

ON DECOMPOSITIONS OF A COMMUTATIVE SEMIGROUP

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If there exists a homomorphism of a semigroup S onto a semigroup S^* having special type, all elements of S are decomposed into the class sum of mutually disjoint subsets. Then we say that the decomposition of S to S^* is obtained. In particular the decomposition to a semilattice is of importance, i.e., $S = \bigcup_{\alpha \in P} S_{\alpha}$ where $S_{\alpha} \cap S_{\beta} = \emptyset$ ($\alpha \neq \beta$), every S_{α} is a restrictive subsemigroup, and for any α, β , there is a unique γ such that $S_{\alpha} S_{\beta} \subset S_{\gamma}$ as well as $S_{\beta} S_{\alpha} \subset S_{\gamma}$. In §1 we argue that there is greatest decomposition of a semigroup to a semilattice; particularly in §2 we show a decomposition of a commutative semigroup by method different from Mr. Numakura's, and in §3 our decomposition is proved to be greatest.

§1 Greatest decomposition

In this paragraph S is assumed to be a general semigroup. A decomposition of S to an idempotent semigroup gives an equivalence relation; and an equivalence relation \sim in S raises a decomposition of S to an idempotent semigroup if and only if

- (1) $a \sim b, c \sim d$ imply $ac \sim bd$,
- (2) if $a \sim b$ then $a \sim ab$.

Lemma 1. (1) and (2) are equivalent to (1') and (2'),

- (1') $a \sim b$ implies $ac \sim bc$ and $ca \sim cb$ for every c ,
- (2') $a \sim a^2$ for every a .

Proof. (1') \rightarrow (1): For, from $a \sim b$, follows $ac \sim bc$; and from $c \sim d$, $bc \sim bd$. By transitivity, $ac \sim bd$. (1) \rightarrow (1'): evident. (1') & (2') \rightarrow (2): from $a \sim b$, it follows that $a \sim a^2 \sim ab$. (2) \rightarrow (2'): evident.

We denote by \mathcal{D} the set of all decompositions φ of S to a semilattice, and by \mathcal{L} the congruence relation which gives φ . The relation

\mathcal{L} and \mathcal{L}' are equal if they give the same decomposition. Obviously \mathcal{D} is not empty, because it includes at least a trivial decomposition, a partition of all elements of S into one class.

Now we introduce the ordering into \mathcal{D} : i.e. $\varphi \geq \psi$ means that $x \mathcal{L}' y$ if $x \mathcal{L} y$. The ordering is clearly a partial ordering. Then we have the below lemmas.

Lemma 2. \mathcal{D} forms a complete semilattice.

Proof. Since \mathcal{D} is a partly ordered set, we show that any subset \mathcal{D}_1 of \mathcal{D} has a least upper bound. We define a relation ξ as follows. $x \xi y$ means that $x \mathcal{L}' y$ for every $\varphi \in \mathcal{D}_1$. It is not hard to verify that ξ is an equivalence relation and satisfies the condition (1') and (2') (in Lemma 1). Clearly $\xi \geq \varphi$ for all $\varphi \in \mathcal{D}_1$. Take up any $\xi' \geq \varphi$ for all $\varphi \in \mathcal{D}_1$, then from $x \mathcal{L}' y$ follows $x \mathcal{L}'' y$ for all $\varphi \in \mathcal{D}_1$, i.e., $x \mathcal{L} y$; hence $\xi' \geq \xi$, and so ξ is the least upper bound of \mathcal{D}_1 . Consequently

Theorem 1. There is a greatest element of \mathcal{D} . In other words, there exists the greatest decomposition of a semigroup to a semilattice.

In another article we shall relate what is an equivalence relation giving the greatest decomposition of a general semigroup.

§2 A decomposition of a commutative semigroup

Let S be a commutative semigroup. We define an ordering $a \geq b$ between elements a and b of S to mean that a certain element $x \in S$ and a positive integer m are found such that

$$a^m = bx$$

The definition is obviously equivalent to the following:

$a^m = b^n y$ for some positive integers m, n , and an element $y \in S$.

