On D-Preopen Sets in D-Matric Spaces

Hussain Wahish
Department of Mathematics,Faculty of Education, University of Saba, Region, Mareb, Yemen

Amin Saif
Department of Mathematics, Faculty of Applied Sciences, Taiz University, Taiz, Yemen

ABSTRACT

The purpose of this paper is to introduce and investigate weak form of D-open sets in D-metric spaces, namely D-preopen sets. The relationships among this form with the other known sets are introduced. We give the notions of the interior operator, the closure operator and frontier operator via D-preopen sets.

AMS classification: Primary 54A05, 54E35.

Keywords

Open set, Metric spaces.

1. INTRODUCTION

Metric spaces is one of the most important spaces in mathematics there are various type of generalization of metric spaces, [5]. The axiomatic approach to the metric spaces is given by a french mathematician M. Frechet in year 1812, [7]. In 1984, Dhage, [3], introduced a new notion of a new structure of D-metric space which is a natural generalization of the notion of ordinary metric space to higher dimensional metric spaces, [4]. In 2000, Dhage, [2], introduced some results in D-metric spaces are obtained and the notion of open and closed balls. In 2013, [6], exhibited methods of generating D-metrics from certain types of real valued partial functions on the three dimensional Euclidean space. In 2017, Ali Fora, Massadeh and Bataineh, [1], introdused and a new topological structure of D-closed set.

This paper is organized as follows. Section 2 is devoted to some preliminaries. Section 3 introduces the concept of D-preopen sets by utilizing the D-open balls. Furthermore, the relationship with the other known sets will be studied. In Section 4 we introduce the concepts of the interior operator, the closure operator and frontier operator via D-preopen sets.

2. PRELIMINARIES

DEFINITION 2.1. [7]. Let X be any nonempty set. A function $d : X \times X \to [0,\infty)$ is called a metric function on X if it satisfies the following three conditions for all $x, y, z \in X$:

(1) (positive property) $d(x, y) \geq 0$ with equality if and only if $x = y$;
(2) (symmetric property) $d(x, y) = d(x, y)$;
(3) (triangle inequality) $d(x, z) \leq d(x, y) + d(y, z)$.

A pair (X, d), where d is a metric on X is called a metric space. By $O_\varepsilon(x)$, we mean the open ball with center x and radius $\varepsilon > 0$, that is,

$$O_\varepsilon(x) = \{y \in X : d(x, y) < \varepsilon\}.$$

By $C_\varepsilon(x)$, we mean the closed ball with center x and radius $\varepsilon > 0$, that is,

$$C_\varepsilon(x) = \{y \in X : d(x, y) \leq \varepsilon\}.$$

For metric space (X, d) and $G \subseteq X$, the set G said to be open set if for any point $x \in G$, there exists $\varepsilon > 0$ such that $O_\varepsilon(x) \subseteq G$. The set G is called closed set in metric space (X, d) if $X - G$ is an open set in metric space (X, D). For the set of real numbers R, we mean by the usual metric space (R, d),

$$d(x, y) = |x - y| \text{ for all } x, y \in R$$

For metric space (X, d) and $G \subseteq X$, the interior operator of G is denoted by $Int(G)$ and the clouser operator of G is denoted by $Cl(G)$.

DEFINITION 2.2. [4]. A nonempty set X, together with a function $D : X \times X \times X \to [0,\infty)$ is called a D-metric space, denoted by (X, D) if D satisfies the following $x, y, z, u \in X$:

(1) $D(x, y, z) = 0 \to x = y = z$ (coincidence);
(2) $D(x, y, z) = D(p(x, y, z))$, where p is a permutation of x, y, z (symmetry);
(3) $D(x, y, z) \leq D(x, y, a) + D(x, a, z) + D(a, y, z)$ for all $x, y, z, u \in X$ (tetrahedral inequality).

