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On Decentralized Optimal Control and Information Structures†

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Abstract— **A canonical decentralized optimal control problem with quadratic cost criteria can be cast as an LQR problem in which the stabilizing controller is restricted to lie in a constraint set. We characterize a wide class of systems and constraint sets for which the canonical problem is tractable. We employ the notion of operator algebras to study the structural properties of the canonical problem. Examples of some widely used operator algebras in the context of distributed control include the subspace of infinite and finite dimensional spatially decaying operators, lower (or upper) triangular matrices, and circulant matrices. For a given operator algebra, we prove that if the trajectory of the solution of an operator differential equation starts inside the operator algebra, it will remain inside for all times. Using this result, we show that if the constraint set is an operator algebra, the canonical problem is solvable and equivalent to the standard LQR problem without the information constraint.**

I. Introduction

One of the initial inspirations of decentralized optimal control can be traced back to team decision theory. A team comprises of many members each of whom takes decisions about something different, but linked to each other through a common single goal or payoff. The team problem is to maximize the entire profit or minimize the payoff. The theory of "teams" and decentralized organizations motivated by Economics was first formulated by Marschak [1], more than half a century ago. Perhaps, the first connection between team theory and control theory was made by Radner [2] in 1962, where he derived a sufficient condition under which a linear controller can achieve the minimal quadratic cost for a linear system. Prior to the seminal result of Ho and Chu [3] in 1972, all the results in this area were mainly limited to static teams [2], [4], where information is only the function of some random variable but is independent of what other team players have done. Ho and Chu studied the Linear Quadratic Gaussian (LQG) team problem with nested information structure and showed that the optimal LQG controller is linear.

In this setting, a *canonical decentralized optimal control problem* involving linear quadratic (LQ) criteria can be cast as an LQR (or LQG) problem in which the stabilizing controller is restricted to lie in a particular subspace $\mathcal{S}$. This subspace of admissible controllers is often referred to as the *information constraint* [5]. For a general linear system and subspace $\mathcal{S}$ there is no known tractable algorithm for computing the optimum. In fact, certain cases have been shown to be intractable [6], [7]. It is also known that in presence of information constraints, the cost function is no longer convex in the controller variables [8], [9].

Despite some successes, a general theory of optimal control for linear systems with information (sparsity) constraints on the optimal feedback law is lacking. This is not surprising, as it is well known that very simple-looking linear quadratic stochastic optimal control problems with sparsity or decentralization constraints on the feedback structure can have complicated nonlinear optimal solutions [8].

In the context of spatially distributed control systems, there is an abundance of results where optimal and robust control theory is used to analyze and synthesize spatially distributed dynamical systems with sparsity constraints. In particular, Fagnani and Willems [10], [11] showed that stability of linear dynamical plants with certain symmetries can be accomplished with controllers that have the same symmetry. In [12], the authors studied optimal control of linear spatially-invariant systems with quadratic performance criteria and showed that the resulting optimal controllers have an inherent spatial locality similar to the underlying system. In other words, optimal controllers for spatially-invariant systems are spatially invariant themselves. Another related result is that of [13], where the problem of distributed controller design with a "funnel causality" constraint is shown to be a convex problem, provided that the plant has a similar funnel-causality structure, and the propagation speed in the controller is at least as fast as those in the plant. In their recent work, Rotkowitz and Lall [5] introduced the notion of quadratic invariance for the constraint set $\mathcal{S}$. Roughly speaking, quadratic invariance relates the plant to the constraint set through a simple algebraic condition. Using this notion, the authors show that the problem of finding optimal controllers (in an input-output setting) for an information constraint set that has the quadratic invariance property can be cast as a convex optimization problem, although the resulting controller might have a very high order. It turns out that many (but not all) tractable decentralized optimal control problems satisfy the quadratic invariance property. In [14], the spatial structure of the optimal control of spatially distributed dynamical systems with linear quadratic (LQ) performance criteria and arbitrary interconnection topologies was studies. By introducing the notion of *spatially decaying* (SD) operators [15], the authors showed that the space of

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SD operators form a Banach algebra and that solutions of Lyapunov and Riccati equations for spatially decaying systems are themselves SD. It was shown that for a wide class of infinite-dimensional spatially distributed dynamical systems the canonical problem is equivalent to the standard LQR problem without the information constraint when $\mathcal{S}$ is the Banach algebra of SD operators.

