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On decidability of monadic logic of order over the naturals extended by monadic predicates ☆

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Abstract

A fundamental result of Büchi states that the set of monadic second-order formulas true in the structure $(Nat, <)$ is decidable. A natural question is: what monadic predicates (sets) can be added to $(Nat, <)$ while preserving decidability? Elgot and Rabin found many interesting predicates $\mathbf{P}$ for which the monadic theory of $\langle Nat, <, \mathbf{P}\rangle$ is decidable. The Elgot and Rabin automata theoretical method has been generalized and sharpened over the years and their results were extended to a variety of unary predicates. We give a sufficient and necessary model-theoretical condition for the decidability of the monadic theory of $(Nat, <, \mathbf{P}_1, ..., \mathbf{P}_n)$. We reformulate this condition in an algebraic framework and show that a sufficient condition proposed previously by O. Carton and W. Thomas is actually necessary. A crucial argument in the proof is that monadic second-order logic has the selection and the uniformization properties over the extensions of $(Nat, <)$ by monadic predicates. We provide a self-contained proof of this result.

1. Introduction

In this paper, we provide necessary and sufficient conditions for the decidability of monadic (second-order) theory of expansions of the linear order of the naturals ω by unary predicates.

☆ This is the extended version of the talk presented at Logic Colloquium 2005 [10].

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The fundamental work of Büchi [1] shows that the monadic theory of $\omega = \langle Nat, <\rangle$ is decidable. Even before the decidability of the monadic theory of ω has been proved, it was shown that the expansions of ω by "interesting" functions have undecidable monadic theory. In particular, the monadic theory of $\langle Nat, <, +\rangle$ and the monadic theory of $\langle Nat, <, \lambda x.2 \times x\rangle$ are undecidable [12,18]. Therefore, most efforts to find decidable expansions of ω deal with expansions of ω by monadic predicates.

It is clear that if $M = \langle Nat, <, \mathbf{P}_1, \ldots, \mathbf{P}_n\rangle$ is an expansion of ω by definable predicates, then the monadic theory of M is decidable. However, only very simple monadic predicates are definable in ω. It is well known that a monadic predicate $\mathbf{S}$ is definable in ω iff it is *ultimately periodic*, i.e., there are $m, d \in Nat$ such that for all $i > m$: $i \in \mathbf{S}$ iff $i + d \in \mathbf{S}$.

In order to prove decidability of the monadic theory of ω, Büchi introduced finite automata over ω-words. He provided a computable reduction from formulas to finite automata. More precisely, he proved that for every monadic formula $\varphi(X)$ with a free second-order monadic variable X there is a finite automaton $\mathcal{A}$ such that a monadic predicate $\mathbf{P}$ satisfies $\varphi(X)$ if and only if $\mathcal{A}$ accepts the characteristic ω-word $u_{\mathbf{P}}$ associated with $\mathbf{P}$ (the ith letter of $u_{\mathbf{P}}$ is 1 if $i \in \mathbf{P}$ and it is 0 if $i \notin \mathbf{P}$). Hence, the monadic theory of $\langle Nat, <, \mathbf{P}\rangle$ is decidable iff for the corresponding ω-string $u_{\mathbf{P}}$ the following decision problem is decidable

$$(\mathrm{Acc}_{u_{\mathbf{P}}})\text{: Given an automaton } \mathcal{A}\text{, does } \mathcal{A} \text{ accepts } u_{\mathbf{P}}?$$

Elgot and Rabin [6] found many interesting predicates $\mathbf{P}$ for which the problem $\mathrm{Acc}_{u_{\mathbf{P}}}$ and hence the monadic theory of $\langle Nat, <, \mathbf{P}\rangle$ are decidable. Among these predicates are the set of factorial numbers $\{n! : n \in Nat\}$, the sets of kth powers $\{n^k : n \in Nat\}$ and the sets $\{k^n : n \in Nat\}$ (for $k \in Nat$).

The Elgot and Rabin automata theoretical method has been generalized and sharpened over the years and their results were extended to a variety of unary predicates (see e.g., [5,15,16,13,3,4]).

In [3,4] a class of effectively profinitely ultimately periodic predicates was introduced by Catron and Thomas. Many examples of effectively profinitely ultimately periodic predicates were provided. Catron and Thomas used an algebraic (semigroup) approach to show that for every effectively profinitely ultimately periodic predicate $\mathbf{P}$ the corresponding problem $\mathrm{Acc}_{u_{\mathbf{P}}}$ is decidable. Consequently, they derived that if $\mathbf{P}$ is an effectively profinitely ultimately periodic predicate then the monadic theory of $\langle Nat, <, \mathbf{P}\rangle$ is decidable. We show that this is a necessary condition for the decidability of the monadic theory of $\langle Nat, <, \mathbf{P}\rangle$.

Unlike previous proofs of the decidability of monadic expansions of ω our proof is based on model theoretical methods developed by Shelah [14].

Let $M = \langle Nat, <, \mathbf{P}\rangle$, where $\mathbf{P}$ is a unary predicate. For an interval $[i, j)$, we denote by $M_{[i,j)}$ the substructure of M over the set $\{k : i \leqslant k < j\}$. Structures are said to be $\equiv_k$-equivalent if they satisfy the same monadic second-order sentences of the quantifier depth at most k. A subset $\mathbf{S} = \{s_1 < s_2 < \cdots < s_i < \cdots\}$ of Nat is said to be k-homogeneous for $M = \langle Nat, <, \mathbf{P}\rangle$ if $\mathbf{S}$ is infinite and $M_{[s_i,s_{i'})} \equiv_k M_{[s_j,s_{j'})}$ for all pairs $i < i'$ and $j < j'$. A set $\mathbf{S}$ is said to be homogeneous for M if for every k the set $\mathbf{S}_k = \mathbf{S} \cap \{n : n \geqslant s_k\}$ is k-homogeneous.

Our main technical result is the following necessary and sufficient condition for the decidability of $M = \langle Nat, <, \mathbf{P}\rangle$:

Theorem A. *The monadic theory of* $M = \langle Nat, <, \mathbf{P} \rangle$ *is decidable if and only if* $\mathbf{P}$ *is recursive and there is a recursive set* $\mathbf{S}$ *which is homogeneous for* M.

We also provide necessary and sufficient algebraic conditions for the decidability of $M = \langle Nat, <, \mathbf{P} \rangle$. An ω-sequence a_i is said to be ultimately constant with lag l if $a_i = a_j$ for $i, j > l$.

Let $M = \langle Nat, <, \mathbf{P} \rangle$ and let $u_{\mathbf{P}} = a_0 a_1 \ldots$ be the characteristic ω-word over $\{0, 1\}$ associated with $\mathbf{P}$. For an infinite set $\mathbf{S} = \{s_1 < s_2 < \cdots < s_i < \cdots\} \subseteq Nat$ define an ω-sequence $w_i = w_i^{\mathbf{S}} = a_{s_i} a_{s_i+1} a_{s_i+2} \ldots a_{s_{i+1}-1}$ of finite words over alphabet $\{0, 1\}$. A set $\mathbf{S} \subseteq Nat$ is ultimately constant for M if for every finite semigroup G and for every morphism h from the semigroup of finite non-empty words over $\{0, 1\}$ into G the sequence $\{h(w_i)\}_{i \in Nat}$ is ultimately constant. A set $\mathbf{S}$ is effectively ultimately constant for M if $\mathbf{S}$ is recursive and ultimately constant for M and there is a recursive function which for every finite semigroup G and morphism h computes a lag of the ultimately constant sequence $\{h(w_i)\}_{i \in Nat}$.

Theorem B. *The monadic theory of* $M = \langle Nat, <, \mathbf{P} \rangle$ *is decidable if and only if* $\mathbf{P}$ *is recursive and there is a set which is effectively ultimately constant for* M.

The "only if" direction is a difficult part of Theorems A and B; the "if" direction of these theorems is easy.

The paper is organized as follows. In Section 2, we fix notations and terminology. In Section 3, elements of the composition method are presented. Though our use of the composition method is not very deep, it is unlikely that these results would have been found in the automata theoretical framework. In Section 4, we give a proof of Theorem A. In Section 5, we reformulate this theorem in the algebraic framework and we prove Theorem B. The algebraic condition of Theorem B is a special case of Carton–Thomas condition shown to be sufficient for decidability [3,4]; hence, we obtain that the monadic theory of $M = \langle Nat, <, \mathbf{P} \rangle$ is decidable if and only if $\mathbf{P}$ is effectively profinitely ultimately periodic. A negative answer to a question raised in [3] is also provided here. Section 6 compares our results with the Semenov theorem [13] which provides another necessary and sufficient condition for the decidability of the monadic theory of M. We also state here the generalization of our results to other logics [11]. A proof of Theorem 17 is given in the Appendix. This theorem is closely related to the uniformizatiion property for $\langle Nat, < \rangle$, which was stated without proof in [9]. The Appendix also compares the uniformization property with the Church uniformization problem.

2. Preliminaries

2.1. Notations and terminology

We use k, l, m, n, i for natural numbers; Nat for the set of natural numbers and capital bold letters $\mathbf{P}$, $\mathbf{S}$, $\mathbf{R}$ for subsets of Nat. We identify subsets of a set A and the corresponding unary (monadic) predicates on A. We use standard notations for ordinals, e.g., ω is the order type of natural numbers.

As usual in set theory, a natural number n can be viewed as a linear order, namely the initial segment $(\{0,\ldots,n-1\},<)$ of the standard ordering of natural numbers.