By $O^D_\varepsilon(x)$, we mean the D-open ball with center x and radius $\varepsilon > 0$, that is,

$$O^D_\varepsilon(x) = \{y \in X : d(x, y, y) < \varepsilon\}.$$

By $C^D_\varepsilon(x)$, we mean the D-closed ball with center x and radius $\varepsilon > 0$, that is,

$$C^D_\varepsilon(x) = \{y \in X : d(x, y, y) \leq \varepsilon\}.$$

The set $G \subseteq X$ is called D-open set in D-metric space (X, D) if for every $x \in G$, there is $\varepsilon > 0$ such that $O^D_\varepsilon(x) \subseteq G$. The set G is called D-closed set in D-metric space (X, D) if $X - G$ is D-open set in D-metric space (X, D). For D-metric space (X, D) and $G \subseteq X$, the interior set of G is denoted by $Int_D(G)$ and the clouser set of G is denoted by $Cl_D(G)$.

THEOREM 2.3. [2]. Let (R, D) be D-metric space where

$$D(x, y, z) = max\{d(x, y), d(y, z), d(z, x)\}$$

and (R, d) is usual metric space. Then for a fixed $x \in R$, the D-open balls $O_\varepsilon^D(x)$ and $O_\varepsilon^D(x)$ are the sets in given by: $O_\varepsilon^D(x) = (x-\varepsilon, x+\varepsilon)$.

THEOREM 2.4. [2]. Let (R, D) be D-metric space, where

$$D(x,y,z) = d(x,y) + d(y,z) + d(z,x)$$

and (R, d) is usual metric space. Then for a fixed $x \in R$, the D-open balls $O_\varepsilon^D(x)$ and $O_\varepsilon^D(x)$ are the sets in given by: $O_\varepsilon^D(x) = (x-\varepsilon/2, x+\varepsilon/2)$.

THEOREM 2.5. [2]. Every D-open O_ε^D(x) , $x \in X, \varepsilon > 0$ is a D-open set in X (i.e., it contains a ball of each of its points).

THEOREM 2.6. [1]. Every a finite set in a D-metric space (X, D) must be D-closed set.

THEOREM 2.7. [2]. Every ball $C_\varepsilon^D(x)$ in a D-metric space (X, D) is D-closed set.

THEOREM 2.8. [2]. Arbitrary union and finite intersection of D-open balls $O_\varepsilon^D(x), x \in X$ is D-open set.

THEOREM 2.9. [1]. Let $D : X \times X \times X \times X \to [0, \infty)$ be a D-metric on X having a finite range. Then every subset A of X is D-closed set.

3. D-PREOPEN SETS

DEFINITION 3.1. Let (X, D) be a D-metric space. A subset $G \subseteq X$ is called a D-preopen set in D-metric space (X, D) if for every $x \in G$, there is $\delta > 0$ such that for every $y \in O_\delta^D(x)$, $O_\varepsilon^D(y) \cap G \neq \emptyset$ for every $\varepsilon > 0$.A subset $G \subseteq X$ is called a D-preclosed set in D-metric space (X, D) if $X - G$ is a D-preopen set in D-metric space (X, D).

The set of all D-preopen sets in X denoted by $D_pO(X, D)$ and the set of all D-preclosed sets in X denoted by $D_pC(X, D)$.

EXAMPLE 3.2. Let (R, D) be D-metric space given by

$$D(x,y,z) = max\{d(x,y), d(y,z), d(z,x)\},$$

where (R, d) is usual metric space on the set of real number R. An open interval $G = (0, 2)$ is D-preopen set in (R, D). For every $x \in G$, take $\delta = min\{|x|, |2-x|\} > 0$. If $y \in O_\delta^D(x)$, then $O_\varepsilon^D(y) \cap G \neq \emptyset$ for every $\varepsilon > 0$.

EXAMPLE 3.3. In Example(3.2), a closed interval $G = [-1, 1]$ is not D-preopen set, since at x=1, take $y = (2+\delta)/2 \in O_\delta^D(1)$ and $\varepsilon = \delta/2 > 0$. Note that $O_{\delta/2}^D((2+\delta)/2) \cap G = \emptyset$. That is, $G = [-1, 1]$ is not D-preopen set in (R, D).

