On $D = 5$ super Yang-Mills theory and $(2, 0)$ theory

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ABSTRACT: We discuss how $D = 5$ maximally supersymmetric Yang-Mills theory (MSYM) might be used to study or even to define the $(2,0)$ theory in six dimensions. It is known that the compactification of $(2,0)$ theory on a circle leads to $D = 5$ MSYM. A variety of arguments suggest that the relation can be reversed, and that all of the degrees of freedom of $(2,0)$ theory are already present in $D = 5$ MSYM. If so, this relation should have consequences for $D = 5$ SYM perturbation theory. We explore whether it might imply all orders finiteness, or else an unusual relation between the cutoff and the gauge coupling. S-duality of the reduction to $D = 4$ may provide nonperturbative constraints or tests of these options.

Contents

1 Introduction

Among the many discoveries of the second superstring revolution which we have not yet fully understood, is the existence of interacting local quantum field theories in spacetime dimensions $4 < D \leq 6$ [27]. So far as we know, such theories must be supersymmetric, and the most intriguing examples are the so-called $(2, 0)$ theories, with 2 chiral supersymmetries (16 supercharges) and superconformal invariance in $D = 6$. These theories are classified by a choice of simply laced Dynkin diagram (so, A_n, D_n, E_6, E_7, E_8 and their direct sums) and can be obtained as low energy decoupling limits of IIB strings compactified on $K3$. The A_n series can also be obtained as the low energy limit of the world-sheet theory of $n + 1$ coincident M5 branes.

No action is known for these theories, and according to the string/M theory definition they have no dimensionless parameters in which to expand, suggesting that perturbative computations would not be possible even given an action. Thus, not much is known about these theories beyond the predictions of supersymmetry and duality arguments.

In this note, we discuss the possibility of using the compactification of $(2, 0)$ theory on a circle (or S^1), which leads to $D = 5$ maximally supersymmetric Yang-Mills theory (MSYM), to learn more about both theories. Following [23, 24], we review the basics of this relation in section 2. The main point which we need for our introductory discussion is that the (squared) gauge coupling, which in $D = 5$ has dimensions of length, is equal to the radius of the S^1 in compactified $(2, 0)$ theory,

$$g_5^2 = R_6 \,. \tag{1.1}$$

Thus, the equivalence to $D = 5$ SYM should be valid in the low energy limit $E \ll 1/g_5^2$, with corrections controlled by the dimensionless parameter $g_5^2 E$.

Of course, the first problem in making sense of this relation is that by the usual power counting arguments, $D = 5$ super Yang-Mills is perturbatively nonrenormalizable. Thus one expects an infinite series of divergences requiring an infinite series of counterterms, and an expansion with little or no predictive power. However, the situation is better in supersymmetric theories. Superspace arguments preclude divergences at low orders, and in D-dimensional MSYM the first divergences appear at ℓ loops with $D = 4 + 6/\ell$ [5]. And, as the technology for higher loop perturbation theory has advanced, remarkable cancellations have been discovered which go beyond the predictions of the superspace arguments. Until a computation which could have diverged consistently with these arguments is done, for example at $\ell = 6$, it is hard to evaluate the situation here. But, at present it is not ruled out that $D = 5$ MSYM is perturbatively finite to all orders, just as is true for $D = 4$.

There is a close analogy to the old question of to whether $N = 8$ supergravity in space-time dimension $D = 4$ is a perturbatively finite quantum field theory, or not. Superspace arguments predict the first divergences at seven loops, which is still out of reach of explicit verification. However, the same arguments predict other divergences in higher dimensions, which recently have been shown to cancel at four loops [6]. At present the systematics of these divergences is unclear; see [8, 9, 13] for recent discussions.

Suppose that $D = 4$ supergravity were finite to all orders. Then the mystery would be far deeper, as there is no known candidate for a fully consistent $D = 4$ quantum field theory (i.e., defined nonperturbatively and at all energy scales) for which the scattering amplitudes would have $N = 8$ supergravity as an asymptotic expansion.

Let us briefly recall why this is. As in almost all interacting quantum field theories with a finite number of fields, the perturbative expansion is expected to have zero radius of convergence (the ℓ loop term behaves as $\ell!$) and by itself does not define a theory. One must propose either a resummation prescription, or (better) a physical picture for how the exact amplitudes behave in the UV, or (ideally) a way to obtain the theory as the low energy limit of some quantum theory which is known to exist — in other words a UV completion — to claim that a fully consistent QFT exists.

This is especially important and nontrivial for a perturbatively nonrenormalizable theory, such as quantum gravity in $D > 2$ and quantum Yang-Mills in $D > 4$. While the formal loop expansion parameter g^2 has dimensions of length to a positive power, the actual behavior of perturbation theory is controlled by a dimensionless effective coupling. Consider a scattering amplitude at energy E; this will be $g^2 E^\alpha$ for some $\alpha > 0$, so at high energies perturbation theory must break down.

At present, the only candidate for a UV completion of quantum gravity is compactified superstring theory, which is perturbatively finite for reasons having no relation to field theoretic perturbation theory. It is not clear what significance perturbative finiteness of supergravity would have in this context; superstring theory would be finite with or without it. See [14] for a recent discussion.

By contrast, we know a UV completion of $D = 5$ maximally supersymmetric Yang-Mills theory, namely compactified $(2, 0)$ theory, so the question of whether it is perturbatively

finite or not should have direct consequences for this theory. In this note we begin to explore this idea.