The set of non-empty finite words (strings) over an alphabet Σ is denoted by Σ^+; the semigroup of finite non-empty words over Σ with the concatenation operation will be also denoted by Σ^+; this semigroup is the *free semigroup generated by* Σ, i.e., every function from Σ into a semigroup G can be extended in a unique way to a morphism from Σ^+ into G.

There exists a one-one correspondence between the set of all ω-strings over the alphabet $\{0,1\}^n$ and the set of all n-tuples $\langle \mathbf{P}_1,\ldots,\mathbf{P}_n\rangle$ of unary predicates over the set of natural numbers. With an n-tuple $\langle \mathbf{P}_1,\ldots,\mathbf{P}_n\rangle$ of unary predicates over *Nat*, we associate the ω-string $a_0a_1\ldots a_k\ldots$ over alphabet $\{0,1\}^n$ defined by $a_k =_{def} \langle b_1^k,\ldots,b_n^k\rangle$ where b_i^k is 1 if $\mathbf{P}_i(k)$ holds and b_i^k is 0 otherwise.

Similarly, there is a one-one correspondence between the set of all strings of length m over the alphabet $\{0,1\}^n$ and the set of all n-tuples $\langle \mathbf{P}_1,\ldots,\mathbf{P}_n\rangle$ of unary predicates over the set $\{0,\ldots,m-1\}$.

A linearly ordered set will be called a chain. A chain with n monadic predicates over its domain will be called an n-labelled chain; whenever n is clear from the context an n-labelled chain will be called a labelled chain.

We will sometimes identify an n-labelled chain $M = \langle Nat,<,\mathbf{P}_1,\ldots,\mathbf{P}_n\rangle$ with the ω-string over the alphabet $\{0,1\}^n$ which corresponds to the n-tuples $\langle \mathbf{P}_1,\ldots,\mathbf{P}_n\rangle$; this ω-string will be called the *characteristic* ω-string (or ω-word) of M. Similarly, we will identify finite n-labelled chains with corresponding strings over $\{0,1\}^n$.

We use standard notations for intervals, e.g., $[a,b)$ denotes the set $\{c : a \leqslant c < b\}$. For a labelled chain M and its interval $[a,b)$ we denote by $M_{[a,b)}$ the substructure of M over $[a,b)$; we denote by $M_{<a}$ the substructure of M over the interval $\{c : c < a\}$.

2.2. Monadic logic of order

The syntax of the monadic second-order logic of order—*MLO* has in its vocabulary *individual* (first order) variables $t_0,t_1,\ldots$, monadic *second-order* variables $X_0,X_1,\ldots$, monadic *predicate* names $P_0,P_1\ldots$ and one binary relation $<$ (the order).

Atomic formulas are of the form $X(t), P(t), t_1 < t_2$ and $t_1 = t_2$. Well-formed formulas of the monadic logic *MLO* are obtained from atomic formulas using Boolean connectives $\neg,\vee,\wedge,\rightarrow$ and the first-order quantifiers $\exists t$ and $\forall t$, and the second-order quantifiers $\exists X$ and $\forall X$. The quantifier depth of a formula φ is denoted by $\mathrm{qd}(\varphi)$.

A structure for *MLO* is $M = \langle A,<,\mathbf{P}_1^M,\ldots,\mathbf{P}_n^M\rangle$, where $<^M$ is a linear order over A and $\mathbf{P}_1^M,\ldots,\mathbf{P}_n^M$ are one-place predicates (sets) which are the interpretations of the monadic predicate names $P_1,\ldots,P_n$ in the structure M. Such a structure is called an n-labelled chain, or for brevity simply a chain.

We shall not repeat the standard inductive definition saying when a formula is satisfied. We recall the notation:

$$M,\tau_1,\ldots\tau_k;\mathbf{S}_1,\ldots,\mathbf{S}_m \models \varphi(t_1,\ldots,t_k;X_1,\ldots,X_m)$$

which we also abbreviate to $M \models \varphi[\tau_1,\ldots,\tau_k;\mathbf{S}_1,\ldots,\mathbf{S}_m]$, where M is a structure, $\tau_1,\ldots,\tau_k$ are elements of M, $\mathbf{S}_1,\ldots,\mathbf{S}_m$ are unary predicates (i.e., sets) over the domain of M, φ is a formula and $t_1,\ldots,t_k;X_1,\ldots,X_m$ include all the free variables of φ.

We use standard abbreviations, e.g., we write $X \subseteq X'$ for $\forall t. X(t) \to X'(t)$; we write $X = X'$ for $\forall t. X(t) \leftrightarrow X'(t)$; symbols "$\exists^{\leqslant 1}$" and "$\exists!$" stands for "there is at most one" and "there is a unique".

Let $M = \langle A, <, \mathbf{P}_1^M, \ldots, \mathbf{P}_n^M \rangle$ be a labelled chain and $\alpha(X)$ be a formula. We say that a set $\mathbf{S} \subseteq A$ is *definable* by $\alpha(X)$ in M if $M \models \alpha[\mathbf{S}]$ and $M \models \exists! X\alpha(X)$. We say that a set $\mathbf{S} \subseteq A$ is definable (in M) if it is definable by a *MLO* formula.

The *monadic theory* of a labelled chain M is the set of all *MLO* sentences which hold in M.

We will deal with decidability questions for the monadic theory of expansions of ω by monadic predicates (i.e., of structures of the form $M = \langle Nat, <, \mathbf{P}_1, \ldots, \mathbf{P}_n \rangle$).

We say that a chain $M = \langle Nat, <, \mathbf{P}_1, \ldots, \mathbf{P}_n \rangle$ is *recursive* if all $\mathbf{P}_i$ are recursive subsets of *Nat*.

3. Elements of composition method

Composition theorems are tools which reduce sentences about some compound structure to sentences about its parts. A seminal example of such a result is the Feferman–Vaught Theorem [7] which reduces the first-order theory of generalized products to the first-order theory of its factors and the monadic second-order theory of index structure.

Shelah [14] used the composition theorem for linear orders as one of the main tools for obtaining very strong decidability results for the monadic theory of linear orders.

In this section, definitions and theorems which will be used later are collected. They are adaptations of more general results proved by Shelah [14]. The proofs of the theorems stated here can be easily extracted from the results in [14] or from surveys by Gurevich [8] and Thomas [17].

3.1. $\equiv_k$-equivalence

We use the notation $\equiv_k$ to say that two labelled chains cannot be distinguished by an *MLO* sentence of quantifier depth k. More precisely, let M and M' be two n-labelled chains. We write $M \equiv_k M'$ if and only if for every sentence φ with $\mathrm{qd}(\varphi) \leqslant k$ we have $M \models \varphi$ iff $M' \models \varphi$.

Theorem 1.

(1) *For every n and every k, the relation $\equiv_k$ defines finitely many equivalence classes $\mathbb{T}_1, \mathbb{T}_2, \ldots, \mathbb{T}_m$ over the set of all n-labelled chains.*
(2) *For each equivalence class $\mathbb{T}_i$ there is a MLO sentence β_i with $\mathrm{qd}(\beta_i) \leqslant k$ which characterizes it; i.e., $M \in \mathbb{T}_i$ iff $M \models \beta_i$. Moreover, there is an algorithm which for every k and n computes the set $Char_n^k$ of the characteristic sentences for $\equiv_k$ over the n-labelled chains.*
(3) *Every MLO sentence φ in the signature $\langle <, P_1, \ldots, P_n \rangle$ with $\mathrm{qd}(\varphi) \leqslant k$ is equivalent to a (finite) disjunction of characteristic sentences from $Char_n^k$. Moreover, there is an algorithm which for every sentence φ computes a finite set D of characteristic sentences such that φ is equivalent to the disjunction of all the sentences from D.*

3.2. Lexicographical sum

Definition 2. (Lexicographic sum) Let $Ind = \langle I, <^{Ind} \rangle$ be a chain and let $M_i = \langle A_i, <^{M_i}, \mathbf{P}_1^{M_i}, \ldots, \mathbf{P}_n^{M_i} \rangle$ for $i \in I$ be a family of (disjoint) n-labelled chains. The *lexicographic* sum of $M_i =$

$\langle A_i, <^{M_i}, \mathbf{P}_1^{M_i}, \ldots, \mathbf{P}_n^{M_i}\rangle$ with respect to the chain *Ind* (notation $\sum_{i\in Ind} M_i$) is the n-labelled chain $M = \langle A, <, \mathbf{P}_1, \ldots, \mathbf{P}_n\rangle$, where

(1) $A = \cup_{i\in I} A_i$ and
(2) An element $\tau \in A_i$ precedes $\tau' \in A_j$ in M if i precedes j in *Ind* or $i = j$ and τ precedes τ' in $\mathcal{A}_i$.
(3) For $l = 1, \ldots, n$, an element $\tau \in \mathbf{P}_l$ if $\tau \in \mathbf{P}_l^{M_i}$ for some $i \in I$.

We refer to M_i as the summand chains and to *Ind* as the indices chain of $\sum_{i\in Ind} M_i$. We write $M_0 + M_1$ for $\sum_{i\in\{0,1\}} M_i$.

Theorem 3 (*Composition theorem for chains*). *Let* $Ind = \langle I, <^{Ind}\rangle$ *be a chain and let* M_i $(i \in I)$ *and* M_i' $(i \in I)$ *be two families of n-labelled chains such that* $M_i \equiv_k M_i'$ *for all* $i \in I$. *Then,* $\sum_{i\in Ind} M_i$ *is* $\equiv_k$*-equivalent to* $\sum_{i\in Ind} M'_i$.