THEOREM 3.4. Every D-open set is a D-preopen set.

PROOF. Let G be any D-open set in D-metric space (X, D). Let $x \in G$ be arbitrary point. Then there is $\delta > 0$ such that $O_\delta^D(x) \subseteq G$. For every $y \in O_\delta^D(x)$, $y \in O_\varepsilon^D(y)$ and $y \in G$ for every $\varepsilon > 0$. That is, $O_\varepsilon^D(y) \cap G \neq \emptyset$ for every $\varepsilon > 0$. Hence G is D-preopen set. □

The converse of above theorem need not be true.

EXAMPLE 3.5. In Example(3.2), the set of rational numbers Q is a D-preopen set but not D-open set in (R, D).

Note that the intersection of two D-preopen sets no need to be D-preopen set. In Example(3.2), the set of rational numbers Q is a D-preopen set but not D-open set in (R, D) and the set $IR \cup \{q\}$ is a D-preopen set in (R, D), where IR is the set of irrational numbers and q is any rational number, but $Q \cap (IR \cup \{q\}) = \{q\}$ is not D-preopen set.

The following theorem shows that the intersection of a D-open set and a D-preopen set is a D-preopen set.

THEOREM 3.6. The intersection of a D-open set and a D-preopen set is a D-preopen set.

PROOF. Let A be D-open set and B be D-preopen set in D-metric space in (X, D). Let $x \in A \cap B$ be arbitrary point. Then $x \in A$ and $x \in B$. Then there are $\delta_1 > 0$ and $\delta_2 > 0$ such that $O_{\delta_1}^D(x) \subseteq A$ and for every $y \in O_{\delta_2}^D(x)$, $O_\varepsilon^D(y) \cap B \neq \emptyset$ for every $\varepsilon > 0$. Take $\delta = min\{\delta_1, \delta_2\} > 0$. Then $O_\delta^D(x) \subseteq A$ and for every $y \in O_\delta^D(x)$, $O_\varepsilon^D(y) \cap B \neq \emptyset$ for every $\varepsilon > 0$. Now for every $y \in O_\delta^D(x)$ and since A is D-open set, then there is $\varepsilon_y > 0$ such that $O_{\varepsilon_y}^D(y) \subseteq A$ and $O_{min\{\varepsilon_y, \varepsilon\}}^D(y) \cap B \neq \emptyset$. Since $O_{min\{\varepsilon_y, \varepsilon\}}^D(y) \cap B \subseteq O_\varepsilon^D(y) \cap A \cap B$, then $O_\varepsilon^D(y) \cap (A \cap B) \neq \emptyset$ for every $\varepsilon > 0$. That is $A \cap B$ is D-preopen set. □

THEOREM 3.7. The union of any family of D-preopen sets is D-preopen set.

PROOF. Let G_λ be a D-preopen subset of D-metric space (X, D) for all $\lambda \in \Delta$. Let $x \in \bigcup_{\lambda \in \Delta} G_\lambda$ be an arbitrary point. Then there is at least $\lambda_0 \in \Delta$ such that $x \in G_{\lambda_0}$. Since G_{λ_0} is a D-preopen then for every $x \in G_{\lambda_0}$, there is $\delta > 0$ such that for every $y \in O_\delta^D(x)$, $O_\varepsilon^D(y) \cap G_{\lambda_0} \neq \emptyset$ for every $\varepsilon > 0$. Since $G_{\lambda_0} \subseteq \bigcup_{\lambda \in \Delta} G_\lambda$, then for every x $\in \bigcup_{\lambda \in \Delta} G_\lambda$, there is $\delta > 0$ such that for every $y \in O_\delta^D(x)$, $O_\varepsilon^D(y) \cap \bigcup_{\lambda \in \Delta} G_\lambda \neq \emptyset$ for every $\varepsilon > 0$. That is $\bigcup_{\lambda \in \Delta} G_\lambda$ is D-preopen set. □

4. D-PREOPEN OPERATORS

In this section, we define the interior operator, the closure operator and frontier operator via D-preopen sets.