Let us begin with a temptingly simple argument as to why $D = 5$ MSYM must be perturbatively finite. Suppose it were not. In this case, we would need to cut off the theory at a scale Λ. This would introduce a new dimensionless parameter $g_5^2 \Lambda$. On the other hand, the $(2,0)$ theory does not have dimensionless parameters, leading to a contradiction.

To make this a bit more precise, the assumption is that the low energy behavior of compactified $(2,0)$ amplitudes looks like $D = 5$ MSYM below some adjustable scale Λ, and is different above this scale, perhaps because of additional particles, perhaps because the interaction has a form factor, or for some other reason. It does not really matter what the nature of the cutoff is for this argument, only that the scale Λ can be extracted from the compactified $(2,0)$ amplitudes, and is adjustable independently from g_5^2.

While this argument has loopholes which we will explain shortly, it should be taken seriously, as this is how many analogous examples work. For example, while perturbative string theory cuts off the divergences of quantum gravity, the cutoff is not at the Planck scale but rather at the string scale; the ratio of these parameters indeed defines a dimensionless parameter, the string coupling. As an even more basic example, the UV divergences of the Fermi theory of weak interactions are cutoff at the W and Z boson masses; comparing these to G_{Fermi} defines new dimensionless parameters, the Standard Model gauge couplings.

Not all nonrenormalizable theories work this way. For example, eleven dimensional M theory has a single dimensionful scale M_{p11}, which determines both the gravitational coupling and the scale at which perturbation theory breaks down. In this example, it seems natural that the two scales should be related. The same might be true for $D = 5$ MSYM, evading the contradiction.

1.1 Cutoff and extra states

An important difference between M theory and $(2,0)$ theory is that in the latter case, the underlying UV theory is scale invariant. Thus, the cutoff Λ only appears upon compactification. On S^1, this defines a single new scale $R_6 = g_5^2$, so the only option is that

$$\Lambda = \frac{c}{g_5^2}\,, \tag{1.2}$$

where c is an order one constant, whose precise definition would depend on the renormalization scheme in $D = 5$ SYM. Conceivably, c might not be constant; it might turn out to depend on loop order or other details of the diagrams being cutoff. Our arguments will not depend on its value, and often we will set $c = 1$ below.

We should think of eq. (1.2) as modeling the effect of some precise cutoff provided by the compactified $(2,0)$ theory. Since this theory has KK modes at the scale $1/R_6 = 1/g_5^2$, the simplest hypothesis is that adding these modes cuts off the UV divergences.

In fact, these KK modes can already be seen in $D = 5$ MSYM. As we review in section 2, they are the self-dual solutions which would have been instantons in $D = 4$. In fact, the other BPS states of $(2,0)$ theory can also be seen in $D = 5$ — for example, the

solutions which would have been monopoles in $D = 4$ become strings, which are precisely the strings which are supposed to arise from M2-branes stretched between M5-branes in this construction of $(2, 0)$ theory.

This is all consistent with the hypothesis that the new degrees of freedom which are required to define the UV completion, are already present in the nonperturbative physics of the $D = 5$ theory.[1] If so, a nonperturbative formulation of $D = 5$ MSYM would *ipso facto* be a formulation of the $(2, 0)$ theory, by taking the infinite coupling limit.

What would this mean? A somewhat more precise claim would be that we can define the quantum Hilbert space and Hamiltonian of $D = 5$ MSYM in more or less the same way as in dimensions $D \leq 4$, using the perturbative and semiclassical degrees of freedom, and then deal with the strongly coupled dynamics at energies $g_5^2 E \sim 1$ by rediagonalizing the Hamiltonian, to obtain the (2,0) Hilbert space. Thus, the Hilbert space would be the space of wave functionals of the gauge fields and their superpartners, which could be thought of as a direct sum of sectors of definite self-dual soliton number (and the other soliton numbers), each defined as a Fock space of excitations of the gauge fields. Features specific to $(2, 0)$ theory, starting with the fact that as $g_5^2 = R_6 \to \infty$ we recover $5 + 1$ dimensional Lorentz invariance, would thus be nonmanifest consequences of nonperturbative dynamics. While this claim sounds rather complicated and is not fully precise either, it is at least a definite approach to defining $(2, 0)$ theory, which puts its relation to $D = 5$ MSYM at center stage. Thus, we would like to evaluate whether it is consistent with the evidence.

Although we do not have a nonperturbative formulation of $D = 5$ MSYM, one can nevertheless get nonperturbative information about a quantum theory by close examination of the perturbative expansion. For example, the $(2g)!$ asymptotics of closed string perturbation theory suggested the importance of objects with mass $1/g_s$, before the realization that these were D-branes [25]. The general principle [19, 20] is that the large order behavior of perturbation theory is controlled by configurations in which the fields are 'large', and more specifically by certain classical solutions. For example, in bosonic $D = 4$ gauge theory, the asymptotic behavior has been argued to be controlled by a certain complex instanton [21]. An interesting application of these ideas to multiparticle production in $D = 4$ MSYM appears in [17].

Conversely, if one tries to define a field theory nonperturbatively by resumming the perturbative expansion, one often finds that the resummation is ambiguous in a way that is resolved by adding in the effects of the classical solution. Mathematically, this is the case in which the Borel transform has singularities on the positive real axis. While in practice it is very difficult to make such resummation procedures precise, such an argument can give important clues about nonperturbative physics.

