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On defining the distributions δ^r and $(\delta')^r$ by conformable derivatives

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Abstract

In this paper, starting from a fixed δ-sequence, we use the generalized Taylor's formula based on conformable derivatives and the neutrix limit to find the powers of the Dirac delta function δ^r and $(\delta')^r$ for any $r \in \mathbb{R}$.

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1 Introduction

Scientists have been facing some difficulties in describing complex systems by using the classical calculus. To overcome these difficulties, some scientists started to use fractional calculus that studies the integration and differentiation of any order. See, for example, [1, 2] and the references therein. The conventional fractional derivatives, such as Riemann–Liouville, Caputo, Hadamard, and Grunwald–Letnikov, have found numerous applications in science and engineering by giving the memory and hereditary effects. The so-called Hausdorff derivative or the fractal derivative (see [3, 4] and the references therein) are also derivatives that can be used to describe complex systems. Other derivatives, such as the conformable derivatives which were introduced in [5, 6], may also be candidates to be used in order to describe complex real world phenomena.

On the other hand, the powers of distribution functions play important roles in a variety of branches of applied mathematics and physics (see [7, 8] and the references therein). Consequently, finding the powers of such functions have attracted the attention of many scientists. In [9], the author defined the powers of the Dirac-delta distribution and the powers of its first order derivative for positive integers. In [10], the author utilized the Temple delta sequence and the neutrix limit to show that $\delta^r(x) = 0$ for all $r \in \mathbb{Z}^-$. Recently, C.K. Li and C.P. Li in [8] and C.K. Li in [11] defined the powers of the delta function using Caputo fractional derivatives.

Recently, there have been some works on conformable derivatives and their applications in various fields. For example in [12], the Newton mechanics in the frame of conformable derivatives were discussed, and in [13] the physical interpretation of generalized conformable derivatives was tackled. In [14], the Sturm–Liouville problems in the frame of conformable derivatives were discussed.

In this paper, we investigate a new approach to finding the distributions δ^r and $(\delta')^r$ for $r \in \mathbb{R}$ using the Galapon property with the conformable derivative. We propose to apply the neutrix calculus developed by J.G. van der Corput [15] and to utilize the conformable derivatives [5, 6].

This article is organized as follows: In Sect. 2, we recall some basic definitions, lemmas, and theorems. In Sect. 3, we present the powers of the delta function. In Sect. 4, we present the powers of derivatives of the delta function. In Sect. 5, we present $\delta^r(x^\lambda)$ for some values of λ. The last section is devoted to the conclusion.

2 Preliminaries

Let $\mathcal{D}(\mathbb{R})$ be the space of infinitely differentiable function with compact support in $\mathbb{R}$, and let $\mathcal{D}'(\mathbb{R})$ be the space of distributions defined on $\mathcal{D}(\mathbb{R})$. The Dirac distribution is defined by the formal property

$$(\delta, \phi) = \int_{-\infty}^{+\infty} \delta(x)\phi(x)\,dx = \phi(0) \tag{1}$$

for sufficiently well-behaved functions $\phi(x)$. Relation (1) is called the sifting property or the reproducing property of the Dirac distribution (delta function). This definition of the δ-function stands for a limit of a sequence of inner products, namely

$$\lim_{n\to\infty} \int_{-\infty}^{+\infty} \delta_n(x)\phi(x)\,dx = \phi(0), \tag{2}$$

where $\delta_n(x)$ is a sequence of ordinary (classical) functions. Such a sequence is referred to as a delta-convergent sequence or simply a δ-sequence. For a given δ-sequence, $\delta_n(x)$, we have what is known as a limit representation of the δ-function

$$\delta(x) = \lim_{n\to\infty} \delta_n(x). \tag{3}$$

Known δ-sequences have either increasing positive or infinite values at the origin; that is, for sufficiently large n, $0 < \delta_{n_1}(0) < \delta_{n_2}(0)$ when $n_1 < n_2$ or $\delta_n(0) = \infty$ for all n. Certain sequences of classical functions exist and have property (2). For instance, the well-known Dirichlet formula

$$\lim_{n\to\infty} \int_{-\infty}^{+\infty} \frac{n}{\sqrt{\pi}} \exp\bigl(-n^2x^2\bigr)\phi(x)\,dx = \phi(0). \tag{4}$$

Theorem 1 ([16]) *Let $\rho(x)$ be a nonnegative, locally integrable function in the d-dimensional space $\mathbb{R}^d$ and*

$$\int_{\mathbb{R}^d} \rho(x)\,dx = 1.$$

Then the sequence of functions

$$g_\epsilon(x) = \epsilon^{-d}\rho(x/\epsilon)$$

converges to $\delta(x)$, $\epsilon \to \infty$ (in the sense of generalized functions).

