KYUNGPOOK Math. J. 56(2016), 357-366
http://dx.doi.org/10.5666/KMJ.2016.56.2.357
pISSN 1225-6951 eISSN 0454-8124

On Deferred Statistical Convergence of Sequences

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Abstract. In this paper, deferred statistical convergence is defined by using deferred Cesàro mean instead of Cesàro mean in the definition of statistical convergence. The obtained method is compared with strong deferred Cesàro mean and statistical convergence under some certain assumptions. Also, some inclusion theorems and examples are given.

1. Introduction and Definitions

The concept of statistical convergence was introduced by I.J. Steinhaus in [17] and H. Fast in [6] independently in the same year. Nowadays, this subject has became one of the most active research area in the theory of summability. It was applied in different areas of mathematics such as number theory by P. Erdös-G.Tenenbaum [5] and summability theory by A. R. Freedman-J. J. Sember-M. Raphael [7].

Furthermore, this subject was studied in [3], [4], [8], [9], [10], [15], [16] etc.

Statistical convergence is also closely related to the subject of asymptotic density(or natural density) of the subset of natural numbers (see, [2]) and its root goes back to A. Zygmund [19].

In 1932, R.P.Agnew [1] defined the deferred Cesaro mean as a generalization of Cesàro mean of real (or complex) valued sequence $x = (x_k)$ by

$$(D_{p,q}x)_n := \frac{1}{q(n)-p(n)} \sum_{k=p(n)+1}^{q(n)} x_k, \quad n = 1, 2, 3, ..., \tag{1.1}$$

where $p = \{p(n) : n \in \mathbb{N}\}$ and $q = \{q(n) : n \in \mathbb{N}\}$ are the sequences of nonnegative

* Corresponding Author.
Received September 26, 2012; revised May 8, 2013; accepted March 28, 2016.
2010 Mathematics Subject Classification: 40A05, 40C05.
Key words and phrases: statistical convergence, deferred statistical convergence, summability of sequences, strongly summability.

integers satisfying

$$p(n) < q(n) \text{ and } \lim_{n\to\infty} q(n) = \infty. \tag{1.2}$$

R.P. Agnew also showed that the method in (1.1) has some important properties besides regularity(see for reqularity [11, Theorem 3]).

A sequence $x = (x_k)$ is said to be strong $D_{p,q}-$convergent to l if

$$\lim_{n\to\infty} \frac{1}{q(n)-p(n)} \sum_{k=p(n)+1}^{q(n)} |x_k - l| = 0,$$

holds and it is denoted by

$$\lim_{n\to\infty} x_n = l \ (D\,[p,q]).$$

Recall that a sequence $x = (x_k)$ is said to be statistically convergent to l if for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{n} |\{k : k \leq n, |x_k - l| \geq \varepsilon\}| = 0,$$

satisfied where the vertical bars indicate the numbers of elements inside the set and it is denoted by $\lim_{n\to\infty} x_n = l(S)$.

There is a natural relationship between statistical convergence and strong summability of sequences. This relation has been investigated in [3], [4], [12], [13], [14] and etc.

Definition 1.1 (Deferred Statistical Convergence (DS)) A sequence $x = (x_k)$ is said to be deferred statisticaly convergent to $l \in \mathbb{R}$ if for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{q(n)-p(n)} |\{k : p(n) < k \leq q(n), \ \ |x_k - l| \geq \varepsilon\}| = 0,$$

holds and it is denoted by

$$\lim_{n\to\infty} x_n = l \ (DS\,[p,q]).$$

It is clear that;

(i) If $q\,(n) = n$ and $p\,(n) = 0$, then Definition 1.1. is coincide with the definition of statistical convergence,

(ii) If we consider $q\,(n) = k_n$ and $p\,(n) = k_{n-1}$ (for any lacunary sequence of nonnegative integers with $k_n - k_{n-1} \to \infty$ as $n \to \infty$), then Definition 1.1. is turned to Lacunary Statistical convergence [9],

(iii) If $q(n) = \lambda_n$ and $p(n) = 0$ (where λ_n is a strictly increasing sequence of natural numbers such that $\lim_n \lambda_n = \infty$), then Definition 1.1. is coincide $\lambda-$statistical convergence of sequences which is given by Osikievich [18] and Mursaleen [13].