As we explain in section 2, the large order behavior of $D = 5$ scattering amplitudes, at least if they behave as in other field theories, suggests that it is somehow controlled by the self-dual particle solution. This suggests that a close analysis of the physics of this solution might be key to understanding $(2, 0)$ theory, as has been hinted in previous works [2, 11].

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[1]In Lambert et al. [18], which appeared after this work was substantially completed, this idea is suggested independently, and new evidence is given. They also believe this suggests that $D = 5$ MSYM could be perturbatively finite.

To conclude this subsection, we should point out that one could make different hypotheses about the relation between $D = 5$ SYM and $(2,0)$ theory, in which the $(2,0)$ theory has additional or different microscopic degrees of freedom. A relation like eq. (1.2) could arise naturally in such an effective low energy Lagrangian, for example the chiral Lagrangian which describes pions works this way [26]. Thus it is important to look for evidence distinguishing the scenario described here, from other possibiities.

1.2 Consequences of divergences for reduction to $D = 4$

How can we test the ideas we just described, and more specifically eq. (1.2)? Since we cannot do the necessary nonperturbative computations directly in $D = 5$ and $D = 6$, let us instead consider its consequences in $D = 4$.

Thus, let us consider the compactification $D = 5$ to $D = 4$ on a circle, or equivalently compactify the $(2,0)$ theory on T^2 with a flat metric. This leads to $N = 4$ SYM, as we will review in section 3, but let us again state the basics for purposes of our introduction. Consider the special case of a rectangular torus, then the radii of the torus are related to the $D = 4$ gauge coupling as $R_5 = L/g_4$ and $R_6 = g_4 L$, where L is a parameter with dimensions of length. Taking the limit $L \to 0$ at fixed g_4, we recover the finite $D = 4$ theory, while taking small but finite L produces corrections to the $D = 4$ amplitudes. These will be described by irrelevant operators such as $L^4 \text{tr} \, F^4$, $L^6 \text{tr} \, D^2 F^4$ and so on, whose coefficients are each a function of g_4.

Now, if there are divergences in $D = 5$, we argued earlier that there must be a nonperturbative consistency condition relating Λ and g_5^2. A natural source of such a nonperturbative consistency condition is S-duality of $D = 4$ MSYM. Specifically, the scattering amplitudes of states which map into themselves under S-duality (the $U(1)^r$ gauge multiplets) must be invariant under S-duality, including any corrections controlled by L. For example, the scattering of four photons gets a one loop contribution from charged $D = 5$ KK gauge bosons. Extending this to a sum including 6d KK modes will suggest an S-duality invariant extension eq. (3.9) of this amplitude. The resulting $\text{tr} \, F^4$ term only receives contributions at one loop and nonperturbatively, and is finite. But it seems reasonable to believe that, if we could compute a higher correction of this type from $D = 5$, and if the computation required cutting off UV divergences, it would be invariant under S-duality only for a very specific cutoff prescription, and for a unique value of Λ.

To understand the Λ dependence of these corrections, as we will discuss in section 3, one must address the following issue. An implicit assumption made in all work on these theories, and supported by the results, is that in a situation like this, the $L \to 0$ limit is regular. Suppose we have a higher dimensional field theory, compactify it, and take a limit which sends the KK and other higher dimensional states off to infinite mass; then the corrections to the lower dimensional field theory go smoothly to zero. Here, as we take $L \to 0$, the corrections must vanish as positive powers of L, for any fixed g_4.

While this is evident at the classical level, and even with finite quantum corrections, it can be spoiled by UV divergences in the higher dimensional theory. Let us consider the case at hand, thought of as a compactification of $D = 5$ SYM to $D = 4$, and suppose that there were a $D = 5$ perturbative UV divergence. Since $D = 4$ MSYM is finite, the

compactified $D = 5$ counterterms would come entirely from diagrams involving $D = 5$ KK states, which have masses between $1/R_5$ and the cutoff Λ. Although as $L \rightarrow 0$ both of these scales become large, if $\Lambda \gg 1/R_5$ (as it is here), these quantum effects will survive the limit. Since in a nonrenormalizable theory one expects corrections which come as positive powers of Λ, this could easily violate the regularity assumption.

In compactification of string theory, or of a (presumably) finite field theory such as $(2, 0)$ theory, this potential problem should be solved by whatever physics makes the theory finite, as this will suppress the contributions of KK states with masses above the cutoff. In $D = 5$ terms, this potential problem must be solved by our assumption eq. (1.2).

Let us consider a $D = 5$ UV divergent counterterm. As we discuss in section 3, these can be estimated by expanding the $D = 5$ Feynman diagrams in sums over the 5d KK momentum. They take the general form $(\Lambda/M_5)^n$ or perhaps $\log \Lambda/M_5$, where $M_5 = 1/R_5$ is the KK scale. We then take the counterterm and, granting that $\Lambda = 1/g_5^2 = 1/R_6$, express it in $D = 4$ terms.

Since $\Lambda/M_5 = 1/g_4^2$, this procedure introduces negative powers of g_4. A UV correction with an overall dependence $1/g_4^{2n}$ or $\log g_4$ would be very problematic, as it would become arbitrarily large at small g_4, and at least naively would contradict the regularity assumption. While one could escape the contradiction by postulating that higher order corrections dominate this one, then the perturbative expansion would completely break down.