Definition 2 ([17]) A sequence of functions $(\delta_n(x))$ is a delta-convergent sequence: (i) if for any $M > 0$ and for $|a|, |b| \leq M$, then $\int_a^b \delta_n(x)\,dx$ are bounded by a constant depending only on M; (ii) for any fixed non-vanishing a and b,

$$\lim_{n\to\infty} \int_a^b \delta_n(x)\,dx = \begin{cases} 0 & \text{for } 0 < a < b \text{ and } a < b < 0, \\ 1 & \text{for } a < 0 < b. \end{cases}$$

We now wish to show that for any positive integer m the set of functions with the Galapon property

$$\delta_n(m,x) = \frac{1}{2^{2m+1}\Gamma(m+1/2)} n^{m+1/2} x^{2m} \exp\bigl(-nx^2/4\bigr) \tag{5}$$

is a δ-sequence in n so that

$$\lim_{n\to\infty} \delta_n(m,x) = \delta(x).$$

These functions satisfy

$$\delta_n(m,0) = 0 \quad \text{for all } n \in \mathbb{N}.$$

At any other point x, the sequence also tends to zero. Nevertheless, the sequence generates the δ-function. The known limit representations of the δ-function involve delta sequences that do not vanish at the support of the limit δ-function. However, in [18], Galapon assumes that a δ-sequence may vanish at the support of the limit δ-function for all finite values of the limit parameter. It seems impossible to define products of two arbitrary distributions in general [17, 19]. However, the product of an infinitely differentiable function $\phi(x)$ with a distribution g is given by

$$(\phi g, \psi) = (g, \phi\psi), \tag{6}$$

which is well defined since $\phi\psi \in \mathcal{D}(\mathbb{R})$ if $\psi \in \mathcal{D}(\mathbb{R})$.

Definition 3 ([16, 17]) Let $g \in \mathcal{D}'(\mathbb{R})$. The distributions g' given by (1) are called the first order derivatives of g, then

$$\bigl(g', \phi\bigr) = -\bigl(g, \phi'\bigr), \quad \phi \in \mathcal{D}(\mathbb{R}), \bigl(*' = d/dx\bigr). \tag{7}$$

A distribution has derivatives of all orders. For one can iterate (7) to obtain

$$\bigl(g^{(n)}, \phi\bigr) = (-1)^n \bigl(g, \phi^{(n)}\bigr). \tag{8}$$

The product of $\phi\delta^{(n)}$ exists by equation (6) for $n = 0, 1, 2, \ldots$, and

$$\phi(x)\delta^{(n)}(x) = (-1)^n \sum_{k=0}^{n} \binom{n}{k} (-1)^k \phi^{(n-k)}(0)\delta^{(k)}(x), \qquad \binom{n}{k} = \frac{n!}{k!(n-k)!}. \tag{9}$$

Recently, in [5, 6], the definition of conformable derivatives of functions of order $\alpha \in (0,1]$ have been formulated and given as follows.

Definition 4 ([5, 6]) Given any function $f : [0,\infty] \to \mathbb{R}$. Then the conformable derivative of f of order $\alpha \in (0,1]$ is defined by

$$\left(T_a^\alpha g\right)(x) = \lim_{\epsilon\to 0} \frac{g(x+\epsilon x^{1-\alpha}) - g(x)}{\epsilon}.$$

Note that if g is differentiable, then

$$\left(T_a^\alpha g\right)(x) = x^{1-\alpha} g'(x), \tag{10}$$

where

$$g'(x) = \lim_{\epsilon\to 0}\left[g(x+\epsilon) - g(x)\right]/\epsilon.$$

Reciprocally, if $(T_a^\alpha g)(x)$ exists, then for $x \neq 0$ we have

$$g'(x) = \lim_{\epsilon\to 0}\left[g(x+\epsilon) - g(x)\right]/\epsilon = \lim_{\epsilon\to 0}\left[g\left(x+\epsilon x^{1-\alpha}\right) - g(x)\right]/\left(\epsilon x^{1-\alpha}\right) = \left(T_a^\alpha g\right)(x).$$

Definition 5 ([5]) Let $\alpha \in (n, n+1]$ and g be an n-differentiable function at $x > 0$, then the left conformable derivative of order α at $x > 0$ is given by

$$\left(T_a^\alpha g\right)(x) = \left(T_a^\alpha\right) g^{(n)}(x) = \lim_{\epsilon\to 0}\left[g^{(n)}\left(x+\epsilon x^{n+1-\alpha}\right) - g^{(n)}(x)\right]/\left(\epsilon x^{n+1-\alpha}\right)$$

provided the limit of the right-hand side exists. If g is α-differentiable in some $(0,a)$, $a > 0$, and $\lim_{x\to 0^+}(T_a^\alpha g)(x)$ exists, then define $(T^0 g)(x) = \lim_{x\to 0^+}(T_a^\alpha g)(x)$.

The properties of $(T_a^\alpha g)$ can be found in [5].

Lemma 6 ([5]) *Let $x > 0$, $\alpha \in (n, n+1]$. The function g is $(n+1)$-differentiable if and only if g is α-differentiable; moreover, $(T_a^\alpha g)(x) = x^{n+1-\alpha} g^{(n+1)}(x)$.*

The generalized Taylor's formula and Taylor's theorem in the frame of conformable derivatives are given in the following theorems.

Theorem 7 ([5]) *Assume that g is an infinitely α-differentiable function for some $0 < \alpha \le 1$ at a neighborhood of a point a. Then g has the fractional power series expansion*

$$g(x) = \sum_{k=0}^{\infty} \frac{(x-a)^{k\alpha}(T_a^\alpha g)^{(k)}(a)}{\alpha^k k!}, \quad \textit{for all } 0 < a < x < a + R^{\frac{1}{\alpha}}, R > 0,$$

here $(T_a^\alpha g)^{(k)}(x) = T_a^\alpha T_a^\alpha \cdots T_a^\alpha g(x)$ means the application of the conformable derivative k times.