This theorem implies the following corollary:

Corollary 4.

(1) *For every n and k there is a function* $\bigoplus_{n,k}$ *from* $Char_n^k \times Char_n^k$ *into* $Char_n^k$ *such that for all characteristic sentences* $\beta_1, \beta_2, \beta_3 \in Char_n^k$ *and for all n-labelled chains* M_1 *and* M_2 *such that* $M_1 \models \beta_1$ *and* $M_2 \models \beta_2$ *the following equivalence holds:*

$$\beta_3 = \bigoplus_{n,k}(\beta_1, \beta_2) \text{ if and only if } M_1 + M_2 \models \beta_3.$$

(2) *For every n and k there is a function* $\bigotimes_{n,k}^{\omega}$ *from* $Char_n^k$ *into* $Char_n^k$ *such that for every characteristic sentences* $\beta_1, \beta_2 \in Char_n^k$ *and for every family* M_i $(i \in Nat)$ *of n-labelled chains such that* $M_i \models \beta_1$ *for every i, the following equivalence holds*:

$$\beta_2 = \bigotimes_{n,k}^{\omega}(\beta_1) \text{ if and only if } \left(\sum_{i\in\omega} M_i\right) \models \beta_2.$$

Observe that $M_1 + (M_2 + M_3)$ *is isomorphic to* $(M_1 + M_2) + M_)$. *Therefore,* $\bigoplus_{n,k}$ *is associative and* $\langle Char_n^k, \bigoplus_{n,k}\rangle$ *is a semigroup.*

Note that for every n and k the domain and the range of the functions $\bigoplus_{n,k}$ and $\bigotimes_{n,k}^{\omega}$ are finite. Our proofs will often rely on the following theorem of Shelah.

Theorem 5. *The functions* $\lambda n \lambda k \bigoplus_{n,k}$ *and* $\lambda n \lambda k \bigotimes_{n,k}^{\omega}$ *are recursive.*

4. Homogeneous sets

In this section, we state and prove our main technical result—Theorem 8— which provides necessary and sufficient conditions for the decidability of the monadic (second-order) theory of $M = \langle Nat, <, \mathbf{P}_1, \ldots, \mathbf{P}_n\rangle$. We start with definitions.

Definition 6 (*k-homogeneous sets*). A set $\mathbf{R} = \{r_1 < r_2 < \cdots < r_i < \cdots\} \subseteq Nat$ is k-homogeneous for $M = \langle Nat, <, \mathbf{P}_1, \ldots, \mathbf{P}_n\rangle$ if $\mathbf{R}$ is infinite and for all $i < i'$ and $j < j'$ the substructures of M over intervals $[r_i, r_{i'})$ and $[r_j, r_{j'})$ are $\equiv_k$-equivalent.

Definition 7 (*homogeneous sets*). A set $\mathbf{R} = \{r_1 < r_2 < \cdots < r_i < \cdots\}$ is homogeneous for $M = \langle Nat, <, \mathbf{P}_1, \ldots, \mathbf{P}_n\rangle$ if for all $m \in Nat$ the set $\{r_m < r_{m+1} < \cdots\}$ is m-homogeneous.

In other words, a set $\mathbf{R} = \{r_1 < r_2 < \cdots\}$ is homogeneous for M if for all $m \in Nat$ and for all $i, j \geqslant m$ and all $i' > i$ and $j' > j$ the substructures of M over intervals $[r_i, r_{i'})$ and $[r_j, r_{j'})$ are $\equiv_m$-equivalent.

Assume that $\mathbf{P}$ is periodic with a period d, i.e., $\forall i(i \in \mathbf{P}) \leftrightarrow (i + d \in \mathbf{P})$. Then, every periodic set with a period kd is homogeneous for $\langle Nat, <, \mathbf{P}\rangle$.

Now we are ready to state our main technical result:

Theorem 8. *The monadic theory of $M = \langle Nat, <, \mathbf{P}_1, \ldots, \mathbf{P}_n\rangle$ is decidable if and only if M is recursive and there is a recursive homogeneous set for M.*

The "if" direction of this theorem is easy and is proved in Theorem 10. The "only if" direction is more difficult and it is proved in Theorem 11.

Remark 9. Instead of homogeneous sets one can consider *weakly* homogeneous sets. A set $\mathbf{R} = \{r_1 < r_2 < \cdots < r_i < \cdots\}$ is weakly homogeneous for $M = \langle Nat, <, \mathbf{P}_1, \ldots, \mathbf{P}_n\rangle$ if it is infinite and for all $m \in Nat$ and for all $i, j \geqslant m$ the substructures of M over intervals $[r_i, r_{i+1})$ and $[r_j, r_{j+1})$ are $\equiv_m$-equivalent. Theorem 8 as well as all the results stated in this paper hold when "homogeneous" is replaced by "weakly homogeneous." Moreover, the proofs require only minor modifications.

Theorem 10. *If $M = \langle Nat, <, \mathbf{P}_1, \ldots, \mathbf{P}_n\rangle$ is recursive and there is a recursive homogeneous set for M, then the monadic theory of M is decidable.*

Proof. Assume that $\mathbf{R} = \{r_1 < r_2 < \cdots\}$ is a recursive set which is homogeneous for M. Assume given a sentence φ of quantifier depth k. In order to verify whether φ holds in M we first find a characteristic sentence $\beta \in Char_n^k$ of M, i.e., $\beta \in Char_n^k$ such that $M \models \beta$.

Let $M_{<r_k}$ and $M_{[r_k, r_{k+1})}$ be the substructure of M over the interval $[0, r_k)$ and $[r_k, r_{k+1})$, respectively. The structures $M_{<r_k}$ and $M_{[r_k, r_{k+1})}$ are finite and computable from (the decision algorithm for) M and $\mathbf{R}$, therefore we can compute the characteristic sentences α, γ from the (finite) set of sentences $Char_n^k$ such that $M_{<r_k} \models \alpha$ and $M_{[r_k, r_{k+1})} \models \gamma$. By Corollary 4(2) and the fact that $\mathbf{R}$ is homogeneous we obtain that $\delta = \bigotimes^{\omega}(\gamma)$ is the characteristic sentence of the substructure $M_{[r_k, \infty)}$ of M over $[r_k, \infty)$. Hence, by Corollary 4(1), $\beta = \alpha \bigoplus \delta$ is the characteristic sentence of M. Moreover, by Theorem 5, β is computable from α and δ, therefore it is computable from M and $\mathbf{R}$.

By Theorem 1(3), we can compute $D \subseteq Char_n^k$ such that φ is equivalent to the disjunction of the sentences from D. Finally, $M \models \varphi$ if and only if $\beta \in D$.

This completes the description of our algorithm for the monadic theory of M. □

Actually, our proof of Theorem 10 shows that the monadic theory of $M = \langle Nat, <, \mathbf{P}_1, \ldots, \mathbf{P}_n\rangle$ is recursive relative to $\mathbf{P}_1, \ldots, \mathbf{P}_n, \mathbf{R}$, where $\mathbf{R}$ is any set homogeneous for M. Similar assertions hold for many results stated in this paper, but to avoid a tiresome exposition they will not be mentioned.

In the rest of this section we are going to prove the following theorem (more difficult direction of Theorem 8):

Theorem 11. *If the monadic theory of $M = \langle Nat, <, \mathbf{P}_1, \ldots, \mathbf{P}_n\rangle$ is decidable, then M is recursive and there is a recursive homogeneous set for M.*

First observe:

Lemma 12. *Every set* $\mathbf{S} \subseteq Nat$ *definable in* $M = \langle Nat, <, \mathbf{P}_1, \ldots, \mathbf{P}_n \rangle$ *is recursive relative to the monadic theory of* M. *In particular, if the monadic theory of* $M = \langle Nat, <, \mathbf{P}_1, \ldots, \mathbf{P}_n \rangle$ *is decidable and a set* $\mathbf{S}$ *is definable, then* $\mathbf{S}$ *is recursive.*

Proof. For every natural number n we compute a formula $\varphi_n(t)$ which defines n in $\omega = \langle Nat, < \rangle$, i.e., $\omega \models \varphi_n[m]$ iff $m = n$.

Let $\mathbf{S}$ be a set definable by a formula $\alpha(X)$.

In order to decide whether n is in $\mathbf{S}$ we can check whether $\exists t \exists X \varphi_n(t) \wedge \alpha(X) \wedge X(t)$ holds in M. □

Definition 13 (γ-*colored sets*)**.** Let γ be an MLO sentence and let $M = \langle Nat, <, \mathbf{P}_1, \ldots, \mathbf{P}_n \rangle$ be a labelled chain. A subset $\mathbf{R}$ of Nat is γ-colored for M if for all $i, j \in \mathbf{R}$, γ holds in the substructure of M over the interval $[i, j)$.

Observe that $\mathbf{R}$ is $\equiv_k$-homogeneous for M iff $\mathbf{R}$ is infinite and γ-colored for some $\gamma \in Char^k_n$.

Lemma 14.

