On $\mathbb{R}^d$-valued peacocks

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Abstract: In this paper, we consider $\mathbb{R}^d$-valued integrable processes which are increasing in the convex order, i.e. $\mathbb{R}^d$-valued peacocks in our terminology. After the presentation of some examples, we show that an $\mathbb{R}^d$-valued process is a peacock if and only if it has the same one-dimensional marginals as an $\mathbb{R}^d$-valued martingale. This extends former results, obtained notably by V. Strassen (1965), J.L. Doob (1968) and H. Kellerer (1972).

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1 Introduction

1.1 Terminology

First we fix the terminology. In the sequel, d denotes a fixed integer and $\mathbb{R}^d$ is equipped with a norm which is denoted by $|\cdot|$.

We say that two $\mathbb{R}^d$-valued processes: $(X_t\ ,\ t \geq 0)$ and $(Y_t\ ,\ t \geq 0)$ are *associated*, if they have the same one-dimensional marginals, i.e. if:

$$\forall t \geq 0, \quad X_t \overset{\text{(law)}}{=} Y_t\ .$$

A process which is associated with a martingale is called a 1-*martingale*.

An $\mathbb{R}^d$-valued process $(X_t\ ,\ t \geq 0)$ will be called a *peacock* if:

i) it is *integrable*, that is:

$$\forall t \geq 0, \quad \mathbb{E}[|X_t|] < \infty$$

ii) it *increases in the convex order*, meaning that, for every convex function $\psi : \mathbb{R}^d \longrightarrow \mathbb{R}$, the map:

$$t \geq 0 \longrightarrow \mathbb{E}[\psi(X_t)] \in (-\infty, +\infty]$$

is increasing.

This terminology was introduced in [HPRY]. We refer the reader to this monograph for an explanation of the origin of the term: "peacock", as well as for a comprehensive study of this notion in the case $d = 1$.

Actually, it may be noted that, in the definition of a peacock, only the family $(\mu_t \ , \ t \geq 0)$ of its one-dimensional marginals is involved. This makes it natural, in the following, to also call a *peacock*, a family $(\mu_t \ , \ t \geq 0)$ of probability measures on $\mathbb{R}^d$ such that:

i) $\quad \forall t \geq 0, \quad \displaystyle\int |x| \ \mu_t(\mathrm{d}x) < \infty$,

ii) for every convex function $\psi : \mathbb{R}^d \longrightarrow \mathbb{R}$, the map:

$$t \geq 0 \longrightarrow \int \psi(x) \ \mu_t(\mathrm{d}x) \in (-\infty, +\infty]$$

is increasing.

Likewise, a family $(\mu_t \ , \ t \geq 0)$ of probability measures on $\mathbb{R}^d$ and an $\mathbb{R}^d$-valued process $(Y_t \ , \ t \geq 0)$ will be said to be *associated* if, for every $t \geq 0$, the law of Y_t is μ_t , i.e. if $(\mu_t \ , \ t \geq 0)$ is the family of the one-dimensional marginals of $(Y_t \ , \ t \geq 0)$.
Obviously, the above notions also are meaningful if one considers processes and families of measures indexed by a subset of $\mathbb{R}_+$ (for example $\mathbb{N}$) instead of $\mathbb{R}_+$.

It is an easy consequence of Jensen's inequality that an $\mathbb{R}^d$-valued process which is a 1-martingale, is a peacock. So, a natural question is whether the converse holds.

1.2 Case $d = 1$

A remarkable result due to H. Kellerer ([K], 1972) states that, actually, any $\mathbb{R}$-valued process which is a peacock, is a 1-martingale. More precisely, Kellerer's result states that any $\mathbb{R}$-valued peacock admits an associated martingale which is *Markovian.*

Two more recent results now complete Kellerer's theorem.

i) G. Lowther ([L], 2008) states that if $(\mu_t \ , \ t \geq 0)$ is an $\mathbb{R}$-valued peacock such that the map: $t \longrightarrow \mu_t$ is weakly continuous (i.e. for any $\mathbb{R}$-valued, bounded and continuous function f on $\mathbb{R}$, the map: $t \longrightarrow \int f(x) \ \mu_t(\mathrm{d}x)$ is continuous), then $(\mu_t \ , \ t \geq 0)$ is associated with a strongly Markovian martingale which moreover is "almost-continuous" (see [L] for the definition).

ii) In a previous paper ([HR], 2011), we presented a new proof of the above mentioned theorem of H. Kellerer. Our method, which is inspired from the "Fokker-Planck Equation Method" ([HPRY, Section 6.2, p.229]), then appears as a new application of M. Pierre's uniqueness theorem for a Fokker-Planck equation ([HPRY, Theorem 6.1, p.223]). Thus, we show that a martingale which is associated to an $\mathbb{R}$-valued peacock, may be obtained as a limit of solutions of stochastic differential equations. However, we do not obtain that such a martingale is Markovian.