Definition 8 ([5]) Assume that g is an infinitely α-differentiable function for some $0 < \alpha \leq 1$ at a neighborhood of a point a. Then g has the fractional power series expansion

$$g(x) = \sum_{k=0}^{s} \frac{(x-a)^{k\alpha}(T_a^\alpha g)^{(k)}(a)}{\alpha^k k!} + \frac{(x-a)^{(s+1)\alpha}(T_a^\alpha g)^{(s+1)}(\zeta)}{\alpha^{(s+1)}(s+1)!} \tag{11}$$

for all $0 < a \leq \zeta \leq x < a + R^{\frac{1}{\alpha}}$, $R > 0$.

Next, following van der Corput [15], we define a neutrix as a class of negligible functions defined in a domain, which satisfy the following two conditions:

(i) the neutrix is an additive group;
(ii) it does not contain any constant except 0.

Definition 9 ([15, 20]) Let N' be a nonempty set, and let N be a commutative, additive group of functions mapping N' into a commutative, additive group N''. The group N is called neutrix if the function which is identically equal to zero is the only constant function occurring in N. The function which belongs to N is called negligible function in N.

Let N' be a set contained in a topological space with a limit point a not belonging to N' and N be a commutative additive group of functions defined on N' with the following property:

$$g \in N, \quad \lim_{x\to a} g(x) = l \text{ (constant)} \quad \text{for } x \in N', \text{then } l = 0.$$

Then this group N is a neutrix. Let g be a real-valued function defined on N', and suppose that it is possible to find a constant l such that $g(x) - l$ is negligible in N. Then l is called the neutrix limit of $g(x)$ as x tends to a and is denoted by

$$N\text{-}\lim_{x\to a} g(x) = l.$$

In the following, we let N be the neutrix having domain $N' = \{1, 2, \ldots, n, \ldots\}$, the positive integers and range N'' as the real numbers with negligible functions being finite linear sums of the functions

$$n^\lambda \ln^{\mu-1} n, \qquad \ln^\mu n, \quad \lambda > 0, \mu = 1, 2, 3, \ldots, \tag{12}$$

and all being functions which converge to zero in the usual sense as n tends to infinity. More mathematical properties of neutrices and the neutrix limit can be found in [15]. Note that taking the neutrix limit of a function $g(n)$ is equivalent to taking the usual limit of Hadamard's finite part of $g(n)$. In [9], the authors used the neutrix limit in order to define the powers δ-function and its derivatives as

$$\bigl(\delta^r(x), \phi(x)\bigr) = N\text{-}\lim_{n\to\infty} \bigl(\delta_n^r(x), \phi(x)\bigr) \tag{13}$$

for $r \in (0, 1)$ and $r = 2, 3, \ldots$.

E.L. Koh and C.K. Li [9] define (δ^r) and $(\delta')^r$ for the fixed δ-sequence and use the Gaussian sequence to give meaning to the distributions (δ^r) and $(\delta')^r$ for $r \in (0, 1)$ and $r = 2, 3, \ldots$, as follows.

Theorem 10 ([9])
- $\delta^0(x) = 1$,
- $\delta^r(x) = 0$ *for* $0 < r < 1$,
- $\delta^{2k}(x) = 0$ *for* $k = 1, 2, \ldots$,
- $\delta^{2k+1}(x) = C_k \delta^{(2k)}(x)$, *where* $C_k = \frac{1}{2^{2k} k! \pi^k (2k+1)^{\frac{2k+1}{2}}}$, *for* $k = 0, 1, 2, \ldots$.

Theorem 11 ([9])
- $(\delta')^r(x) = 0$ *for* $r \in (0, \frac{1}{2})$,
- $(\delta')^{\frac{1}{2}}(x) = \sqrt{2} e^{i\frac{\pi}{4}} (\frac{2}{\pi})^{\frac{1}{4}} \Gamma(\frac{3}{4}) \delta(x)$,
- $(\delta')^{2k}(x) = 0$ *for* $k = 1, 2, \ldots$,
- $(\delta')^{2k+1}(x) = C'_k \delta^{(4k+1)}$, *where* $C'_k = \frac{1 \times 2 \times 3 \times 4 \times \cdots \times (6k+1)}{2^k k! \pi^{\frac{k}{2}} (2k+1)^{\frac{6k+3}{2}} (4k+1)!}$, *for* $k = 0, 1, 2, \ldots$.

In [10], the definition of the distribution $(\delta)^r$ for negative integers was given by

$$\delta^{-r}(x) = 0 \quad \text{for } r = 1, 2, \ldots.$$

In [8], a definition for a product of distributions (δ^r) and $(\delta')^r$, $r \in \mathbb{R}$, is given using the Gaussian sequence and the sequential Caputo fractional derivative.

