')} \lambda_\Xi(t) = \lambda_\beta(x')$ for any $x' \in Y$. For

any $x', y', z' \in Y$, let $x_0 \in f^{-1}(x')$, $y_0 \in f^{-1}(y')$, and $z_0 \in f^{-1}(z')$ be such that

$$\begin{aligned} \tilde{\mu}_{\Xi}(x_0 * z_0) &= \text{rsup}_{t \in f^{-1}(x' * z')} \tilde{\mu}_{\Xi}(t), \\ \tilde{\mu}_{\Xi}(y_0) &= \text{rsup}_{t \in f^{-1}(y')} \tilde{\mu}_{\Xi}(t), \\ \tilde{\mu}_{\Xi}(x_0 * (y_0 * z_0)) \\ &= \tilde{\mu}_{\beta}\{f(x_0 * (y_0 * z_0))\} \quad (28) \\ &= \tilde{\mu}_{\beta}(x' * (y' * z')) \\ &= \text{rsup}_{(x_0 * (y_0 * z_0)) \in f^{-1}(x' * (y' * z'))} \tilde{\mu}_{\Xi}(x_0 * (y_0 * z_0)) \\ &= \text{rsup}_{t \in f^{-1}(x' * (y' * z'))} \tilde{\mu}_{\Xi}(t). \end{aligned}$$

Also

$$\begin{aligned} \lambda_{\Xi}(x_0 * z_0) &= \inf_{t \in f^{-1}(x' * z')} \lambda_{\Xi}(t), \\ \lambda_{\Xi}(y_0) &= \inf_{t \in f^{-1}(y')} \lambda_{\Xi}(t), \\ \lambda_{\Xi}(x_0 * (y_0 * z_0)) \\ &= \lambda_{\beta}\{f(x_0 * (y_0 * z_0))\} \quad (29) \\ &= \lambda_{\beta}(x' * (y' * z')) \\ &= \inf_{(x_0 * (y_0 * z_0)) \in f^{-1}(x' * (y' * z'))} \lambda_{\Xi}(x_0 * (y_0 * z_0)) \\ &= \inf_{t \in f^{-1}(x' * (y' * z'))} \lambda_{\Xi}(t). \end{aligned}$$

Then

$$\begin{aligned} \tilde{\mu}_{\beta}(x' * z') &= \text{rsup}_{t \in f^{-1}(x' * z')} \tilde{\mu}_{\Xi}(t) = \tilde{\mu}_{\Xi}(x_0 * z_0) \\ &\geq \text{rmin}\{\tilde{\mu}_{\Xi}(x_0 * (y_0 * z_0)), \tilde{\mu}_{\Xi}(y_0)\} \\ &= \text{rmin}\left\{\text{rsup}_{t \in f^{-1}(x' * (y' * z'))} \tilde{\mu}_{\Xi}(t), \text{rsup}_{t \in f^{-1}(y')} \tilde{\mu}_{\Xi}(t)\right\} \\ &= \text{rmin}\{\tilde{\mu}_{\beta}(x' * (y' * z')), \tilde{\mu}_{\beta}(y')\}, \\ \lambda_{\beta}(x' * z') &= \inf_{t \in f^{-1}(x' * z')} \lambda_{\Xi}(t) = \lambda_{\Xi}(x_0 * z_0) \\ &\leq \max\{\lambda_{\Xi}(x_0 * (y_0 * z_0)), \lambda_{\Xi}(y_0)\} \\ &= \max\left\{\inf_{t \in f^{-1}(x' * (y' * z'))} \lambda_{\Xi}(t), \inf_{t \in f^{-1}(y')} \lambda_{\Xi}(t)\right\} \\ &= \max\{\lambda_{\beta}(x' * (y' * z')), \lambda_{\beta}(y')\}. \quad (30) \end{aligned}$$

Hence, β is cubic KU-ideal of Y . \square

5. Cartesian Product of Cubic KU-Ideals

In this section we will provide some new definitions on cartesian product of cubic KU-ideals in KU-algebras.

Definition 24. Let $\Xi_1 = \langle \tilde{\mu}_{\Xi_1}, \lambda_{\Xi_1} \rangle$ and $\Xi_2 = \langle \tilde{\mu}_{\Xi_2}, \lambda_{\Xi_2} \rangle$ be two cubic subsets of KU-algebras X_1 and X_2 , respectively. Then cartesian product of cubic subsets Ξ_1 and Ξ_2 is denoted by $\Xi_1 \times \Xi_2 = \langle \tilde{\mu}_{\Xi_1 \times \Xi_2}, \lambda_{\Xi_1 \times \Xi_2} \rangle$ and is defined as

$$\begin{aligned} \tilde{\mu}_{\Xi_1 \times \Xi_2}(x, y) &= \text{rmin}\{\tilde{\mu}_{\Xi_1}(x), \tilde{\mu}_{\Xi_2}(y)\}, \\ \lambda_{\Xi_1 \times \Xi_2}(x, y) &= \max\{\lambda_{\Xi_1}(x), \lambda_{\Xi_2}(y)\}, \end{aligned} \quad (31)$$

for all $(x, y) \in X_1 \times X_2$.

Remark 25. Let X and Y be KU-algebras. We define $*$ on $X \times Y$ by $(x, y)*(u, v) = (x*u, y*v)$ for every $(x, y), (u, v)$ belong to $X \times Y$, then clearly $(X \times Y, *, (0, 0))$ is a KU-algebra.

