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ON CURVATURE HOMOGENEITY OF RIEMANNIAN MANIFOLDS

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1. Introduction. Let M be a Riemannian manifold with metric g. Let ∇, ∇_X and R be the Riemannian connection, the covariant differentiation with respect to a vector X or a vector field X and the Riemannian curvature tensor respectively. k-th covariant differential of a tensor field T is denoted by $\nabla^k T$ and $\nabla^0 T = T$ by definition. For each $x, y \in M$, a linear isomorphism A of the tangent space M_x onto M_y is naturally extended to a linear isomorphism of the tensor algebra $T(M_x)$ onto $T(M_y)$, which is also denoted by A.

Now, we assume that M is (Riemannian) homogeneous i.e. that M admits a transitive group of isometries. Then, for every integer $m \geqq 0$, the following condition $P(m)$ is satisfied. $P(m)$: "For each $x, y \in M$, there exists a linear isometry A of M_x onto M_y such that

$$A(\nabla^k R)_x = (\nabla^k R)_y \quad \text{for} \quad k = 0, 1, 2, \cdots, m \text{ ."}$$

In fact, A is given by putting $A = df_x$, where f is an isometry which maps x to y. Of course, the condition $P(m_1)$ implies $P(m_2)$ if $m_1 \geqq m_2$.

I. M. Singer [4] dealt with a converse problem and he proved that if M is a complete simply connected Riemannian manifold which satisfies the condition $P(m)$ for a certain m, then M is homogeneous. In this theorem, the minimum of such integers m depends on M, though it is smaller than $n(n-1)/2+1$, where n is the dimension of M. And so, he put a question among others that "do there exist curvature homogeneous spaces which are not homogeneous?". A curvature homogeneous space is, by definition, a Riemannian manifold satisfying the condition $P(0)$. The answer to his question is trivial, unless we assume that the space in consideration is complete, simply connected.

The purpose of the present paper is to give an example of curvature homogeneous space which is not homogeneous.

2. Curvature homogeneous spaces. First, we recall the following three types of curvature homogeneous spaces which may be not homogeneous.

1. Let M be an n-dimensional Riemannian manifold which is immersed isometrically into an $(n+1)$-dimensional space form. If the n principal

curvatures are constant on M, then M is curvature homogeneous.

2. Let M be a 3-dimensional Riemannian manifold. If three characteristic roots of the Ricci tensor are constant on M, then M is curvature homogeneous.

3. Let M be an n-dimensional conformally flat Riemannian manifold. If n characteristic roots of the Ricci tensor are constant on M, then M is curvature homogeneous.

K. Sekigawa [3] gave an example of 3-dimensional complete simply connected Riemannian manifold M° with the following properties:

(1) The homogeneous holonomy group of ∇ is irreducible,

(2) The scalar curvature S is negative constant on M°,

(3) M° has two distributions D_0 and D_1 such that

(a) $\dim D_0 = 1$ and $\dim D_1 = 2$,

(b) D_0 and D_1 are mutually orthogonal,

(c) for each $x \in M^\circ$, if $X, Y \in D_1(x)$ and $Z \in D_0(x)$, then

$$\begin{cases} R(X, Y) = S/2X \wedge Y , \\ R(X, Z) = 0 , \end{cases} \tag{2.1}$$

where $X \wedge Y$ maps $W \in M_x^\circ$ to $g(Y, W)X - g(X, W)Y$.

M° is a curvature homogeneous space of type 2. We shall show that M° admits no transitive group of isometries. For this purpose, we use the following theorem of W. Ambrose and I. M. Singer.

THEOREM. ([1]) *Let M be a simply connected, complete Riemannian manifold. A necessary and sufficient condition for M to admit a transitive group of isometries is that there exists a metric linear connection $\overset{*}{\nabla}$ satisfying the following two conditions*:

$$\overset{*}{\nabla} R = 0 \tag{A)'$$

$$\overset{*}{\nabla} T = 0 , \tag{B)'$$

where T is a tensor field of type (1, 2) *defined by* $T(X) = \nabla_X - \overset{*}{\nabla}_X$.