DEFINITION 4.1. Let (X, D) be a D-metric space and $G \subseteq X$. The D_P-closure operator of G is denoted by $Cl_P^D(G)$ and defined by

$$Cl_P^D(G) = \cap\{H \subseteq X : G \subseteq H \text{ and } H \text{ is D-preclosed set}\}.$$

The D_P-interior functor of G is denoted by $Int_P^D(G)$ and defined by

$$Int_P^D(G) = \cup\{H \subseteq X : H \subseteq G \text{ and } H \text{ is D-preopen set}\}.$$

REMARK 4.2.

(1) From Theorem(3.7), $Cl_P^D(G)$ is a D-preclosed set and $Int_P^D(G)$ is D-preopen set in D-metric space (X, D).
(2) For a D-metric space (X, D) and $G \subseteq X$, it is clear from the definition of $Cl_P^D(G)$ and $Int_P^D(G)$ that $G \subseteq Cl_P^D(G)$ and $Int_P^D(G) \subseteq G$.

THEOREM 4.3. For a D-metric space (X, D) and $G \subseteq X$, $Cl_P^D(G) = G$ if and only if G is a D-preclosed set.

PROOF. Let $Cl_P^D(G) = G$. Then from definition of $Cl_P^D(G)$ and Theorem(3.7), $Cl_P^D(G)$ is a D-preclosed set and G is a D-preclosed set. Conversely, we have $G \subseteq Cl_P^D(G)$ by Remark(4.2). Since G is a D-preclosed set, then it is clear from the definition of $Cl_P^D(G)$, $Cl_P^D(G) \subseteq G$. Hence $G = Cl_P^D(G)$. □

THEOREM 4.4. For a D-metric space (X, D) and $G \subseteq X$, and $Int_P^D(G) = G$ if and only if G is a D-preopen set.

PROOF. Let G be D-preopen set. Then for all $x \in G$, we have $x \in G \subseteq G$. That is, $G \subseteq Int_P^D(G)$. Then $G = Int_P^D(G)$ from Remark(4.2). The converse is trivial. □

THEOREM 4.5. For a D-metric space (X, D) and $G \subseteq X$, $x \in Cl_P^D(G)$ if and only if for all D-preopen set M containing x, $M \cap G \neq \emptyset$.

PROOF. Let $x \in Cl_P^D(G)$ and M be any D-preopen set containing x. If $M \cap G = \emptyset$ then $G \subseteq X - M$. Since $X - M$ is a D-preclosed set containing G, then $Cl_P^D(G) \subseteq X - M$ and so $x \in Cl_P^D(G) \subseteq X - M$. Hence this is contradiction, because $x \in M$. Therefore $M \cap G \neq \emptyset$.
Conversely, Let $x \notin Cl_P^D(G)$. Then $X - Cl_P^D(G)$ is a D-preopen set containing x. Hence by hypothesis, $[X - Cl_P^D(G)] \cap G \neq \emptyset$. But this is contradiction, because $X - Cl_P^D(G) \subseteq X - G$. □

THEOREM 4.6. For a D-metric space (X, D) and $G \subseteq X$, $x \in Int_P^D(G)$ if and only if there is D-preopen set M such that $x \in M \subseteq G$.

PROOF. Let $x \in Int_P^D(G)$ and take $M = Int_P^D(G)$. Then by Theorem(4.5) and definition of $Int_P^D(G)$ we get that M is a D-preopen set and by Remark(4.2), $x \in M \subseteq G$. Conversely, let there is D-preopen set M such that $x \in M \subseteq G$ Then by definition of $Int_P^D(G)$, $x \in M \subseteq Int_P^D(G)$. □

THEOREM 4.7. For a D-metric space (X, D) and $G, M \subseteq X$, the following hold:

(1) If $G \subseteq M$ then $Cl_P^D(G) \subseteq Cl_P^D(M)$.
(2) $Cl_P^D(G) \cup Cl_P^D(M) \subseteq Cl_P^D(G \cup M)$.
(3) $Cl_P^D(G \cap M) \subseteq Cl_P^D(G) \cap Cl_P^D(M)$.
(4) $Cl_P^D(G) \subseteq Cl_D(G)$.