Definition 26. A cubic subset $\Xi_1 \times \Xi_2 = \langle \tilde{\mu}_{\Xi_1 \times \Xi_2}, \lambda_{\Xi_1 \times \Xi_2} \rangle$ of $X_1 \times X_2$ is called a cubic KU-subalgebra of $X_1 \times X_2$ if

$$\begin{aligned} (\text{CP1}) \quad \tilde{\mu}_{\Xi_1 \times \Xi_2}(0, 0) &\geq \tilde{\mu}_{\Xi_1 \times \Xi_2}(x, y) \text{ and } \lambda_{\Xi_1 \times \Xi_2}(0, 0) \leq \\ &\lambda_{\Xi_1 \times \Xi_2}(x, y), \\ (\text{CP2}) \quad \tilde{\mu}_{\Xi_1 \times \Xi_2}(x_1, y_1) &\geq \text{rmin}\{\tilde{\mu}_{\Xi_1 \times \Xi_2}((x_1, y_1) * (x_2, y_2)), \\ &\tilde{\mu}_{\Xi_1 \times \Xi_2}(x_2, y_2)\}, \\ (\text{CP3}) \quad \lambda_{\Xi_1 \times \Xi_2}(x_1, y_1) &\leq \max\{\lambda_{\Xi_1 \times \Xi_2}((x_1, y_1) * (x_2, y_2)), \\ &\lambda_{\Xi_1 \times \Xi_2}(x_2, y_2)\}, \end{aligned}$$

for all $(x_1, y_1), (x_2, y_2) \in X_1 \times X_2$.

Definition 27. A cubic subset $\Xi_1 \times \Xi_2 = \langle \tilde{\mu}_{\Xi_1 \times \Xi_2}, \lambda_{\Xi_1 \times \Xi_2} \rangle$ of $X_1 \times X_2$ is called a cubic KU-ideal of $X_1 \times X_2$ if

$$\begin{aligned} (\text{CP4}) \quad \tilde{\mu}_{\Xi_1 \times \Xi_2}(0, 0) &\geq \tilde{\mu}_{\Xi_1 \times \Xi_2}(x, y) \text{ and } \lambda_{\Xi_1 \times \Xi_2}(0, 0) \leq \\ &\lambda_{\Xi_1 \times \Xi_2}(x, y), \\ (\text{CP5}) \quad \tilde{\mu}_{\Xi_1 \times \Xi_2}((x_1, y_1) * (x_3, y_3)) &\geq \text{rmin}\{\tilde{\mu}_{\Xi_1 \times \Xi_2}((x_1, y_1) * \\ &(x_2, y_2) * (x_3, y_3)), \tilde{\mu}_{\Xi_1 \times \Xi_2}(x_2, y_2)\}, \\ (\text{CP6}) \quad \lambda_{\Xi_1 \times \Xi_2}((x_1, y_1) * (x_3, y_3)) &\leq \max\{\lambda_{\Xi_1 \times \Xi_2}((x_1, y_1) * \\ &(x_2, y_2) * (x_3, y_3)), \lambda_{\Xi_1 \times \Xi_2}(x_2, y_2)\}, \end{aligned}$$

for all $(x_1, y_1), (x_2, y_2), (x_3, y_3) \in X_1 \times X_2$.

Theorem 28. Let $\Xi_1 = \langle \tilde{\mu}_{\Xi_1}, \lambda_{\Xi_1} \rangle$ and $\Xi_2 = \langle \tilde{\mu}_{\Xi_2}, \lambda_{\Xi_2} \rangle$ be two cubic KU-subalgebras of KU-algebras X_1 and X_2 , respectively. Then $\Xi_1 \times \Xi_2 = \langle \tilde{\mu}_{\Xi_1 \times \Xi_2}, \lambda_{\Xi_1 \times \Xi_2} \rangle$ is a cubic KU-subalgebra of KU-algebra $X_1 \times X_2$.

Proof. For any $(x, y) \in X_1 \times X_2$,

$$\begin{aligned} \tilde{\mu}_{\Xi_1 \times \Xi_2}(0, 0) &= \text{rmin}\{\tilde{\mu}_{\Xi_1}(0), \tilde{\mu}_{\Xi_2}(0)\} \\ &\geq \text{rmin}\{\tilde{\mu}_{\Xi_1}(x), \tilde{\mu}_{\Xi_2}(y)\} = \tilde{\mu}_{\Xi_1 \times \Xi_2}(x, y), \\ \lambda_{\Xi_1 \times \Xi_2}(0, 0) &= \max\{\lambda_{\Xi_1}(0), \lambda_{\Xi_2}(0)\} \\ &\leq \max\{\lambda_{\Xi_1}(x), \lambda_{\Xi_2}(y)\} = \lambda_{\Xi_1 \times \Xi_2}(x, y). \end{aligned} \quad (32)$$