2. Inclusion Theorems

Throughout the paper, we consider the sequence of nonnegative natural numbers $p = \{p(n) : n \in \mathbb{N}\}$ and $q = \{q(n) : n \in \mathbb{N}\}$ satisfying (1.1). Any other restrictions on (if needed) $p(n)$ and $q(n)$ will be given in related theorems.

2.1 Comparison of D with DS

In this section, strong deferred Cesàro mean $D[p,q]$ and deferred statistical convergence $DS[p,q]$ will be compared. It is going to show that these two methods are equivalent only for bounded sequences.

Theorem 2.1.1. *If* $x_n \to l\ \ (D[p,q])$, *then* $x_n \to l\ \ (DS[p,q])$.

Proof. Assume $x_n \to l\ (D[p,q])$. For an arbitrary $\varepsilon > 0$, following inequality

$$\frac{1}{q(n)-p(n)} \sum_{k=p(n)+1}^{q(n)} |x_k - l|$$

$$= \frac{1}{q(n)-p(n)} \left(\sum_{\substack{k=p(n)+1 \\ |x_k-l|\geq\varepsilon}}^{q(n)} + \sum_{\substack{k=p(n)+1 \\ |x_k-l|<\varepsilon}}^{q(n)} \right) |x_k - l| \ \geq \ \frac{1}{q(n)-p(n)} \sum_{\substack{k=p(n)+1 \\ |x_k-l|\geq\varepsilon}}^{q(n)} |x_k - l|$$

$$\geq \ \varepsilon \frac{1}{q(n)-p(n)} \left|\{k : p(n) < k \leq q(n),\ \ |x_k - l| \geq \varepsilon\}\right|$$

holds. After taking limit when $n \to \infty$, we have

$$\lim_{n\to\infty} \frac{1}{q(n)-p(n)} \left|\{k : p(n) < k \leq q(n),\ \ |x_k - l| \geq \varepsilon\}\right| = 0.$$

Therefore, desired result is obtained. □

Corollary 2.1.2. If $x_n \to l\ (n \to \infty)$, then $x_n \to l\ \ (DS[p,q])$.

Remark 2.1.3. The converse of Theorem 2.1.1 and Corollary 2.1.2 are not true, in general.

For this, consider a sequence $x = (x_k)$ as

$$x_k := \begin{cases} k^2, & \left[\left|\sqrt{q(n)}\right|\right] - m_0 < k \leq \left[\left|\sqrt{q(n)}\right|\right],\ n = 1, 2, 3, ..., \\ 0, & \text{otherwise}, \end{cases}$$

where $q(n)$ is a monotone increasing sequence and $m_0 \neq 0$ is an arbitrary fixed natural number.

If we consider $D[p,q]$ for the sequence $p(n)$ satisfying

$$0 < p(n) \leq \left[\left|\sqrt{q(n)}\right|\right] - m_0,$$

then for an arbitrary $\varepsilon > 0$ we have

$$\frac{1}{q(n)-p(n)}\left|\{k : p(n) < k \le q(n),\ \ |x_k - 0| \ge \varepsilon\}\right| = \frac{m_0}{q(n)-p(n)} \to 0,$$

when $n \to \infty$, i.e., $x_k \to 0\big(DS\,[p,q]\big)$.
On the other hand,

$$\frac{1}{q(n)-p(n)}\sum_{p(n)+1}^{q(n)} |x_k - 0| \ \ge\ \frac{m_0\left(\left[\left|\sqrt{q(n)}\right|\right] - m_0\right)^2}{q(n)-p(n)} \to m_0,$$

when $n \to \infty$, i.e., (x_k) is not $D\,[p,q]$ convergent to zero. It is also clear that the sequence does not convergent to zero in usual case.

Let us recall that l_∞ is the set of all bounded sequences.

Theorem 2.1.4. *If* $x = (x_n) \in l_\infty$ *and* $x_n \to l\ (DS\,[p,q])$ *then* $x_n \to l\ \ (D\,[p,q])$.

Proof. Suppose that $x = (x_n) \in l_\infty$ and $x_n \to l\ (DS\,[p,q])$. Under the assumption on (x_n) there exists positive reel number M such that $|x_n - l| \le M$ hods for all n.