Theorem 12 ([8])
- $\delta^0(x) = 1$,
- $\delta^r(x) = 0$ *for* $r < 1$ *and* $r \neq 0$,
- $(\delta^r(x), \phi(x)) = \frac{(1+(-1)^{r-1})\Gamma(\frac{r}{2})}{2\Gamma(r)} (\frac{1}{r\pi})^{r/2} ({}^c\hat{D}_0^{[r-1]} \phi)(0)$ *for* $r \geq 1$, *where* ${}^c\hat{D}_0^{[r-1]} \equiv ({}^c\hat{D}_0^{s\alpha}) = ({}^cD_0^{\alpha})({}^cD_0^{\alpha}) \cdots ({}^cD_0^{\alpha})$ (*s times and* $s \in \mathbb{Z}^+$) *is the sequential Caputo derivative*,
- $\delta^{2k}(x) = 0$ *for* $k = 1, 2, \ldots$,
- $\delta^{2k+1}(x) = \frac{1}{2^{2k}(k!)(2k+1)^{\frac{2k+1}{2}} \pi^k} \delta^{(2k)}(x)$ *for* $k = 0, 1, 2, \ldots$.

where

$$\Gamma\left(k + \frac{1}{2}\right) = \frac{1.3.5 \cdots (2k-1)}{2^k} \sqrt{\pi} = \frac{(2k)!}{4^k k!} \sqrt{\pi} \quad \text{for } k = 0, 1, 2, \ldots.$$

Theorem 13 ([8])
- $(\delta')^0(x) = 1$,
- $(\delta')^r(x) = 0$ *for* $r < \frac{1}{2}$ *and* $r \neq 0$,
- $(\delta')^{\frac{1}{2}}(x) = \sqrt{2} e^{i\frac{\pi}{4}} (\frac{2}{\pi})^{\frac{1}{4}} \Gamma(\frac{3}{4}) \delta(x)$,
- $((\delta')^r(x), \phi(x)) = \frac{((-1)^r + (-1)^{2r-1})\Gamma(\frac{3r}{2})}{2^{1-r} r^r \Gamma(2r)} (\frac{1}{r\pi})^{r/2} ({}^c\hat{D}_0^{[2r-1]} \phi)(0)$ *for* $r > 1/2$,
- $(\delta')^{2k}(x) = 0$ *for* $k = 1, 2, \ldots$,
- $(\delta')^{2k+1}(x) = \frac{1.3.5 \cdots (6k+1)}{2^k \pi^k (2k+1)^{\frac{6k+3}{2}} (4k+1)!} \delta^{(4k+1)}$ *for* $k = 0, 1, 2, \ldots$.

In [11], a definition for a product of distributions (δ^r) and $(\delta')^r$, $r \in \mathbb{R}$ is given using the Orentzian sequence and the sequential Caputo derivative.

Theorem 14 ([11])
- $\delta^0(x) = 1$,
- $\delta^r(x) = 0$ *for* $\frac{1}{2} < r < 1$ *and* $r \neq 0$,

- $(\delta^r(x), \phi(x)) = \frac{(1+(-1)^{r-1})\Gamma^2(\frac{r}{2})}{2\pi^r\Gamma^2(r)}({}^c\hat{D}_0^{[r-1]}\phi)(0)$ *for* $r \geq 1$, *where* ${}^c\hat{D}_0^{[r-1]} \equiv ({}^c\hat{D}_0^{s\alpha}) = ({}^cD_0^\alpha)({}^cD_0^\alpha)\cdots({}^cD_0^\alpha)$ (*s times and* $s \in \mathbb{Z}^+$) *is the sequential Caputo derivative*,
- $\delta^{2k}(x) = 0$ *for* $k = 1, 2, \ldots$,
- $\delta^{2k+1}(x) = \frac{1}{2^{4k}(k!)^2\pi^{2k}}\delta^{(2k)}(x)$ *for* $k = 0, 1, 2, \ldots$.

Theorem 15 ([11])

- $(\delta')^r(x) = 1$ *for* $r = 0$,
- $(\delta')^r(x) = 0$ *for* $\frac{1}{3} < r < \frac{1}{2}$,
- $(\delta')^{\frac{1}{2}}(x) = (i+1)\frac{\Gamma(\frac{3}{4})\Gamma(\frac{1}{4})}{\sqrt{2\pi}}\delta(x)$,
- $((\delta')^r(x), \phi(x)) = \frac{((-1)^r+(-1)^{2r-1})\Gamma(\frac{3r}{2})\Gamma(\frac{r}{2})}{2^{1-r}(\pi)^r\Gamma^2(2r)}({}^c\hat{D}_0^{[2r-1]}\phi)(0)$ *for* $r > 1/2$,
- $(\delta')^{2k}(x) = 0$ *for* $k = 1, 2, \ldots$,
- $(\delta')^{2k+1}(x) = \frac{2^{2k+1}\Gamma(k+\frac{1}{2})\Gamma(3k+\frac{3}{2})}{\pi^{2k+1}[(4k+1)!]^2}\delta^{(4k+1)}$ *for* $k = 0, 1, 2, \ldots$.

3 Defining $\delta^r(x)$ for all $r \in \mathbb{R}$ using the conformable derivatives

In this section, we define $(\delta)^r$ for all $r \in \mathbb{R}$ using the δ-sequence defined in (5) and the conformable derivatives in the following theorem.

Theorem 16 *For any positive integer m, we have*

1. $\delta^0(x) = 1$,
2. $\delta^r(x) = 0$ *for* $r < 1$, $r \neq 0$,
3. $(\delta^r(x), \phi(x)) = \frac{(1+(-1)^{r-1})}{2}(\frac{1}{r})^{(m+1/2)r}(\frac{\Gamma((m+1/2)r)}{(\Gamma(m+1/2))^r})\frac{(T_0^\alpha\phi)^{[\frac{r-1}{\alpha}]}(0)}{\alpha^{[\frac{r-1}{\alpha}]}[\frac{r-1}{\alpha}]!}$ *for* $r > 1$, *where* T_0^α *is the conformable derivative at* $a = 0$ *and* $[r]$ *is the smallest integer greater than or equal to* r.