The goal of this paper is to analyze the structural properties of the canonical problem for a broader range of systems with infinite and finite dimensions. Our goal here is to prove that the canonical problem is equivalent to the standard LQR problem if the information constraint set $\mathcal{S}$ is an *operator algebra*. Simply put, an operator algebra is a set of continuous linear operators on a topological vector space such as a Banach space, which is typically required to be closed in a specified operator topology. Examples of operator algebras that play important roles in cooperative control and networked control problems include the Banach algebra of infinite-dimensional spatially decaying (SD) operators, the subspace of finite-dimensional spatially decaying operators, the subspace of lower (or upper) triangular matrices, and the subspace of circulant matrices. We study the structural properties of solutions of first-order operator differential equations. Specifically, we prove that if the underlying operator algebra is *invariant* (see theorem 4.3 for details) and the initial condition lies inside the operator algebra then trajectory of the solution of the operator differential equation stays inside the operators algebra for all times. Using this result, we consider systems whose state-space operators belong to an operator algebra and prove that solution of Sylvester equation and the unique solution of Lyapunov and algebraic Riccati equations (ARE) corresponding to these systems indeed lie in the operator algebra.

The implication of these result is that if the information constraint set $\mathcal{S}$ in the canonical problem is an operator algebra and that the state-space operators of the system as well as the weighting operators in the quadratic cost functional belong to $\mathcal{S}$, then the optimal feedback law also belong to $\mathcal{S}$. In other words, the information constraint LQR problem is equivalent to the standard LQR problem.

This paper is organized as follows: We introduce the notation and mathematical preliminaries in Section II. The canonical decentralized optimal control problem is discussed in Section III. In IV, we introduce the notion of operator algebra and present our main results on the structure of solutions to an operator differential equations. Then using tools from IV, we study the structural properties of the canonical problem in section V. Our concluding remarks are presented in Section VI.

II. Mathematical Preliminaries

$\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}^+$ the set of positive real numbers, $\mathbb{Z}$ the set of integer numbers, and $\mathbb{N}$ the set of natural numbers. The inner product on $\mathbb{R}^n$ is denoted by $\langle\ ,\ \rangle$ with corresponding norm $\|x\| = \sqrt{\langle x, x\rangle}$ for all $x \in \mathbb{R}^n$. For notational simplicity, the matrix norm induced by $\|.\|$ is also denoted by $\|.\|$. A subset $\mathbb{G}$ of $\mathbb{N}^d$ or $\mathbb{R}^d$ is referred to as the *spatial domain* if it consists of countably many d-tuples $i = (i_1, \ldots, i_d)$.

Definition 2.1: A distance function on a discrete topology with a set of nodes $\mathbb{G}$ is defined as a single-valued function $\mathrm{dis} : \mathbb{G} \times \mathbb{G} \to \mathbb{R}^+$ which has the following properties for all $k, i, j \in \mathbb{G}$:

(1) $\quad \mathrm{dis}(k,i) = 0 \quad \text{iff} \quad k = i$.
(2) $\quad \mathrm{dis}(k,i) = \mathrm{dis}(i,k)$.
(3) $\quad \mathrm{dis}(k,i) \le \mathrm{dis}(k,j) + \mathrm{dis}(j,i)$.

The Banach space $\ell_p(\mathbb{G})$ for $1 \le p < \infty$ is defined as the set of all sequences $x = (x_i)_{i\in\mathbb{G}}$ in which $x_i \in \mathbb{R}^n_i$ for some $n_i \ge 1$ satisfying

$$\sum_{i\in\mathbb{G}} \|x_i\|^p < \infty,$$

and endowed with the norm

$$\|x\|_p^p := \sum_{i\in\mathbb{G}} \|x_i\|^p.$$

The Banach space $\ell_\infty(\mathbb{G})$ denotes the set of all bounded sequences endowed with the norm

$$\|x\|_\infty := \sup_{i\in\mathbb{G}} \|x_i\|.$$

Throughout the paper, we will use the shorthand notation ℓ_p for $\ell_p(\mathbb{G})$. The space ℓ_2 is a Hilbert space with inner product