(1) *For every MLO sentence* γ *there is a formula* $col_\gamma(X)$ *such that for every labelled chain* $M = \langle Nat, <, \mathbf{P}_1, \ldots, \mathbf{P}_n \rangle$*:*

$$M \models col_\gamma[\mathbf{S}] \text{ if and only if } \mathbf{S} \text{ is an infinite } \gamma\text{-colored set.}$$

(2) *For every n and k there is a formula* $Hom^n_k(X)$ *such that for every n-labelled chain M*

$$M \models Hom^n_k[\mathbf{S}] \text{ if and only if } \mathbf{S} \text{ is } k\text{-homogeneous for } M.$$

Proof. (1) is immediate; (2) follows from (1) and the observation that $\mathbf{R}$ is $\equiv_k$-homogeneous for M iff $\mathbf{R}$ is infinite and γ-colored for some $\gamma \in Char^k_n$. □

Whenever n is clear from the context, we will write "Hom_k" for "Hom^n_k".

Lemma 15. *For every n-labelled chain* $M = \langle Nat, <, \mathbf{P}_1, \ldots, \mathbf{P}_n \rangle$ *and every k,every infinite set contains a subset which is* $\equiv_k$*-homogeneous for M.*

Proof. Let $\mathbf{V} \subseteq Nat$ be an infinite set. Let Col be a function which assigns to every pair $v_1 < v_2$ of elements from $\mathbf{V}$ a sentence in $Char^k_n$ as follows:

$$Col(v_1, v_2) = \gamma_i \text{ iff } M_{[v_1, v_2)} \models \gamma_i.$$

Recall that $Char^k_n$ is finite. Hence, by the Ramsey theorem, there exist $\gamma_i \in Char^k_n$ and an infinite γ_i-colored subset $\mathbf{S}$ of $\mathbf{V}$. Therefore, $\mathbf{S}$ is k-homogeneous infinite subset of $\mathbf{V}$. □

As a consequence of Lemma 15 we obtain that there is a $\equiv_k$-homogeneous set for M. Moreover, by Lemma 14(2), the family of all $\equiv_k$-homogeneous sets for M is definable by Hom_k(X). We are going to show that if the monadic theory of M is decidable, then there is a recursive k-homogeneous set for M. In order to prove it we will show that there is a definable $\equiv_k$-homogeneous set and therefore by Lemma 12 this set is recursive.

First, we need the following definition and theorem.

Definition 16 (*selector*)**.** $\alpha(X)$ is a selector for $\beta(X)$ over a class C of n-labelled chains iff the following conditions hold:

(1) $C \models \exists^{\leqslant 1} X\ \alpha(X)$.
(2) $C \models \forall X\big(\alpha(X) \to \beta(X)\big)$.
(3) $C \models \big(\exists Y\ \beta(Y)\big) \to \big(\exists X\ \alpha(X)\big)$.

Here and below "$\exists^{\leqslant 1}$" stands for "there is at most one". We say that $\alpha(X)$ is a selector for $\beta(X)$ over a chain $M = \langle Nat, <, \mathbf{P}_1, \ldots, \mathbf{P}_n\rangle$ if $\alpha(X)$ is a selector for $\beta(X)$ over the class $\{M\}$.

Theorem 17 (*computability of selector*)**.** *There is an algorithm that for every formula $\beta(X)$ constructs a formula $\alpha(X)$ such that $\alpha(X)$ is a selector for $\beta(X)$ over the class $C = \{\langle Nat, <, \mathbf{P}_1, \ldots, \mathbf{P}_n\rangle : \mathbf{P}_i \subseteq Nat\}$ of labelled chains.*

Proof. This theorem follows from [9]; its proof will be given in the Appendix. □

Now we are ready to prove Theorem 11. We are going to construct a sequence $\alpha_0(X), \alpha_1(X), \alpha_2(X), \cdots, \alpha_i(X), \cdots$ of formulas such that for all $i \in Nat$ and $M = \langle Nat, <, \mathbf{P}_1, \ldots, \mathbf{P}_n\rangle$ the following conditions hold:

(A) $M \models \exists! X\ \alpha_i(X)$.
(B) $M \models \forall XY\big(\alpha_i(X) \wedge \alpha_{i+1}(Y)\big) \to Y \subseteq X$.
(C) The set U_i definable by $\alpha_i(X)$ in M is i-homogeneous for M.

Basis. Let $\alpha_0(X)$ be $\forall t X(t)$.
Inductive step. $i \rightarrowtail i+1$. Let $\alpha_{i+1}(X)$ be a selector for $\exists Y\big(Hom_{i+1}(X) \wedge \alpha_i(Y) \wedge X \subseteq Y\big)$.

Note $M \models \exists^{\leqslant 1} X \alpha_i(X)$ holds because $\alpha_i(X)$ are selectors. By the induction on i it is easy to derive from Lemma 15 that there is an infinite set U_i that satisfies $\alpha_i(X)$ in M. Therefore, (A) above holds. From the definition of α_i, it immediately follows that (B) and (C) hold.

Now we are ready to finish the proof of Theorem 11. Assume that the monadic theory of M is decidable. Then, by Lemma 12, all U_i defined above are recursive; by (B) and (C), we have that $U_{i+1} \subseteq U_i$ and U_i is i homogeneous for all i. Therefore, the set U defined as

$$U = \{a_i : a_i \text{ is the } i\text{th element of } U_i \text{ for } i \in Nat\}$$

is infinite, recursive and homogeneous for M.

Hence, we proved that if the monadic theory of M is decidable, then there is a recursive homogeneous set for M.

Remark 18. For our proof we need a theorem much weaker than Theorem 17. We only used the following: if the monadic theory of M is decidable then there is an algorithm that for every formula $\beta(X)$ constructs a formula $\alpha(X)$ such that $\alpha(X)$ is a selector for $\beta(X)$ over M.

5. Algebraic conditions for decidability

The purpose of this section is to provide necessary and sufficient algebraic conditions for the decidability of the monadic theory of $M = \langle Nat, <, \mathbf{P}_1, \ldots, \mathbf{P}_n\rangle$.

Recall that an ω-sequence a_i is said to be ultimately constant with lag l if $a_i = a_j$ for $i, j > l$.

For $M = \langle Nat, <, \mathbf{P}_1, \ldots, \mathbf{P}_n\rangle$ and an infinite set $\mathbf{S} = \{s_1 < s_2 < \cdots < s_i < \cdots\} \subseteq Nat$ define an ω-sequence $w_i = w_i^{\mathbf{S}}$ of finite words over the alphabet $\{0,1\}^n$, which correspond to the substructures $M_{[s_i, s_{i+1})}$ of M (see Subsection 2.1): the length of w_i is $l_i = s_{i+1} - s_i$ and for $j < l_i$: the jth letter of w_i is $\langle b_1, \ldots, b_m, \ldots, b_n\rangle$, where b_m is 1 if $s_i + j \in \mathbf{P}_m$, and b_m is 0, otherwise.

A set $\mathbf{S} \subseteq Nat$ is ultimately constant for M if for every finite semigroup G and for every morphism h from the semigroup of finite non-empty words over $\Sigma = \{0,1\}^n$ into G the sequence $\{h(w_i)\}_{i \in Nat}$ is ultimately constant. A set $\mathbf{S}$ is effectively ultimately constant for M if $\mathbf{S}$ is recursive and ultimately constant for M and there is a recursive function which for every finite semigroup G and morphism $h : \Sigma^+ \to G$ computes a lag of the ultimately constant sequence $\{h(w_i)\}_{i \in Nat}$.

Theorem 19. *The monadic theory of $M = \langle Nat, <, \mathbf{P}_1, \ldots, \mathbf{P}_n\rangle$ is decidable if and only if M is recursive and there is effectively ultimately constant set for M.*

The “only if” direction of Theorem 19 immediately follows from Theorem 8 and Lemma 20(2); the “if” direction immediately follows from Lemma 21.

In the next lemma, we identify words and the corresponding labelled chains.

Lemma 20.

(1) For every $n, l \in Nat$ there is $k \in Nat$ such that:
for all finite non-empty words u and v over $\Sigma = \{0,1\}^n$ and for every semigroup G of size at most l and every morphism $h : \Sigma^+ \to G$ if $u \equiv_k v$ then $h(u) = h(v)$.
Moreover, k is computable from n and l.

(2) If a recursive set $\mathbf{S}$ is homogeneous for M, then $\mathbf{S}$ is effectively ultimately constant for M.

Proof. Observe that for every finite semigroup G, an element $a \in G$ and a morphism $h : \Sigma^+ \to G$ one can construct an *MLO* sentence $\xi_{G,a,h}$ such that for every word $u \in \Sigma^+$

$$u \models \xi_{G,a,h} \text{ if and only if } h(u) = a.$$

For every $l \in Nat$ let $k_{l,n}$ be defined as the maximum of quantifier ranks of $\xi_{G,a,h}$ over the semigroups G of size at most l, elements $a \in G$ and morphisms $h : \Sigma^+ \to G$.

Observe that the function $\lambda n \lambda l.\ k_{l,n}$ is recursive. Observe also that for words u_1, u_2:

$$\text{if } u_1 \equiv_{k_{l,n}} u_2 \text{ then } h(u_1) = h(u_2)$$

for every semigroup G of size $\leqslant l$ and every morphism $h : \Sigma^+ \to G$.

Lemma 20(1) follows from these two observations. Lemma 20(2) is an immediate consequence of Lemma 20(1). □

Lemma 21. *If $M = \langle Nat, <, \mathbf{P}_1, \ldots, \mathbf{P}_n \rangle$ is recursive and there is a set which is effectively ultimately constant for M, then the monadic theory of M is decidable.*

Proof. The proof is similar to the proof of Theorem 10.