In particular, when $\alpha = 1$, *we have, for* $k = 1, 2, \ldots$, $\delta^{2k}(x) = 0$, *and for* $k = 0, 1, 2, \ldots$,

$$\delta^{2k+1}(x) = \frac{((-1)^{2k}+1)}{2(2k)!(2k+1)^{(2k+1)(m+1/2)}}\frac{\Gamma((2k+1)(m+1/2))}{(\Gamma(m+1/2))^{2k+1}}\delta^{(2k)}(x).$$

Proof For all $r \in \mathbb{R}$,

$$\begin{aligned}(\delta^r(x), \phi(x)) &= N\text{-}\lim_{n\to\infty}(\delta_n^r(m,x), \phi(x)) \\ &= N\text{-}\lim_{n\to\infty}\int_{-\infty}^{+\infty}\left(\frac{1}{2^{2m+1}\Gamma(m+1/2)}n^{m+1/2}x^{2m}e^{-nx^2/4}\right)^r\phi(x)\,dx, \qquad (14)\end{aligned}$$

where N is the neutrix defined by (12).

Case 1. For $r < 0$,we make the substitution $x = \sqrt{\frac{-4}{rn}}y$ in (14), and since $\phi(x) \in \mathcal{D}(\mathbb{R})$, hence

$$(\delta^r(x), \phi(x)) = N\text{-}\lim_{n\to\infty}(Qn^{m+1/2})^r\left(\frac{-4}{rn}\right)^{\frac{2mr+1}{2}}\int_{a\sqrt{\frac{-4}{rn}}}^{b\sqrt{\frac{-4}{rn}}}y^{2mr}e^{y^2}\phi\left(\sqrt{\frac{-4}{rn}}y\right)dy, \qquad (15)$$

where

$$Q = \frac{1}{2^{2m+1}\Gamma(m+1/2)}.$$

When n tends to infinity, one has $n^{\frac{r-1}{2}} \to 0$, and $\operatorname{supp}\phi \in [a,b]$. Thus, in view of (15), $\delta^r(x) = 0$ for all $r < 0$.

Case 2. For $r = 0$, it is obvious that

$$\begin{aligned}\bigl(\delta^0(x), \phi(x)\bigr) &= N\text{-}\lim_{n\to\infty}\bigl(\delta_n^0(x), \phi(x)\bigr) = N\text{-}\lim_{n\to\infty}\int_{-\infty}^{+\infty}\phi(x)\,dx \\ &= \int_{-\infty}^{+\infty}\phi(x)\,dx = \bigl(1, \phi(x)\bigr).\end{aligned} \tag{16}$$

Thus, in view of (16), $\delta^0 = 1$.

Case 3. For $0 < r < 1$, making the substitution $x = \sqrt{\frac{4}{rn}}y$ in (14), we obtain

$$\bigl(\delta_n^r(x), \phi(x)\bigr) = \bigl(Qn^{m+1/2}\bigr)^r\left(\frac{4}{rn}\right)^{\frac{2mr+1}{2}}\int_{-\infty}^{+\infty} y^{2mr}e^{-y^2}\phi\left(\sqrt{\frac{4}{rn}}y\right)dy.$$

Since $\phi \in \mathcal{D}(\mathbb{R})$, there exists a positive real number M_1 such that $\sup_{x\in\mathbb{R}}|\phi(x)| = M_1$, we have

$$\begin{aligned}\bigl|\bigl(\delta_n^r(x), \phi(x)\bigr)\bigr| &\le M_1\bigl(Qn^{m+1/2}\bigr)^r\left(\frac{4}{rn}\right)^{\frac{2mr+1}{2}}\int_{-\infty}^{+\infty} y^{2mr}e^{-y^2}\,dy \\ &\le M_1Q^r\left(\frac{4}{r}\right)^{\frac{2mr+1}{2}} n^{\frac{r-1}{2}}\frac{1}{2}\bigl(1 + (-1)^{2mr}\bigr)\Gamma(mr + 1/2),\end{aligned} \tag{17}$$

where

$$\int_{-\infty}^{+\infty} y^{2mr}e^{-y^2}\,dy = \frac{1}{2}\bigl(1 + (-1)^{2mr}\bigr)\Gamma(mr + 1/2).$$

The right-hand side of (17) tends to zero as n tends to infinity because $n^{\frac{r-1}{2}} \to 0$, hence

$$\delta^r(x) = 0 \quad \text{for all } 0 < r < 1.$$

Case 4. For $r = 1$, it is easily seen that

$$\bigl(\delta^1(x), \phi(x)\bigr) = N\text{-}\lim_{n\to\infty}\bigl(\delta_n^1(x), \phi(x)\bigr) = \bigl(\delta(x), \phi(x)\bigr).$$

Therefore, we have $\delta^1 = \delta$.