The conditions (A)′ and (B)′ are equivalent to

$$\nabla_X R = T(X) \cdot R \tag{A}$$

and

$$\nabla_X T = T(X) \cdot T \qquad \text{for all } X , \tag{B}$$

respectively, where $T(X)$ operates on R and T as a derivation of tensor algebra at each point of M. And so, we show that M° can not admit such a T.

Now, we take a local field of orthonormal basis $\{X_1, X_2, X_3\}$ on some neighborhood U_x of each $x \in M^\circ$ and put

$$\nabla_{X_i} X_j = \sum_{k=1}^{3} B_{ij}{}^k X_k \quad \text{for} \quad i, j = 1, 2, 3 .$$

Then, we have $B_{ij}{}^k = -B_{ik}{}^j$ for $i, j, k = 1, 2, 3$. K. Sekigawa [3] showed that as such a $\{X_1, X_2, X_3\}$, we can choose one which satisfies the following conditions:

$$X_1, X_2 \in D_1, X_3 \in D_0 , \tag{2.2}$$

$$B_{3i}{}^j = B_{23}{}^1 = B_{13}{}^1 = B_{22}{}^3 = 0, B_{12}{}^3 \neq 0 \quad \text{on} \quad U_x \tag{2.3}$$

for $i, j = 1, 2, 3$.

Next, we assume that there exists a metric connection $\overset{*}{\nabla}$ satisfying (A)′ and (B)′ on M°. Put

$$T(X_i)X_j = \sum_{k=1}^{3} T_{ij}{}^k X_k \quad \text{for} \quad i, j = 1, 2, 3 ,$$

then $T_{ij}{}^k = -T_{ik}{}^j$ since $\overset{*}{\nabla}$ is a metric connection. From (A), we have

$$(\nabla_{X_i} R_1)(X_j, X_k) = (T(X_i)\cdot R_1)(X_j, X_k) \quad \text{for} \quad i, j, k = 1, 2, 3 , \tag{2.4}$$

where R_1 is the Ricci tensor. On the other hand, from (2.1), we have

$$R_1(X_i, X_j) = \lambda_i \delta_{ij} \quad \text{for} \quad i, j = 1, 2, 3 , \tag{2.5}$$

where $\lambda_1 = \lambda_2 = S/2 \neq 0$ and $\lambda_3 = 0$. Substituting (2.5) into (2.4), we have

$$B_{ij}{}^k(\lambda_j - \lambda_k) = T_{ij}{}^k(\lambda_j - \lambda_k) \quad \text{for} \quad i, j, k = 1, 2, 3 ,$$

which show that

$$B_{i3}{}^1 = T_{i3}{}^1, B_{i2}{}^3 = T_{i2}{}^3 \quad \text{for} \quad i = 1, 2, 3 . \tag{2.6}$$

From (B), we have

$$\begin{aligned} X_i T_{jk}{}^l &+ \sum_{s=1}^{3} (B_{is}{}^l - T_{is}{}^l) T_{jk}{}^s \\ &- \sum_{s=1}^{3} (B_{ik}{}^s - T_{ik}{}^s) T_{js}{}^l \\ &- \sum_{s=1}^{3} (B_{ij}{}^s - T_{ij}{}^s) T_{sk}{}^l = 0 \end{aligned} \tag{2.7}$$

for $i, j, k, l = 1, 2, 3$. Taking account of (2.3) and (2.6), (2.7) reduces to

$$(2.8)\qquad \begin{cases} X_iT_{11}{}^2 - (B_{i1}{}^2 - T_{i1}{}^2)T_{21}{}^2 = 0\ , \\ X_iT_{12}{}^3 = 0\ , \\ (B_{i2}{}^1 - T_{i2}{}^1)T_{13}{}^2 = 0\ , \\ X_iT_{21}{}^2 - (B_{i2}{}^1 - T_{i2}{}^1)T_{11}{}^2 = 0\ , \\ X_iT_{31}{}^2 = 0 \quad \text{for} \quad i = 1, 2, 3\ . \end{cases}$$