Carrying this our more carefully in section 3, we find that the UV corrections all come with positive powers of g_4 and L, and in this sense are consistent with our interpretation of the $D = 5$ divergences as cut off in compactified $(2, 0)$ theory. Thus there seems to be no *a priori* argument, at least from these considerations, that $D = 5$ MSYM must be finite. However, log UV divergences leave an unusual signature in $D = 4$.

This is as far as we have gotten with general arguments, and clearly the question of divergences in $D = 5$ MSYM at a given (low) loop order would be better settled by direct computation. Whatever comes out, we believe we have made the point that the answer is quite relevant to the structure of $(2, 0)$ theory.

2 The $(2, 0)$ theory and $D = 5$ MSYM

Let us review the basic properties of the $(2, 0)$ theories [24]. Their original definition was in terms of superstring theory. One can compactify the IIb string on a K3 surface, leading to a theory whose low energy limit is a six-dimensional supergravity. The K3 surface has metric moduli (volumes of two-cycles) which can be taken to singular limits, classified by the simply laced (ADE) Dynkin diagrams. In these limits, a new energy scale $\langle \phi \rangle$ appears, parametrically lower than the string and Planck scales. One can argue that the dynamics at this scale decouples from supergravity, leaving a nontrivial interacting six-dimensional field theory. Taking the limit $\langle \phi \rangle \rightarrow 0$, one obtains a scale invariant theory.

In six dimensions, $(2, 0)$ supersymmetry is maximal (for a non-gravitational theory), and there is a unique matter multiplet with this supersymmetry, containing a self-dual two-form potential B, five scalars, and 4 chiral fermions. The theory with a single matter multiplet can be realized as the low energy limit of the world-volume theory of a five-brane

in M theory. Since the only parameter of the five-brane theory is the eleven-dimensional Planck scale, in this limit there are no adjustable parameters, dimensionful or dimensionless. The presence of a self-dual field makes a covariant Lagrangian description tricky (see [22]), but a noncovariant Lagrangian can be written [1]; it is unique with no free parameters.

One can also define the A series theories by taking the decoupling limit of the worldvolume theory of n coincident five-branes in M theory. By separating the branes (going out of the Coulomb branch), and taking a further low energy limit, one obtains a sum of the single multiplet theories we just described. This supports the idea that the interacting theories have no free parameters; and that the interaction is (in some sense) order one. This fits with the fact that objects charged under self-dual gauge fields, satisfying

$$ H \equiv dB = *dB \,, $$

must satisfy the Dirac quantization condition simultaneously as electric and as magnetic objects, forcing their charge to be order one.

2.1 Reduction of the $(2,0)$ theory to $D = 5$

We now compactify x^6 on a circle of circumference $2\pi R_6$. For reasons we explain shortly, we give this parameter several names:

$$ \frac{g_5^2}{8\pi} \equiv \frac{1}{M_6} \equiv R_6 \,. \tag{2.1} $$

While we have no action for the $(2,0)$ theory, as we discussed above, by turning on appropriate expectation values $\Delta_{ij}\phi \equiv \phi_i - \phi_j \neq 0 \forall i,j$ for the scalars and taking a low energy limit, it reduces to a sum of $D = 6$ self-dual tensor theories. Applying standard dimensional reduction to these theories, we find a sum of $D = 5$ super-Maxwell actions, where the 5d gauge field strength is $F_{\mu\nu} = H_{6\mu\nu}$. The other components of H are determined by the self-duality condition. Then, given that the compactification to $D = 4$ is MSYM, we conclude that the limit $\phi_i \to 0$ in $D = 5$ should also be MSYM.

We next discuss the perturbative expansion of MSYM in $D = 5$, so that we can make the arguments of section 1.1 precise, in particular to examine the claim that Λ/M_6 could be extracted from the $(2,0)$ observables. Suppose for the moment that $D = 5$ MSYM were UV convergent, then by dimensional analysis and the standard asymptotics of perturbation theory, the ℓ loop contribution to a fixed angle scattering amplitude at energy E is expected to grow as

$$ \mathcal{A}_\ell \sim \ell! \left(\frac{E}{M_6} \right)^\ell \,. $$

Of course, there could be UV divergent terms. Such a term will come with an additional factor of $(\Lambda/E)^k$ (or perhaps logarithms). To make the finiteness argument we stated in the introduction, we need to show that this expansion can be written in terms of Λ/M_6, and argue that it is plausible that terms in the expansion can be reconstructed from the

exact amplitudes.[2] Now, the maximal possible UV divergence in a $D = 5$ bosonic gauge theory at ℓ loops[3] appears in front of the $\operatorname{tr} F^2$ term and is Λ^ℓ, so already in the bosonic case one can reformulate the series expansion as

$$\mathcal{A}_\ell \sim \ell! \sum_{k=0}^{\ell} \left(\frac{E}{M_6} \right)^{\ell-k} \left(\frac{\Lambda}{M_6} \right)^k .$$

with all powers of M_6 in the denominator. Of course, the UV divergences in the supersymmetric case are far milder, so the same is true.

This corresponds to an expansion in positive powers of g_5^2, and thus one can isolate functions of the dimensionless parameter Λ/M_6 at each order in such an expansion. This is consistent with the idea that one can determine Λ/M_6 from an asymptotic expansion around $g_5 = 0$ of the (unknown) exact $D = 5$ amplitudes, and that the terms in this expansion are the terms computed in $D = 5$ perturbation theory.