Case 5. For $r > 1$, we have

$$\begin{aligned}\bigl(\delta^r(x), \phi(x)\bigr) &= N\text{-}\lim_{n\to\infty}\int_{-\infty}^{+\infty}\delta_n^r(m,x)\phi(x)\,dx \\ &= N\text{-}\lim_{n\to\infty}\left(\int_0^{+\infty}\delta_n^r(m,x)\phi(x)\,dx + \int_{-\infty}^0\delta_n^r(m,x)\phi(x)\,dx\right) \\ &= N\text{-}\lim_{n\to\infty}\left(\int_0^{+\infty}\delta_n^r(m,x)\phi(x)\,dx + \int_0^{+\infty}\delta_n^r(m,-x)\phi(-x)\,dx\right) \\ &= N\text{-}\lim_{n\to\infty}(I_1 + I_2).\end{aligned} \tag{18}$$

Using the generalized Taylor's formula from (11), we have

$$\phi(x)=\sum_{k=0}^{s-1}c_kx^{k\alpha}+c_sx^{s\alpha}+\frac{(T_0^\alpha\phi)^{(s+1)}(\zeta)}{\alpha^{(s+1)}(s+1)!}x^{(s+1)\alpha},\quad 0<\zeta\le x, \tag{19}$$

where

$$c_k=\frac{(T_0^\alpha\phi)^{(k)}(0)}{\alpha^k k!},\quad k=0,\dots,s.$$

We denote $s\alpha=r-1$ and let $0<\alpha\le 1$. Utilizing (18) gives

$$\begin{aligned}
I_1 &= \int_0^\infty\left(\frac{1}{2^{2m+1}\Gamma(m+1/2)}n^{m+1/2}x^{2m}\exp\bigl(-nx^2/4\bigr)\right)^r\phi(x)\,dx\\
&= \int_0^\infty \delta_n^r(m,x)\left[\sum_{k=0}^{s-1}c_kx^{k\alpha}+c_sx^{s\alpha}+\frac{(T_0^\alpha\phi)^{(s+1)}(\zeta)}{\alpha^{(s+1)}(s+1)!}x^{(s+1)\alpha}\right]dx\\
&:= I_{11}+I_{12}+I_{13}.
\end{aligned}\tag{20}$$

Setting $x=\sqrt{\frac{4}{rn}}y$ and interchanging the order of integration and summation, we get

$$I_{11}=\sum_{k=0}^{s-1}c_k\bigl(Qn^{m+1/2}\bigr)^r\left(\frac{4}{rn}\right)^{\frac{2mr+k\alpha+1}{2}}\int_0^\infty y^{2mr+k\alpha}e^{-y^2}\,dy, \tag{21}$$

$$I_{12}=c_s\bigl(Qn^{m+1/2}\bigr)^r\left(\frac{4}{rn}\right)^{\frac{2mr+s\alpha+1}{2}}\int_0^\infty y^{2mr+s\alpha}e^{-y^2}\,dy, \tag{22}$$

$$I_{13}=\bigl(Qn^{m+1/2}\bigr)^r\left(\frac{4}{rn}\right)^{\frac{2mr+(s+1)\alpha+1}{2}}\int_0^\infty \frac{(T_0^\alpha\phi)^{(s+1)}(\zeta)}{\alpha^{(s+1)}(s+1)!}y^{2mr+(s+1)\alpha}e^{-y^2}\,dy, \tag{23}$$

where

$$\int_0^\infty y^{2mr+s\alpha}e^{-y^2}\,dy=\frac{1}{2}\Gamma\left(mr+\frac{s\alpha+1}{2}\right).$$

It follows immediately that

$$I_{11}=\sum_{k=0}^{s-1}c_k\left(\frac{1}{2^{2m+1}\Gamma(m+1/2)}\right)^r n^{r-k\alpha-1}\left(\frac{4}{r}\right)^{\frac{2mr+k\alpha+1}{2}}\int_0^\infty y^{2mr+k\alpha}e^{-y^2}\,dy.$$

Now taking the neutrix limit with

$$\mu=1,\qquad \lambda=r-k\alpha-1>0\quad\text{for all } k=0,\dots,s-1,$$

we conclude that

$$N\text{-}\lim_{n\to\infty}I_{11}=0. \tag{24}$$

Since $\phi(x) \in \mathcal{D}(\mathbb{R})$, there exists $M_2 > 0$ such that $\sup_{x\in\mathbb{R}} |(T_0^\alpha \phi)^{(s+1)}(x)| \le M_2$. Thus, it is not difficult to see that

$$|I_{13}| \le \frac{M_2}{\alpha^{(s+1)}(s+1)!}\bigl(Qn^{m+1/2}\bigr)^r \left(\frac{4}{rn}\right)^{\frac{2mr+(s+1)\alpha+1}{2}} \int_0^\infty y^{2mr+(s+1)\alpha} e^{-y^2}\, dy,$$

and therefore

$$|I_{13}| \to 0 \quad \text{as } n \to \infty, \qquad N\text{-}\lim_{n\to\infty} I_{13} = 0.$$

Now, we have

$$\begin{aligned} N\text{-}\lim_{n\to\infty} I_1 &= \frac{2^{(s\alpha-r)}}{r^{(mr+\frac{s\alpha+1}{2})}} \frac{\Gamma(mr+\frac{s\alpha+1}{2})}{\Gamma(m+1/2)^r \alpha^s s!}\bigl(T_0^\alpha \phi\bigr)^{(s)}(0) \\ &= \left(\frac{1}{2}\right)\left(\frac{1}{r}\right)^{(m+1/2)r} \left(\frac{\Gamma((m+1/2)r)}{(\Gamma(m+1/2))^r}\right) \frac{(T_0^\alpha \phi)^{[\frac{r-1}{\alpha}]}(0)}{\alpha^{[\frac{r-1}{\alpha}]}[\frac{r-1}{\alpha}]!}. \end{aligned} \tag{25}$$