$$\langle x, y\rangle := \sum_{i\in\mathbb{G}} \ \langle x_i, y_i\rangle,$$

for all $x, y \in \ell_2$. An operator $A : \ell_p \to \ell_p$ is bounded if it has a finite induced norm, i.e., the following quantity

$$\|A\|_{p,p} := \sup_{\|x\|_p = 1} \|Ax\|_p, \tag{1}$$

is bounded. The set of all bounded linear operators of ℓ_p into ℓ_p is denoted by $\mathscr{B}(\ell_p)$. The identity operator is denoted by I. An operator $A \in \mathscr{B}(\ell_p)$ has an *algebraic* inverse if it has an inverse A^{-1} in $\mathscr{B}(\ell_p)$.

Definition 2.2: The adjoint operator of $A \in \mathscr{B}(\ell_2)$ is the operator A^* in $\mathscr{B}(\ell_2)$ such that

$$\langle Ax, y\rangle = \langle x, A^* y\rangle\,,$$

for all $x, y \in \ell_2$.
An operator A is self-adjoint if $A = A^*$.

Definition 2.3: An operator $A \in \mathscr{B}(\ell_2)$ is *positive definite*, shown as $A \succ 0$, if there exists a number $\alpha > 0$ such that

$$\langle x, Ax\rangle > \ \alpha\ \|x\|_2^2,$$

for all nonzero $x \in \ell_2$.
Throughout this paper we are interested in linear operators $A : \ell_p \to \ell_p$ which have a matrix representation.

The set of all functions from $D \subseteq \mathbb{R}$ into $\mathbb{R}$ is a vector space $\mathscr{F}$ over $\mathbb{R}$. For χ' , $\chi'' \in \mathscr{F}$, the notation $\chi' \preceq \chi''$ will be used to mean the pointwise inequality $\chi'(x) \le \chi''(x)$ for all $x \in D$. A family of *seminorms* on $\mathscr{F}$ is defined as $\{\|.\|_T \mid T \in \mathbb{R}^+\}$ in which

$$\|\chi\|_T := \sup_{x\le T} |\chi(x)|,$$

for all $\chi \in \mathscr{F}$. The topology generated by all open $\|.\|_T$-balls is called the topology generated by the family of seminorms. Continuity of a function χ in this topology is equivalent to continuity in every seminorm in the family [5].

III. A Canonical Decentralized Optimal Control Problem

In this section, we discuss the optimal control of spatially distributed linear systems. Spatially distributed dynamical systems are a general class of systems comprised of a countably large, possibly infinite, number of subsystems coupled either through their dynamics or through a common objective, shared cooperatively with other subsystems to achieve a global task.

As discussed earlier, a canonical decentralized optimal control problem with a quadratic performance criteria can be formulated in the following form:

$$\underset{K}{\text{minimize}} \quad \int_0^\infty \langle \psi, Q\psi\rangle + \langle u, Ru\rangle \; dt \tag{2}$$

$$\text{subject to:} \quad \frac{d}{dt}\psi \;=\; A\psi \;+\; Bu \tag{3}$$

$$u \;=\; K\psi \tag{4}$$

$$K \in \mathcal{S}, \tag{5}$$

with initial condition $\psi(0) = \psi_0$. The state and input variables are denoted by $\psi = (\psi_k)_{k\in\mathbb{G}}$ and $u = (u_k)_{k\in\mathbb{G}}$, respectively. The state-space operators A, B and the weighting operators $Q \succeq 0$ and $R \succ 0$ are assumed to be constant functions of time and linear from ℓ_2 to itself. We assume the existence and uniqueness of solutions of the system defined by (3) (cf. [16] for more details). In the canonical problem, the goal is to find a state feedback operator K that minimizes the quadratic cost functional (2) and satisfies the constraint (5). The subspace of admissible controllers $\mathcal{S}$ is often referred to as the *information constraint* [5]. For a general linear system and subspace $\mathcal{S}$ there is no known tractable algorithm for computing the optimum. In some cases, the canonical problem may not have a solution. In other words, the optimal feedback policy could be a nonlinear function of the state variables [8]. In Section V, we will prove that if $\mathcal{S}$ exhibits some specific topological and algebraical structures, the canonical problem (2)-(5) is solvable and equivalent to the standard LQR problem, without the information constraint 5. In the following section, we will expound how to impose sufficient mathematical structures on the constraint set $\mathcal{S}$ to enable for a thorough analysis of the canonical problem.