Let us very briefly review what this means, to make the point that, given certain hypotheses about the exact amplitudes, we could determine the perturbative expansion and thus Λ/M_6. Very generally in quantum theories, an amplitude $\mathcal{A}(g)$ will be analytic in some wedge around the origin $g = 0$ containing the positive real axis. Within this wedge, it has an asymptotic expansion in g such that

$$\mathcal{A} = \sum_{k=0}^{r-1} a_k g^k + \mathcal{R}_k(g) \tag{2.2}$$

with a remainder term $\mathcal{R}$ satisfying a bound

$$|\mathcal{R}_k(g)| \leq C\sigma^r r! |g|^r \tag{2.3}$$

uniformly in r and g. Because of this, by taking derivatives at suitably small g, the terms a_k in the expansion can be estimated to arbitrary accuracy.

While we do not know how to argue that exact compactified $(2, 0)$ theory amplitudes have the required analyticity, if negative powers of g_5 had appeared, the claim would be evident nonsense, so we can at least observe that a nontrivial consistency check has been passed, and continue on this assumption.

Granting it, if $D = 5$ MSYM has perturbative UV divergences, these plausibly signal the presence of a dimensionless parameter Λ/M_6 in the exact amplitudes. Since the compactified $(2, 0)$ theory has no such parameter, we have a reason to think that $D = 5$ MSYM is UV finite. If not, there must be some consistency condition which prevents us from varying the $(\Lambda/M_6)^k$ terms. We now model this by setting all such terms to 1.

Next, we can make a plausible guess for the behavior of the exact amplitude by imagining that the expansion can be Borel resummed [19], leading to

$$\mathcal{A} \sim \exp -\frac{M_6}{E} .$$

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[2]Given our ignorance of $(2, 0)$ theory, one can even question whether an S-matrix can be defined at all. One way to argue that it can is to turn on linearly varying scalar vevs (in the brane language, put the branes at small angles), to turn off the interactions in the asymptotic region. We thank Greg Moore for a discussion on this point.

[3]This leaves out the vacuum energy, but this is zero for MSYM.

Now, such resummation arguments are not easy to make precise in practice, as the physical effects of solitons and instantons tend to be associated with singularities of the Borel transform on the positive real coupling axis, and the resummation procedure becomes ambiguous beyond the one-instanton or one-soliton level [19]. Still, many examples are known of theories in which the asymptotic behavior of perturbation theory is controlled by simple nonperturbative solutions, so we can take this as suggesting an important nonperturbative role for a soliton associated with the mass M_6.

2.2 Instantonic particles

An important clue to the UV completion of the 5d theory [23] is the fact that the familiar self-dual solution of Yang-Mills theory, satisfying $F = *F$, describes a particle in $D = 5$, which classically has mass $M_6 = 8\pi/g_5^2$, as in eq. (2.1).

These particles carry a conserved charge, with current

$$J_6^\lambda = \epsilon^{\mu\nu\rho\sigma\lambda} \text{tr} \, F_{\mu\nu} F_{\rho\sigma}$$

related to the instanton number in $D = 4$. It enters into a central charge of the supersymmetry algebra and thus by BPS arguments, the particle mass cannot receive quantum corrections.

From the compactified 6d point of view, this conserved charge is the momentum in the sixth dimension. This can be seen by reducing the 6d expression for the appropriate component of the stress tensor,

$$T_{06} = H_0^{ij} H_{6ij} = \epsilon^{0ijkl} F_{kl} F_{ij} \, .$$

To summarize, $D = 5$ super Yang-Mills indeed contains a particle at the mass scale M_6 at which perturbation theory breaks down, which can be identified with a Kaluza-Klein mode of the $(2, 0)$ theory. It is an attractive hypothesis that adding this particle to the $D = 5$ theory, a soliton made up from the original fields of the theory, gives us the full nonperturbative particle spectrum, and determines the nonperturbative completion. Of course, it is not immediately obvious how to do this.

In particular, compared to better understood solitons, these solutions have rather peculiar properties. For example, one of their bosonic zero modes is a scale size ρ; classical solutions exist with arbitrary size. Supersymmetry arguments imply that this mode will not be lifted in perturbation theory, so apparently the energy-momentum of the solution can be arbitrarily widely dispersed. One might well question whether 'particle' is an appropriate name for such a thing. Possibly, as suggested in [11], this is a signal that these solutions are bound states of several particles, the so-called 'fractional instantons' or 'partons'.

If we turn on scalar vevs, this scale invariance is broken, and (by analogy with discussions of constrained instantons in $D = 4$) the scale mode ρ is confined to small values. One could then quantize it to get a more conventional looking particle.[4] In addition, a second soliton in the theory appears, the analog of the 't Hooft-Polyakov monopole solution, which in $D = 5$ is a string with tension $T = \langle \Delta\phi \rangle M_6$. This string has chiral zero modes, with

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[4]In [18], this is demonstrated for the closely related dyonic solution.

a very interesting anomaly cancellation mechanism discussed in [10]. See [15] for further analysis of the interpretation of these objects in $(2, 0)$ theory.

It would be interesting to further explore the Coulomb phase $0 \neq \Delta\phi \ll M_6$, which exhibits the same UV puzzles, but avoids many of the other puzzles of the nonabelian theory.