1.3 Case $d \geq 1$

Concerning the case $\mathbb{R}^d$ with $d \geq 1$, and even much more general spaces, we would like to mention the following three important papers.

i) In [CFM] (1964), P. Cartier, J.M.G. Fell and P.-A. Meyer study the case of two probability measures (μ_1, μ_2) on a metrizable convex compact K of a locally convex space. They prove, using the Hahn-Banach theorem, that, if (μ_1, μ_2) is a K-valued peacock (indexed by $\{1, 2\}$), then there exists a Markovian kernel P on K such that: $\theta(\mathrm{d}x_1, \mathrm{d}x_2) := \mu_1(\mathrm{d}x_1) \, P(x_1, \mathrm{d}x_2)$ is the law of a K-valued martingale (Y_1, Y_2) associated to (μ_1, μ_2) .

ii) In [S] (1965), V. Strassen extends the Cartier-Fell-Meyer result to $\mathbb{R}^d$-valued peacocks without making the assumption of compact support. Then he proves that, if $(\mu_n \ , \ n \geq 0)$ is an $\mathbb{R}^d$-valued peacock (indexed by $\mathbb{N}$), there exists an associated martingale which is obtained as a Markov chain.

iii) In [D] (1968), J.L. Doob studies, in a very general extended framework, peacocks indexed by $\mathbb{R}_+$ and taking their values in a fixed compact set. In particular, he proves that they admit associated martingales. Note that in [D], the Markovian character of the associated martingales is not considered.

1.4 Organization

The remainder of this paper is organised as follows:

- In Section 2, we present some basic facts concerning the $\mathbb{R}^d$-valued peacocks and we describe some examples, thus extending results of [HPRY].
- In Section 3, starting from Strassen's theorem, we prove that a family $(\mu_t\ ,\ t \geq 0)$ of probability measures on $\mathbb{R}^d$, is associated to a *right-continuous* martingale, if and only if, $(\mu_t\ ,\ t \geq 0)$ is a peacock such that the map: $t \longrightarrow \mu_t$ is *weakly right-continuous* on $\mathbb{R}_+$.
- In Section 4, by approximation from the previous result, we extend this result to the case of general $\mathbb{R}^d$-valued peacocks.

2 Generalities, Examples

2.1 Notation

In the sequel, d denotes a fixed integer, $\mathbb{R}^d$ is equipped with a norm which is denoted by $|\cdot|$, and we adopt the terminology of Subsection 1.1.

We also denote by $\mathcal{M}$ the set of probability measures on $\mathbb{R}^d$, equipped with the topology of weak convergence (with respect to the space $C_b(\mathbb{R}^d)$ of $\mathbb{R}$-valued, bounded, continuous functions on $\mathbb{R}^d$). We denote by $\mathcal{M}_f$ the subset of $\mathcal{M}$ consisting of measures $\mu \in \mathcal{M}$ such that $\int |x|\ \mu(\mathrm{d}x) < \infty$. $\mathcal{M}_f$ is also equipped with the topology of weak convergence.

$C_c(\mathbb{R}^d)$ denotes the space of $\mathbb{R}$-valued continuous functions on $\mathbb{R}^d$ with compact support, and $C_c^+(\mathbb{R}^d)$ is the subspace consisting of all the nonnegative functions in $C_c(\mathbb{R}^d)$.

2.2 Basic facts

Proposition 2.1 *Let* $(X_t\ ,\ t \geq 0)$ *be an* $\mathbb{R}^d$*-valued integrable process. Then* $(X_t\ ,\ t \geq 0)$ *is a peacock if (and only if) the map:* $t \longrightarrow \mathbb{E}[\psi(X_t)]$ *is increasing, for every function* $\psi : \mathbb{R}^d \longrightarrow \mathbb{R}$ *which is convex, of* C^∞ *class and such that the derivative* ψ' *is bounded on* $\mathbb{R}^d$*.*

Proof Let $\psi : \mathbb{R}^d \longrightarrow \mathbb{R}$ be a convex function. For every $a \in \mathbb{R}^d$, there exists an affine function h_a such that:

$$\forall x \in \mathbb{R}^d, \quad \psi(x) \geq h_a(x) \quad \text{and} \quad \psi(a) = h_a(a)\ .$$

Let $\{a_n\ ;\ n \geq 1\}$ be a countable dense subset of $\mathbb{R}^d$. We set:

$$\forall n \geq 1, \quad \psi_n(x) = \sup_{1 \leq j \leq n} h_{a_j}(x)\ .$$

Then:

$$\forall x \in \mathbb{R}^d, \quad \lim_{n \uparrow \infty} \uparrow \psi_n(x) = \psi(x)\ .$$

The functions ψ_n are convex and Lipschitz continuous.

Let ϕ be a nonnegative function, of C^∞ class, with compact support and such that $\int \phi(x)\, \mathrm{d}x = 1$. We set, for $n, p \geq 1$,

$$\forall x \in \mathbb{R}^d, \quad \psi_{n,p}(x) = \int \psi_n \left(x - \frac{1}{p}\, y \right) \phi(y)\, \mathrm{d}y\ .$$

Clearly, $\psi_{n,p}$ is convex, of C^∞ class and Lipschitz continuous. Consequently, its derivative is bounded on $\mathbb{R}^d$. Moreover, $\lim_{p \to \infty} \psi_{n,p} = \psi_n$ uniformly on $\mathbb{R}^d$.