IV. Operator Algebra

The concept of an operator algebra was introduced in a 1913 book by Riesz, where he studied the algebra of bounded operators on the Hilbert space ℓ_2. An operator algebra is a set of continuous linear operators on a topological vector space such as a Banach space, which is typically required to be closed in a specified operator topology. In this paper, operator algebra is employed to develop a mathematical framework to analyze spatially distributed dynamical systems. Using this framework, results that are applicable to a wide class of such systems are proved. In this paper, we refer to the notion of operator algebra in the following sense.

Definition 4.1: A vector space of bounded linear operators $\mathcal{S}$ is called operator algebra if it is closed in the norm topology of operators, $I \in \mathcal{S}$, and for every $A, B \in \mathcal{S}$,

(i) $A + B \in \mathcal{S}$.
(ii) $AB \in \mathcal{S}$.

The closedness of $\mathcal{S}$ implies that for a convergent sequence $A_n \to A$ (in the sense of the norm topology) as $n \to \infty$, if $A_n \in \mathcal{S}$ for all $n \geq 0$, then $A \in \mathcal{S}$.

A. Examples of Operator Algebras

There are many instances in the context of spatially distributed control systems where the space of matrices stemmed from analysis of various types of couplings form operator algebras. In the following, we illustrate some of these examples.

1) Infinite-Dimensional Spatially Decaying Operators

In the following, we will briefly review some of the definitions and results from [14].

Definition 4.2: A nondecreasing continuous function $\chi : \mathbb{R}^+ \to [1, \infty)$ is called a *coupling characteristic* function if $\chi(0) = 1$ and $\chi(x + y) \leq \chi(x)\;\chi(y)$ for all $x, y \in \mathbb{R}^+$.

In order to be able to characterize rates of decay we define a one-parameter family of coupling characteristic functions as follows.

Definition 4.3: A one-parameter family of coupling characteristic functions $\mathscr{C}$ is defined to be an ordered set of all coupling characteristic functions χ_α for $\alpha \in \mathbb{R}^+$ such that

(i) $\chi_0(x) = 1$ for all $x \in \mathbb{R}^+$.
(ii) $\chi_\alpha(x)\chi_\beta(x) = \chi_{\alpha+\beta}(x)$ for all $x \in \mathbb{R}^+$.
(iii) For $\alpha \leq \beta$, relation $\chi_\alpha \preceq \chi_\beta$ holds.
(iv) χ_α is a continuous function of α in topology generated by all $\|.\|_T$.

Suppose that a parameterized family of coupling characteristic functions $\mathscr{C}$ is given. For all coupling characteristic function $\chi_\alpha \in \mathscr{C}$, we assume that the following condition holds

$$\sup_{k\in\mathbb{G}} \sum_{i\in\mathbb{G}} \; \chi_\alpha(\mathrm{dis}(k, i))^{-1} < \infty.$$

The class of spatially decaying (SD) operators with respect to $\mathscr{C}$ with decay margin $\tau > 0$ is defined as follows

$$\mathcal{S}_\tau^\infty(\mathscr{C}) := \{A \;:\; |||A|||_{(\mathscr{C},\tau)} \;<\; \infty\},$$

where the operator norm is defined as

$$|||A|||_{(\mathscr{C},\tau)} := \max\left\{ \sup_{\alpha\in[0,\tau)} \sup_{k\in\mathbb{G}} \sum_{i\in\mathbb{G}} \|A_{ki}\|\chi_\alpha(\mathrm{dis}(k, i)), \right.$$
$$\left. \sup_{\alpha\in[0,\tau)} \sup_{i\in\mathbb{G}} \sum_{k\in\mathbb{G}} \|A_{ki}\|\chi_\alpha(\mathrm{dis}(k, i)) \right\},$$

and $\chi_\alpha \in \mathscr{C}$ for all $0 \leq \alpha < \tau$. For a given family of coupling characteristic function, the corresponding class of SD operators can be characterized. In the sequel, some important types of SD operators are given:

- The class of *exponentially decaying* operators where $\chi_\alpha(x) = e^{\alpha x}$.