For any $(x_1, y_1), (x_2, y_2) \in X_1 \times X_2$. Then

$$\begin{aligned}
& \tilde{\mu}_{\Xi_1 \times \Xi_2}(x_1, y_1) \\
&= \text{rmin}\{\tilde{\mu}_{\Xi_1}(x_1), \tilde{\mu}_{\Xi_2}(y_1)\} \\
&\succeq \text{rmin}\{\text{rmin}\{\tilde{\mu}_{\Xi_1}(x_1 * x_2), \tilde{\mu}_{\Xi_1}(x_2)\}, \\
&\quad \text{rmin}\{\tilde{\mu}_{\Xi_2}(y_1 * y_2), \tilde{\mu}_{\Xi_2}(y_2)\}\} \\
&= \text{rmin}\{\text{rmin}\{\tilde{\mu}_{\Xi_1}(x_1 * x_2), \tilde{\mu}_{\Xi_2}(y_1 * y_2)\}, \\
&\quad \text{rmin}\{\tilde{\mu}_{\Xi_1}(x_2), \tilde{\mu}_{\Xi_2}(y_2)\}\} \\
&= \text{rmin}\{\tilde{\mu}_{\Xi_1 \times \Xi_2}(x_1 * x_2, y_1 * y_2), \tilde{\mu}_{\Xi_1 \times \Xi_2}(x_2, y_2)\} \\
&\succeq \text{rmin}\{\tilde{\mu}_{\Xi_1 \times \Xi_2}((x_1 * y_1)(x_2 * y_2)), \tilde{\mu}_{\Xi_1 \times \Xi_2}(x_2, y_2)\}, \\
&\lambda_{\Xi_1 \times \Xi_2}(x_1, y_1) \\
&= \max\{\lambda_{\Xi_1}(x_1), \lambda_{\Xi_2}(y_1)\} \\
&\leq \max\{\max\{\lambda_{\Xi_1}(x_1 * x_2), \lambda_{\Xi_1}(x_2)\}, \\
&\quad \max\{\lambda_{\Xi_2}(y_1 * y_2), \lambda_{\Xi_2}(y_2)\}\} \\
&= \max\{\max\{\lambda_{\Xi_1}(x_1 * x_2), \lambda_{\Xi_1}(y_1 * y_2)\}, \\
&\quad \max\{\lambda_{\Xi_2}(x_2), \lambda_{\Xi_2}(y_2)\}\} \\
&= \max\{\lambda_{\Xi_1 \times \Xi_2}(x_1 * x_2, y_1 * y_2), \lambda_{\Xi_1 \times \Xi_2}(x_2, y_2)\} \\
&\leq \max\{\lambda_{\Xi_1 \times \Xi_2}((x_1 * y_1)(x_2 * y_2)), \lambda_{\Xi_1 \times \Xi_2}(x_2, y_2)\}. \tag{33}
\end{aligned}$$

Hence, for all $(x_1, y_1), (x_2, y_2) \in X_1 \times X_2$, $\Xi_1 \times \Xi_2 = \langle \tilde{\mu}_{\Xi_1 \times \Xi_2}, \lambda_{\Xi_1 \times \Xi_2} \rangle$ is a cubic KU-subalgebra of KU-algebra $X_1 \times X_2$. \square

Theorem 29. Let $\Xi_1 = \langle \tilde{\mu}_{\Xi_1}, \lambda_{\Xi_1} \rangle$ and $\Xi_2 = \langle \tilde{\mu}_{\Xi_2}, \lambda_{\Xi_2} \rangle$ be two cubic KU-ideals of KU-algebras X_1 and X_2 , respectively. Then $\Xi_1 \times \Xi_2 = \langle \tilde{\mu}_{\Xi_1 \times \Xi_2}, \lambda_{\Xi_1 \times \Xi_2} \rangle$ is a cubic KU-ideal of KU-algebra $X_1 \times X_2$.

Proof. For any $(x, y) \in X_1 \times X_2$,

$$\begin{aligned}
\tilde{\mu}_{\Xi_1 \times \Xi_2}(0, 0) &= \text{rmin}\{\tilde{\mu}_{\Xi_1}(0), \tilde{\mu}_{\Xi_2}(0)\} \\
&\succeq \text{rmin}\{\tilde{\mu}_{\Xi_1}(x), \tilde{\mu}_{\Xi_2}(y)\} = \tilde{\mu}_{\Xi_1 \times \Xi_2}(x, y), \\
\lambda_{\Xi_1 \times \Xi_2}(0, 0) &= \max\{\lambda_{\Xi_1}(0), \lambda_{\Xi_2}(0)\} \\
&\leq \max\{\lambda_{\Xi_1}(x), \lambda_{\Xi_2}(y)\} = \lambda_{\Xi_1 \times \Xi_2}(x, y). \tag{34}
\end{aligned}$$

Now for any $(x_1, y_1), (x_2, y_2), (x_3, y_3) \in X_1 \times X_2$,

$$\begin{aligned}
& \tilde{\mu}_{\Xi_1 \times \Xi_2}((x_1, y_1) * (x_3, y_3)) \\
&= \tilde{\mu}_{\Xi_1 \times \Xi_2}(x_1 * x_3, y_1 * y_3) \\
&= \text{rmin}\{\tilde{\mu}_{\Xi_1}(x_1 * x_3), \tilde{\mu}_{\Xi_2}(y_1 * y_3)\} \\
&\succeq \text{rmin}\{\text{rmin}\{\tilde{\mu}_{\Xi_1}(x_1 * (x_2 * x_3)), \tilde{\mu}_{\Xi_1}(x_2)\}, \\
&\quad \text{rmin}\{\tilde{\mu}_{\Xi_2}(y_1 * (y_2 * y_3)), \tilde{\mu}_{\Xi_2}(y_2)\}\} \\
&= \text{rmin}\{\text{rmin}\{\tilde{\mu}_{\Xi_1}(x_1 * (x_2 * x_3)), \tilde{\mu}_{\Xi_2}(y_1 * (y_2 * y_3))\}, \\
&\quad \text{rmin}\{\tilde{\mu}_{\Xi_1}(x_2), \tilde{\mu}_{\Xi_2}(y_2)\}\} \\
&= \text{rmin}\{\tilde{\mu}_{\Xi_1 \times \Xi_2}((x_1 * (x_2 * x_3)), (y_1 * (y_2 * y_3))), \\
&\quad \tilde{\mu}_{\Xi_1 \times \Xi_2}(x_2, y_2)\} \\
&\succeq \text{rmin}\{\tilde{\mu}_{\Xi_1 \times \Xi_2}((x_1, y_1) * ((x_2, y_2) * (x_3, y_3))), \\
&\quad \tilde{\mu}_{\Xi_1 \times \Xi_2}(x_2, y_2)\}, \\
&\lambda_{\Xi_1 \times \Xi_2}((x_1, y_1) * (x_3, y_3)) \\
&= \lambda_{\Xi_1 \times \Xi_2}(x_1 * x_3, y_1 * y_3) \\
&= \max\{\lambda_{\Xi_1}(x_1 * x_3), \lambda_{\Xi_2}(y_1 * y_3)\} \\
&\leq \max\{\max\{\lambda_{\Xi_1}(x_1 * (x_2 * x_3)), \lambda_{\Xi_1}(x_2)\}, \\
&\quad \min\{\lambda_{\Xi_2}(y_1 * (y_2 * y_3)), \lambda_{\Xi_2}(y_2)\}\} \\
&= \max\{\max\{\lambda_{\Xi_1}(x_1 * (x_2 * x_3)), \lambda_{\Xi_2}(y_1 * (y_2 * y_3))\}, \\
&\quad \max\{\lambda_{\Xi_1}(x_2), \lambda_{\Xi_2}(y_2)\}\} \\
&= \max\{\lambda_{\Xi_1 \times \Xi_2}((x_1 * (x_2 * x_3)), (y_1 * (y_2 * y_3))), \\
&\quad \lambda_{\Xi_1 \times \Xi_2}(x_2, y_2)\} \\
&\leq \max\{\lambda_{\Xi_1 \times \Xi_2}((x_1, y_1) * ((x_2, y_2) * (x_3, y_3))), \\
&\quad \lambda_{\Xi_1 \times \Xi_2}(x_2, y_2)\}. \tag{35}
\end{aligned}$$