So, the inequality

$$\frac{1}{q(n)-p(n)}\sum_{k=p(n)+1}^{q(n)} |x_k - l| \ = \frac{1}{q(n)-p(n)}\left(\sum_{\substack{k=p(n)+1\\ |x_k-l|\ge\varepsilon}}^{q(n)} + \sum_{\substack{k=p(n)+1\\ |x_k-l|<\varepsilon}}^{q(n)}\right)|x_k - l|$$

$$\le \frac{1}{q(n)-p(n)}\left(M\sum_{\substack{k=p(n)+1\\ |x_k-l|\ge\varepsilon}}^{q(n)} 1 + \varepsilon \sum_{\substack{k=p(n)+1\\ |x_k-l|<\varepsilon}}^{q(n)} 1\right)$$

$$\le M\frac{1}{q(n)-p(n)}\left|\{k : p(n) < k \le q(n),\ \ |x_k - l| \ge \varepsilon\}\right|$$

$$+\varepsilon\frac{1}{q(n)-p(n)}\left|\{k : p(n) < k \le q(n),\ \ |x_k - l| < \varepsilon\}\right|,$$

is hold. From the limit relation we have

$$\lim_{n\to\infty}\frac{1}{q(n)-p(n)}\sum_{k=p(n)+1}^{q(n)} |x_k - l| = 0.$$

So, the proof is completed. □

2.2. Comparison of S with $DS\,[p,q]$

In this section, statistical convergence and deferred statistical convergence will be compared under some restrictions on $p(n)$ or $q(n)$.

Theorem 2.2.1. *If the sequence* $\left\{\frac{p(n)}{q(n)-p(n)}\right\}_{n\in\mathbb{N}}$ *is bounded, then* $x_n \to l\ (S)$ *implies* $x_n \to l\ \ (DS\,[p,q])$.

Proof. Let's give a note about the sequences of positive natural numbers $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ without proof: if $\lim\limits_{n\to\infty} a_n = a$, $a \in \mathbb{R}$ and $\lim\limits_{n\to\infty} b_n = +\infty$, then $\lim\limits_{n\to\infty} a_{b_n} = a$.

From the assumption on (x_n), the limit relation

$$\lim_{n\to\infty} \frac{1}{n} \left|\{k : k \le n,\ \ |x_k - l| \ge \varepsilon\}\right| = 0, \tag{2.1}$$

holds for every $\varepsilon > 0$. Since the sequence $q(n)$ satisfies (1.2), then the sequence

$$\left\{\frac{|\{k : k \le q(n),\quad |x_k - l| \ge \varepsilon\}|}{q(n)}\right\}_{n\in\mathbb{N}}$$

is convergent to zero.

Therefore, the inclusion

$$\{k : p(n) < k \le q(n),\ \ |x_k - l| \ge \varepsilon\} \subset \{k : k \le q(n),\ \ |x_k - l| \ge \varepsilon\},$$

and the inequality

$$\left|\{k : p(n) < k \le q(n),\ \ |x_k - l| \ge \varepsilon\}\right| \le \left|\{k : k \le q(n),\ \ |x_k - l| \ge \varepsilon\}\right|,$$

are hold. From the last inequality we have

$$\frac{1}{q(n)-p(n)} \left|\{k : p(n) < k \le q(n),\ \ |x_k - l| \ge \varepsilon\}\right|$$

$$\le \Big(1 + \frac{p(n)}{q(n)-p(n)}\Big).\frac{1}{q(n)} \left|\{k : k \le q(n),\ \ |x_k - l| \ge \varepsilon\}\right|,$$

and from the limit relation we get

$$x_k \to l(DS\,[p,q]).$$

So, desired result is obtained. □

Corollary 2.2.2. *Let* $\{q(n)\}_{n\in\mathbb{N}}$ *be an arbitrary sequence with* $q(n) < n$ *for all* $n \in \mathbb{N}$ *and* $\left\{\frac{n}{q(n)-p(n)}\right\}_{n\in\mathbb{N}}$ *be a bounded sequence. Then,* $x_n \to l\ (S)$ *implies* $x_n \to l\ \ (DS\,[p,q])$.

Remark 2.2.3. The converse of Theorem 2.2.1 is not true even if $\left\{\frac{p(n)}{q(n)-p(n)}\right\}_{n\in\mathbb{N}}$ is bounded.

Example 2.2.4. Let us consider $p(n) = 2n,\ q(n) = 4n$ and a sequence $x = (x_n)$ as

$$x_n = \begin{cases} \frac{n+1}{2}, & n \text{ is odd}, \\ -\frac{n}{2}, & n \text{ is even}. \end{cases}$$

It is clear that the assumption of Theorem 2.2.1 is fulfilled and $x_n \to 0\ (D[2n, 4n])$.