PROOF. (1) Let $x \in Cl_P^D(G)$. Then by Theorem(4.5), for all D-preopen set N containing x, $N \cap G \neq \emptyset$. Since $G \subseteq M$ then $N \cap M \neq \emptyset$. Hence $x \in Cl_P^D(M)$. That is, $Cl_P^D(G) \subseteq Cl_P^D(M)$.
(2) Since $G \subseteq G \cup M$ and $M \subseteq G \cup M$, then by part(1), $Cl_P^D(G) \subseteq Cl_P^D(G \cup M)$ and $Cl_P^D(M) \subseteq Cl_P^D(G \cup M)$. Hence $Cl_P^D(G) \cup Cl_P^D(M) \subseteq Cl_P^D(G \cup M)$.
(3) Since $G \cap M \subseteq G$ and $G \cap M \subseteq M$, then by part(1), $Cl_P^D(G \cap M) \subseteq Cl_P^D(G)$ and $Cl_P^D(G \cap M) \subseteq Cl_P^D(M)$. Hence $Cl_P^D(G \cap M) \subseteq Cl_P^D(G) \cap Cl_P^D(M)$.
(4) It is clear from Theorem(4.5) and from every D-open set is D-preopen set.

□

In the above theorem $Cl_P^D(G \cup M) \neq Cl_P^D(G) \cup Cl_P^D(M)$ as it is shown in the following example.

EXAMPLE 4.8. Let (R, D) be D-metric space, where

$$D(x, y, z) = d(x, y) + d(y, z) + d(z, x)$$

and (R, d) is usual metric space. Let $G = IR$ and $M = Q - [\{q\}]$, where Q is the set of rational numbers, IR is the set of irrational numbers and q is any rational number. Since G and M are D-precloced sets in R. Then $Cl_P^D(G) \cup Cl_P^D(M) = G \cup M = R - \{q\}$. If $R - \{q\}$ is D-precloced set in R then $\{q\}$ is D-preopen set but $\{q\}$ is not D-preopen set and this contradiction. Hence $R - \{q\}$ is not D-precloced set in R. Since $R - \{q\} \subseteq Cl_P^D(R - \{q\})$ then

$$Cl_P^D(G \cup M) = Cl_P^D(R - \{q\}) = R.$$

THEOREM 4.9. For a D-metric space (X, D) and $G, M \subseteq X$, the following hold:

(1) If $G \subseteq M$ then $Int_P^D(G) \subseteq Int_P^D(M)$.
(2) $Int_P^D(G) \cup Int_P^D(M) \subseteq Int_P^D(G \cup M)$.
(3) $Int_P^D(G \cap M) \subseteq Int_P^D(G) \cap Int_P^D(M)$.
(4) $Int_D(G) \subseteq Int_P^D(G)$.

PROOF. (1) Let $x \in Int_P^D(G)$. Then by Theorem(4.6), there is D-preopen set N such that $x \in N \subseteq G$ Since $G \subseteq M$ then $x \in N \subseteq M$. Hence $x \in Int_P^D(M)$. That is, $Int_P^D(G) \subseteq Int_P^D(M)$.
(2) Since $G \subseteq G \cup M$ and $M \subseteq G \cup M$, then by part(1), $Int_P^D(G) \subseteq Int_P^D(G \cup M)$ and $Int_P^D(M) \subseteq Int_P^D(G \cup M)$. Hence $Cl_P^D(G) \cup Int_P^D(M) \subseteq Int_P^D(G \cup M)$.
(3) Since $G \cap M \subseteq G$ and $G \cap M \subseteq M$, then by part(1), $Int_P^D(G \cap M) \subseteq Int_P^D(G)$ and $Int_P^D(G \cap M) \subseteq Int_P^D(M)$. Hence $Int_P^D(G \cap M) \subseteq Int_P^D(G) \cap Int_P^D(M)$.
(4) It is clear from Theorem(4.5) and from every D-open set is D-preopen set.