Assume that $\mathbf{R} = \{r_1 < r_2 < \cdots\}$ is a (recursive) set which is effectively ultimately constant for M. Given a sentence φ of quantifier depth k. In order to verify whether φ holds in M we first find a characteristic sentence $\beta \in Char_n^k$ of M, i.e., $\beta \in Char_n^k$ such that $M \models \beta$.

Let $\Sigma = \{0,1\}^n$. Define $\mu_k : \Sigma \to Char_n^k$ as follows: $\mu_k(\langle b_1, \ldots, b_n \rangle) = \gamma$, if γ is the characteristic sentence of the one element chain $\langle \{0\}, <, \mathbf{P}_1, \ldots, \mathbf{P}_n \rangle$, where $0 \in \mathbf{P}_i$ if b_i is 1 and $0 \notin \mathbf{P}_i$ otherwise. (In other words, $\mu_k(a) = \gamma$ iff γ is the characteristic sentence of the chains that corresponds to the letter $a = \langle b_1, \ldots, b_n \rangle$.) Let $h_k : \Sigma^+ \to Char_n^k$ be the (unique) morphism which extends μ_k.

From Corollary 4(1) it follows that for every string $u \in \Sigma^+$: if $h_k(u) = \gamma$, then γ is the characteristic sentence of the labelled chain which corresponds to u.

Let u_i (for $i \in Nat$) be the string that corresponds to $M_{[r_i, r_{i+1})}$—the substructure of M over the interval $[r_i, r_{i+1})$, respectively.

Since R is effectively ultimately constant, we can compute $l \in Nat$ and $\gamma \in Char_n^k$ such that $h_k(u_i) = \gamma$ for all $i > l$. By Corollary 4(2) we obtain that $\delta = \bigotimes^{\omega}(\gamma)$ is the characteristic sentence of the substructure $M_{[r_l, \infty)}$ of M over $[r_l, \infty)$. Let v be the string that corresponds to the substructure of M over the interval $[0, r_l)$; note that α defined as $h_k(v)$ is the characteristic sentence of this substructure. Hence, by Corollary 4(1), $\beta = \alpha \bigoplus \delta$ is the characteristic sentence of M. Moreover, by Theorem 5, β is computable from α and δ.

By Theorem 1(3), we can compute $D \subseteq Char_n^k$ such that φ is equivalent to the disjunction of the sentences from D. Finally, $\beta \in D$ if and only if $M \models \varphi$.

This completes the description of our algorithm for the monadic theory of M. □

We conclude this section with a discussion on a class of effectively profinitely ultimately periodic structures which was introduced by Carton and Thomas. In [3,4] numerous examples of effectively profinitely ultimately periodic structures were provided and it was shown that if $M = \langle Nat, <, \mathbf{P}_1, \ldots, \mathbf{P}_n \rangle$ is effectively profinitely ultimately periodic, then the monadic theory of M is decidable. The algebraic condition of Theorem 19 is a special case of the effectively profinitely ultimately periodic condition. Hence, we derive that the effectively profinitely ultimately periodic condition is a necessary condition for the decidability of the monadic theory of M.

Let as recall that an ω-sequence a_i is said to be ultimately periodic with lag l and period d if $a_i = a_{i+d}$ for $i > l$.

For $M = \langle Nat, <, \mathbf{P}_1, \ldots, \mathbf{P}_n \rangle$ and an infinite set $\mathbf{S} = \{s_1 < s_2 < \cdots < s_i < \cdots\} \subseteq Nat$ define an ω-sequence $w_i = w_i^{\mathbf{S}}$ of finite words over the alphabet $\{0,1\}^n$ which correspond to the substructures $M_{[s_i, s_{i+1})}$ of M (see Section 2.1).

A set $\mathbf{S} \subseteq Nat$ is ultimately periodic for M if it is infinite and for every finite semigroup G and for every morphisms h from the semigroup of finite non-empty words over $\{0,1\}^n$ into G the sequence $\{h(w_i)\}_{i \in Nat}$ is ultimately periodic. A set $\mathbf{S}$ is effectively ultimately periodic for M if $\mathbf{S}$ is recursive and ultimately periodic for M and there is a recursive function which for every finite semigroup G and morphism h computes a lag and a period of the ultimately periodic sequence $\{h(w_i)\}_{i \in Nat}$. Note that if $\mathbf{S}$ is an effectively ultimately constant set for M, then it is effectively ultimately periodic for M.

A labelled chain $M = \langle Nat, <, \mathbf{P}_1, \ldots, \mathbf{P}_n \rangle$ is effectively profinitely ultimately periodic if M is recursive and there is a set $\mathbf{S}$ which is effectively ultimately periodic for M.

Theorem 22. *The monadic theory of $M = \langle Nat, <, \mathbf{P}_1, \ldots, \mathbf{P}_n \rangle$ is decidable if and only if M is effectively profinite ultimately periodic.*

Proof. The "only if" part: assume that the monadic theory of M is decidable. By Theorem 19, M is recursive and there is effectively ultimately constant set for M. Every effectively ultimately constant set is effectively ultimately periodic (with period one). Hence, M is effectively profinite ultimately periodic.

The "if" part: this direction was proved in [3]. A proof similar to the proofs of Theorems 10 and 21 can also be easily obtained. □

Carton and Thomas asked in [3], whether the monadic theory of $\langle Nat, <, \mathbf{P}_1, \mathbf{P}_2 \rangle$ is decidable if it is known that $\langle Nat, <, \mathbf{P}_1 \rangle$ and $\langle Nat, <, \mathbf{P}_2 \rangle$ are effectively profinitely ultimately periodic.

The negative answer to this question can be obtained as follows. Semenov [13] constructed $\mathbf{P}_1$ and $\mathbf{P}_2$ such that the monadic theories of $\langle Nat, <, \mathbf{P}_1 \rangle$ and $\langle Nat, <, \mathbf{P}_2 \rangle$ are decidable, yet the monadic theory of $\langle Nat, <, \mathbf{P}_1, \mathbf{P}_2 \rangle$ is undecidable. Hence, by Theorem 22 it follows that $\langle Nat, <, \mathbf{P}_1 \rangle$ and $\langle Nat, <, \mathbf{P}_2 \rangle$ are effectively profinitely ultimately periodic and $\langle Nat, <, \mathbf{P}_1, \mathbf{P}_2 \rangle$ has undecidable monadic theory.

6. Related works and further results

We proved that the following conditions are equivalent:

(1) $M = \langle Nat, <, \mathbf{P}_1, \ldots, \mathbf{P}_n \rangle$ is recursive and effectively ultimately constant.
(2) $M = \langle Nat, <, \mathbf{P}_1, \ldots, \mathbf{P}_n \rangle$ is recursive and there is a recursive homogeneous set for M.
(3) The monadic theory of $M = \langle Nat, <, \mathbf{P}_1, \ldots, \mathbf{P}_n \rangle$ is decidable.

The equivalence (1) ⇔ (2) is easy. The implication (2) ⇒ (3) is easy as well. The difficult part of our proof was the implication (3) ⇒ (2).

In [13], Semenov provided another necessary and sufficient condition for the decidability of the monadic theory of M. The Semenov theorem states that (3) is equivalent to

(4) There is an algorithm that for every regular expression E finds whether every suffix of the ω-word $u = a_1a_2\ldots$ which corresponds to M contains a word from the language L definable by E or returns n such that no word from L occurs in the nth suffix $a_na_{n+1}\ldots$ of u.

The implication (3) ⇒ (4) is an easy part of the Semenov theorem. The difficult part is the implication (4) ⇒ (3).

In [11] we extended the results of this paper and obtained alternative proofs. We provided a Ramsey type condition which is equivalent to the four conditions above and generalized results of this paper to a class of logics for which the composition theorem holds.

For two k-characteristic sentences α, γ of n-labelled chains, consider the following condition:

Hom$_{\alpha,\gamma}$:
There is a k-homogeneous set $H = \{h_0 < h_1 < \ldots\}$ such that α is the characteristic sentence of $M[0, h_0)$ and γ is the characteristic sentence of $M[h_0, h_1)$.

We proved in [11] that each of the conditions (1)–(4) above are equivalent to the following condition:

RecRamsey(M):
There is a recursive function assigning to each k a pair of k-characteristic sentences α, γ such that $\text{Hom}_{\alpha,\gamma}$ holds in M.

Now we illustrate the extension of our results to the first-order monadic logic of order. This logic is defined like *MLO* , but the quantifications over monadic variables are not allowed.

We use the notation $\equiv_k^{FO}$ to say that two labelled chains cannot be distinguished by an first-order *MLO* sentence of quantifier depth k. More precisely, let M and M' be two n-labelled chains. We write $M \equiv_k^{FO} M'$ if and only if for every first-order *MLO* sentence φ with $\text{qd}(\varphi) \leqslant k$ we have $M \models \varphi$ iff $M' \models \varphi$.

All the results of Section 3 hold when $\equiv_k$ is replaced by $\equiv_k^{FO}$. In particular, for every k and n there are first-order characteristic sentences for the $\equiv_k^{FO}$-equivalence classes over n-labelled chains. The notions of first-order k-homogeneous set for M and of first-order homogeneous set for M are defined like in Definitions 6 and 7.

It was proved in [11] that the following conditions are equivalent:

First-order Decidability. The first-order monadic theory of $M = \langle Nat, <, \mathbf{P}_1, \ldots, \mathbf{P}_n \rangle$ is decidable.

