On Current and Power Injection Models for Angle and Voltage Stability Analysis of Power Systems

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Abstract— This letter provides a comparison between the current and power injection models for angle and voltage stability analysis of power systems. Traditionally, it is believed that the current injection model is computationally more efficient and, for this reason, this is the model that is implemented in most commercial software tools for dynamic studies and transient analysis of power systems. On the other hand, the power injection model requires computing the products of currents and voltages and, hence, is expected to be computationally more demanding than the current injection one. This letter compares the two approaches using a unique framework. The two formulations are compared through a dynamic 1479-bus model of the all-island Irish transmission system.

Index Terms— Current injection model, power injection model, transient stability analysis, power system dynamics.

I. INTRODUCTION

THE conventional power system model used for angle and voltage stability analysis is a set of nonlinear differential algebraic equations (DAE) with inclusion of discrete variables:

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{p}, \boldsymbol{u}, t) \qquad (1)$$
$$\boldsymbol{0} = \boldsymbol{g}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{p}, \boldsymbol{u}, t) \ ,$$

where $\boldsymbol{x}$ ($\boldsymbol{x} \in \mathbb{R}^{n_x}$) indicates the vector state variables, $\boldsymbol{y}$ ($\boldsymbol{y} \in \mathbb{R}^{n_y}$) are the algebraic variables, $\boldsymbol{p}$ ($\boldsymbol{p} \in \mathbb{R}^{n_p}$) are input variables, $\boldsymbol{u}$ ($\boldsymbol{u} \in \mathbb{R}^{n_u}$) the vector of discrete variables, t ($t \in \mathbb{R}^+$) the time, $\boldsymbol{f}$ ($\boldsymbol{f} : \mathbb{R}^{n_x} \times \mathbb{R}^{n_y} \times \mathbb{R}^{n_p} \times \mathbb{R}^{n_u} \times \mathbb{R}^+ \mapsto \mathbb{R}^{n_x}$) are the differential equations, and $\boldsymbol{g}$ ($\boldsymbol{g} : \mathbb{R}^{n_x} \times \mathbb{R}^{n_y} \times \mathbb{R}^{n_p} \times \mathbb{R}^{n_u} \times \mathbb{R}^+ \mapsto \mathbb{R}^{n_y}$) are the algebraic equations. If discrete variables $\boldsymbol{u}$ are substituted for if-then rules, (1) becomes an *hybrid dynamical system*, i.e., a collection of continuous DAE, one per each discrete variable change [1].

Differential equations $\boldsymbol{f}$ depends on machine, regulator and load dynamic models. While the formulation of differential equations $\boldsymbol{f}$ is not unique, the structure of $\boldsymbol{f}$ and, in particular, its nonlinearity is unequivocally defined. On the other hand, the subset of algebraic equations $\boldsymbol{g}$ that describes the flows in transmission lines can be written in terms of currents or complex powers. The two approaches lead to different structures, i.e., nonlinearity, and hence computational complexity of the overall set of DAEs.

The main purpose of the letter is to show that the power injection model is not necessarily more computationally expensive than the current injection one. With this aim, the letter provides a fair comparison of the two models through the simulation of a large power system. This comparison cannot be found in the literature, as common power system software tools provide either the current or the power injection model, but not both. Hence, the discussion and the case study provided in the letter can be useful to the power system community and, in particular, to power system model developers.

The two models, namely, the current and power injection ones, are briefly recalled below to allow a clear comparison in Section II.