From (2.3), (2.6) and (2.8), we have

$$T_{i1}{}^2 = B_{i1}{}^2,\ X_iT_{11}{}^2 = X_iT_{12}{}^3 = X_iT_{21}{}^2 = X_iT_{31}{}^2 = 0$$

for $i = 1, 2, 3$. Summing up the above results, we get

$$(2.9)\qquad B_{ij}{}^k = T_{ij}{}^k,\ X_iB_{jk}{}^l = 0 \quad \text{for} \quad i, j, k, l = 1, 2, 3.$$

By (2.1), (2, 3) and (2.9), we have

$$\begin{aligned} 0 = R(X_1, X_2)X_3 &= \nabla_{X_1}\nabla_{X_2}X_3 - \nabla_{X_2}\nabla_{X_1}X_3 - \nabla_{[X_1, X_2]}X_3 \\ &= (B_{13}{}^2B_{22}{}^1)X_1 + (B_{12}{}^1B_{13}{}^2)X_2\ , \end{aligned}$$

which shows that

$$(2.10)\qquad B_{22}{}^1 = B_{12}{}^1 = 0\ .$$

But, by (2.1), (2.3), (2.9) and (2.10), we have

$$-S/2X_2 = R(X_1, X_2)X_1 = \nabla_{X_1}\nabla_{X_2}X_1 - \nabla_{X_2}\nabla_{X_1}X_1 - \nabla_{[X_1, X_2]}X_1 = 0\ ,$$

which is a contradiction.

Finally, we write down the example of Sekigawa: $R^3 = \{(u, v, w): u, v, w$ are real$\}$ with line element

$$ds^2 = (f(u, v, w))^2du^2 + dv^2 + dw^2\ ,$$

where

$$\begin{gathered} f(u, v, w) = c_1 \exp(t\sqrt{-S/2}) + c_2 \exp(-t\sqrt{-S/2})\ , \\ t = v\cos u - w \sin u\ , \end{gathered}$$

c_1, c_2: positive constants, S: a negative constant .

This is complete, since $f^2 \geqq c$ for some positive constant c. If we put

$$\begin{cases} X_1 = 1/f\ \ \partial/\partial u\ , \\ X_2 = (\cos u)\partial/\partial v - (\sin u)\partial/\partial w\ , \\ X_3 = (\sin u)\partial/\partial v + (\cos u)\partial/\partial w\ , \end{cases}$$

then

$$\begin{gathered} B_{3i}{}^j = B_{23}{}^1 = B_{22}{}^3 = B_{13}{}^1 = B_{21}{}^2 = 0,\ B_{12}{}^3 \neq 0\ , \\ R(X_1, X_2) = S/2\ X_1 \wedge X_2,\ R(X_1, X_3) = R(X_2, X_3) = 0\ . \end{gathered}$$

REFERENCES

[1] W. AMBROSE AND I. M. SINGER, On homogeneous Riemannian manifolds, Duke Math. J., 25 (1958), 647-669.

[2] S. KOBAYASHI AND K. NOMIZU, Foundations of Differential Geometry, Vol. I, Interscience Publisher, New York, 1963.

[3] K. SEKIGAWA, On some 3-dimensional Riemannian manifolds, Hokkaido Math. J. 2, No. 2 (1973), 259-270.

[4] I. M. SINGER, Infinitesimally homogeneous spaces, Comm. Pure Appl. Math., 13 (1960), 685-697.

[5] S. TANNO, 3-dimensional Riemannian manifolds satisfying $R(X, Y)\cdot R=0$, to appear.

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