Similarly, we have

$$N\text{-}\lim_{n\to\infty} I_2 = N\text{-}\lim_{n\to\infty}\left(\int_0^{+\infty} \delta_n^r(m,-x)\phi(-x)\, dx\right), \tag{26}$$

where

$$\begin{aligned} I_2 &= \int_0^\infty \left(\frac{1}{2^{2m+1}\Gamma(m+1/2)} n^{m+1/2} x^{2m} \exp\bigl(-nx^2/4\bigr)\right)^r \phi(-x)\, dx \\ &= \int_0^\infty \delta_n^r(m,x)\left[\sum_{k=0}^{s-1} c_k(-1)^{k\alpha} x^{k\alpha} + c_s(-1)^{s\alpha}x^{s\alpha} + (-1)^{(s+1)\alpha}\frac{(T_0^\alpha \phi)^{(s+1)}(\zeta)}{\alpha^{(s+1)}(s+1)!} x^{(s+1)\alpha}\right] dx. \end{aligned}$$

Using similar arguments, we derive that

$$N\text{-}\lim_{n\to\infty} I_2 = (-1)^{r-1} 2^{(2m+1)r}\left(\frac{1}{2}\right)^{r+1} \frac{\Gamma((m+1/2)r)}{(2^{2m}\Gamma(m+1/2))^r}\left(\frac{1}{r}\right)^{(m+1/2)r} \frac{(T_0^\alpha \phi)^{[\frac{r-1}{\alpha}]}(0)}{\alpha^{[\frac{r-1}{\alpha}]}[\frac{r-1}{\alpha}]!}. \tag{27}$$

Further, it is easily seen that from (18), (25), and (27), we have

$$\begin{aligned} \bigl(\delta^r(x), \phi(x)\bigr) &= N\text{-}\lim_{n\to\infty}(I_1 + I_2) \\ &= \frac{(1+(-1)^{r-1})}{2(r)^{(m+1/2)r}}\left(\frac{\Gamma((m+1/2)r)}{(\Gamma(m+1/2))^r}\right)\frac{(T_0^\alpha \phi)^{[\frac{r-1}{\alpha}]}(0)}{\alpha^{[\frac{r-1}{\alpha}]}[\frac{r-1}{\alpha}]!}. \end{aligned} \tag{28}$$

In particular, for $r = 1$, $\alpha = 1$,we have

$$\bigl(\delta^1(x), \phi(x)\bigr) = \phi(0).$$

For $r = 2k$, we have

$$\delta^{2k}(x) = 0.$$

For $r = 2k + 1$,

$$\delta^{2k+1}(x) = \frac{((-1)^{2k} + 1)}{2(2k)!(2k+1)^{(2k+1)(m+1/2)}} \left(\frac{\Gamma((2k+1)(m+1/2))}{(\Gamma(m+1/2))^{2k+1}} \right) \delta^{(2k)}(x)$$

for all $k = 0, 1, 2, \ldots$. This completes the proof. □

Remark 17 We would like to point out that Theorem 16 is a generalization of Theorem 10 obtained in [9], where the case for $r \in \mathbb{R}$ is mainly discussed. In both theorems, the even powers of $\delta(x)$ turn out to be zero, while the odd powers are expressible as a constant multiple of a derivative of $\delta(x)$.

We conclude this section by the following example.

Example 18 The choice of $\alpha \in (0,1]$, r, and m in Theorem 16, we have

$$\sqrt{\delta(x)} = 0, \qquad \delta^2(x) = 0,$$

and if $r = 3$, then

$$\delta^3(x) = \frac{1}{3^{3m+\frac{3}{2}}} \left(\frac{\Gamma(3(m+1/2))}{(\Gamma(m+1/2))^3} \right) \frac{(T_0^\alpha \phi)^{[\frac{2}{\alpha}]}(0)}{\alpha^{[\frac{2}{\alpha}]}[\frac{2}{\alpha}]!}.$$

Using the asymptotic formula (see [21, p. 257])

$$\Gamma(pz+q) \simeq \sqrt{2\pi} e^{-pz} (pz)^{pz+q-1/2} \quad \text{for all } p > 0, q \in \mathbb{R}, |\arg z| < \pi, \tag{29}$$

with $z = 1$, we have

$$\delta^3(x) = \frac{m}{2\pi\sqrt{3}} \frac{(T_0^\alpha \phi)^{[\frac{2}{\alpha}]}(0)}{\alpha^{[\frac{2}{\alpha}]}[\frac{2}{\alpha}]!}.$$

When $\alpha = 1$,

$$\delta^3(x) = \frac{m}{4\pi\sqrt{3}} \phi''(0), \quad m \in \mathbb{N}.$$

If $r = \frac{3}{2}$, we have

$$\delta^{\frac{3}{2}}(x) = (1+i) \left(\frac{2^{\frac{3}{2}m-\frac{1}{4}}}{3^{\frac{3}{2}(m+1/2)}} \right) \left(\frac{\Gamma(\frac{3}{2}(m+1/2))}{(\Gamma(m+1/2))^{\frac{3}{2}}} \right) \frac{(T_0^\alpha \phi)^{[\frac{1}{2\alpha}]}(0)}{\alpha^{[\frac{1}{2\alpha}]}[\frac{1}{2\alpha}]!}.$$