The desired result now follows directly.

□

The next result will be useful in the sequel.

Proposition 2.2 *Let* $(X_t\ ,\ t \geq 0)$ *be an* $\mathbb{R}^d$*-valued peacock. Then:*

1. *the map:* $t \longrightarrow \mathbb{E}[X_t]$ *is constant;*

2. *the map:* $t \longrightarrow \mathbb{E}[|X_t|]$ *is increasing, and therefore, for every* $T \geq 0$*,*

$$\sup_{0 \leq t \leq T} \mathbb{E}[|X_t|] = \mathbb{E}[|X_T|] < \infty\ ;$$

3. *for every $T \geq 0$, the random variables $(X_t \ ; \ 0 \leq t \leq T)$ are uniformly integrable.*

Proof Properties 1 and 2 are obvious.

If $c \geq 0$,

$$|x| \, 1_{\{|x| \geq c\}} \leq (2\,|x| - c)^+ \ .$$

As the function $x \longrightarrow (2\,|x| - c)^+$ is convex,

$$\sup_{t \in [0,T]} \mathbb{E}\left[|X_t| \, 1_{\{|X_t| \geq c\}}\right] \leq \mathbb{E}[(2\,|X_T| - c)^+] \ .$$

Now, by dominated convergence,

$$\lim_{c \to +\infty} \mathbb{E}[(2\,|X_T| - c)^+] = 0 \ .$$

Hence, property 3 holds.

□

2.3 Examples

The following examples are given in [HPRY] for $d = 1$. The proofs given below are essentially the same as in [HPRY].

Proposition 2.3 *Let X be a centered $\mathbb{R}^d$-valued random variable. Then $(t\,X \ , \ t \geq 0)$ is a peacock.*

Proof Let $\psi : \mathbb{R}^d \longrightarrow \mathbb{R}$ be a convex function, and $0 \leq s < t$. Then,

$$\psi(s\,X) \leq \left(1 - \frac{s}{t}\right) \psi(0) + \frac{s}{t}\,\psi(t\,X) \ .$$

Since X is centered, by Jensen's inequality:

$$\psi(0) = \psi\left(\mathbb{E}[t\,X]\right) \leq \mathbb{E}[\psi(t\,X)] \ .$$

Hence,

$$\mathbb{E}[\psi(s\,X)] \leq \left(1 - \frac{s}{t}\right) \mathbb{E}[\psi(t\,X)] + \frac{s}{t}\,\mathbb{E}[\psi(t\,X)] = \mathbb{E}[\psi(t\,X)] \ .$$

□

Proposition 2.4 *Let* $(X_t\ ,\ t \geq 0)$ *be a family of centered,* $\mathbb{R}^d$*-valued, Gaussian variables. We denote by* $C(t) = (c_{i,j}(t))_{1\leq i,j\leq d}$ *the covariance matrix of* X_t *. Then,* $(X_t\ ,\ t \geq 0)$ *is a peacock if and only if the map:* $t \longrightarrow C(t)$ *is increasing in the sense of quadratic forms, i.e:*

$$\forall a = (a_1, \cdots, a_d) \in \mathbb{R}^d, \quad t \longrightarrow \sum_{1\leq i,j\leq d} c_{i,j}(t)\, a_i a_j \quad \textit{is increasing.}$$

Proof

1) For every $a \in \mathbb{R}^d$, the function:

$$x \in \mathbb{R}^d \longrightarrow \sum_{1\leq i,j\leq d} a_i\, a_j\, x_i\, x_j = \left(\sum_{i=1}^{d} a_i\, x_i\right)^2$$

is convex. This entails that, if $(X_t\ ,\ t \geq 0)$ is a peacock, then the map: $t \longrightarrow C(t)$ is increasing in the sense of quadratic forms.

2) Conversely, suppose that the map: $t \longrightarrow C(t)$ is increasing in the sense of quadratic forms. By the proof of [HPRY, Theorem 2.16, p.132], there exists a centered $\mathbb{R}^d$-valued Gaussian process: $(\Gamma_t = (\Gamma_{1,t}, \cdots, \Gamma_{d,t})\ ,\ t \geq 0)$, such that:

$$\forall s, t \geq 0, \ \ \forall 1 \leq i, j \leq d, \quad \mathbb{E}[\Gamma_{i,s}\, \Gamma_{j,t}] = c_{i,j}(s \wedge t)\ .$$

Therefrom we deduce that $(\Gamma_t\ ,\ t \geq 0)$ is a martingale which is associated to $(X_t\ ,\ t \geq 0)$, and consequently, $(X_t\ ,\ t \geq 0)$ is a peacock.