Hence, for all $(x_1, y_1), (x_2, y_2), (x_3, y_3) \in X_1 \times X_2$, $\Xi_1 \times \Xi_2 = \langle \tilde{\mu}_{\Xi_1 \times \Xi_2}, \lambda_{\Xi_1 \times \Xi_2} \rangle$ is a cubic KU-ideal of KU-algebra $X_1 \times X_2$. \square

Lemma 30. If $\Xi_1 \times \Xi_2 = \langle \tilde{\mu}_{\Xi_1 \times \Xi_2}, \lambda_{\Xi_1 \times \Xi_2} \rangle$ is a cubic KU-ideal of KU-algebra $X_1 \times X_2$ and if $(x_1, y_1) \leq (x_2, y_2)$, we have $\tilde{\mu}_{\Xi_1 \times \Xi_2}(x_2, y_2) \preceq \tilde{\mu}_{\Xi_1 \times \Xi_2}(x_1, y_1)$ and $\lambda_{\Xi_1 \times \Xi_2}(x_2, y_2) \geq \lambda_{\Xi_1 \times \Xi_2}(x_1, y_1)$, for all $(x_1, y_1), (x_2, y_2) \in X_1 \times X_2$.

Proof. Let $(x_1, y_1), (x_2, y_2) \in X_1 \times X_2$, such that $(x_1, y_1) \leq (x_2, y_2) \Rightarrow (x_2, y_2) * (x_1, y_1) = (0, 0)$. This together with

$(0, 0) * (x_1, y_1) = (x_1, y_1)$ and $\tilde{\mu}_{\Xi_1 \times \Xi_2}(x_2, y_2) \leq \tilde{\mu}_{\Xi_1 \times \Xi_2}(0, 0)$
also $\lambda_{\Xi_1 \times \Xi_2}(x_2, y_2) \geq \lambda_{\Xi_1 \times \Xi_2}(0, 0)$. Consider

$$\begin{aligned}
& \tilde{\mu}_{\Xi_1 \times \Xi_2}((0, 0) * (x_1, y_1)) \\
&= \tilde{\mu}_{\Xi_1 \times \Xi_2}(x_1, y_1) \\
&\geq \text{rmin}\{\tilde{\mu}_{\Xi_1 \times \Xi_2}((0, 0) * ((x_2, y_2) * (x_1, y_1))), \\
&\quad \tilde{\mu}_{\Xi_1 \times \Xi_2}(x_2, y_2)\} \\
&= \text{rmin}\{\tilde{\mu}_{\Xi_1 \times \Xi_2}((0, 0) * (0, 0)), \tilde{\mu}_{\Xi_1 \times \Xi_2}(x_2, y_2)\} \\
&= \text{rmin}\{\tilde{\mu}_{\Xi_1 \times \Xi_2}(0, 0), \tilde{\mu}_{\Xi_1 \times \Xi_2}(x_2, y_2)\} \\
&= \tilde{\mu}_{\Xi_1 \times \Xi_2}(x_2, y_2), \\
&\lambda_{\Xi_1 \times \Xi_2}((0, 0) * (x_1, y_1)) \\
&= \lambda_{\Xi_1 \times \Xi_2}(x_1, y_1) \\
&\leq \max\{\lambda_{\Xi_1 \times \Xi_2}((0, 0) * ((x_2, y_2) * (x_1, y_1))), \\
&\quad \lambda_{\Xi_1 \times \Xi_2}(x_2, y_2)\} \\
&= \max\{\lambda_{\Xi_1 \times \Xi_2}((0, 0) * (0, 0)), \lambda_{\Xi_1 \times \Xi_2}(x_2, y_2)\} \\
&= \max\{\lambda_{\Xi_1 \times \Xi_2}(0, 0), \lambda_{\Xi_1 \times \Xi_2}(x_2, y_2)\} \\
&= \lambda_{\Xi_1 \times \Xi_2}(x_2, y_2).
\end{aligned} \tag{36}$$

This shows that $\tilde{\mu}_{\Xi_1 \times \Xi_2}(x_2, y_2) \leq \tilde{\mu}_{\Xi_1 \times \Xi_2}(x_1, y_1)$ and $\lambda_{\Xi_1 \times \Xi_2}(x_2, y_2) \geq \lambda_{\Xi_1 \times \Xi_2}(x_1, y_1)$, for all $(x_1, y_1), (x_2, y_2) \in X_1 \times X_2$. \square