From Theorem 2.1.1 we get $x_n \to 0\ (DS[2n, 4n])$. But, for an arbitrary small $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{n} \left|\{k : k \le n,\ \ |x_n - 0| \ge \varepsilon\}\right| \ne 0.$$

Definition 2.2.5. A method $DS[p, q]$ is called properly deferred when $\{p(n)\}$ and $\{q(n)\}$ satisfy in addition to (1.2) the condition $\left\{\frac{p(n)}{q(n)-p(n)}\right\}$ is bounded for all n.

Remark 2.2.6. Two properly deferred statistically convergence method must not be include each other. Let

$$x_n := \begin{cases} k+1, & n = 2k+1, \\ -k, & n = 2k. \end{cases}$$

It is clear that $x_n \to 0(DS[2n, 4n])$ and $x_n \to \frac{1}{2}(DS[2n-1, 4n-1])$.

Theorem 2.2.7. *Let $q(n) = n$ for all $n \in \mathbb{N}$. Then, $x_n \to l\ (DS\,[p, n])$ if and only if $x_n \to l\ (S)$.*

Proof. ($\Longrightarrow$) Let us assume that $x_k \to l(DS\,[p, n])$. We shall apply the technique which was used by Agnew in [1]. Then, for any $n \in \mathbb{N}$,

$$p(n) = n^{(1)} > p(n^{(1)}) = n^{(2)} > p(n^{(2)}) = n^{(3)} > \dots\ ,$$

and we may write the set $\{k \le n : |x_k - l| \ge \varepsilon\}$ as

$$\{k \le n : |x_k - l| \ge \varepsilon\} = \left\{k \le n^{(1)} : |x_k - l| \ge \varepsilon\right\} \cup \left\{n^{(1)} < k \le n : |x_k - l| \ge \varepsilon\right\},$$

and the set $\left\{1 < k \le n^{(1)} : |x_k - l| \ge \varepsilon\right\}$ as

$$\left\{1 < k \le n^{(1)} : |x_k - l| \ge \varepsilon\right\}$$
$$= \left\{k \le n^{(2)} : |x_k - l| \ge \varepsilon\right\} \cup \left\{n^{(2)} < k \le n^{(1)} : |x_k - l| \ge \varepsilon\right\},$$

and the set $\left\{k \le n^{(2)} : |x_k - l| \ge \varepsilon\right\}$ as

$$\left\{k \le n^{(2)} : |x_k - l| \ge \varepsilon\right\} = \left\{k \le n^{(3)} : |x_k - l| \ge \varepsilon\right\} \cup \left\{n^{(3)} < k \le n^{(2)} : |x_k - l| \ge \varepsilon\right\},$$

and if this process is continued we obtain

$$\left\{k \le n^{(h-1)} : |x_k - l| \ge \varepsilon\right\}$$

$$= \left\{k \le n^{(h)} : |x_k - l| \ge \varepsilon\right\} \cup \left\{n^{(h)} < k \le n^{(h-1)} : |x_k - l| \ge \varepsilon\right\}$$

for a certain positive integer $h > 0$ depending on n such that $n^{(h)} \ge 1$ and $n^{(h+1)} = 0$. From the above discussion, the relation

$$\frac{1}{n} \left|\{k \le n : |x_k - l| \ge \varepsilon\}\right|$$

$$= \sum_{m=0}^{h} \frac{n^{(m)} - n^{(m+1)}}{n} \frac{1}{n^{(m)} - n^{(m+1)}} \left|\left\{n^{(m+1)} < k \le n^{(m)} : |x_k - l| \ge \varepsilon\right\}\right|$$

holds for every n. This relation gives that statistical convergency of the sequence (x_n) to l is a linear combination of following sequence.

$$\left\{\frac{1}{n^{(m)} - n^{(m+1)}} \left|\left\{n^{(m+1)} < k \le n^{(m)} : |x_k - l| \ge \varepsilon\right\}\right|\right\}_{m\in\mathbb{N}}.$$

Let us consider the matrix

$$b_{n,m} := \begin{cases} \frac{n^{(m)} - n^{(m+1)}}{n}, & m = 0, 1, 2, \dots, h\ , \\ 0, & \text{otherwise.} \end{cases}$$

where $n^{(0)} := n$.