□

Corollary 2.1 *Let* A *be a* $d \times d$ *matrix. We consider the* $\mathbb{R}^d$*-valued Ornstein-Uhlenbeck process* $(U_t\ ,\ t \geq 0)$ *, defined as (the unique) solution, started from 0, of the SDE:*

$$\mathrm{d}U_t = \mathrm{d}B_t + A\, U_t\, \mathrm{d}t$$

where $(B_t\ ,\ t \geq 0)$ *denotes a* d*-dimensional Brownian motion. Then,* $(U_t\ ,\ t \geq 0)$ *is a peacock.*

Proof One has:

$$U_t = \int_0^t \exp((t-s)\,A)\,\mathrm{d}B_s\ .$$

Hence, for every $t \geq 0$, U_t is a centered, $\mathbb{R}^d$-valued Gaussian variable whose covariance matrix is:

$$C(t) = \int_0^t \exp(s\,A)\,\exp(s\,A^*)\,\mathrm{d}s$$

where A^* denotes the adjoint matrix of A . Therefrom it is clear that the map: $t \longrightarrow C(t)$ is increasing in the sense of quadratic forms, and Proposition 2.4 applies.

□

Proposition 2.5 *Let* $(M_t\ ,\ t \geq 0)$ *be an* $\mathbb{R}^d$*-valued, right-continuous martingale such that:*

$$\forall T > 0, \quad \mathbb{E}\left[\sup_{0\leq t\leq T} |M_t|\right] < \infty\ .$$

Then,

1. $\left(X_t := \dfrac{1}{t}\displaystyle\int_0^t M_s\,\mathrm{d}s\ ;\ t \geq 0\right)$ *is a peacock,*
2. $\left(\widetilde{X}_t := \displaystyle\int_0^t (M_s - M_0)\,\mathrm{d}s\ ;\ t \geq 0\right)$ *is a peacock.*

Proof Using Proposition 2.1, we may use the proof of [HPRY, Theorem 1.4, p.26]. For the convenience of the reader, we reproduce this proof below.

1) Let $\psi : \mathbb{R}^d \longrightarrow \mathbb{R}$ be a convex function, of C^∞ class and such that the derivative ψ' is bounded on $\mathbb{R}^d$. Setting:

$$\widehat{M}_t = \int_0^t s\,\mathrm{d}M_s\ ,$$

one has, by integration by parts:

$$X_t = M_t - t^{-1}\widehat{M}_t \quad \text{and} \quad \mathrm{d}X_t = t^{-2}\widehat{M}_t\,\mathrm{d}t\ .$$

Denoting by $\mathcal{F}_s$ the σ-algebra generated by $\{M_u\ ;\ 0 \leq u \leq s\}$, one gets, for $0 \leq s \leq t$,

$$\mathbb{E}[X_t \mid \mathcal{F}_s] = X_s + (s^{-1} - t^{-1})\,\widehat{M}_s\ .$$

Consequently, by Jensen's inequality,

$$\mathbb{E}[\psi(X_t)] \geq \mathbb{E}[\psi(X_s + (s^{-1} - t^{-1})\,\widehat{M}_s)]\ .$$

Using again the fact that ψ is convex, one obtains:

$$\mathbb{E}[\psi(X_t)] \geq \mathbb{E}[\psi(X_s)] + (s^{-1} - t^{-1})\,\mathbb{E}[\psi'(X_s) \cdot \widehat{M}_s]\ .$$

Now,

$$\psi'(X_s) \cdot \widehat{M}_s = \int_0^s u^{-2}\psi''(X_u)(\widehat{M}_u, \widehat{M}_u)\,\mathrm{d}u + \int_0^s u\,\psi'(X_u) \cdot \mathrm{d}M_u$$

and therefore

$$\mathbb{E}[\psi(X_t)] - \mathbb{E}[\psi(X_s)] \geq (s^{-1} - t^{-1})\,\mathbb{E}[\psi'(X_s) \cdot \widehat{M}_s] \geq 0\ ,$$

which, by Proposition 2.1, yields the desired result.

2) Let ψ be as above. One may suppose that $M_0 = 0$. One has, for $0 \leq s \leq t$,

$$\mathbb{E}[\widetilde{X}_t \mid \mathcal{F}_s] = \widetilde{X}_s + (t - s)\,M_s\ .$$

Consequently, by Jensen's inequality,

$$\mathbb{E}[\psi(\widetilde{X}_t)] \geq \mathbb{E}[\psi(\widetilde{X}_s + (t - s)\,M_s)]\ .$$

Using again the fact that ψ is convex, one obtains:

$$\mathbb{E}[\psi(\widetilde{X}_t)] \geq \mathbb{E}[\psi(\widetilde{X}_s)] + (t - s)\,\mathbb{E}[\psi'(\widetilde{X}_s) \cdot M_s]\ .$$

Now,

$$\psi'(\widetilde{X}_s) \cdot M_s = \int_0^s \psi''(\widetilde{X}_u)(M_u, M_u)\,\mathrm{d}u + \int_0^s \psi'(\widetilde{X}_u) \cdot \mathrm{d}M_u$$

and therefore

$$\mathbb{E}[\psi(\widetilde{X}_t)] - \mathbb{E}[\psi(\widetilde{X}_s)] \geq (t - s)\,\mathbb{E}[\psi'(\widetilde{X}_s) \cdot M_s] \geq 0\ ,$$

which, by Proposition 2.1, yields the desired result.