- The class of *algebraically decaying* operators where $\chi_\alpha(x) = (1 + \lambda x)^\alpha$ for some $\lambda > 0$.
- The class of *banded operators* that are SD with respect to all types of coupling characteristic functions.

An important example of exponentially decaying operators is the class of translation invariant operators defined on group $(\mathbb{Z}, +)$ (see [14] for more details).

Theorem 4.1: Given a one-parameter family of coupling characteristic functions $\mathscr{C}$ and $\tau > 0$, the operator space $\mathcal{S}_\tau^\infty(\mathscr{C})$ forms a *Banach Algebra* with respect to $|||.|||_{(\mathscr{C},\tau)}$ under the operator composition operation.

Proof: We refer to [14] for a proof. ■

According to definition 4.1, a Banach algebra is also an operator algebra. We emphasize that the applications of SD operators in modeling various types of couplings in networked dynamical systems cover a wide class of such systems (see [14], [17] for further details).

2) Finite-Dimensional Spatially Decaying Operators

According to the definition, all finite-dimensional matrices automatically satisfy the membership condition of the Banach algebra $\mathcal{S}_\tau^\infty(\mathscr{C})$. Therefore, in the finite-dimensional case, theorem 4.1 can not be applied directly. However, the results can be extended to finite-dimensional operators by appropriately adjusting the notion of an SD operator to the finite dimensional case as follows. Suppose that a spatial domain $\mathbb{G}$ with cardinality $N < \infty$ and a parameterized family of coupling characteristic functions $\mathscr{C}$ are given.

Definition 4.4: The subspace of spatially decaying matrices $\mathcal{S}_\tau^N(\mathscr{C})$ with decay margin $\tau > 0$ is defined to be the set of all matrices A for which there exist constants $C, C' > 0$ and $0 < \alpha < \tau$ such that each block submatrix of A satisfies

$$\|A_{ki}\| \ \leq \ C \ \chi_\alpha(\mathrm{dis}(k,i))^{-1},$$

for all $k, i \in \mathbb{G}$. The number α is referred to as the decay rate of matrix A.

Intuitively, a matrix is spatially decaying, if the size of each blocks decays faster than inverse of a coupling characteristic function.

Theorem 4.2: The subspace of spatially decaying matrices $\mathcal{S}_\tau^N(\mathscr{C})$ form an operator algebra under the matrix composition operation.

Proof: Given $A \in \mathcal{S}_\tau^N(\mathscr{C})$ with decay rate $0 < \alpha < \tau$ and $B \in \mathcal{S}_\tau^N(\mathscr{C})$ with decay rate $0 < \beta < \tau$, we have

$$\begin{aligned} \|A_{ki} + B_{ki}\| &\leq \|A_{ki}\| + \|B_{ki}\| \\ &\leq C_A\chi_\alpha(\mathrm{dis}(k,i))^{-1} + C_B\chi_\beta(\mathrm{dis}(k,i))^{-1} \\ &\leq C \ \chi_\gamma(\mathrm{dis}(k,i))^{-1}, \end{aligned} \tag{6}$$

where $\gamma = \min(\alpha, \beta)$ and C is a number greater than $C_A + C_B$. This shows that $A + B \in \mathcal{S}_\tau^N(\mathscr{C})$. In the next step, consider the (k, i) block submatrix of AB, we have

$$\begin{aligned} \|\sum_{j\in\mathbb{G}} A_{kj}B_{ji}\| &\leq \sum_{j\in\mathbb{G}} \|A_{kj}\|\|B_{ji}\| \\ &\leq C_AC_B\sum_{j\in\mathbb{G}}\chi_\alpha(\mathrm{dis}(k,j))^{-1}\chi_\beta(\mathrm{dis}(j,i))^{-1} \\ &\leq C \ \chi_\gamma(\mathrm{dis}(k,i))^{-1}. \end{aligned} \tag{7}$$