2.3 Non simply laced gauge groups

There are good string theory arguments that only the ADE theories exist in $D = 6$. Thus, to get the BCFG gauge groups in $D = 5$, we must modify the dimensional reduction. This can be done by positing twisted boundary conditions on the S^1. Although we do not have a concrete $D = 6$ definition, presumably automorphisms of the Dynkin diagram correspond to symmetries of the $(2, 0)$ theory. Thus one twists by such an automorphism. After reduction, this corresponds to twisting the gauge fields by an outer automorphism of the gauge group. It is known that the fixed points of such symmetries generate all of the non simply laced groups. One might be able to test this idea by reducing to $D = 4$ and making contact with the discussion of S-duality for BCFG groups in [16].

3 Reduction of the $(2, 0)$ theory and $D = 5$ MSYM to $D = 4$

Let us turn to T^2 compactification. A flat metric on T^2 has three real parameters: two radii R_5 and R_6, and a relative angle between the A and B cycles. We will group these into a dimensionless complex structure modulus τ which becomes the complexified $D = 4$ gauge coupling $\tau = \theta/2\pi + 8\pi i/g_4^2$, and the volume $L^2 = \text{Vol}\,(T^2)$, a parameter with dimensions of length squared.

Our basic strategy will be to consider this theory as the compactification of $D = 5$ SYM on a circle of radius R_5. Thus, we will restrict attention to a rectangular T^2 with $\theta = 0$. Before doing so, let us briefly discuss how θ_{QCD} arises in $D = 4$, and its $D = 5$ analog. In the self-dual tensor theory, one has a term in the the $D = 6$ action $g^{56} H_{5\mu\nu} H_{6\mu\nu}$, which using self-duality reduces to $\theta F \wedge F$. In $D = 5$, the metric components $g_{\mu 6}$ become a vector coupling to J_6^μ. Thus the θ angle shifts the quantization of the momenta of the instantonic particles. The $D = 4$ instantons arise as Euclidean winding configurations of these particles.

We now take $\theta = 0$; then there is an evident (geometric) symmetry under the interchange

$$R_5 \leftrightarrow R_6\,.$$

Taking the low energy limit, this is the origin of S-duality in $D = 4$. Evidently it extends to an exact symmetry of the UV completion.

In $D = 5$, this interchange corresponds to

$$R_5 \leftrightarrow \frac{g_5^2}{8\pi}\,. \tag{3.1}$$

As one might expect for an S-duality, this is a highly nontrivial relation, involving all orders in perturbation theory. At least at first sight, it looks different in $D = 5$ than in $D = 4$.

However, it is not essentially different, as we can see by developing the perturbative expansion for the compactified 5d theory. To do this, we restrict the momentum along the new S^1 to quantized values

$$\int dp_5\, f(p_5) \to \frac{1}{R_5} \sum_{n\in\mathbb{Z}} f\left(\frac{n}{R_5}\right). \tag{3.2}$$

Taking into account the normalization of the loop integrals, the perturbative series is actually a series in $g_4^2 = R_6/R_5$, just as in $D = 4$. Furthermore, if we take $R_5 \to 0$ with fixed R_6/R_5, we are sending the 5d momentum states to infinite energy, so with the naive prescription of dropping them, we find that the relation eq. (3.1) does reduce to 4d S-duality in the naive way.

This is an example of the regularity of the $R_5 \to 0$ limit which we referred to in the introduction. Although it is intuitive that this should work, one can imagine scenarios in which it did not. In particular, we might worry about the consequences of UV divergences in $D = 5$. Such divergences would imply that the compactified $D = 4$ amplitudes get contributions from arbitrarily large momenta, so that $D = 5$ KK states at energies between $1/R_5$ and the cutoff Λ would contribute. In the $(2,0)$ theory, the S-duality partners of these states are KK states in the compactified x^6 direction. Thus, S-duality requires us to add these states. This was evident at finite L, but if $D = 5$ MSYM theory has UV divergences, then we must add them even for arbitrarily small but non-zero L.

To continue the comparison between $D = 5$ and $D = 4$, let us rewrite the $D = 5$ expansion in terms of $D = 4$ parameters,

$$g_4^2 \equiv \frac{R_6}{R_5}; \qquad L^2 \equiv R_5 R_6\,.$$

Now the prescription for compactification turns into

$$g_5^2 \int dp_5 \to g_4^2 \sum_n; \qquad \frac{1}{p_i^2} \to \frac{1}{p_{4i}^2 + g_4^2 n_i^2/L^2}\,. \tag{3.3}$$

The terms with all $n_i = 0$ are the $D = 4$ amplitudes, while in principle we could sum the rest of the expansion to identify the finite L corrections to $D = 4$ amplitudes. Of course, there are subtleties in doing this, analogous to those in finite temperature computations in conventional gauge theory, but now with the possibility of additional nonrenormalizable UV divergences.

3.1 Compactified $D = 5$ at one loop

To get a sense of the resulting structures, let us consider a one-loop amplitude with $k = 4$ external particles. In MSYM in any dimension, this amplitude is the product of a scalar box integral with a kinematic factor,

$$\mathcal{A} = \mathcal{K} \times \sum_{n\in\mathbb{Z}} I(s,t)\,. \tag{3.4}$$

Since $p_5^2 = n^2 M_5^2$ for a KK mode on S^1, the $D = 5$, $k = 4$ amplitude is obtained by summing a series of $D = 4$ amplitudes computed with massive box integrals,

$$\mathcal{A}_{D=5} = \mathcal{K} \times \sum_{n \in \mathbb{Z}} I(s, t; m^2 = n^2 M_5^2). \tag{3.5}$$

An explicit expression for this can be found in ([4], eq. A.8).[5]

As one can verify from the explicit result, because the $m \neq 0$ terms are both UV and IR convergent, one can just scale out m, so it has a non-zero $s = t = 0$ limit of order $1/m^4$ and a series expansion in powers of s/m^2 and t/m^2 around zero. Thus the sum will look like

$$\zeta(4) \frac{1}{M_5^4} + \zeta(6) \frac{s+t}{M_5^6} + \dots \tag{3.6}$$

plus the $n = 0$ term $\sim 1/st$.