□

3 Right-continuous peacoks

In this section, we shall show that any right continuous peacock admits an associated right-continuous martingale. For this, we start from Strassen's theorem, which we now recall.

Theorem 3.1 (Strassen [S], Theorem 8) *Let* $(\mu_n\ ,\ n \in \mathbb{N})$ *be a sequence in* $\mathcal{M}$. *Then* $(\mu_n\ ,\ n \in \mathbb{N})$ *is a peacock if and only if there exists a martingale* $(M_n\ ,\ n \in \mathbb{N})$ *which is associated to* $(\mu_n\ ,\ n \in \mathbb{N})$ *.*

We shall extend this theorem to right-continuous peacocks indexed by $\mathbb{R}_+$. In the case $d = 1$, the following theorem is proven in [HR], by a quite different method. In particular, in [HR], we do not use Strassen's theorem, nor the Hahn-Banach theorem, but an explicit approximation by solutions of SDE's.

Theorem 3.2 *Let* $(\mu_t\ ,\ t \geq 0)$ *be a family in* $\mathcal{M}$. *Then the following properties are equivalent:*

i) There exists a right-continuous martingale associated to $(\mu_t\ ,\ t \geq 0)$ *.*

ii) $(\mu_t\ ,\ t \geq 0)$ *is a peacock and the map:*

$$t \geq 0 \longrightarrow \mu_t \in \mathcal{M}$$

is right-continuous.

Proof

1) We first assume that property i) is satisfied. Then, the fact that $(\mu_t\ ,\ t \geq 0)$ is a peacock follows classically from Jensen's inequality. Let $(M_t\ ,\ t \geq 0)$ be a right-continuous martingale associated to $(\mu_t\ ,\ t \geq 0)$. Then, if $f \in C_b(\mathbb{R}^d)$, dominated convergence yields that, for any $t \geq 0$,

$$\lim_{s \to t, s > t} \int f(x)\ \mu_s(\mathrm{d}x) = \lim_{s \to t, s > t} \mathbb{E}[f(M_s)] = \mathbb{E}[f(M_t)] = \int f(x)\ \mu_t(\mathrm{d}x)\ .$$

Therefore, the map:

$$t \geq 0 \longrightarrow \mu_t \in \mathcal{M}$$

is right-continuous, and property ii) is satisfied.

2) Conversely, we now assume that property ii) is satisfied. For every $n \in \mathbb{N}$, we set:

$$\mu_k^{(n)} = \mu_{k2^{-n}} \quad , \quad k \in \mathbb{N}\ .$$

By Strassen's theorem (Theorem 3.1), there exists a martingale $(M_k^{(n)}\ ,\ k \in \mathbb{N})$ which is associated to $(\mu_k^{(n)}\ ,\ k \in \mathbb{N})$. We set:

$$X_t^{(n)} = M_k^{(n)} \ \text{ if } \ t = k\,2^{-n} \quad \text{and} \quad X_t^{(n)} = 0 \ \text{ otherwise.}$$

Consequently, the law of $X_t^{(n)}$ is μ_t if $t \in \{k\,2^{-n}\ ;\ k \in \mathbb{N}\}$, and is δ (the Dirac measure at 0) if $t \notin \{k\,2^{-n}\ ;\ k \in \mathbb{N}\}$.
Note that, due to the lack of uniqueness in Strassen's theorem, the law of $(X_{k2^{-n}}^{(n)}\ ,\ k \in \mathbb{N})$ may be not the same as the law of $(X_{k2^{-n}}^{(n+1)}\ ,\ k \in \mathbb{N})$. Only the one-dimensional marginals are identical.

3) Let $D = \{k\,2^{-n}\ ;\ k, n \in \mathbb{N}\}$ the set of dyadic numbers. For every $n \in \mathbb{N}$, for every $r \geq 1$ and $\tau_r = (t_1, t_2, \cdots, t_r) \in D^r$, we denote by $\Pi_{\tau_r}^{(r,n)}$ the law of $(X_{t_1}^{(n)}, \cdots, X_{t_r}^{(n)})$, a probability on $(\mathbb{R}^d)^r$.

Lemma 3.1 *For every* $\tau_r \in D^r$ *, the set of probability measures:* $\{\Pi_{\tau_r}^{(r,n)}\ ;\ n \in \mathbb{N}\}$ *is tight.*