Lemma 31. If $\Xi_1 \times \Xi_2 = \langle \tilde{\mu}_{\Xi_1 \times \Xi_2}, \lambda_{\Xi_1 \times \Xi_2} \rangle$ is a cubic KU-ideal of KU-algebra $X_1 \times X_2$ and if $(x_1, y_1) * (x_2, y_2) \leq (x_3, y_3)$ holds in $X_1 \times X_2$, then we have $\tilde{\mu}_{\Xi_1 \times \Xi_2}(x_2, y_2) \geq \text{rmin}\{\tilde{\mu}_{\Xi_1 \times \Xi_2}(x_1, y_1), \tilde{\mu}_{\Xi_1 \times \Xi_2}(x_3, y_3)\}$ and $\lambda_{\Xi_1 \times \Xi_2}(x_2, y_2) \leq \max\{\lambda_{\Xi_1 \times \Xi_2}(x_1, y_1), \lambda_{\Xi_1 \times \Xi_2}(x_3, y_3)\}$, for all $(x_1, y_1), (x_2, y_2), (x_3, y_3) \in X_1 \times X_2$.

Proof. Let $(x_1, y_1), (x_2, y_2), (x_3, y_3) \in X_1 \times X_2$ and let $(x_1, y_1) * (x_2, y_2) \leq (x_3, y_3)$ holds in $X_1 \times X_2$, then

$$(x_3, y_3) * ((x_1, y_1) * (x_2, y_2)) = (0, 0). \tag{37}$$

Now for $(x_3, y_3) = (0, 0)$ and from (CP5)

$$\begin{aligned}
& \tilde{\mu}_{\Xi_1 \times \Xi_2}((x_3, y_3) * (x_2, y_2)) \\
&\geq \text{rmin}\{\tilde{\mu}_{\Xi_1 \times \Xi_2}((x_3, y_3) * ((x_1, y_1) * (x_2, y_2))),
\end{aligned} \tag{38}$$

$$\tilde{\mu}_{\Xi_1 \times \Xi_2}(x_1, y_1)\},$$

we have

$$\begin{aligned}
& \tilde{\mu}_{\Xi_1 \times \Xi_2}((0, 0) * (x_2, y_2)) \\
&= \tilde{\mu}_{\Xi_1 \times \Xi_2}(x_2, y_2) \\
&\geq \text{rmin}\{\tilde{\mu}_{\Xi_1 \times \Xi_2}((x_1, y_1) * (x_2, y_2)), \tilde{\mu}_{\Xi_1 \times \Xi_2}(x_1, y_1)\} \\
&\geq \text{rmin}\{\text{rmin}\{\tilde{\mu}_{\Xi_1 \times \Xi_2}((x_1, y_1) * ((x_3, y_3)(x_2, y_2))), \\
&\quad \tilde{\mu}_{\Xi_1 \times \Xi_2}(x_3, y_3)\}, \tilde{\mu}_{\Xi_1 \times \Xi_2}(x_1, y_1)\} \\
&= \text{rmin}\{\text{rmin}\{\tilde{\mu}_{\Xi_1 \times \Xi_2}((x_3, y_3) * ((x_1, y_1)(x_2, y_2))), \\
&\quad \tilde{\mu}_{\Xi_1 \times \Xi_2}(x_3, y_3)\}, \tilde{\mu}_{\Xi_1 \times \Xi_2}(x_1, y_1)\} \\
&= \text{rmin}\{\tilde{\mu}_{\Xi_1 \times \Xi_2}(x_3, y_3), \tilde{\mu}_{\Xi_1 \times \Xi_2}(x_1, y_1)\} \\
&= \text{rmin}\{\tilde{\mu}_{\Xi_1 \times \Xi_2}(x_1, y_1), \tilde{\mu}_{\Xi_1 \times \Xi_2}(x_3, y_3)\},
\end{aligned} \tag{39}$$

and from (CP6)

$$\begin{aligned}
& \lambda_{\Xi_1 \times \Xi_2}((x_3, y_3) * (x_2, y_2)) \\
&\leq \max\{\lambda_{\Xi_1 \times \Xi_2}((x_3, y_3) * ((x_1, y_1) * (x_2, y_2))), \\
&\quad \lambda_{\Xi_1 \times \Xi_2}(x_1, y_1)\},
\end{aligned} \tag{40}$$

we have

$$\begin{aligned}
& \lambda_{\Xi_1 \times \Xi_2}((0, 0) * (x_2, y_2)) \\
&= \lambda_{\Xi_1 \times \Xi_2}(x_2, y_2) \\
&\leq \max\{\lambda_{\Xi_1 \times \Xi_2}((x_1, y_1) * (x_2, y_2)), \lambda_{\Xi_1 \times \Xi_2}(x_1, y_1)\} \\
&\leq \max\{\max\{\lambda_{\Xi_1 \times \Xi_2}((x_1, y_1) * ((x_3, y_3)(x_2, y_2))), \\
&\quad \lambda_{\Xi_1 \times \Xi_2}(x_3, y_3)\}, \lambda_{\Xi_1 \times \Xi_2}(x_1, y_1)\} \\
&= \max\{\max\{\lambda_{\Xi_1 \times \Xi_2}((x_3, y_3) * ((x_1, y_1)(x_2, y_2))), \\
&\quad \lambda_{\Xi_1 \times \Xi_2}(x_3, y_3)\}, \lambda_{\Xi_1 \times \Xi_2}(x_1, y_1)\} \\
&= \max\{\max\{\lambda_{\Xi_1 \times \Xi_2}(0, 0), \lambda_{\Xi_1 \times \Xi_2}(x_3, y_3)\}, \\
&\quad \lambda_{\Xi_1 \times \Xi_2}(x_1, y_1)\} \\
&= \max\{\lambda_{\Xi_1 \times \Xi_2}(x_3, y_3), \lambda_{\Xi_1 \times \Xi_2}(x_1, y_1)\} \\
&= \max\{\lambda_{\Xi_1 \times \Xi_2}(x_1, y_1), \lambda_{\Xi_1 \times \Xi_2}(x_3, y_3)\}.
\end{aligned} \tag{41}$$

