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ON CUSPIDAL REPRESENTATIONS OF p-ADIC REDUCTIVE GROUPS

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Abstract. Let k be a p-adic field, and G a reductive connected algebraic group over k. Fix a maximal torus T of G which splits in an unramified extension of k, and which has the same split rank as the center of G. For each character θ of $T(k)$, satisfying some conditions, there is a cuspidal representation γ_θ of $G(k)$ which is a sum of a finite number of irreducible representations; the correspondence $\theta \mapsto \gamma_\theta$ is one-to-one on the orbits of such characters by the little Weyl group of T; furthermore, the formulas for the formal degree of γ_θ and its character for sufficiently regular elements of $T(k)$ are given: they are formally the same as is the discrete series for real reductive groups.

1. **Unramified maximal tori.** Let k be a p-adic field, that is a finite extension of $\mathbf{Q}_p$ or a field of formal series over a finite extension of $\mathbf{F}_p$. We denote by $\bar{k}$ the residue field of order q.

Let G be a reductive connected algebraic group defined over k, the derived group G_{der} of which is simply connected. A maximal torus of G defined over k is called *minisotropic* if it normalizes no (proper) horocyclic subgroup of G defined over k.

LEMMA. *Suppose there exists a minisotropic maximal torus T of G which splits in a finite unramified extension L of G. Then the Galois group Γ of L over k has a unique fixed point v in the apartment of T in the building of $G_{\mathrm{der}}(L)$* [2]; *moreover, the face of v is minimal amongst the faces in this apartment which are invariant by Γ.*

2. **Characters.** We conserve notations and hypotheses of §1 and the Lemma. Let θ be a continuous character of $T(k)$. For each $\lambda \in X^{\vee}(T)$, the lattice of rational one-parameter subgroups of T, we define a character θ_λ of $L^\times$ by

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$$\theta_\lambda(z) = \theta\left(\prod_\Gamma^\gamma (z^\lambda)\right), \qquad z \in L^\times.$$

DEFINITION 1. The character θ is called *regular* if, for every root α of (G, T), the character $\theta_{\alpha^\vee}$ of $L^\times$ associated to the coroot $\alpha^\vee$ is nontrivial.

For each root α, we denote by $|\alpha|_\theta$ the conductor of $\theta_{\alpha^\vee}$ and let R_f be the set of roots of (G, T) such that $|\alpha|_\theta \leqslant f$.

DEFINITION 2. The character θ is called *good* if it is regular and if, for every f, the set R_f is a convex set of roots.

We need the following form of *Macdonald's conjecture* (cf. [1, C-6.7]):

Let $\bar{S}$ be a reductive connected algebraic group over $\bar{k}$, $\bar{T}$ a minisotropic maximal torus of $\bar{S}$; fix a finite extension $\bar{L}$ of $\bar{k}$ which splits $\bar{T}$; let Γ be its Galois group. A character $\bar{\theta}$ of $\bar{T}(\bar{k})$ is called regular if, for every root α of $(\bar{S}, \bar{T})$, the character $z \longmapsto \bar{\theta}(\Pi_\Gamma^\gamma(z^{\alpha^\vee}))$ of $\bar{L}^\times$ is nontrivial; if $\bar{\theta}$ is a regular character of $\bar{T}(\bar{k})$, then there exists a unique class $\bar{\sigma}_{\bar{\theta}}$ of representations of $\bar{S}(\bar{k})$ such that, if $\mathrm{St}_{\bar{S}}$ denotes the Steinberg representation of $\bar{S}(\bar{k})$, $\mathrm{St}_{\bar{S}} \otimes \bar{\sigma}_{\bar{\theta}}$ is the induced representation $\mathrm{Ind}(\bar{S}(\bar{k}), \bar{T}(\bar{k}), \bar{\theta})$. Moreover $\bar{\sigma}_{\bar{\theta}}$ is cuspidal, and the intertwining number of two such representations $\bar{\sigma}_{\bar{\theta}_1}$ and $\bar{\sigma}_{\bar{\theta}_2}$ is equal to the number of elements in the little Weyl group of $\bar{T}$ in $\bar{G}(\bar{k})$ which send $\bar{\theta}_1$ on $\bar{\theta}_2$.

3. Cuspidal representations.

THEOREM. *Let G be a reductive connected algebraic group over the p-adic field k, the derived group G_{der} of which is simply connected. Let T be a minisotropic maximal torus which splits in an unramified extension of k. Fix a good character θ of $T(k)$. Assume Macdonalds's conjecture and one of the following conditions*:

(i) $|\alpha|_\theta > 1$ *for every root α of $(G(L), T(L))$*

(ii) *the residual characteristic of k is not 2 and there exists a rational representation ρ of G_{der} such that the corresponding bilinear form* $\mathrm{Tr}\, \rho(X)\rho(Y)$ *on the Lie algebra* $\mathrm{Lie}\ \bar{T}_{\mathrm{der}}(\bar{k})$ *is nondegenerate.*

Then there exists a class γ_θ of representations of $G(k)$ such that:

(a) γ_θ *is a finite sum of irreducible representations of $G(k)$, the coefficients of which have compact support modulo the center*;

(b) *the intertwining number of two such representations γ_{θ_1} and γ_{θ_2} is the number of elements in the little Weyl group $W(T)$ of T in $G(k)$ which send θ_1 on θ_2*;

(c) *there exists a Haar measure on $G(k)$, independent of θ, such that the formal degree of γ_θ is*

$$d(\theta) = \left(\prod_R q^{|\alpha|_\theta - 1}\right)^{1/2}$$

where R is the set of roots of (G, T);

(d) *for $t \in T(k)$ such that* $\mathrm{val}(t^\alpha - 1) \geqslant |\alpha|_\theta/3$ *for every $\alpha \in R$, the value on t of the character of γ_θ is given by*

$$\operatorname{Tr} \gamma_\theta(t) = (-1)^{l(G)}(-1)^{\sum_{R/\Gamma}(|\alpha|_\theta - 1)} \sum_{W(T)} \frac{\theta}{\Delta}({}^w t)$$

where $l(G)$ is the split semisimple rank of G, Δ is the $W(T)$-invariant function on the regular elements of $T(k)$ given by

$$\Delta(t) = (-1)^{\sum_{R/\Gamma} \mathrm{val}(t^\alpha - 1)} |\mathrm{Det}_{\mathrm{Lie}\, G/\mathrm{Lie}\, T}(\mathrm{Ad}\, t - 1)|^{1/2},$$

and Γ is the Galois group over $\bar{k}$ of an unramified extension which splits T.

4. **Remarks.** 1. The proof is based upon an explicit construction of γ_θ (assuming Macdonald's conjecture), obtained by inducing a finite dimensional representation of a compact open subgroup of $G(k)$ naturally associated to T and θ; the essential tool is given by Weil's paper about Heisenberg groups **[5]**; we used too an argument given by R. Howe **[4]**.

2. In the case where $|\alpha|_\theta$ is constant and strictly greater than 1, and if the point v of the lemma is special, the proofs are given in [3].

3. G. Lusztig has just proved Macdonald's conjecture.

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