As a function of M_5 this is regular at $M_5 \to \infty$, and using $M_5 = g_4/L$ we get a correction

$$C_8 \, L^4 \mathrm{tr}\, F^4 \tag{3.7}$$

to the $D = 4$ effective Lagrangian. This is a sensible "mock tree order" term; although it is generated at one loop, the g_4 dependence is cancelled by the g_4 dependence of M_5.[6]

One could go on to try to compute an S-dual version of this interaction, by adding in the $D = 6$ KK modes. A natural guess is that the coefficient is

$$C_8 L^4 = \sum_{m,n \neq 0} \frac{g_{n_5,n_6}^4}{M_{n_5,n_6}^4} \tag{3.8}$$

where g_{n_5,n_6} and M_{n_5,n_6} are the effective coupling (to a photon) and mass for a KK mode with quantized momenta $p_i = n_i/L$. One can check that $M_{n_5,n_6}^2 = |\tau n_6 + n_5|^2$. While the effective coupling to a 6d KK mode is not known, under the simple assumption that g_{n_5,n_6} is independent of n_i, the sum turns into a nonholomorphic Eisenstein series,

$$C_8 = \zeta(4) E(\tau, 2) \equiv \sum_{m,n \neq 0} \left(\frac{\mathrm{Im}\,\tau}{|m\tau + n|^2} \right)^2, \tag{3.9}$$

and one obtains an S-duality invariant coefficient function.

It would be interesting to predict this term from some string theory argument. Usually such corrections are controlled by α' or l_{p11}, but here we dropped this dependence in defining the $(2, 0)$ theory. On the other hand, one might find some duality which relates our volume of T^2 to these parameters.

The next term in the expansion in s/m and t/m is more problematic:

$$\frac{L^6}{g_4^2} \mathrm{tr}\, D^2 F^4. \tag{3.10}$$

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[5]One takes $M = 0$ in the result quoted there, which drops the last term.

[6]Note that this is in conventions with $F = dA + g_4 A^2$. To discuss S-duality one would take $S = g_4^{-2} \mathrm{tr}\, \tilde{F}^2$ and $\tilde{F} = g_4 dA + g_4^2 A^2$, and $\mathrm{tr}\, F^4$ becomes $\mathrm{tr}\, (\tilde{F}/g_4)^4$; this is the convention in eq. (3.9).

While this was a sensible $1/M_5^2$ correction to effective field theory, it does not make much sense as a term in a perturbative g_4 expansion, and spoils the $g_4 \to 0$ limit at fixed L. What is going on?

In fact, the $g_4 \to 0$ limit at fixed L corresponds to taking $R_5 \to \infty$ and $M_5 \to 0$, so it is not surprising that it does not have a $D = 4$ interpretation. We need to take $L \to 0$ faster than $g_4 \to 0$ so that $M_5 \to \infty$ to have such an interpretation. Keeping this in mind, we can still ask whether particular $D = 4$ corrections are regular.

3.2 Structure of $D = 5$ UV divergences

Let us now consider the $D = 4$ effective Lagrangian obtained by integrating out KK modes. We can apply the same strategy to all orders in the perturbative expansion. Denote the loop order as ℓ. We assume the external momenta all satisfy $p_i \cdot p_j \ll M_5$; then we make the replacements eq. (3.3) and integrate out all lines with $p_5 \neq 0$, to get an expression for the corrections to the $D = 4$ theory as a sum over $D = 4$ loops of particles with masses $m_a = M_5 n_a$,

$$\mathcal{A}_n = \sum_\ell g_4^{2+2\ell} \sum_{n_1,...,n_\ell} \mathcal{A}_{n,D=4}(p_i; \; m_a) \,. \tag{3.11}$$

We would then expand this amplitude in external momenta and identify terms as due to operators in an effective Lagrangian,

$$\mathcal{L} = \sum_\Delta C_\Delta \mathcal{O}_\Delta \,, \tag{3.12}$$

where for brevity we label the coefficient of a generic operator with dimension Δ as C_Δ.

Since the particles in these loops are all massive, the $D = 4$ momentum integrals are IR convergent. Furthermore, at large momentum we can ignore the masses, so since this is MSYM in $D = 4$ the integrals are UV convergent. Thus we can again estimate them using dimensional analysis, in terms of the unique scale in the computation, M_5. The only difference with the one loop computation above is that we can produce more general operators, and the final sum is over ℓ independent KK momenta. Thus,

$$C_\Delta \sim \sum_\ell g_4^{2+2\ell} \sum_{n_1,...,n_\ell} \frac{1}{(n \cdot M_5)^{\Delta-4}} \tag{3.13}$$