□

In the above theorem $Int_P^D(G \cap M) \neq Int_P^D(G) \cap Int_P^D(M)$ as it is shown in the following example.

EXAMPLE 4.10. In Example(4.8), take $G = Q \cup \{r\}$ and $M = IR$, where Q is the set of rational numbers, IR is the set of irrational numbers and r is any irrational number. Since G and M are D-preopen sets in R. Then $Int_P^D(G) \cap Int_P^D(M) = G \cap M = (Q \cup \{r\}) \cap IR = \{r\}$. Since $\{r\}$ is not D-preopen set and $Int_P^D(\{r\}) \subseteq \{r\}$ then $Int_P^D(G \cap M) = Int_P^D(\{r\}) = \emptyset$.

THEOREM 4.11. For a D-metric space (X, D) and $G \subseteq X$, the following hold:

(1) $Int_P^D(X - G) = X - Cl_P^D(G)$.
(2) $Cl_P^D(X - G) = X - Int_P^D(G)$.

PROOF. (1) Since $G \subseteq Cl_P^D(G)$, then $X - Cl_P^D(G) \subseteq X - G$. Since $Cl_P^D(G)$ is a D-preclosed set then $X - Cl_P^D(G)$ is a D-preopen set. Then

$$X - Cl_P^D(G) = Int_P^D[X - Cl_P^D(G)] \subseteq Int_P^D(X - G).$$

For the other side, let $x \in Int_P^D(X - G)$. Then there is D-preopen set N such that $x \in N \subseteq X - G$. Then $X - N$ is a D-preclosed set containing G and $x \notin X - N$. Hence $x \notin Cl_P^D(G)$, that is, $x \in X - Cl_P^D(G)$.
(2) Since $Int_P^D(G) \subseteq G$, then $X - G \subseteq X - Int_P^D(G)$. Since $Int_P^D(G)$ is a D-preopen set then $X - Int_P^D(G))$ is a D-preclosed set. Then

$$Cl_P^D(X - G) \subseteq Cl_P^D[X - Int_P^D(G)] = X - Int_P^D(G)$$

. For the other side, let $x \notin Cl_P^D(X - G)$. Then by Theorem(4.5), there is a D-preopen set N containing x such that $N \cap (X - G) = \emptyset$. Then $x \in N \subseteq G$, that is, $x \in Int_P^D(G)$. Hence $x \notin X - Int_P^D(G)$. Therefore $X - Int_P^D(G) \subseteq Cl_P^D(X - G)$.

□

THEOREM 4.12. For a subset $G \subseteq X$ of D-metric space (X, D) the following hold:

(1) If M is a D-open set in X then $Cl_P^D(G) \cap M \subseteq Cl_P^D(G \cap M)$.

(2) If M is a D-closed set in X then $Int_P^D(G\cup M) \subseteq Int_P^D(G)\cup M$.

PROOF. (1) Let $x \in Cl_P^D(G)\cap M$. Then $x \in Cl_P^D(G)$ and $x \in M$. Let V be any D-preopen set in (X, D) containing x. By Theorem(3.6), $V \cap M$ is D-preopen set containing x. Since $x \in Cl_P^D(G)$ then by Theorem(4.5), $(V\cap M)\cap G \neq \emptyset$. This implies, $V \cap (M \cap G) \neq \emptyset$. Hence by Theorem(4.5), $x \in Cl_P^D(G\cap M)$. That is, $Cl_P^D(G)\cap M \subseteq Cl_P^D(G\cap M)$.

(2) Since M is a D-closed set X then by the part(1) and Theorem(4.11),

$$\begin{aligned} X-[Int_P^D(G)\cup M] &= [X-Int_P^D(G)]\cap[X-M] \\ &= [Cl_P^D(X-G)]\cap[X-M] \\ &\subseteq Cl_P^D[(X-G)\cap(X-M)] \\ &= Cl_P^D(X-(G\cup M)) \\ &= X-(Int_P^D(G\cup M)). \end{aligned}$$

Hence $Int_P^D(G\cup M) \subseteq Int_P^D(G)\cup M$.