In inequality (7), if $\alpha = \beta$, the decay rate γ is a positive number less than α. Otherwise, $\gamma = \min(\alpha, \beta)$. This proves that $AB \in \mathcal{S}_\tau^N(\mathscr{C})$. Finally, given a convergent sequence $A_n \to A$ as $n \to \infty$ with property $A_n \in \mathcal{S}_\tau^N(\mathscr{C})$ for all $n \geq 0$. It is straightforward to show that $A \in \mathcal{S}_\tau^N(\mathscr{C})$. Therefore, $\mathcal{S}_\tau^N(\mathscr{C})$ is an operator algebra. ■

Note that $\mathcal{S}_\tau^N(\mathscr{C})$ is closed under the multiplication of finite number of matrices.

3) Lower and Upper Triangular Matrices

The subspace of all $Nm \times Nm$ lower triangular matrices consists of all matrices of the following structure

$$A = \begin{bmatrix} \star & 0 & 0 & \cdots & & 0 \\ \star & \star & 0 & 0 & \cdots & 0 \\ \star & \star & \star & 0 & \cdots & 0 \\ \star & \star & \star & \star & \cdots & 0 \\ \vdots & & & \ddots & \ddots & \vdots \\ \star & \star & \star & \star & \star & \star \end{bmatrix},$$

where the symbol $\star$ represents a $m \times m$. The formal representation of a lower triangular matrix A is as follows

$$A_{ki} = \left\{ \begin{array}{ll} \star & \text{if} \quad k \geq i \\ 0 & \text{if} \quad k < i \end{array} \right. .$$

Similarly, the subspace of all $Nm \times Nm$ upper triangular matrices consists of all matrices of the following structure

$$A = \begin{bmatrix} \star & \star & \star & \star & \cdots & \star \\ 0 & \star & \star & \star & \cdots & \star \\ 0 & 0 & \star & \star & \cdots & \star \\ 0 & & 0 & \star & \cdots & \star \\ \vdots & & & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & \star \end{bmatrix}.$$

The formal representation of the upper triangular matrix A is

$$A_{ki} = \left\{ \begin{array}{ll} \star & \text{if} \quad k \leq i \\ 0 & \text{if} \quad k > i \end{array} \right. .$$

Proposition 4.1: The subspace of lower (or upper) triangular matrices with the same dimensions form an operator algebra under the matrix composition operation.

4) Circulant Matrices

An $Nm \times Nm$ circulant matrix has the following form

$$A = \begin{bmatrix} A_0 & A_{N-1} & \cdots & A_2 & A_1 \\ A_1 & A_0 & A_{N-1} & \cdots & A_2 \\ \vdots & A_1 & A_0 & \ddots & \cdots \\ A_{N-2} & & \ddots & \ddots & A_{N-1} \\ A_{N-1} & A_{N-2} & \cdots & A_1 & A_0 \end{bmatrix},$$

where $A_k \in \mathbb{R}^{m\times m}$ for all $k \in \{1, ..., N\}$. This class of matrices can be used to represent the coupling between subsystems in a networked dynamic system whose structure has a ring topology.

Proposition 4.2: The subspace of all circulant matrices with the same dimensions form an operator algebra under the matrix composition operation.

Indeed, the set of all circulant matrices form a commutative algebra, since for any two given circulant matrices A and B, the sum $A + B$ is circulant, the product AB is circulant, and $AB = BA$.

B. Closure Under Inversion

The following result is from [17] and it shows that under some mild assumptions, an operator algebra could be closed under inversion operation. We assume that assumption 5.1 holds.

Lemma 4.1: Suppose that $\mathcal{S}$ is an operator algebra and $A \in \mathcal{S}$ has an algebraic inverse on $\mathscr{B}(\ell_2)$. Then $A^{-1} \in \mathcal{S}$.

C. Operator Differential Equations

The main result of this paper is presented in theorem 4.3 and it provides a unified approach to study structural properties of solutions of a wide range of operator differential equations that normally arise in analysis of spatially distributed control systems. Examples of such operator differential equations include operator Sylvester and Riccati differential equations.