Lemma 3. This ordering is a quasi-ordering.

Proof. (1) $a \geq a$ for all a , because $a^m = a a^{m-1}$ for $m > 1$. (2) $a \geq b$ and $b \geq c$ imply $a \geq c$. For, from $a^m = b^k x$ and $b^n = c y$, we get $a^{mn} = c z$ where $z = y x^n$.

Lemma 4. $a \geq b$ implies $ac \geq bc$ for every $c \in S$.

Proof. By the assumption $a^m = b^k x$ for some m and x . Multiply c^m by both sides of the equality, we get $(ac)^m = (bc)(c^{m-1}x)$ where m may be supposed to be greater than 1. This shows $ac \geq bc$.

Now, if we define a relation $a \geq b$ and $b \geq a$, the relation is an equivalence relation.

Lemma 5. $a \sim b$ implies that $ac \sim bc$.

Proof. Use Lemma 4.

Lemma 6. $a \sim a^2$ for every $a \in S$.

Proof. Obvious by the definition.

From Lemma 1, 5, and 6, we have

Theorem 2. We have a decomposition of a commutative semigroup S by introducing the equivalence relation $a \sim b$, or $a \geq b$ and $b \geq a$, into S .

Next, we investigate the property of the subsemigroup S_α whose class sum is S .

Lemma 7. Let e be an idempotent element of S . If $e \geq a$, there exists x of S such that $e \geq x \geq a$ and $ax = e$.

Proof. By the definition of the ordering, $e = a^n y$ for some $y \in S$ where we may assume $n > 1$. Set $x = a^{n-1} y$ then $e = ax$ and $e \geq x \geq a$.

Lemma 8. If $a \sim e$ where e is an idempotent, there is x such that $ax = e$ and $e \sim x$.

Proof. Since $e \geq a$ by Lemma 7, there is x such that $ax = e$ and $e \geq x \geq a$. On the other hand $a \geq e$; hence $e \sim x$.

Now, let D be the set of all idempotents of a commutative semigroup S .

Lemma 9. D is not only a subsemigroup of S but a semilattice.

The partial ordering \preceq is introduced into D in usual way:

$$e \preceq f \quad \text{if} \quad e = fe$$

Lemma 10. As far as elements of D are concerned, it holds that $e \preceq f$ if and only if $e \geq f$.

Proof. Suppose $e \preceq f$ i.e. $e = fx$ for some $x \in S$. Then $fe = f^2 x = fx = e$. Hence $e \geq f$. The converse is trivial.

Lemma 11. Let $e, f \in D$. $e \sim f$ implies $e = f$.

Proof. From $e \sim f$, we have $e \preceq f$ and $f \preceq e$ by Lemma 10. Since \preceq is a partial ordering, $e = f$ is concluded.

From Lemma 11 we have the interesting theorem.

Theorem 3. In the decomposition of a commutative semigroup as Theorem 2, S_α is a subsemigroup having at most one idempotent.

Furthermore, if S_α contains an idempotent, S_α is a unipotent invertible semigroup [1]. Then $S_\alpha e$, in which e is an idempotent of S_α , is the greatest group of S_α and S_α has the property that

For $x \in S_\alpha$ there is a positive integer n such that $x^n \in S_\alpha e$.

The structure of a commutative nonpotent semigroups such as S_α will be argued precisely in another paper.

{ 3 Two decompositions

Mr. K. Numakura obtained a decomposition of a commutative semigroup S by the following equivalence relation \approx [2] as follows.

$a \approx b$ if and only if $\bigcap_{n=1}^{\infty} (Sp a^n) = \bigcap_{n=1}^{\infty} (Sp b^n)$ for all $p \in S$.

The decomposition due to \sim (§ 2) and \approx are denoted by φ_1 and φ_2 respectively. We shall discuss the relations between φ_1 and φ_2 .