Proof We set, for $x = (x^1, \cdots, x^r) \in (\mathbb{R}^d)^r$, $|x|_r = \sum_{j=1}^r |x^j|$. Then, for $p > 0$,

$$\Pi_{\tau_r}^{(r,n)}(|x|_r \geq p) \leq \frac{1}{p}\,\Pi_{\tau_r}^{(r,n)}(|x|_r) = \frac{1}{p}\sum_{j=1}^{r} \mathbb{E}[|X_{t_j}^{(n)}|] \leq \frac{1}{p}\sum_{j=1}^{r} \mu_{t_j}(|x|)$$

since, by point 2), the law of $X_{t_j}^{(n)}$ is either μ_{t_j} or δ . Hence,

$$\lim_{p\to\infty} \sup_{n\geq 0} \Pi_{\tau_r}^{(r,n)}(|x|_r \geq p) = 0\ .$$

□

4) As a consequence of the previous lemma, and with the help of the diagonal procedure, there exists a subsequence $(n_l)_{l\geq 0}$ such that, for every $\tau_r \in D^r$, the sequence of probabilities on $(\mathbb{R}^d)^r$: $(\Pi_{\tau_r}^{(r,n_l)}\ ,\ l \geq 0)$, weakly converges to a probability which we denote by $\Pi_{\tau_r}^{(r)}$. We remark that, for l large enough, the law of $X_{t_j}^{(n_l)}$ is μ_{t_j} . Then, there exists an $\mathbb{R}^d$-valued process $(X_t\ ,\ t \in D)$ such that, for every $r \in \mathbb{N}$ and every $\tau_r = (t_1, \cdots, t_r) \in D^r$, the law of $(X_{t_1}, \cdots, X_{t_r})$ is $\Pi_{\tau_r}^{(r)}$, and $\Pi_t^{(1)} = \mu_t$ for every $t \in D$.

Lemma 3.2 *The process* $(X_t\ ,\ t \in D)$ *is a martingale associated to* $(\mu_t\ ,\ t \in D)$ *.*

Proof As we have already seen, the process $(X_t\ ,\ t \in D)$ is associated to $(\mu_t\ ,\ t \in D)$. We now prove that it is a martingale. We set:

$$\forall p > 0,\ \ \forall x \in \mathbb{R}^d, \quad \varphi_p(x) = \left(1 \vee \frac{|x|}{p}\right)^{-1} x\ .$$

Then,

$$\varphi_p \in C_b(\mathbb{R}^d;\mathbb{R}^d) \quad \text{and} \quad \varphi_p(x) = x \ \text{ for } \ |x| \le p\ .$$

Let $0 \le s_1 < s_2 < \cdots < s_r \le s \le t$ be elements of D, and let $f \in C_b((\mathbb{R}^d)^r)$. We set: $\| f \|_\infty = \sup\{|f(x)|\ ;\ x \in (\mathbb{R}^d)^r\}$. Then, for l large enough,

$$\mathbb{E}[f(X^{(n_l)}_{s_1}, \cdots, X^{(n_l)}_{s_r})\, X^{(n_l)}_t] = \mathbb{E}[f(X^{(n_l)}_{s_1}, \cdots, X^{(n_l)}_{s_r})\, X^{(n_l)}_s]\ .$$

On the other hand,

$$\begin{aligned} &|\mathbb{E}[f(X_{s_1}, \cdots, X_{s_r})\, \varphi_p(X_t)] - \mathbb{E}[f(X_{s_1}, \cdots, X_{s_r})\, X_t]| \\ \le\ & \| f \|_\infty\, \mu_t\left(|x|\, 1_{\{|x| \ge p\}}\right), \ \text{ for every } p > 0, \end{aligned}$$

$$\begin{aligned} &\left|\mathbb{E}[f(X^{(n_l)}_{s_1}, \cdots, X^{(n_l)}_{s_r})\, \varphi_p(X^{(n_l)}_t)] - \mathbb{E}[f(X^{(n_l)}_{s_1}, \cdots, X^{(n_l)}_{s_r})\, X^{(n_l)}_t]\right| \\ \le\ & \| f \|_\infty\, \mu_t\left(|x|\, 1_{\{|x| \ge p\}}\right), \ \text{ for every } l \text{ and every } p > 0, \end{aligned}$$

and likewise, replacing t by s. Moreover,

$$\lim_{l \to \infty} \mathbb{E}[f(X^{(n_l)}_{s_1}, \cdots, X^{(n_l)}_{s_r})\, \varphi_p(X^{(n_l)}_t)] = \mathbb{E}[f(X_{s_1}, \cdots, X_{s_r})\, \varphi_p(X_t)]\ ,$$

and likewise, replacing t by s. Finally, we obtain, for $p > 0$,

$$\begin{aligned} &|\mathbb{E}[f(X_{s_1}, \cdots, X_{s_r})\, X_t] - \mathbb{E}[f(X_{s_1}, \cdots, X_{s_r})\, X_s]| \\ \le\ & 2\, \| f \|_\infty \left[\mu_t\left(|x|\, 1_{\{|x| \ge p\}}\right) + \mu_s\left(|x|\, 1_{\{|x| \ge p\}}\right)\right]\ , \end{aligned}$$

and the desired result follows, letting p go to ∞.

□

5) By the classical theory of martingales (see, for example, [DM]), almost surely, for every $t \geq 0$,

$$M_t = \lim_{s \to t, s \in D, s > t} X_s$$

is well defined, and $(M_t\ ,\ t \geq 0)$ is a right-continuous martingale. Besides, since, by hypothesis, the map: $t \geq 0 \longrightarrow \mu_t \in \mathcal{M}$ is right-continuous, we deduce from Lemma 3.2 that this martingale $(M_t\ ,\ t \geq 0)$ is associated to $(\mu_t\ ,\ t \geq 0)$.