A. Current Injection Model

The classical formulation of algebraic equations for transient stability analysis is the current injection model [2]. According to this model, the algebraic variables are *exclusively* the bus voltage phasors $\bar{\boldsymbol{v}}$. The algebraic equations express the current injections at network buses:

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \bar{\boldsymbol{v}}, \boldsymbol{p}, \boldsymbol{u}, t) \qquad (2)$$
$$\boldsymbol{0} = \bar{\boldsymbol{i}}(\boldsymbol{x}, \bar{\boldsymbol{v}}, \boldsymbol{p}, \boldsymbol{u}, t) - \bar{\boldsymbol{Y}}(\boldsymbol{u})\bar{\boldsymbol{v}} \ .$$

where $\bar{\boldsymbol{i}}$ are the current injections at network buses, $\bar{\boldsymbol{v}}$ are the bus voltage phasors, and $\bar{\boldsymbol{Y}}$ is the system admittance matrix. Note that, in this classical formulation, $\bar{\boldsymbol{Y}}$ depends on system events $\boldsymbol{u}$ (e.g., transmission line outages) but not on other system variables.

Equation (2) can be simplified if loads are modelled as constant admittances. In fact, for a constant admittance, load currents are given by $\bar{i}_L = \bar{y}_L \bar{v}_L$ and $\bar{y}_L$ can be added to the diagonal element of the admittance matrix $\bar{\boldsymbol{Y}}$ corresponding to load bus L. Similarly, pure transit nodes are special loads for which $\bar{y}_L = 0$. By defining the reduced generator admittance matrix $\bar{\boldsymbol{Y}}_G$ as:

$$\bar{\boldsymbol{Y}}_G = \bar{\boldsymbol{Y}}_{GG} - \bar{\boldsymbol{Y}}_{GL}[\bar{\boldsymbol{Y}}_{LL} + \bar{\boldsymbol{Y}}_L]^{-1}\bar{\boldsymbol{Y}}_{LG} \ , \qquad (3)$$

where $\bar{\boldsymbol{Y}}_{GG}$, $\bar{\boldsymbol{Y}}_{GL}$, $\bar{\boldsymbol{Y}}_{LG}$ and $\bar{\boldsymbol{Y}}_{LL}$ are the partitions of $\bar{\boldsymbol{Y}}$ corresponding to generator (G) and load (L) buses, and $\bar{\boldsymbol{Y}}_L$ is the diagonal matrix of load admittances, one can rewrite (2) as:

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \bar{\boldsymbol{v}}_G, \boldsymbol{p}, \boldsymbol{u}, t) \qquad (4)$$
$$\boldsymbol{0} = \bar{\boldsymbol{i}}_G(\boldsymbol{x}, \bar{\boldsymbol{v}}_G, \boldsymbol{p}, \boldsymbol{u}, t) - \bar{\boldsymbol{Y}}_G(\boldsymbol{u})\bar{\boldsymbol{v}}_G \ .$$

The model above is the most commonly used in transient stability analysis, especially in proprietary software packages.

B. Power Injection Model

The power injection model can be obtained from (2) by multiplying the conjugate of algebraic equations by bus voltages phasors:

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \bar{\boldsymbol{v}}, \boldsymbol{p}, \boldsymbol{u}, t) \qquad (5)$$
$$\boldsymbol{0} = \bar{\boldsymbol{s}}(\boldsymbol{x}, \bar{\boldsymbol{v}}, \boldsymbol{p}, \boldsymbol{u}, t) - \bar{\boldsymbol{V}}\bar{\boldsymbol{Y}}^*(\boldsymbol{u})\bar{\boldsymbol{v}}^* \ ,$$

where $\bar{\boldsymbol{V}} = \mathrm{diag}(\bar{v}_1, \bar{v}_2, \dots, \bar{v}_{n_b})$, where n_b is the number of buses. The term $\bar{\boldsymbol{V}}\bar{\boldsymbol{Y}}^*\bar{\boldsymbol{v}}^*$ are the power flow equations, and $\bar{\boldsymbol{s}} = \bar{\boldsymbol{V}}\bar{\boldsymbol{i}}^*$ are the complex powers injected at network buses. Equations (5) are equivalent to (2) but are intrinsically nonlinear and, as discussed in [3], are expected to be computationally more demanding than (2). The latter statement is further discussed and partially contradicted by simulation results presented in Section III. Note also that in (5) there is no need to assume a constant admittance matrix as the nonlinearity of algebraic equations prevent removing load buses as in (2).