Lemma 4.2: Let $\mathcal{S}$ be an operator algebra and $F : [a, b] \to \mathscr{B}(\ell_p)$ a continuous map where $a < b$ are some real numbers. If $F(t) \in \mathcal{S}$ for all $t \in [a, b]$, then

$$\int_a^b F(t)\, dt \ \in \ \mathcal{S}.$$

Theorem 4.3: Suppose that $\mathcal{S}$ is an operator algebra and map $F : \mathbb{R} \times \mathscr{B}(\ell_p) \to \mathscr{B}(\ell_p)$ is continuous on $\mathbb{R}$. Assume that under appropriate conditions the following initial value problem

$$\frac{d}{dt}X(t) = F(t, X(t)) \quad , \quad X(0) = X_0, \qquad (8)$$

has a unique solution. If

(i) $\mathcal{S}$ is F-invariant,
(ii) $X_0 \in \mathcal{S}$,

then $X(t) \in \mathcal{S}$ for all $t \geq 0$ where $X(t)$ is the unique solution of problem (8).

Proof: We use fixed-point iteration (also called Picard iteration) method to prove our claim. Define

$$T_0(t) = X_0,$$

and

$$T_n(t) = X_0 + \int_0^t F(\sigma, T_{n-1}(\sigma))\ d\sigma. \qquad (9)$$

According to assumption (ii), it follows that $T_0(t) \in \mathcal{S}$ for all $t \geq 0$. Assumption (i) implies that $F(t, T_0(t)) \in \mathcal{S}$ for all $t \geq 0$. By applying lemma 4.2 and induction on n, one can conclude from equation (9) that $T_n(t) \in \mathcal{S}$ for all $n \geq 1$ and $t \geq 0$. By applying the Banach fixed point theorem, one can show that the infinite series of T_n is uniformly convergent. Define the uniform norm as follows

$$\|X\|_{\text{unif}} := \sup\ \{\ \|X(t)\|_{p,p} \mid t \in \mathbb{R}^+\}.$$

Thus, we have $T_n \to X$ as $n \to \infty$ in the sense of the uniform norm. The limit X is the unique solution of problem (8). The closedness property of $\mathcal{S}$ ensures that $X(t) \in \mathcal{S}$ for all $t \geq 0$. ■

Example 4.1: An immediate application of theorem 4.3 is to study the structural properties of the solution of Sylvester differential equations of the form

$$\frac{d}{dt}X(t) = A(t)X(t) + X(t)B(t) + Q(t)\ ,\ X(0) = X_0, \quad (10)$$

where the coefficient $A(t)$, $B(t)$, $Q(t)$, and the unknown $X(t)$ are complex matrices of appropriate dimensions. The Sylvester differential equation appears in several applications such as large-space flexible structures [18], jump linear systems [19], control of linear systems with non-Markovian modal changes [20], or when one uses semi-discretization techniques to solve scalar partial differential equations [21]. Let assume that $\mathcal{S}$ be an operator algebra and that $A(t), B(t), Q(t) \in \mathcal{S}$ for all $t \geq 0$ as well as $X_0 \in \mathcal{S}$. It is straightforward to verify that $\mathcal{S}$ is invariant under the following map

$$F(t, X) = A(t)X + XB(t) + Q(t).$$

According to theorem 4.3, it follows that $X(t) \in \mathcal{S}$ for all $t \geq 0$. For instance, if we assume that $\mathcal{S}$ is the operator algebra of lower triangular matrices, then the solution of the Sylvester differential equation $X(t)$ is also lower triangular for all $t \geq 0$. This result might be useful in developing numerical methods to solve equation (10).

V. Analysis of The Canonical Problem

In this section, we study the structural properties of the canonical decentralized optimal control problem (2)-(5). We show that the canonical problem (2)-(5) is equivalent to the standard LQR problem if we assume that the information constraint set $\mathcal{S}$ is an operator algebra along with the following assumption.

Assumption 5.1: If $X \in \mathcal{S}$, then $X^* \in \mathcal{S}$.

A. Lyapunov Equation

Consider the operator Lyapunov equation

$$A^*\overline{P} + \overline{P}A + Q = 0, \qquad (11)$$

where A is the infinitesimal generator of an exponentially stable C_0-semigroup on ℓ_2 and $Q \succeq 0$.