This completes the proof. \square

Definition 32. Let $\Xi_1 \times \Xi_2 = \langle \tilde{\mu}_{\Xi_1 \times \Xi_2}, \lambda_{\Xi_1 \times \Xi_2} \rangle$ be a cubic subset of KU-algebra $X_1 \times X_2$, and for any $\tilde{t} \in D[0, 1]$ and $s \in [0, 1]$, the set

$$\begin{aligned} U(\Xi_1 \times \Xi_2; \tilde{t}, s) \\ = \{(x, y) \in X_1 \times X_2 : \tilde{\mu}_{\Xi_1 \times \Xi_2}(x, y) \geq \tilde{t}, \\ \lambda_{\Xi_1 \times \Xi_2}(x, y) \leq s\} \end{aligned} \quad (42)$$

is called the cubic level set of $\Xi_1 \times \Xi_2 = \langle \tilde{\mu}_{\Xi_1 \times \Xi_2}, \lambda_{\Xi_1 \times \Xi_2} \rangle$.

Theorem 33. Let $\Xi_1 \times \Xi_2 = \langle \tilde{\mu}_{\Xi_1 \times \Xi_2}, \lambda_{\Xi_1 \times \Xi_2} \rangle$ be a cubic subset of KU-algebra $X_1 \times X_2$. Then $\Xi_1 \times \Xi_2 = \langle \tilde{\mu}_{\Xi_1 \times \Xi_2}, \lambda_{\Xi_1 \times \Xi_2} \rangle$ is a cubic KU-subalgebra of KU-algebra $X_1 \times X_2$ if and only if for any $\tilde{t} \in D[0, 1]$ and $s \in [0, 1]$, the set $U(\Xi_1 \times \Xi_2; \tilde{t}, s)$ is either empty or a KU-subalgebra of $X_1 \times X_2$. \square

Proof. The proof is straightforward. \square

Theorem 34. Let $\Xi_1 \times \Xi_2 = \langle \tilde{\mu}_{\Xi_1 \times \Xi_2}, \lambda_{\Xi_1 \times \Xi_2} \rangle$ be a cubic subset of KU-algebra $X_1 \times X_2$. Then $\Xi_1 \times \Xi_2 = \langle \tilde{\mu}_{\Xi_1 \times \Xi_2}, \lambda_{\Xi_1 \times \Xi_2} \rangle$ is a cubic KU-ideal of KU-algebra $X_1 \times X_2$ if and only if for any $\tilde{t} \in D[0, 1]$ and $s \in [0, 1]$, the set $U(\Xi_1 \times \Xi_2; \tilde{t}, s)$ is either empty or a KU-ideal of $X_1 \times X_2$.

Proof. Let $\Xi_1 \times \Xi_2 = \langle \tilde{\mu}_{\Xi_1 \times \Xi_2}, \lambda_{\Xi_1 \times \Xi_2} \rangle$ be a cubic KU-ideal of KU-algebra $X_1 \times X_2$. For any $\tilde{t} \in D[0, 1]$ and $s \in [0, 1]$, define the sets

$$\begin{aligned} U(\Xi_1 \times \Xi_2; \tilde{t}, s) \\ = \{(x, y) \in X_1 \times X_2 : \tilde{\mu}_{\Xi_1 \times \Xi_2}(x, y) \geq \tilde{t}, \\ \lambda_{\Xi_1 \times \Xi_2}(x, y) \leq s\}. \end{aligned} \quad (43)$$

Since $U(\Xi_1 \times \Xi_2; \tilde{t}, s) \neq \emptyset$, let $(x, y) \in U(\Xi_1 \times \Xi_2; \tilde{t}, s)$. This implies $\tilde{\mu}_{\Xi_1 \times \Xi_2}(x, y) \geq \tilde{t}$ and $\lambda_{\Xi_1 \times \Xi_2}(x, y) \leq s$. So

$$\begin{aligned} \tilde{\mu}_{\Xi_1 \times \Xi_2}(0, 0) \geq \tilde{\mu}_{\Xi_1 \times \Xi_2}(x, y) \geq \tilde{t} \implies \tilde{\mu}_{\Xi_1 \times \Xi_2}(0, 0) \geq \tilde{t}, \\ \lambda_{\Xi_1 \times \Xi_2}(0, 0) \leq \lambda_{\Xi_1 \times \Xi_2}(x, y) \leq s \implies \lambda_{\Xi_1 \times \Xi_2}(0, 0) \leq s. \end{aligned} \quad (44)$$

This shows that $(0, 0) \in U(\Xi_1 \times \Xi_2; \tilde{t}, s)$. Let $(x_1, y_1) * ((x_2, y_2) * (x_3, y_3)) \in U(\Xi_1 \times \Xi_2; \tilde{t}, s)$ and $(x_2, y_2) \in U(\Xi_1 \times \Xi_2; \tilde{t}, s)$. This implies

$$\begin{aligned} \tilde{\mu}_{\Xi_1 \times \Xi_2}((x_1, y_1) * ((x_2, y_2) * (x_3, y_3))) &\geq \tilde{t}, \\ \tilde{\mu}_{\Xi_1 \times \Xi_2}(x_2, y_2) &\geq \tilde{t}, \\ \lambda_{\Xi_1 \times \Xi_2}((x_1, y_1) * ((x_2, y_2) * (x_3, y_3))) &\leq s, \\ \lambda_{\Xi_1 \times \Xi_2}(x_2, y_2) &\leq s. \end{aligned} \quad (45)$$