The matrix $(b_{n,m})$ is satisfied the Silverman Toeplitz theorem (see in [11]). So, we have

$$\lim_{n\to\infty} \frac{1}{n} \left|\{k \le n : |x_k - l| \ge \varepsilon\}\right| = 0$$

since

$$\frac{1}{n^{(m)} - n^{(m+1)}} \left|\left\{n^{(m+1)} < k \le n^{(m)} : |x_k - l| \ge \varepsilon\right\}\right| \to 0,$$

when $n \to \infty$.

($\Longleftarrow$) Since $q(n) = n$ is satisfied (1.2), then the inverse of the theorem is a simple consequence of Theorem 2.2.1. □

Corollary 2.2.8. *Assume that* $\{q(n)\}_{n\in\mathbb{N}}$ *contains almost all positive integers. Then,* $x_n \to l(DS\,[p,q])$ *implies* $x_n \to l(S)$.

Proof. Let $x_n \to l(DS\,[p,q])$ for an arbitrary $\{p(n)\}$ and choose sufficiently large positive integer m such that the set $\{q(n)\}$ contains all positive integers which is greater than m. Then, it can be constructed a sequence (k_n) as follows:

$$k_1 = k_2 = \cdots = k_m = 1$$

and for each $n > m$ an index k_n such that $q_{k_n} = n$.

It is clear from the construction that (k_n) is a monotone increasing sequence. So, from the assumption $x_n \to l(DS\,[p_{k_n}, q_{k_n}])$. Hence, the proof of Corollary follows from Theorem 2.2.5. □

Corollary 2.2.9. *Let us assume* $\{q(n)\}$ *contains almost all positive integers. If* $x = (x_n)$ *is a sequence such that* $x_n \to l(DS\,[p,q])$ *for an arbitrary* $\{p(n)\}_{n\in\mathbb{N}}$ *and* $\Delta x_n = O(\frac{1}{n})$ *then* $x_n \to l$ $(n \to \infty)$.

Proof. It is clear from Corollary 2.2.6 and the assumption on $q(n)$ that $x_n \to l(S)$. Therefore, if we use [9, Theorem 3] we obtained the proof. □

2.3. Comparison of $DS\,[p',q']$ with $DS\,[p,q]$

In this section, the methods $DS\,[p,q]$ and $DS\,[p',q']$ will be compared under the following restriction

$$p(n) \le p'(n) < q'(n) \le q(n) \tag{3.1}$$

for all $n \in \mathbb{N}$.

Theorem 2.3.1. *Let* $p' = \{p'(n)\}_{n\in\mathbb{N}}$ *and* $q' = \{q'(n)\}_{n\in\mathbb{N}}$ *be sequences of positive natural numbers satisfying* (3.1) *such that the sets*

$$\{k : p(n) < k \le p'(n)\} \;\; and \;\; \{k : q'(n) < k \le q(n)\}$$

are finite sets for all $n \in \mathbb{N}$. *Then,* $x_k \to l(DS\,[p',q'])$ *implies* $x_k \to l(DS\,[p,q])$.

Proof. Let us consider the sequence $x = (x_k)$ such that $x_k \to l(DS\,[p',q'])$. For an arbitrary $\varepsilon > 0$, the equality

$$\{k : p(n) < k \le q(n),\;\; |x_k - l| \ge \varepsilon\} = \{k : p(n) < k \le p'(n),\;\; |x_k - l| \ge \varepsilon\}$$

$$\cup \{k : p'(n) < k \le q'(n),\;\; |x_k - l| \ge \varepsilon\} \cup \{k : q'(n) < k \le q(n),\;\; |x_k - l| \ge \varepsilon\},$$

and

$$\frac{1}{q(n)-p(n)} \left|\{k : p(n) < k \le q(n),\;\; |x_k - l| \ge \varepsilon\}\right|$$

$$\le \frac{1}{q'(n)-p'(n)} \left|\{k : p(n) < k \le p'(n),\;\; |x_k - l| \ge \varepsilon\}\right|$$

$$+ \frac{1}{q'(n)-p'(n)} \left|\{k : p'(n) < k \le q'(n),\;\; x\,|x_k - l| \ge \varepsilon\}\right|$$

$$+ \frac{1}{q'(n)-p'(n)} \left|\{k : q'(n) < k \le q(n),\;\; |x_k - l| \ge \varepsilon\}\right|,$$

are hold.