First-order RecHom. $M = \langle Nat, <, \mathbf{P}_1, \ldots, \mathbf{P}_n \rangle$ is recursive and there is a recursive first-order homogeneous set for M.

First-order RecRamsey(M). There is a recursive function assigning to each k a pair of first-order k-characteristic sentences α, γ such that $\text{Hom}_{\alpha,\gamma}$ holds in M.

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Appendix A

The first subsection of appendix discusses the relationship between selectors, the uniformization property and the Church uniformization problem. The second subsection is independent from the first and contains a proof of Theorem 17.

A.1. Selector, uniformization and the Church uniformization problem

In Definition 16, it was defined when $\alpha(X)$ is a selector for $\beta(X)$ over a class of structures. For a sequence of variable $\bar{X}$ one can define in a similar way when $\alpha(\bar{X})$ is a selector for $\beta(\bar{X})$ over a class of structures.

Lemma 23. *Let C be a class of structures. Assume that every formula $\beta(X)$ has a selector over C. Then every formula $\gamma(X_1, \ldots, X_n)$ has a selector over C.*

Proof. By induction on n. The base case is just the assumption of the lemma.

Inductive step: $n \mapsto n + 1$. Assume that every formula with n free variables has a selector over C.

Let $\gamma(X_1, \ldots, X_{n+1})$ be a formula. Let $\gamma_{n+1}(X_1, \ldots, X_n)$ be defined as $\exists X_{n+1}\gamma$. By the inductive assumption γ_{n+1} has a selector $\alpha_{n+1}(X_1, \ldots, X_n)$ over C. Let $\theta(X_{n+1})$ be defined as $\exists X_1 \exists X_2 \ldots \exists X_n(\alpha_{n+1} \wedge \gamma)$. By our assumption $\theta(X_{n+1})$ has a selector $\delta(X_{n+1})$ over C.

It is easy to verify that $\alpha_{n+1}(X_1, \ldots, X_n) \wedge \delta(X_{n+1})$ is a selector for γ over C. $\square$

Note that in the above lemma it was not assumed that C is a class of labelled chains. Observe also that the reduction used in the proof is effective. Therefore, Theorem 17 and Lemma 23 imply.

Theorem 24 (*computability of selector*). *There is an algorithm that for every formula $\beta(X_1, \ldots, X_n)$ constructs a formula $\alpha(X_1, \ldots, X_n)$ such that α is a selector for β over the class $C = \{\langle Nat, <, \mathbf{P}_1, \ldots, \mathbf{P}_n \rangle : \mathbf{P}_i \subseteq Nat\}$ of labelled chains.*

Let us recall now the uniformization problem also known as the Rabin uniformization problem.

Definition 25 (*Uniformization*). $\alpha(\bar{X}, \bar{Y})$ uniformizes $\beta(\bar{X}, \bar{Y})$ over a class C of n-chains iff the following conditions hold:

(1) $C \models \forall \bar{Y} \exists^{\leqslant 1} \bar{X}\ \alpha(\bar{X}, \bar{Y})$.
(2) $C \models \forall \bar{Y} \forall \bar{X} \big(\alpha(\bar{X}, \bar{Y}) \rightarrow \beta(\bar{X}, \bar{Y})\big)$.
(3) $C \models \forall \bar{Y} \Big(\big(\exists \bar{Z} \beta(\bar{Z}, \bar{Y})\big) \rightarrow \big(\exists \bar{X} \alpha(\bar{X}, \bar{Y})\big)\Big)$.

Here and below $\bar{X}$, $\bar{Y}$ and $\bar{Z}$ stand for sequences of monadic variables.

A class C of structures has the uniformization property if for every formula $\beta(\bar{X}, \bar{Y})$ there exists a formula $\alpha(\bar{X}, \bar{Y})$ which uniformizes $\beta(\bar{X}, \bar{Y})$ over C.

The following observation is immediate

Observation 26. Let C be a class of structures and let $E = \{A : A$ is an expansion of a structure in C by monadic predicates$\}$. Then

(1) C has the uniformization property iff E has the uniformization property.
(2) C has the uniformization property iff for every $\beta(X)$ there is a selector over E (see Definition 16).

Lifsches and Shelah (cf. Corollary 6.4 [9]) proved the following theorem:

Theorem 27. *An ordinal μ has the uniformization property iff $\mu < \omega^{\omega}$.*

Moreover, from the proof of [9] an algorithm which for every ordinal $\mu < \omega^\omega$ and every formula β constructs a formula that uniformizes β over μ can be extracted.

These results imply Theorem 17. The proof of the "if" direction of Theorem 27 in [9] consists of three parts: (1) The proof that the class of finite ordinals has the uniformization property; (2) The proof that ω has the uniformization property; (3) The proof that if ordinals μ and ν have the uniformization property, then the ordinals $\mu + \nu$ and $\mu \times \nu$ have the uniformization property.

The Lifsches and Shelah proof that ω has the uniformization property (the base of the induction) consists of just five characters—"By [2]". However, in [2] the Church uniformization problem is considered and it is proved to be decidable. Unfortunately, we could not show that either decidability of the Church uniformization problem or the proof given in [2] imply that ω has the uniformization property. In the next subsection we provide a direct proof of Theorem 17 (which implies the assertion that ω has the uniformization property).

Below, we recall the Church uniformization problem in order to emphasize its difference from the uniformization property of ω.

A function F from the set $P(\omega)$ of all monadic predicates over *Nat* into $P(\omega)$ is called *retrospective* if for every $n \in Nat$ whenever X_1 and X_2 coincide on the interval $[0, n]$ then their images $F(X_1)$ and $F(X_2)$ coincide on the interval $[0, n]$. A function is retrospective iff it can be computed by an (may be infinite state) input–output deterministic automaton.

The Church Uniformization Problem: Given a formula $\beta(X, Y)$. Check whether there is a retrospective function F which uniformizes $\beta(X, Y)$, i.e., $\forall X(\beta(X, F(X)))$.

It was proved in [2] that the Church uniformization problem is decidable. Moreover, the following theorem holds:

Theorem 28. *There is an algorithm which for every $\beta(X, Y)$ checks whether there is a retrospective function which uniformizes $\beta(X, Y)$ and if so, outputs a formula $\alpha(X, Y)$ which defines a retrospective function F which uniformizes $\beta(X, Y)$; moreover, F is computable by a finite state deterministic input–output automaton $\mathcal{A}$ and there is an algorithm that constructs $\mathcal{A}$ from β.*

A.2. Proof of Theorem 17

First recall the following theorem of Lifsches and Shelah (cf. Fact 6.2 in [9]):

Theorem 29 (*computability of selector over the class of finite chains*)**.** *There is an algorithm that for every formula $\beta(X)$ constructs a formula $\alpha(X)$ such that $\alpha(X)$ is a selector for $\beta(X)$ over the class of finite chains.*

Proof. Below $WO(X, Y)$ denotes the formula

$$\exists t\big(\neg X(t) \wedge Y(t) \wedge \forall t'\big(t' < t \rightarrow (X(t') \leftrightarrow Y(t'))\big)\big).$$

For every finite chain M the formula $WO(X, Y)$ defines a well order on the set of all monadic predicates over M. Hence, $\beta(X) \wedge \forall Y\big(\beta(Y) \rightarrow (WO(X, Y) \vee X = Y)\big)$ uniformizes $\beta(X)$ over the class of finite labelled chains. □

Note that the relation definable by $WO(X, Y)$ on the set of all monadic predicates over *Nat* is not a well-order. It is unlikely that there is any definable well order on this (uncountable) set.

We will also use the following simple Lemma.

Lemma 30. *Assume that*

(1) $\alpha_i(Z)$ *is a selector for* $A_i(Z)$ *(for* $i = 1, \dots, m$*) over a class* C *and*
(2) $C \models B(Z) \leftrightarrow \left(\vee_{i=1}^{m} A_i(Z)\right)$.

Then

$$\bigvee_{i=1}^{m} \left(\alpha_i(Z) \wedge \left(\bigwedge_{j=1}^{i-1} \forall Z \neg A_j(Z) \right) \right)$$

is a selector for $B(Z)$ *over* C.

In order to prove Theorem 17, we are going to present the selection algorithm. (Below we often denote by P both a predicate name and its interpretation.)

A.2.0.1. Selection algorithm

Instance: A formula $\beta(Z, P_1, \dots, P_n)$.
Task: Find a selector $\alpha(Z, P_1, \dots, P_n)$ for $\beta(Z, P_1, \dots, P_n)$ over the class of chains $C_n = \{\langle Nat, <, P_1, \dots, P_n\rangle : P_i \subseteq Nat\}$.

Step 1. Let k be the quantifier depth of β and let

$$Char^k_{n+1} = \{\gamma_1(Z, P_1, \dots, P_n), \dots, \gamma_l(Z, P_1, \dots, P_n)\}$$

be the set of characteristic sentences for $\equiv_k$ over the class of $n+1$-labelled chains.
Step 2. For $\gamma(Z, P_1, \dots, P_n), \gamma'(Z, P_1, \dots, P_n) \in Char^k_{n+1}$ let $A_{\gamma,\gamma'}(X, Z, P_1, \dots, P_n)$ be a formula such that for every $n+1$-labelled chain $M = \langle Nat, <, Z, P_1, \dots, P_n\rangle$

$$M \models A_{\gamma,\gamma'}(\mathbf{S}, Z, P_1, \dots, P_n) \text{ iff the following conditions hold}$$

(1) $\mathbf{S}$ is a cofinal subset of $Nat \setminus \{0\}$ which is γ'-colored for M and
(2) γ is the characteristic sentence of the subchain $M_{[0,x_1)}$, where x_1 is the minimal element of $\mathbf{S}$.