□

LEMMA 4.13. For a D-metric space (X, D) and $G \subseteq X$, $x \in Cl_D(G)$ if and only if for all $\varepsilon > 0$, $O_\varepsilon^D(x)\cap G \neq \emptyset$.

PROOF. Let $x \in Cl_D(G)$ and $\varepsilon > 0$. If $O_\varepsilon^D(x)\cap G = \emptyset$ then $G \subseteq X - O_\varepsilon^D(x)$. Since $X - O_\varepsilon^D(x)$ is a D-closed set containing G, then $Cl_D(G) \subseteq X - O_\varepsilon^D(x)$ and $x \in Cl_D(G) \subseteq X - O_\varepsilon^D(x)$. Hence this is contradiction, because $x \in O_\varepsilon^D(x)$. Therefore $O_\varepsilon^D(x)\cap G \neq \emptyset$.
Conversely, Let $x \notin Cl_D(G)$. Then $X - Cl_D(G)$ is a D-open set containing x. Then there is $\varepsilon > 0$ such that $O_\varepsilon^D(x) \subseteq X - Cl_D(G)$ Hence by hypothesis, $O_\varepsilon^D(x)\cap G \neq \emptyset$. But this is contradiction, because $O_\varepsilon^D(x) \subseteq X - Cl_D(G) \subseteq X - G$. □

THEOREM 4.14. A subset $G \subseteq X$ of D-metric space (X, D) is a D-preopen set if and only if $G \subseteq Int_D(Cl_D(G))$.

PROOF. Suppose that G is a D-preopen set. Let $x \in G$ be arbitrary point. Then there is $\delta > 0$ such that for every $y \in O_\delta^D(x)$, $O_\varepsilon^D(y)\cap G \neq \emptyset$ for every $\varepsilon > 0$. By Lemma(4.13), we get that $O_\delta^D(x) \subseteq Cl_D(G)$. That is, $x \in Int_D(Cl_D(G))$. Hence $G \subseteq Int_D(Cl_D(G))$.
Conversely, Suppose that $G \subseteq Int_D(Cl_D(G))$ and $x \in G$ is arbitrary point. Then $x \in Int_D(Cl_D(G))$. That is, there is $\delta > 0$ such that $O_\delta^D(x) \subseteq Cl_D(G)$. Hence for every $y \in O_\delta^D(x)$, $O_\varepsilon^D(y)\cap G \neq \emptyset$ for every $\varepsilon > 0$. Hence G is a D-preopen set. □

For a subset G of D-metric space (X, D) the D-frontier operator of G is defined by

$$\Gamma_P^D(G) = Cl_P^D(G) - Int_P^D(G).$$

THEOREM 4.15. For a subset $G \subseteq X$ of D-metric space (X, D), the following hold:

(1) $Cl_P^D(G) = \Gamma_P^D(G)\cup Int_P^D(G)$.
(2) $\Gamma_P^D(G)\cap Int_P^D(G) = \emptyset$.
(3) $\Gamma_P^D(G) = Cl_P^D(G)\cap Cl_P^D(X-G)$.

PROOF. Note that

(1) $\Gamma_P^D(G)\cup Int_P^D(G)$

$$\begin{aligned} &= (Cl_P^D(G)-Int_P^D(G))\cup Int_P^D(G) \\ &= [Cl_P^D(G)\cap(X-Int_P^D(G))]\cup Int_P^D(G) \\ &= [Cl_P^D(G)\cup Int_P^D(G)]\cap[(X-Int_P^D(G))\cup Int_P^D(G)] \\ &= Cl_P^D(G)\cap X = Cl_P^D(G). \end{aligned}$$

(2) It is clear from the definition of $\Gamma_P^D(G)$.
(3) By Theorem(4.11),

$$\begin{aligned} \Gamma_P^D(G) &= Cl_P^D(G)-Int_P^D(G) = Cl_P^D(G) \\ &\quad \cap(X-Int_P^D(G)) \\ &= Cl_P^D(G)\cap Cl_P^D(X-G). \end{aligned}$$

This is the desired. □

COROLLARY 4.16. For a subset $G \subseteq X$ of D-metric space (X, D), $\Gamma_P^D(G)$ is D-preclosed set in (X, D).