Theorem 4. $\varphi_1 \geq \varphi_2$, in other words, if $a \sim b$ then $a \approx b$.

From $a \sim b$, i.e. $a^m = bx$, $b^n = ay$, for any $p \in S$,

$$\bigcap_{i=1}^{\infty} (Sp a^i) = \bigcap_{k=1}^{\infty} (Sp a^{km}) \subset \bigcap_{i=1}^{\infty} (Sp b^i).$$

Similarly

$$\bigcap_{i=1}^{\infty} (Sp b^i) \subset \bigcap_{i=1}^{\infty} (Sp a^i).$$

Thus we have

$$\bigcap_{i=1}^{\infty} (Sp a^i) = \bigcap_{i=1}^{\infty} (Sp b^i), \text{ i.e., } a \approx b.$$

Let φ_0 be the greatest decomposition of S to a semilattice (for the existence of φ_0 is assured in § 1), and let \equiv be the equivalence relation determined by φ_0 . Evidently $\varphi_2 \leq \varphi_1 \leq \varphi_0$.

Theorem 5. It holds that $\varphi_1 = \varphi_0$, in other words, φ_1 is the greatest decomposition of S to a semilattice.

Proof. It is sufficient to show that $a \sim b$, (or $a^m = bx$ and $b^n = ay$) implies $a \equiv b$. Since each class by φ_0 is a subsemigroup, it follows that $a = a^m$. Let \bar{a} be a class to which a belongs, and \bar{S} be the semilattice which is determined by φ_0 . The multiplication in \bar{S} is denoted by \cup . From $a^m = bx$, we get $\bar{a} = \bar{b} \cup \bar{x}$, and consequently $\bar{a} \succ \bar{b}$ where \succ is a partial ordering in \bar{S} . Similarly, from $b^n = ay$, we have $\bar{a} \prec \bar{b}$. Thus it has proved that $a \sim b$ implies $\bar{a} = \bar{b}$ or $a \equiv b$.

Now, if a semigroup S is decomposed to a semilattice composed of only one element, S is called an s -indecomposable semigroup. We have immediately from Theorem 5 the below theorem.

Theorem 6. A commutative semigroup

S is s -indecomposable, if and only if, for every pair a, b of elements of S , there exist a positive integer m and an element $x \in S$ such that $a^m = bx$.

Finally we show $\varphi_2 < \varphi_0$ by an example. Let S be the set of all pairs (i, j) of non-negative integers except one $(0, 0)$, and the multiplication is defined as

$$(i_1, j_1) (i_2, j_2) = (i_1 + i_2, j_1 + j_2)$$

where $i_1 + i_2, j_1 + j_2$ are usual additions.

S is a commutative semigroup. Now let

$$A = \{(i, 0) ; i \geq 1\},$$

$$B = \{(0, j) ; j \geq 1\},$$

$$C = \{(i, j) ; i \geq 1, j \geq 1\}.$$

A, B and C are mutually disjoint subsemigroups and

$$S = A \cup B \cup C$$

It is easily seen that this is a decomposition, written by φ' , of S to a semilattice. Of course $\varphi' \leq \varphi_0$. On the other hand, we consider the mapping f of S on the additive semigroup I of all natural numbers as follows.

$$(i, j) \xrightarrow{f} i + j$$

f is a homomorphism of S on I . Setting $a = (i, j)$,

$$\begin{aligned} f(x p a^n) &= f(x) + f(p) + n f(a) \\ &\geq 1 + 1 + n(i + j) \\ &> n \end{aligned}$$

Let $I_n = \{i ; i > n\}$. Then $f(S p a^n) \subset I_n$. Since $\bigcap_{n=1}^{\infty} I_n = \emptyset$, $\bigcap_{n=1}^{\infty} (S p a^n) = \emptyset$ for

every $p, a \in S$. It follows that φ_2 decomposes all elements of S into one class. Clearly $\varphi_2 < \varphi'$. At last we arrived at $\varphi_2 < \varphi_0$.

References

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