Compute:

- a selector $\alpha_{\gamma,\gamma'}(X, P_1, \dots, P_n)$ for $\exists Z A_{\gamma,\gamma'}(X, Z, P_1, \dots, P_n)$ over the class $C_n = \{\langle Nat, <, P_1, \dots, P_n\rangle : P_i \subseteq Nat\}$ of n-labelled chains.
- a selector $\beta_{\gamma,\gamma'}(Z, P_1, \dots, P_n)$ for $\exists X A_{\gamma,\gamma'}(X, Z, P_1, \dots, P_n)$ over the class $C_n = \{\langle Nat, <, P_1, \dots, P_n\rangle : P_i \subseteq Nat\}$ of n-labelled chain.

Step 3. Compute $D_1 \subseteq Char^k_{n+1}$ such that

$$\beta(Z,P_1,\ldots,P_n) \leftrightarrow \bigvee_{j\in D_1} \gamma_j(Z,P_1,\ldots,P_n) \tag{A.1}$$

Compute $D_2 \subseteq Char^k_{n+1} \times Char^k_{n+1}$ such that

$$\beta(Z,P_1,\ldots,P_n) \leftrightarrow \bigvee_{\langle\gamma,\gamma'\rangle\in D_2} \exists X A_{\gamma,\gamma'}(X,Z,P_1,\ldots,P_n) \tag{A.2}$$

Step 4. Compute a selector $\alpha(Z,P_1,\ldots,P_n)$ for $\bigvee_{\langle\gamma,\gamma'\rangle\in D_2} \exists X A_{\gamma,\gamma'}(X,Z,P_1,\ldots,P_n)$ over the class $C_n = \{\langle Nat,<,P_1,\ldots,P_n\rangle : P_i \subseteq Nat\}$ of n-labelled chains.

The correctness of the algorithm is immediate by Step 4 and Eq. (A.2).
Below we will show how each step can be algorithmically implemented.

A.2.0.2. Implementation of Step 1

This step of the selection algorithm can be implemented by Theorem 1.

A.2.0.3. Implementation of Step 2

Throughout the proof of this step we sometimes do not display $Z,P_1,\ldots,P_n$ and write "$A_{\gamma,\gamma'}(X)$" or "$A_{\gamma,\gamma'}(X,Z)$" for "$A_{\gamma,\gamma'}(X,Z,P_1,\ldots,P_n)$"; we also will treat a free variable Z as a predicate name.
First we are going to show how to write the formulas $A_{\gamma,\gamma'}(X)$.
For every sentence γ we denote by $\gamma^<(t)$ the formula obtained by replacing all first-order quantifiers "$\forall v$" or "$\exists v$" in γ by their relativized versions "$(\forall v)^{v<t}$" and "$(\exists v)^{v<t}$" which are shorthand for "$\forall v\ v<t \rightarrow \ldots$" and "$\exists v\ v<t \wedge \ldots$", where t is a fresh variable. For every labelled chain M

$$M \models \gamma^<[a] \text{ iff } M_{<a} \models \gamma, \tag{A.3}$$

where $M_{<a}$ is the substructure of M over the elements less than a.
Let $first(t,X)$ be a formula that says that t is the minimal element of X.
For characteristic sentences $\gamma,\gamma' \in Char^k_{n+1}$ let $A_{\gamma,\gamma'}(X)$ be defined as the conjunction of $col_{\gamma'}(X)$ (see Lemma 14), $\forall t \exists t'(t<t' \wedge t' \in X)$ and

$$\exists t\Big((\exists t' t'<t) \wedge \big(first(t,X) \wedge \gamma^<(t)\big)\Big).$$

For every set $\mathbf{S} = \{s_1 < s_2 < \ldots\} \subseteq Nat$:

$$M \models A_{\gamma,\gamma'}[\mathbf{S}] \text{ iff } (\mathbf{S} \text{ is an infinite } \gamma'\text{-colored set and } M_{<s_1} \models \gamma) \tag{A.4}$$

A selector $\alpha_{\gamma,\gamma'}(X,P_1,\ldots,P_n)$ for $\exists ZA_{\gamma,\gamma'}(X,Z,P_1,\ldots,P_n)$ over the class $C_n=\{\langle Nat,<,P_1,\ldots,P_n\rangle : P_i\subseteq Nat\}$ can be defined as the conjunction of $\exists ZA_{\gamma,\gamma'}(X,Z)$ and of the formula which says:

for every Y which satisfies $\exists ZA_{\gamma,\gamma'}(Y,Z)$ and for every t, whenever Y and X coincide on the interval $[0,t)$ and Y contains t then X contains t.

Let us show that $\alpha_{\gamma,\gamma'}(X)$ is a selector for $\exists ZA_{\gamma,\gamma'}(X,Z)$, i.e., that all the requirements of the Definition 16 are fulfilled over the class C_n.

The second conjunct ensures that there is at most one X that satisfies $\alpha_{\gamma,\gamma'}(X)$.

Hence, the first requirement of Definition 16 holds. The second requirement holds because $\exists ZA_{\gamma,\gamma'}(X,Z)$ is the first conjunct in the definition of $\alpha_{\gamma,\gamma'}(X)$.

Finally, assume that $\langle Nat,<,P_1,\ldots,P_n\rangle\models\exists X\exists ZA_{\gamma,\gamma'}(X,Z)$. In order to show that the last requirement of Definition 16 holds, we will show that there is a set $\mathbf{R}=\{r_1<r_2<\cdots\}$ which satisfies $\alpha_{\gamma,\gamma'}(X)$ in $\langle Nat,<,P_1,\ldots,P_n\rangle$. Define

(1) $\mathbf{R}_1=\{m : \langle Nat,<,\mathbf{Z}_0,P_1,\ldots,P_n\rangle_{<m}\models\gamma(Z,P_1,\ldots,P_n)$ for some $\mathbf{Z}_0$ and m is the minimal element of an infinite subset of $Nat\setminus\{0\}$, which is γ'-colored for $\langle Nat,<,\mathbf{Z}_0,P_1,\ldots,P_n\rangle\}$. Let r_1 be the minimal element of $\mathbf{R}_1$ and let $\mathbf{Z}_0$ be any set used for the inclusion of r_1 in $\mathbf{R}_1$.
(2) For $i>0$ let $\mathbf{R}_{i+1}=\big\{m$: for some $\mathbf{Z}_i\subseteq Nat$, m is the $i+1$th element of an infinite set $\mathbf{S}\subseteq Nat\setminus\{0\}$, which is γ'-colored for $\langle Nat,<,\mathbf{Z}_i,P_1,\ldots,P_n\rangle$ and the first i elements of $\mathbf{S}$ are $\{r_1,\ldots,r_i\}\big\}$. Let r_{i+1} be the minimal element of $\mathbf{R}_{i+1}$ and let $\mathbf{Z}_i$ be any set used for the inclusion of r_{i+1} in $\mathbf{R}_{i+1}$.

Let us verify that $\langle Nat,<,P_1,\ldots,P_n\rangle\models\alpha_{\gamma,\gamma'}[\mathbf{R}]$. The only non-immediate part is to show that $\langle Nat,<,P_1,\ldots,P_n\rangle\models\exists ZA_{\gamma,\gamma'}(\mathbf{R},Z)$.

Define M_0 as $\langle Nat,<,\mathbf{Z}_0,P_1,\ldots,P_n\rangle_{[0,r_1)}$ and for $i>0$ define M_i as $\langle Nat,<,\mathbf{Z}_i,P_1,\ldots,P_n\rangle_{[r_i,r_{i+1})}$. Note that γ is the characteristic sentence of M_0 and γ' is the characteristic sentence of M_i for $i>0$.

Observe also that $\gamma'=\gamma'\bigoplus_{n+1,k}\gamma'$. Indeed let a_1,a_2,a_2 be the first three elements of the γ'-colored set for $M'=\langle Nat,<,\mathbf{Z}_0,P_1,\ldots,P_n\rangle$. The substructures of M' over the intervals $[a_1,a_2)$, $[a_2,a_3)$ and $[a_1,a_3)$ have the same characteristic sentence γ'. But the last substructure is the sum of the first two substructures. Therefore, $\gamma'=\gamma'\bigoplus_{n+1,k}\gamma'$.

Let $M=\langle Nat,<,\mathbf{Z},P_1,\ldots,P_n\rangle$ be $\sum_{i\in\omega}M_i$—the lexicographical sum of M_i over ω. Note that $\sum_{j\in m}M_{i+j}$ is the substructure of M over $[r_i,r_{i+m})$ for $m>0$ and all i. Its characteristic sentence is $\gamma'\bigoplus_{n+1,k}\gamma'\cdots\bigoplus_{n+1,k}\gamma'$ (m summands); hence, by the above observation its characteristic sentence is equal to γ' and therefore, $\mathbf{R}$ is γ'-colored for M. Together with the observation that γ is the characteristic sentence of $M_0=M_{[0,r_1)}$, this implies that $M\models A_{\gamma,\gamma'}(\mathbf{R},Z)$.

Therefore, $\alpha_{\gamma,\gamma'}(X)$ is a selector for $\exists ZA_{\gamma,\gamma'}(X,Z)$.