On taking limits when $n \to \infty$, we obtain

$$\lim_{n\to\infty} \frac{1}{q(n)-p(n)} \left|\{k : p(n) < k \le q(n),\;\; |x_k - l| \ge \varepsilon\}\right| = 0.$$

This proves our assertion. □

Theorem 2.3.2. *Let* $\{p(n)\}_{n\in\mathbb{N}}$, $\{q(n)\}_{n\in\mathbb{N}}$ *and* $\{p'(n)\}_{n\in\mathbb{N}}$, $\{q'(n)\}_{n\in\mathbb{N}}$ *be sequences of positive natural numbers satisfying* (3.1) *such that*

$$\lim_{n\to\infty}\frac{q(n)-p(n)}{q'(n)-p'(n)}=d>0.$$

Then, $x_k \to l(DS\,[p,q])$ *implies* $x_k \to l(DS\,[p',q'])$.

Proof. It is easy too see that the inclusion

$$\{k : p'(n)+1\le k\le q'(n),\ \ |x_k-l|\ge\varepsilon\}\subset\{k : p(n)+1\le k\le q(n),\ \ |x_k-l|\ge\varepsilon\}$$

and the inequality

$$|\{k : p'(n)+1\le k\le q'(n),\ \ |x_k-l|\ge\varepsilon\}|\le|\{k : p(n)+1\le k\le q(n),\ \ |x_k-l|\ge\varepsilon\}|$$

are true. So, we have

$$\frac{1}{q'(n)-p'(n)}\left|\{k : p'(n)+1\le k\le q'(n),\ \ |x_k-l|\ge\varepsilon\}\right|$$
$$\le\frac{q(n)-p(n)}{q'(n)-p'(n)}\frac{1}{q(n)-p(n)}\left|\{k : p(n)+1\le k\le q(n),\ \ |x_k-l|\ge\varepsilon\}\right|.$$

Taking limits as $n\to\infty$ the desired results is obtained. □

Acknowledgements. The authors would like to thank to the referees for the helpful suggestions.

References

[1] R. P. Agnew, *On deferred Cesaro Mean*, Comm. Ann. Math., **33**(1932), 413-421.

[2] R. C. Buck, *Generalized asymptotic density*, Amer. J. Math., **75**(1953), 335-346.

[3] J. S. Connor, *The statistical and strong p-Cesaro of sequences*, Analysis, **8**(1988), 47-63.

[4] J. S. Connor, *On strong matrix summability with respect to a modulus and statistical convergence*, Canad. Math. Bull., **32**(1989), 194-198.

[5] P. Erdös and G. Tenenbaum, *Sur les densites de certaines suites d'entiers*, Proc. London Math. Soc., **59(3)**(1989), 417-438.

[6] H. Fast, *Sur la convergence statiistique*, Colloq. Math., **2**(1951), 241-244.

[7] A. R. Freedman, J. J. Sember and M. aphael, *Some Cesarotype summability spaces*, Proc. London Math.soc., **37**(1978), 301-313.

[8] J. A Fridy, *On statistical convergence*, Analysis, **5**(1985), 301-313.

[9] J. A. Fridy, and C. Orhan, *Lacunary statistical convergence*, Pacific. J. Math., **160**(1993), 43-51.

[10] J. A. Fridy and H. I.Miller, *A matrix characterization of statistical convergence*, Analysis, **11**(1991), 59-66.

[11] I. J. Maddox, *Elements of Functional Analysis*, Cambridge at the University Press, (1970).

[12] I. J. Maddox, *Space of strongly summable functions*, Oxford(2), Quart. J. Mah., (1967), 345-355.

[13] M. Mursaleen, $\lambda-$*statistical convergence*, Math. Slovaca, **50(1)**(2000), 111-115.

[14] F. Nuray, $\lambda-$*strongly summable and* $\lambda-$ *statistically convergencet functions*, Iran. J. Sci. Technol. Trans. A Sci., **34(A4)**(2010), 335-338.

[15] T. Salat, *On statistically convergent sequences of real numbers*, Math. Slovaca, **30**(1980), 139-150.

[16] I. J. Schoenberg, *The integrability of certain functions and related summability methods*, Amer. Math. Monthly, **66**(1959), 361-375.

[17] H. Steinhaus, *Sur la convergence ordinaire et la convergence asymptotique*, Colloq. Math., **2**(1951), 73-74.

[18] J. A. Osikiewicz, *Summability of Matrix Submethods and Spliced Sequences*, Ph.D. Thesis, August, (1997).

[19] A. Zygmund, *Trigonometric Series*, Cambiridge Univ. Press Cambridge, U. K., (1979).