Theorem 5.1: Suppose that $\mathcal{S}$ is an operator algebra. If $A, Q \in \mathcal{S}$, then the unique positive definite solution of Lyapunov equation (11) satisfies $\overline{P} \in \mathcal{S}$.

Proof: The unique solution of Lyapunov equation (11) is the unique equilibrium point of the following operator differential equation

$$\frac{d}{dt}P(t) = A^*P(t) + P(t)A + Q, \tag{12}$$

with initial value $P(0) = Q$. By using closure under addition and multiplication of $\mathcal{S}$, one can verify that $A^*X+XA+Q \in \mathcal{S}$ whenever $X \in \mathcal{S}$. According to theorem 4.3, it follows that $P(t) \in \mathcal{S}$ for all $t \geq 0$. On the other hand, we know that $P(t) \to \overline{P}$ as $t \to \infty$ in ℓ_2-norm (see [16]). The closure property of $\mathcal{S}$ implies that $\overline{P} \in \mathcal{S}$. ■

B. Riccati Equation

The canonical decentralized optimal control problem (2)-(5) without the information constraint $K \in \mathcal{S}$ is the standard LQR problem. It is well-known that the associated optimal feedback law to the standard LQR problem is given by

$$K = -R^{-1}B^*\overline{P}, \tag{13}$$

where $\overline{P}$ is the unique solution of the Riccati equation

$$A^*\overline{P} + \overline{P}A + Q - \overline{P}BR^{-1}B^*\overline{P} = 0, \tag{14}$$

and we assume that all standard LQR assumptions hold for (14) to have a unique solution. Note that in all of the examples of operator algebras we discussed in subsection IV-A, it can be shown that the operator algebras are also closed under inversion operation (see [14] and [17] for a complete discussion). Thus, without loss of generality we may assume that $\mathcal{S}$ is closed under inversion operation.

In the canonical problem (2)-(5) if we assume that $A, B, Q, R \in \mathcal{S}$, we will only need to prove that $\overline{P} \in \mathcal{S}$ in order to show that $K \in \mathcal{S}$. The following theorem shows that this is indeed the case.

Theorem 5.2: Consider the canonical problem (2)-(5). Suppose that $\mathcal{S}$ is an operator algebra. If $A, B, Q, R \in \mathcal{S}$, then the unique positive definite solution of Riccati equation (14) satisfies $\overline{P} \in \mathcal{S}$.

Proof: Consider the following operator Riccati differential equation

$$\frac{d}{dt}P(t) = A^*P(t) + P(t)A + Q - P(t)BR^{-1}B^*P(t),$$

with $P(0) = 0$. We denote the unique solution of this Riccati differential equation in the class of strongly continuous, self-adjoint operators in $\mathscr{B}(\ell_2)$ by the one-parameter family of operator-valued function $P(t)$ for $t \geq 0$. By using our assumptions and closure properties of $\mathcal{S}$, if $X \in \mathcal{S}$, then

$$A^*X + XA + Q - XBR^{-1}B^*X \in \mathcal{S}.$$

Therefore, one can conclude from theorem 4.3 that $P(t) \in \mathcal{S}$ for all $t \geq 0$. On the other hand, the nonnegative operator $\overline{P}$, the unique solution of ARE, is the strong limit of $P(t)$ on ℓ_2 as $t \to \infty$ (see theorem 6.2.4 of [16]). Using the fact that $\mathcal{S}$ is closed, one concludes that $\overline{P} \in \mathcal{S}$. ■

Remark 5.1: Among all examples of operator algebras presented in subsection IV-A, the subspaces of lower and upper triangular matrices do not satisfy the assumption 5.1.

VI. Conclusion

The goal of this paper was to offer a rigorous mathematical framework for analysis and synthesis of spatially distributed control systems. Specifically, we studied structural properties of optimal control of spatially distributed dynamical systems with linear quadratic criteria (such as LQR or LQG) where the stabilizing controller was restricted to lie in a constraint set. For a general linear system and constraint set, there is no known tractable algorithm to find the optimum. We applied tools from operator theory such as the notion of operator algebra to specify a wide class of systems and constraint sets for which the constrained optimal control problem is equivalent to the corresponding unconstrained problem.

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