□

4 The general case

Theorem 3.2 shall now be extended, by approximation, to the general case.

Theorem 4.1 *Let* $(\mu_t\ ,\ t \geq 0)$ *be a family in* $\mathcal{M}$*. Then the following properties are equivalent:*

i) There exists a martingale associated to $(\mu_t\ ,\ t \geq 0)$ *.*

ii) $(\mu_t\ ,\ t \geq 0)$ *is a peacock.*

Proof Let $(\mu_t\ ,\ t \geq 0)$ be a peacock.

Lemma 4.1 *There exists a countable set* $\Delta \subset \mathbb{R}_+$ *such that the map:*

$$t \longrightarrow \mu_t \in \mathcal{M}$$

is continuous at any $s \notin \Delta$*.*

Proof Let $\chi : \mathbb{R}^d \longrightarrow \mathbb{R}_+$ be defined by:

$$\chi(x) = (1 - |x|)^+ = (1 \vee |x|) - |x|\ .$$

Then $\chi \in C_c^+(\mathbb{R}^d)$ and χ is the difference of two convex functions. We set: $\chi_m(x) = m^d\, \chi(m\, x)$, and we define the countable set $\mathcal{H}$ by:

$$\mathcal{H} = \left\{ \sum_{j=0}^{r} a_j\, \chi_m(x - q_j)\ ;\ r \in \mathbb{N},\ m \in \mathbb{N},\ a_j \in \mathbb{Q}_+,\ q_j \in \mathbb{Q}^d \right\}\ .$$

For $h \in \mathcal{H}$, the function: $t \longrightarrow \mu_t(h)$ is the difference of two increasing functions, and hence admits a countable set Δ_h of discontinuities. We set $\Delta = \bigcup_{h\in\mathcal{H}} \Delta_h$. Then Δ is a countable subset of $\mathbb{R}_+$, and $t \longrightarrow \mu_t(h)$ is continuous at any $s \notin \Delta$, for every $h \in \mathcal{H}$. Now, it is easy to see that $\mathcal{H}$ is dense in $C_c^+(\mathbb{R}^d)$ in the following sense: for every $\varphi \in C_c^+(\mathbb{R}^d)$, there exist a compact set $K \subset \mathbb{R}^d$ and a sequence $(h_n)_{n\geq 0} \subset \mathcal{H}$ such that:

$$\forall n, \;\; \mathrm{Supp}\, h_n \subset K \quad \text{and} \quad \lim_{n\to\infty} h_n = \varphi \;\text{ uniformly.}$$

Consequently, $t \longrightarrow \mu_t$ is vaguely continuous at any $s \notin \Delta$, and, since measures μ_t are probabilities, $t \longrightarrow \mu_t$ is also weakly continuous at any $s \notin \Delta$.

□

We may write $\Delta = \{d_j \; ; \; j \in \mathbb{N}\}$. For $n \in \mathbb{N}$, we denote by $(k_l^{(n)} \; , \; l \geq 0)$ the increasing rearrangement of the set:

$$\{k\, 2^{-n} \; ; \; k \in \mathbb{N}\} \cup \{d_j \; ; \; 0 \leq j \leq n\} \; .$$

We define $(\mu_t^{(n)} \; , \; t \geq 0)$ by:

$$\mu_t^{(n)} = \mu_{k_l^{(n)}} \qquad \text{if there exists } l \text{ such that } t = k_l^{(n)},$$

and by: $$\mu_t^{(n)} = \frac{k_{l+1}^{(n)} - t}{k_{l+1}^{(n)} - k_l^{(n)}}\, \mu_{k_l^{(n)}} + \frac{t - k_l^{(n)}}{k_{l+1}^{(n)} - k_l^{(n)}}\, \mu_{k_{l+1}^{(n)}} \qquad \text{if } t \in [k_l^{(n)}, k_{l+1}^{(n)}].$$

Lemma 4.2 *The following properties hold:*

1. *For every $n \geq 0$, $(\mu_t^{(n)} \; , \; t \geq 0)$ is a peacock and the map: $t \longrightarrow \mu_t^{(n)} \in \mathcal{M}$ is continuous.*
2. *For any $t \geq 0$, $\sup\{\mu_t^{(n)}(|x|) \; ; \; n \in \mathbb{N}\} < \infty$.*
3. *For any $t \geq 0$, the set $\{\mu_t^{(n)} \; ; \; n \in \mathbb{N}\}$ is uniformly integrable.*
4. *For $t \geq 0$, $\lim_{n\to\infty} \mu_t^{(n)} = \mu_t$ in $\mathcal{M}$.*

Proof Properties 1 and 4 are clear by construction. Property 2 (resp. property 3) follows directly from property 2 (resp. property 3) in Proposition 2.2.

□

By Theorem 3.2, there exists, for each n, a right-continuous martingale

$(M_t^{(n)}\ ,\ t \geq 0)$ which is associated to $(\mu_t^{(n)}\ ,\ t \geq 0)$. For any $r \in \mathbb{N}$ and $\tau_r = (t_1, \cdots, t_r) \in \mathbb{R}_+^r$, we denote by $\Pi_{\tau_r}^{(r,n)}$ the law of $(M_{t_1}^{(n)}, \cdots, M_{t_r}^{(n)})$, a probability measure on $(\mathbb{R}^d)^r$.