Since

$$\begin{aligned} &\tilde{\mu}_{\Xi_1 \times \Xi_2}((x_1, y_1) * (x_3, y_3)) \\ &\geq \text{rmin}\{\tilde{\mu}_{\Xi_1 \times \Xi_2}((x_1, y_1) * ((x_2, y_2) * (x_3, y_3))), \\ &\quad \tilde{\mu}_{\Xi_1 \times \Xi_2}(x_2, y_2)\} \\ &\geq \text{rmin}\{\tilde{t}, \tilde{t}\} = \tilde{t}, \\ &\lambda_{\Xi_1 \times \Xi_2}((x_1, y_1) * (x_3, y_3)) \\ &\leq \max\{\lambda_{\Xi_1 \times \Xi_2}((x_1, y_1) * ((x_2, y_2) * (x_3, y_3))), \\ &\quad \lambda_{\Xi_1 \times \Xi_2}(x_2, y_2)\} \\ &\leq \max\{s, s\} = s. \end{aligned} \quad (46)$$

This implied that $(x_1, y_1) * (x_3, y_3) \in U(\Xi_1 \times \Xi_2; \tilde{t}, s)$. Hence, $U(\Xi_1 \times \Xi_2; \tilde{t}, s)$ is a KU-ideal of $X_1 \times X_2$.

Conversely, suppose $U(\Xi_1 \times \Xi_2; \tilde{t}, s)$ is a KU-ideal of $X_1 \times X_2$, for any $\tilde{t} \in D[0, 1]$ and $s \in [0, 1]$. Assume $(x_1, y_1) \in X_1 \times X_2$, such that

$$\begin{aligned} \tilde{\mu}_{\Xi_1 \times \Xi_2}(0, 0) &< \tilde{\mu}_{\Xi_1 \times \Xi_2}(x_1, y_1), \\ \lambda_{\Xi_1 \times \Xi_2}(0, 0) &> \lambda_{\Xi_1 \times \Xi_2}(x_1, y_1). \end{aligned} \quad (47)$$

Put

$$\begin{aligned} \tilde{t}_o &= \frac{1}{2} \{\tilde{\mu}_{\Xi_1 \times \Xi_2}(0, 0) + \tilde{\mu}_{\Xi_1 \times \Xi_2}(x_1, y_1)\} \\ &\implies \tilde{\mu}_{\Xi_1 \times \Xi_2}(0, 0) < \tilde{t}_o < \tilde{\mu}_{\Xi_1 \times \Xi_2}(x_1, y_1), \\ s_o &= \frac{1}{2} \{\lambda_{\Xi_1 \times \Xi_2}(0, 0) + \lambda_{\Xi_1 \times \Xi_2}(x_1, y_1)\} \\ &\implies \lambda_{\Xi_1 \times \Xi_2}(0, 0) > t_o > \lambda_{\Xi_1 \times \Xi_2}(x_1, y_1). \end{aligned} \quad (48)$$

This implies $(x_1, y_1) \in U(\Xi_1 \times \Xi_2; \tilde{t}, s)$, but $(0, 0) \notin U(\Xi_1 \times \Xi_2; \tilde{t}, s)$, which is contradiction. Therefore, $\tilde{\mu}_{\Xi_1 \times \Xi_2}(0, 0) \geq \tilde{\mu}_{\Xi_1 \times \Xi_2}(x, y)$ and $\lambda_{\Xi_1 \times \Xi_2}(0, 0) \leq \lambda_{\Xi_1 \times \Xi_2}(x, y)$ for all $(x, y) \in X_1 \times X_2$. Assume $(x_1, y_1), (x_2, y_2), (x_3, y_3) \in X_1 \times X_2$, such that

$$\begin{aligned} &\tilde{\mu}_{\Xi_1 \times \Xi_2}((x_1, y_1) * (x_3, y_3)) \\ &< \text{rmin}\{\tilde{\mu}_{\Xi_1 \times \Xi_2}((x_1, y_1) * ((x_2, y_2) * (x_3, y_3))), \\ &\quad \tilde{\mu}_{\Xi_1 \times \Xi_2}(x_2, y_2)\}. \end{aligned} \quad (49)$$

Let

$$\begin{aligned} \tilde{t}_o &= \frac{1}{2} \{\tilde{\mu}_{\Xi_1 \times \Xi_2}((x_1, y_1) * (x_3, y_3)) \\ &\quad + \text{rmin}\{\tilde{\mu}_{\Xi_1 \times \Xi_2}((x_1, y_1) * ((x_2, y_2) * (x_3, y_3))), \\ &\quad \tilde{\mu}_{\Xi_1 \times \Xi_2}(x_2, y_2)\}\}, \end{aligned} \quad (50)$$

then

$$\begin{aligned} & \tilde{\mu}_{\Xi_1 \times \Xi_2}((x_1, y_1) * (x_3, y_3)) \\ & \prec \tilde{t}_o \prec \text{rmin} \left\{ \tilde{\mu}_{\Xi_1 \times \Xi_2}((x_1, y_1) * ((x_2, y_2) * (x_3, y_3))), \right. \\ & \quad \left. \tilde{\mu}_{\Xi_1 \times \Xi_2}(x_2, y_2) \right\}. \end{aligned} \quad (51)$$

Also

$$\begin{aligned} & \lambda_{\Xi_1 \times \Xi_2}((x_1, y_1) * (x_3, y_3)) \\ & > \max \left\{ \lambda_{\Xi_1 \times \Xi_2}((x_1, y_1) * ((x_2, y_2) * (x_3, y_3))), \right. \\ & \quad \left. \lambda_{\Xi_1 \times \Xi_2}(x_2, y_2) \right\}. \end{aligned} \quad (52)$$