C. Augmented Power System Models

The sets of equations (2) and (5) assume that the only algebraic variables are bus voltage phasors. These formulations cannot include causal constraints or auxiliary variables that are often needed to simplify the modelling and simulation of complex systems. Recent studies have identified alternative formulations that offer numerical and/or computational advantages. For the sake of example, see [4] for a formulation based on modified augmented nodal analysis approach and [5] for a formulation that accounts for zero-time constants of differential equations. The augmented version of the power injection model (2) with inclusion of additional algebraic variables $\hat{\boldsymbol{y}}$ and constraints $\hat{\boldsymbol{g}}$:

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \hat{\boldsymbol{y}}, \bar{\boldsymbol{v}}, \boldsymbol{p}, \boldsymbol{u}, t) \qquad (6)$$
$$\boldsymbol{0} = \hat{\boldsymbol{g}}(\boldsymbol{x}, \hat{\boldsymbol{y}}, \bar{\boldsymbol{v}}, \boldsymbol{p}, \boldsymbol{u}, t)$$
$$\boldsymbol{0} = \bar{\boldsymbol{i}}(\boldsymbol{x}, \hat{\boldsymbol{y}}, \bar{\boldsymbol{v}}, \boldsymbol{p}, \boldsymbol{u}, t) - \bar{\boldsymbol{Y}}(\boldsymbol{u})\bar{\boldsymbol{v}} \ .$$

The augmented formulation of (5) can be defined in a similar way:

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \tilde{\boldsymbol{y}}, \bar{\boldsymbol{v}}, \boldsymbol{p}, \boldsymbol{u}, t) \qquad (7)$$
$$\boldsymbol{0} = \bar{\boldsymbol{g}}(\boldsymbol{x}, \tilde{\boldsymbol{y}}, \bar{\boldsymbol{v}}, \boldsymbol{p}, \boldsymbol{u}, t)$$
$$\boldsymbol{0} = \bar{\boldsymbol{s}}(\boldsymbol{x}, \tilde{\boldsymbol{y}}, \bar{\boldsymbol{v}}, \boldsymbol{p}, \boldsymbol{u}, t) - \bar{\boldsymbol{V}}\bar{\boldsymbol{Y}}^*(\boldsymbol{u})\bar{\boldsymbol{v}}^* \ .$$

Note that the algebraic variables and constraints in (6) and (7) are not necessarily the same. This point is further discussed in the case study presented in Section III.

II. REMARKS ON CURRENT AND POWER INJECTION MODELS

The current and power injection models are expected to be mathematically equivalent and to give same results. Arguably, the nonlinearity of the power injections introduce extraneous solutions, but, in practice, only the physically meaningful solution is found. On the other hand, due to the reduced size and linearity with respect to bus voltages of the second equation of (4), the current injection model is commonly considered better performing than the power injection one [3]. While it is true that the power injection model is not common in commercial software, there are some methodological and practical reasons why the current injection model is not necessarily to be preferred even if considering only performance and computational speed, as discussed below.

The two major assumptions in (4) are that loads can be considered constant admittances and the admittance matrix is constant except for few discrete events. These assumptions are correct only for short-term transient stability analysis (i.e., loss of synchronism of synchronous machines following a short-circuit). However, these simplifications are not acceptable for medium- and long-term angle and voltage stability analyses. For longer time scales, in fact, the actions of ULTC transformers and load dynamics cannot be ignored [6]. Moreover, power systems are evolving towards a situation in which the high penetration of distributed generation, FACTS devices, HVDC connections and advanced control schemes, consistently reduces the number of *pure* load buses. The inadequacy of (4) is also methodological. In fact, in (4), the formulation of algebraic constraints relies on a strong modelling assumption. If the hypothesis on load models is removed, (4) cannot be defined. Hence, in the remainder of this letter, only extended versions of the current and power injection models, (6) and (7), respectively, are considered.