Lemma 4.3 *For every $\tau_r \in \mathbb{R}_+^r$, the set of probability measures: $\{\Pi_{\tau_r}^{(r,n)}\ ;\ n \in \mathbb{N}\}$ is tight.*

Proof As in Lemma 3.1, for $p > 0$,

$$\Pi_{\tau_r}^{(r,n)}(|x|_r \geq p) \leq \frac{1}{p} \sum_{j=1}^{r} \mu_{t_j}^{(n)}(|x|),$$

and by property 2 in Lemma 4.2,

$$\lim_{p \to \infty} \sup_{n \geq 0} \Pi_{\tau_r}^{(r,n)}(|x|_r \geq p) = 0\ .$$

□

Let now $\mathcal{U}$ be an ultrafilter on $\mathbb{N}$, which refines Fréchet's filter. As a consequence of the previous lemma, for every $r \in \mathbb{N}$ and every $\tau_r \in \mathbb{R}_+^r$, $\lim_{\mathcal{U}} \Pi_{\tau_r}^{(r,n)}$ exists for the weak convergence and we denote this limit by $\Pi_{\tau_r}^{(r)}$. By property 4 in Lemma 4.2, $\Pi_t^{(1)} = \mu_t$. There exists a process $(M_t\ ,\ t \geq 0)$ such that, for every $r \in \mathbb{N}$ and every $\tau_r = (t_1, \cdots, t_r) \in \mathbb{R}_+^r$, the law of $(M_{t_1}, \cdots, M_{t_r})$ is $\Pi_{\tau_r}^{(r)}$. In particular, this process $(M_t\ ,\ t \geq 0)$ is associated to $(\mu_t\ ,\ t \geq 0)$.

Lemma 4.4 *The process $(M_t\ ,\ t \geq 0)$ is a martingale.*

Proof The proof is quite similar to that of Lemma 3.2, but we give the details for the sake of completeness. We recall the notation:

$$\forall p > 0,\ \ \forall x \in \mathbb{R}^d, \quad \varphi_p(x) = \left(1 \vee \frac{|x|}{p}\right)^{-1} x\ .$$

Let $0 \leq s_1 < s_2 < \cdots < s_r \leq s \leq t$ be elements of $\mathbb{R}_+$, and let $f \in C_b((\mathbb{R}^d)^r)$. We set: $\| f \|_\infty = \sup\{|f(x)|\ ;\ x \in (\mathbb{R}^d)^r\}$. Then, for every n,

$$\mathbb{E}[f(M_{s_1}^{(n)}, \cdots, M_{s_r}^{(n)})\, M_t^{(n)}] = \mathbb{E}[f(M_{s_1}^{(n)}, \cdots, M_{s_r}^{(n)})\, M_s^{(n)}]\ .$$

On the other hand,

$$\begin{aligned} & |\mathbb{E}[f(M_{s_1}, \cdots, M_{s_r})\, \varphi_p(M_t)] - \mathbb{E}[f(M_{s_1}, \cdots, M_{s_r})\, M_t]| \\ \leq\ & \| f \|_\infty\, \mu_t\left(|x|\, 1_{\{|x| \geq p\}}\right), \quad \text{for every } p > 0, \end{aligned}$$

$$\left|\mathbb{E}[f(M_{s_1}^{(n)},\cdots,M_{s_r}^{(n)})\,\varphi_p(M_t^{(n)})]-\mathbb{E}[f(M_{s_1}^{(n)},\cdots,M_{s_r}^{(n)})\,M_t^{(n)}]\right|$$
$$\leq\ \|f\|_\infty\,\mu_t^{(n)}\left(|x|\,1_{\{|x|\geq p\}}\right),\quad\text{for every } n \text{ and every } p>0,$$

and likewise, replacing t by s. Moreover,

$$\lim_{\mathcal{U}}\mathbb{E}[f(M_{s_1}^{(n)},\cdots,M_{s_r}^{(n)})\,\varphi_p(M_t^{(n)})]=\mathbb{E}[f(M_{s_1},\cdots,M_{s_r})\,\varphi_p(M_t)]\ ,$$

and likewise, replacing t by s. Finally, we obtain, for $p>0$,

$$|\mathbb{E}[f(X_{s_1},\cdots,X_{s_r})\,X_t]-\mathbb{E}[f(X_{s_1},\cdots,X_{s_r})\,X_s]|$$
$$\leq\ 2\,\|f\|_\infty\sup_{n\geq 0}\left[\mu_t^{(n)}\left(|x|\,1_{\{|x|\geq p\}}\right)+\mu_s^{(n)}\left(|x|\,1_{\{|x|\geq p\}}\right)\right]\ ,$$

and, by property 3 in Lemma 4.2, the desired result follows, letting p go to ∞.

□

This lemma completes the proof of Theorem 4.1.

□

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