Of course, the explicit functions of the n_a appearing here will be quite complicated, and we cannot hope to do these sums explicitly, but we can estimate their dependence on the cutoff. If we assume genericity, so that the maximal UV divergence appears at each order, we find at loop orders $\ell > \Delta - 4$

$$C_\Delta \sim \sum_\ell g_4^{2+2\ell} \left(\frac{\Lambda}{M_5}\right)^{\ell+4-\Delta} \frac{1}{M_5^{\Delta-4}} \tag{3.14}$$

while at $\ell = \Delta - 4$ we find

$$C_\Delta \sim g_4^{2+2\ell} \log\left(\frac{\Lambda}{M_5}\right) \frac{1}{M_5^{\Delta-4}} \tag{3.15}$$

– 13 –

Now, let us explore the consequences of assuming $\Lambda = 1/R_6 = 1/g_4 L$. Recalling that $M_5 = g_4/L$, we find

$$C_\Delta \sim \sum_\ell g_4^{2\Delta-6} \left(\frac{L}{g_4}\right)^{\Delta-4}$$
$$\sim L^{\Delta-4} \sum_\ell g_4^{\Delta-2} \qquad (3.16)$$

from the powerlike divergences, and

$$C_\Delta \sim L^{\Delta-4} g_4^{\Delta-2} \log g_4 \qquad (3.17)$$

from the log divergence.

Note that the dependence on loop order ℓ cancels out of the leading divergences — the loop counting parameter g_4^2 is compensated by the additional UV divergence $\Lambda/M_5 \sim 1/g_4^2$. Subleading divergences at loop order ℓ could produce terms with an additional $g_4^2, g_4^6, \ldots$ up to $g_4^{2\ell-2}$ and $g_4^{2\ell} \log g_4$.

Finally, a finite contribution will go as $g_4^{2\ell+6-\Delta}$. Although this exponent can be negative at low loop order, this will be for the same reason as in eq. (3.10), that the $g_4 \to 0$ limit at fixed L actually decompactifies, so these are not contributions to the $D = 4$ effective Lagrangian. Presumably, finite contributions are always regular, by the general consistency of duality and compactification.

Turning to actual counterterms, the lowest dimension correction in $D = 4$ MSYM is the $\operatorname{tr} F^4$ term with $\Delta = 8$, which only appears at one loop. At higher loops one can have counterterms with $\Delta \geq 10$, starting with $\operatorname{tr} D^2 F^4$ which can be generated logarithmically at $\ell = 6$. This is a superspace D-term [3, 12] so one expects that it will be generated, as will operators with $\Delta > 10$ at higher loops, presumably continuing without bound.

In general, the dependence on the $D = 4$ parameters g_4 and L looks sensible. L controls the corrections as expected for irrelevant operators, and g_4 appears with positive powers, or as $\log g_4$ multiplied by a positive power, which at least vanishes as $g_4 \to 0$. The problem we observed with the convergent one-loop diagram does not show up in the UV divergent terms.

3.3 Discussion and speculations

The conclusion of this section is that $D = 5$ UV divergences look like they can have a sensible $D = 4$ interpretation, under the assumption that the $(2,0)$ theory provides a cutoff which can be modeled by eq. (1.2). Adding in the subleading divergences and finite terms, we could in principle compute corrections to the $D = 4$ MSYM low energy limit as well-defined functions of g_4. To get S-dual results, we must somehow add in the contibutions of $(2,0)$ KK modes, as in eq. (3.9). One might hope that, instead of explicitly adding in the contributions of instantonic particles, this could be done by finding a resummation prescription consistent with modular invariance.

However, since every leading divergence in the loop expansion appears at the same order in g_4, one must ask whether this is a controlled expansion at all. Clearly this point

must be resolved in order to proceed further. If the ℓ loop term has the generic growth $\ell!$, then it is hard to see how such an expansion can make sense. This might be another argument favoring UV finiteness of $D = 5$ perturbation theory. Conversely, if there are UV divergences, this might be an argument that the relation between $(2, 0)$ theory and $D = 5$ SYM is more complicated than we have assumed, perhaps involving additional states.

Another possibility is that the coefficients of the cutoff dependent terms do not have the generic growth $\ell!$. Perhaps only a finite number of divergences contribute at each order in the coupling. We should also remember that the assumption $\Lambda = c/R_6$ is only a model for some more concrete (and complicated) cutoff provided by $(2, 0)$ theory, such as the loop contributions of 6d KK modes, a form factor, or something else. For the purpose of computing corrections to an effective Lagrangian, the cutoff should still be describable in $D = 5$ terms, and perhaps even in terms of a simple ansatz, which makes the series eq. (3.16) convergent. For example, the coefficient c could fall off with ℓ, compensating the $\ell!$.

If the individual corrections eq. (3.16) and eq. (3.17) do make sense, then the log divergence eq. (3.17) leads to a very unusual contribution to the coefficient C_Δ of a dimension Δ operator, proportional to $g_4^{\Delta-2} \log g_4$. For example, the potential divergence at $\ell = 6$ is a log divergent coefficient of the operator $L^6 \text{tr}\, D^2 F^4$. This would be a very distinctive feature to try to match from some other construction of $(2, 0)$ theory on T^2. Or, perhaps there is some argument that it must vanish, either to satisfy S-duality, or from general properties of another construction.

Perhaps future developments in loop calculations will tell us whether there are UV divergences, and provide information enabling us to continue this discussion. For example, the planar six loop integrand for the log divergent coefficient of $\text{tr}\, D^2 F^4$ has been written out explicitly [7]. It exhibits some cancellations and is not positive definite, in contrast to the cases of five or fewer loops in $D = 4 + 6/\ell$. Numerical integration could help decide whether or not it vanishes.

Acknowledgments

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