Note that from the above definition of $\mathbf{R}$ it follows that

if $\mathbf{R}$ is defined by $\alpha_{\gamma,\gamma'}(X)$ in $\langle Nat,<,P_1,\ldots,P_n\rangle$ then

$$\langle Nat,<,P_1,\ldots,P_n\rangle_{[0,r_1)}\models\exists Z\gamma(Z,P_1,\ldots,P_n)$$

and

$$\langle Nat,<,P_1,\ldots,P_n\rangle_{[r_i,r_{i+i})}\models\exists Z\gamma'(Z,P_1,\ldots,P_n)\text{ for } i>0.$$

Let $\delta_\gamma(Z, P_1, \dots, P_n)$ and $\delta_{\gamma'}(Z, P_1, \dots, P_n)$ be selectors for $\gamma(Z, P_1, \dots, P_n)$ and $\gamma'(Z, P_1, \dots, P_n)$ over the class of finite labelled chains (these selectors exist by Theorem 29).

A selector $\beta_{\gamma,\gamma'}(Z, P_1, \dots, P_n)$ for $\exists X A_{\gamma,\gamma'}(X, Z, P_1, \dots, P_n)$ can be defined by the following instructions:

(1) Select $\mathbf{R} = \{r_1 < r_2 < \cdots\}$ by $\alpha_{\gamma,\gamma'}(X, P_1, \dots, P_n)$.
(2) Let $\mathbf{Z}$ be selected by $\delta_\gamma(Z, P_1, \dots, P_n)$ on the subchain over the interval $[0, r_1)$ and by $\delta_{\gamma'}(Z, P_1, \dots, P_n)$ on the subchain over the intervals $[r_i, r_{i+1})$ for $i > 0$.

This definition can be easily formalized by a formula in the monadic logic.

It is clear that if $\langle Nat, <, P_1, \dots, P_n\rangle \models \exists Z \exists X\, A_{\gamma,\gamma'}(X, Z, P_1, \dots, P_n)$, then there is a unique $\mathbf{Z}$ which is chosen by the above instructions.

Let us demonstrate that the set $\mathbf{Z}$, selected as above, satisfies $\exists X\, A_{\gamma,\gamma'}(X, Z, P_1, \dots, P_n)$. Indeed, let $\mathbf{R}$ and $\mathbf{Z}$ be sets which are selected in $\langle Nat, <, P_1, \dots, P_n\rangle$ by the above instructions. Let $M = \langle Nat, <, \mathbf{Z}, P_1, \dots, P_n\rangle$. Then γ' is the characteristic sentence of $M_{[r_i, r_{i+1})}$ for $i > 0$. Since $\gamma' = \gamma' \bigoplus_{n+1,k} \gamma'$, we obtain that γ' is the characteristic sentence of $M_{[r_i, r_j)}$ for all $j > i > 0$. Therefore, $\mathbf{R}$ is γ'-colored for M. By the choice of $\mathbf{Z}$ on the interval $[0, r_1)$, γ is the characteristic sentence of $M_{[0,r_1)}$; therefore, $\langle Nat, <, \mathbf{R}, \mathbf{Z}, P_1, \dots, P_n\rangle \models A_{\gamma,\gamma'}(X, Z, P_1, \dots, P_n)$. Hence, $\mathbf{Z}$ satisfies $\exists X A_{\gamma,\gamma'}(X, Z, P_1, \dots, P_n)$ in $\langle Nat, <, P_1, \dots, P_n\rangle$.

This completes the proof that $\beta_{\gamma,\gamma'}(Z)$ is a selector for $\exists X A_{\gamma,\gamma'}(X, Z)$.

A.2.0.4. Implementation of Step 3

D_1 can be easily computed by Theorem 1.

Let $D_2 = \{\langle \gamma, \gamma'\rangle : \left(\gamma \bigoplus_{n+1,k} \left(\bigotimes^{\omega}_{n+1,k} \gamma'\right)\right) \in D_1\}$. The set D_2 can be easily computed by Theorem 5.

We have to show that the following formula holds

$$\forall Z \left(\beta(Z, P_1, \dots, P_n) \leftrightarrow \bigvee_{\langle \gamma, \gamma'\rangle \in D_2} \exists X A_{\gamma,\gamma'}(X, Z, P_1, \dots, P_n) \right). \tag{A.5}$$

Let us show first the implication from left to right. Assume that $M = \langle Nat, <, \mathbf{Z}, P_1, \dots, P_n\rangle \models \beta(Z, P_1, \dots, P_n)$. By Lemma 15, there is an infinite set $\mathbf{R} = \{r_1 < r_2 < \cdots\} \subseteq Nat \setminus \{0\}$ which is homogeneous for M. Let γ be the characteristic sentence of $M_{[0,r_1)}$ and γ' be the color of $\mathbf{R}$ in M. Then

$$M \models \exists X A_{\gamma,\gamma'}(X, Z, P_1, \dots, P_n). \tag{A.6}$$

Note that γ' is the characteristic sentence of $M_{[r_i, r_{i+1})}$ for $i > 0$. Hence, $\bigotimes^{\omega}_{n+1,k} \gamma'$ is the characteristic sentence of $M_{[r_1,\infty)}$ and $\gamma \bigoplus_{n+1,k} \left(\bigotimes^{\omega}_{n+1,k} \gamma'\right)$ is the characteristic sentence of M. Since $M \models \beta(Z, P_1, \dots, P_n)$, it follows that $\gamma \bigoplus_{n+1,k} \left(\bigotimes^{\omega}_{n+1,k} \gamma'\right) \to \beta$ and therefore $\langle \gamma, \gamma'\rangle \in D_2$. Together with Eq. (A.6) this shows that the implication from left to right in Eq. (A.6) holds.

The implication from right to left easy follows from the definitions of D_2 and $A_{\gamma,\gamma'}(X, Z, P_1, \dots, P_n)$.

A.2.0.5. Implementation of Step 4

In step 2 we constructed selectors $\beta_{\gamma,\gamma'}(Z,P_1,\ldots,P_n)$ for $\exists X A_{\gamma,\gamma'}(X,Z,P_1,\ldots,P_n)$. In step 3 we found D_2 such that Eq. (A.2) holds. Therefore, by Lemma 30 we can construct a selector for $\beta(Z,P_1,\ldots,P_n)$.

This completes our proof of Theorem 17.

References

[1] J.R. Büchi, On a decision method in restricted second order arithmetic, in: E. Nagel (Ed.), Proceedings the of International Congress on Logic, Methodology and Philosophy of Science, Stanford University Press, Stanford, CA, 1960, pp. 1–11.
[2] J.R. Büchi, L.H. Landweber, Solving sequential conditions by finite-state strategies, Transactions of the American Mathematical Society 138 (1969) 295–311.
[3] O. Carton, W.Thomas, The monadic theory of morphic infinite words and generalizations, in: MFCS, 2000, pp. 275–284.
[4] O. Carton, W. Thomas, The monadic theory of morphic infinite words and generalizations, Information and Computation 176 (1) (2002) 51–65.
[5] Y. Choueka. Finite Automata on Infinite Structure, Ph.D Thesis, Hebrew University, 1970.
[6] C. Elgot, M.O. Rabin, Decidability and undecidability of extensions of second (first) Order theory of (generalized) Successor, Journal of Symbolic Logic 31 (2) (1966) 169–181.
[7] S. Feferman, R.L. Vaught, The first-order properties of products of algebraic systems, Fundamenta Mathematica 47 (1959) 57–103.
[8] Y. Gurevich, Monadic second order theories, in: J. Barwise, S. Feferman (Eds.), Model Theoretical Logics, Springer Verlag, Berlin, 1986, pp. 479–506.
[9] S. Lifsches, S. Shelah, Uniformization and Skolem functions in the class of trees, Journal of Symbolic Logic 63 (1998) 103–127.
[10] A. Rabinovich, On decidability of monadic logic of order over the naturals extended by monadic predicates, 2005 Summer Meeting of the Association for Symbolic Logic, Logic Colloquium 05, The Bulletin of Symbolic Logic 12 (2006) 343–344.
[11] A. Rabinovich, W. Thomas, Decidable Theories of the Ordering of Natural Numbers with Unary Predicates, CSL 2006, 2006, pp. 562–574.
[12] R.M. Robinson, Restricted set-theoretical definitions in arithmetic, in: Proceedings of the American Mathematical Society, vol. 9, No. 2, 1958, pp. 238–242.
[13] A. Semenov. Logical theories of one-place functions on the set of natural numbers. Mathematics of the USSR—Izvestia, vol. 22, pp. 587–618, 1984.
[14] S. Shelah, The monadic theory of order, Annals of Mathematics 102 (1975) 349–419.
[15] D. Siefkies, Decidable extensions of monadic second-order successor arithmetic, in: J. Doerr, G. Hotz (Eds.), Automatentheorie und Formale Sprachen, 1970, pp. 441–472.
[16] W. Thomas, Das Entscheidungsproblem für einige Erweiterungen der Nachfalger-Arithmetic. Ph. D. Thesis Albert-Ludwigs Universität, 1975.
[17] W. Thomas, Ehrenfeucht Games, the composition method, and the monadic theory of ordinal words, in: Structures in Logic and Computer Science: A Selection of Essays in Honor of A. Ehrenfeucht, Lecture Notes in Computer Science, 1261, Springer-Verlag, Berlin, 1997, pp. 118–143.
[18] B.A. Trakhtenbrot, Finite automata and the logic of one-place predicates, in: AMS Transl. 59, 1966, pp. 23–55 (Russian version 1961).