PROOF. By Theorem(4.9) and the part(3) of the last theorem. □

THEOREM 4.17. For a subset $G \subseteq X$ of D-metric space (X, D), the following hold:

(1) G is a D-preopen set if and only if $\Gamma_P^D(G)\cap G = \emptyset$.

(2) G is a D-preclosed set if and only if $\Gamma_P^D(G) \subseteq G$.

(3) G is both D-preopen set and D-preclosed set if and only if $\Gamma_P^D(G) = \emptyset$.

PROOF. (1) Let G be a D-preopen set. Then $Int_P^D(G) = G$. Then by Theorem(4.15),

$$\Gamma_P^D(G)\cap G = \Gamma_P^D(G)\cap Int_P^D(G) = \emptyset$$

Conversely, suppose that $\Gamma_P^D(G)\cap G = \emptyset$. Then

$$\begin{aligned} G-Int_P^D(G) &= [G\cap Cl_P^D(G)]-[G\cap Int_P^D(G)] \\ &= G\cap(Cl_P^D(G)-Int_P^D(G)) \\ &= G\cap\Gamma_P^D(G) = \emptyset. \end{aligned}$$

That is, $Int_P^D(G) = G$. Hence G is a D-preopen set.

(2) Let G be a D-preclosed set. Then $Cl_P^D(G) = G$. Then

$$\Gamma_P^D(G) = Cl_P^D(G)-Int_P^D(G) = G-Int_P^D(G) \subseteq G.$$

Conversely, let $\Gamma_P^D(G) \subseteq G$. Then by Theorem(4.15),

$$Cl_P^D(G) = Int_P^D(G)\cup\Gamma_P^D(G) \subseteq Int_P^D(G)\cup G \subseteq G.$$

That is, $Cl_P^D(G) = G$. Hence G is D-preclosed set.

(3) Let G be both D-preclosed set and D-preopen set. Then $Cl_P^D(G) = G = Int_P^D(G)$. Then

$$\Gamma_P^D(G) = Cl_P^D(G)-Int_P^D(G) = G-G = \emptyset.$$

Conversely, suppose that $\Gamma_P^D(G) = \emptyset$. Then $Cl_P^D(G) - Int_P^D(G) = \emptyset$. Since $Int_P^D(G) \subseteq Cl_P^D(G)$ then $Cl_P^D(G) = Int_P^D(G)$. Since $Int_P^D(G) \subseteq G \subseteq Cl_P^D(G)$ then

$$Cl_P^D(G) = G = Int_P^D(G).$$

That is, $Cl_P^D(G) = G$. Hence G is both D-preclosed set and D-preopen set. □

5. REFERENCES

[1] A. A. Ali Fora, M. O. Massadeh and M. S. Bataineh, A new Structure and Contribution in D-metric Spaces, British Journal of Mathematics and Computer Science, 22(1), (2017), 1-9.

[2] B. C. Dhage, Generalized Metric Space and Topological Structure I, Analele Stiitifice Ale Univ. Cuza Iasi Mat., 46, (2000), 3-24.

[3] B. C. Dhage, (1984), A study of some fixed point theorms, Ph.D.Thesis. Marathwada Univ. Aurangabad, India.

[4] C. D. Bele and U. P. Dolhare, An Extension of Common Fixed Point Theorem in D-Metric Space, International Journal of Mathematics and its Applications, 5, (2017), 13-18.

[5] C. G. Aras, S. Bayramov and M. I. Yazar, Soft D-Metric Spaces, Bol. Soc. Paran. Mat, 38, (2020), 137-147.

[6] I. Ramabhadrasarma and S. Sambasivarao, On D-metric spaces, Journal of Global Research in Mathematical Archives, 12, (2013), 31-39.

[7] J. Muscat, (2016), Function anaylsise, Springer Heidelberg New York, London.