Let

$$\begin{aligned} s_o = \frac{1}{2} & \left\{ \lambda_{\Xi_1 \times \Xi_2}((x_1, y_1) * (x_3, y_3)) \right. \\ & + \max \left\{ \lambda_{\Xi_1 \times \Xi_2}((x_1, y_1) * ((x_2, y_2) * (x_3, y_3))), \right. \\ & \quad \left. \lambda_{\Xi_1 \times \Xi_2}(x_2, y_2) \right\}, \end{aligned} \quad (53)$$

then

$$\begin{aligned} & \lambda_{\Xi_1 \times \Xi_2}((x_1, y_1) * (x_3, y_3)) \\ & > s_o > \max \left\{ \lambda_{\Xi_1 \times \Xi_2}((x_1, y_1) * ((x_2, y_2) * (x_3, y_3))), \right. \\ & \quad \left. \lambda_{\Xi_1 \times \Xi_2}(x_2, y_2) \right\}. \end{aligned} \quad (54)$$

This shows that

$$\begin{aligned} & (x_1, y_1) * ((x_2, y_2) * (x_3, y_3)) \in U(\Xi_1 \times \Xi_2; \tilde{t}, s), \\ & (x_2, y_2) \in U(\Xi_1 \times \Xi_2; \tilde{t}, s). \end{aligned} \quad (55)$$

But $(x_1, y_1) * (x_3, y_3) \notin U(\tilde{\mu}_{\Xi_1 \times \Xi_2}; \tilde{t}_o, s_o)$, which is a contradiction; therefore,

$$\begin{aligned} & \tilde{\mu}_{\Xi_1 \times \Xi_2}((x_1, y_1) * (x_3, y_3)) \\ & \geq \text{rmin} \left\{ \tilde{\mu}_{\Xi_1 \times \Xi_2}((x_1, y_1) * ((x_2, y_2) * (x_3, y_3))), \right. \\ & \quad \left. \tilde{\mu}_{\Xi_1 \times \Xi_2}(x_2, y_2) \right\}. \end{aligned} \quad (56)$$

Similarly

$$\begin{aligned} & \lambda_{\Xi_1 \times \Xi_2}((x_1, y_1) * (x_3, y_3)) \\ & \leq \max \left\{ \lambda_{\Xi_1 \times \Xi_2}((x_1, y_1) * ((x_2, y_2) * (x_3, y_3))), \right. \\ & \quad \left. \lambda_{\Xi_1 \times \Xi_2}(x_2, y_2) \right\}. \end{aligned} \quad (57)$$

Hence, $\Xi_1 \times \Xi_2 = \langle \tilde{\mu}_{\Xi_1 \times \Xi_2}, \lambda_{\Xi_1 \times \Xi_2} \rangle$ is a cubic KU-ideal of KU-algebra $X_1 \times X_2$. \square

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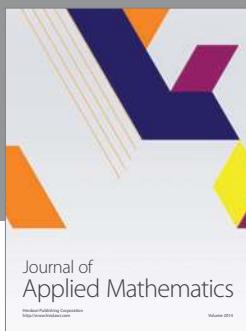
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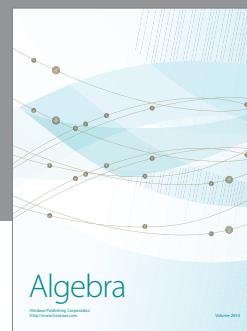
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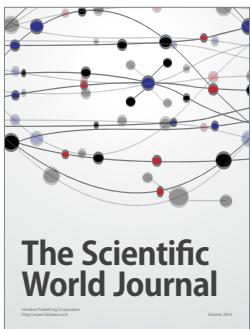
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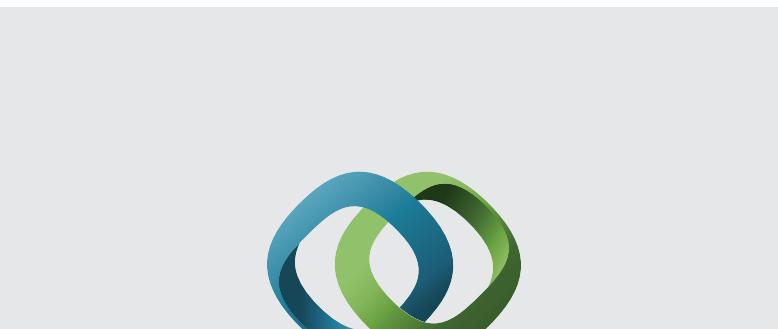
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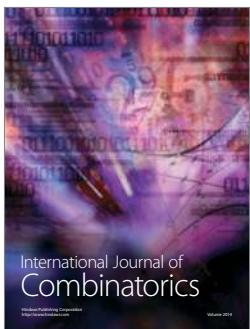


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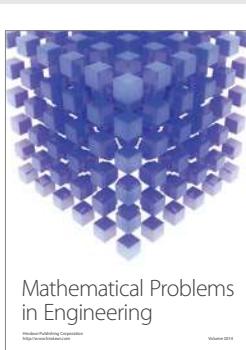
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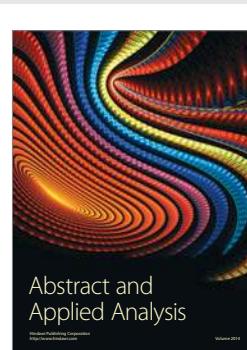
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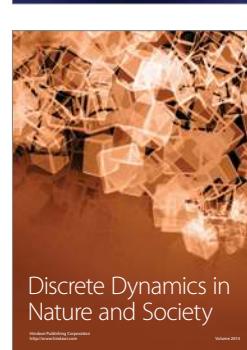
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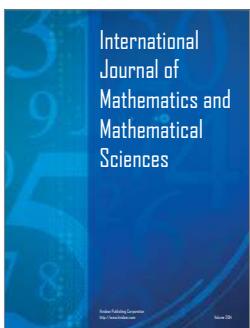
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