It remains the fact that the current injection model is *more* linear than the power injection one. This is certainly an advantage as the number of constant elements of the Jacobian matrix are higher for (6) than for (7). However, both (6) and (7) are nonlinear. In fact, (6) is nonlinear at least in the differential equations of synchronous machines (e.g., mechanical equations, saturations, etc.).

For the reasons above, it is not obvious that the current injection model provides a better performance than the power injection one. Moreover, since software packages implement only one model, a direct and fair comparison of the performance of the two formulations is not a simple task to solve. The following case study provides a quantitative evaluation of these two formulations based on the same software tool.

III. CASE STUDY

In this case study, the augmented current and power injection models (6) and (7) are compared through a dynamic model of the all-island Irish transmission system developed at the UCD Electricity Research Center. The model includes 1479 buses, 1851 transmission lines and transformers, 245 loads, 22 conventional synchronous power plants with AVRs and turbine governors, 6 PSSs and 176 wind power plants. All loads are modelled as ZIP, i.e., a mix of constant impedance, constant current and constant power. The topology and the data of the transmission system are based on the actual real-world system, but dynamic data are guessed based on the knowledge of the technology of power plants. Simulations are solved using Dome [7], which provides the ability to define dynamic models using either the current or the power injection formulations. Both models are solved using a simultaneous implicit trapezoidal scheme.

The all-island Irish system has been simulated for 30 seconds considering a three-phase fault on a 275 kV bus and stochastic variations of wind speeds. The fault is cleared after 80 ms by opening

TABLE I
STATISTICS FOR THE ALL-ISLAND IRISH TRANSMISSION SYSTEM

Model	Current Injection	Power Injection
CPU time	2.49 s	2.25 s
State Variables x	1436	1436
Algeb. Variables y	6594	5470
# of elements of A_C	$64.5 \cdot 10^6$	$47.7 \cdot 10^6$
NNZ of A_C	0.055%	0.068%
Constant NNZ of A_C	0.042%	0.016%

NNZ: number of non-zero elements
A_C: complete Jacobian matrix of the DAE

a transmission line connected to the faulted bus. For the current injection model (6), currents and voltages are represented in rectangular coordinates, whereas for the power injection model (7) voltages are represented in polar coordinates. While other combinations can be used, this choice has been driven by the following considerations: (i) the power injection model is also used for power flow analysis for which the polar representation is more common; and (ii) rectangular coordinates allow simplifying the definition of current injections and increasing the linearity of the equations. Finally, the integration scheme used for both models is an implicit trapezoidal method with a constant time step (0.1 s) and dishonest Newton-Raphson, i.e., the system Jacobian matrix is updated only if structural changes occur or the number of iterations required to solve a time step is above a given threshold.

Statistics for the two models are shown in Table I. Despite the higher nonlinearity, the power injection model is slightly faster than the current injection one. This result is not surprising if one considers that the number of algebraic variables required by (6) is much higher than that of (7). The fact that the number of algebraic variables of the current injection model is bigger than that of the power injection one is due to the following reasons.

- Active and reactive powers, required for synchronous machine and wind power plant controllers, have to be computed on purpose when using the current injection model, as it does not require such variables to solve network equations.
- Since a rectangular coordinate representation of bus voltage phasors has been used for the current injection model, bus voltage magnitudes required for voltage regulators of synchronous machines and variable speed wind power plants as well as for ZIP load models have to be computed apart.
- ZIP load models also require to compute power and current injections into network nodes. This currents are defined as algebraic variables in the current injection model.

The increase of the number of algebraic variables in the current injection model is relevant. This fact balances the higher sparsity and higher number constant elements of the matrix A_C of the current injection model with respect to the power injection one.

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