From (29), we conclude that

$$\delta^{\frac{3}{2}}(x) = (1+i) \left(\frac{1}{2\sqrt{3}} \right) \left(\frac{2m}{\pi} \right)^{1/4} \frac{(T_0^\alpha \phi)^{[\frac{1}{2\alpha}]}(0)}{\alpha^{[\frac{1}{2\alpha}]}[\frac{1}{2\alpha}]!}.$$

4 Defining $(\delta')^r(x)$ for all $r \in \mathbb{R}$

In this section, we give a definition of $(\delta')^r(x)$ for any $r \in \mathbb{R}$.

Theorem 19 *For any positive integer m, the distribution $(\delta')^r(x)$ for $r \in \mathbb{R}$ is defined by*

1. *For $r = 0 : (\delta')(x) = 1$.*
2. *For $r < \frac{1}{2}$ and $r \neq 0 : (\delta')^r(x) = 0$.*
3. *For $r \geq \frac{1}{2}$:*

$$\left(\delta'\right)^r(x) = \frac{(1 + (-1)^{3r-1})}{(\Gamma(m + 1/2))^r}\left(\frac{1}{r}\right)^{(m+1/2)r}\left(\frac{(T_0^\alpha \phi)^{[\frac{2r-1}{\alpha}]}(0)}{\alpha^{[\frac{2r-1}{\alpha}]}[\frac{2r-1}{\alpha}]!}\right)\mathcal{J}(r,m), \quad 0 < \alpha \leq 1,$$

where $\mathcal{J}(r,m)$ is defined below by (36) and T_0^α is the conformable derivative at $a = 0$. In particular, when $\alpha = 1$, we have for $k = 1, 2, \ldots$ $(\delta')^{2k}(x) = 0$ and

$$\left(\delta'\right)^{2k+1}(x) = \frac{((-1)^{6k+2} + 1)}{(4k + 1)!(\Gamma(m + 1/2))^{2k+1}}\left(\frac{1}{2k + 1}\right)^{(m+1/2)(2k+1)}\left(\delta^{(4k+1)}(x)\right)\mathcal{J}(2k + 1, m).$$

Proof Considering the derivative of the δ-sequence (5), we have

$$\delta_n'(m,x) = \frac{1}{2^{2m+1}\Gamma(m + 1/2)}n^{m+1/2}\left(2m - \frac{n}{2}x^2\right)x^{2m-1}e^{-nx^2/4}.$$

For all $r \in \mathbb{R}$, we have

$$\left(\left(\delta'\right)^r(x), \phi(x)\right) = N\text{-}\lim_{n\to\infty}\int_{-\infty}^{+\infty}\left(Qn^{m+1/2}\right)^r\left(2m - \frac{n}{2}x^2\right)^r x^{(2m-1)r}e^{-nx^2/4}\phi(x)\,dx. \tag{30}$$

Case 1. For $r < 0$, making the substitution $x = \sqrt{-\frac{4}{nr}}y$ in (30), we have

$$\begin{aligned}\left(\left(\delta'\right)^r, \phi\right) &= N\text{-}\lim_{n\to\infty}\left(Qn^{m+1/2}\right)^r\left(\frac{-4}{rn}\right)^{\frac{(2m-1)r+1}{2}} \\ &\quad \times \int_{a\sqrt{\frac{-4}{rn}}}^{b\sqrt{\frac{-4}{rn}}}\left(2m - \frac{n}{2}\left(\frac{-4}{nr}\right)y^2\right)^r y^{(2m-1)r}e^{y^2}\phi\left(\sqrt{\frac{-4}{rn}}y\right)dy.\end{aligned}$$

When n tends to ∞, one has $n^{\frac{2r-1}{2}} \to 0$, $\operatorname{supp}\phi \in [a,b]$. Thus, $((\delta')^r(x), \phi(x)) = 0$. Therefore,

$$\left(\delta'\right)^r(x) = 0 \quad \text{for all } r < 0.$$

Case 2. For $r = 0$, clearly, we have

$$\left(\left(\delta'\right)^0(x), \phi(x)\right) = N\text{-}\lim_{n\to\infty}\int_{-\infty}^{+\infty}\phi(x)\,dx = \left(1, \phi(x)\right).$$

Thus,

$$\left(\delta'\right)^0(t) = 1.$$

Case 3. For $0 < r < \frac{1}{2}$,

$$\left((\delta')^r(x),\phi(x)\right) = N\text{-}\lim_{n\to\infty}\int_{-\infty}^{+\infty}\left(Qn^{m+1/2}\right)^r\left(2m-\frac{n}{2}x^2\right)^r x^{(2m-1)r}e^{-nrx^2/4}(x)\,dx.$$

By making the substitution $x = \sqrt{\frac{4}{nr}}y$, it follows that

$$\begin{aligned}\left((\delta')^r,\phi\right) &= N\text{-}\lim_{n\to\infty}\left((\delta'_n)^r(x),\phi(x)\right)\\ &= N\text{-}\lim_{n\to\infty}\left(Qn^{m+1/2}\right)^r\left(\frac{4}{rn}\right)^{\frac{(2m-1)r+1}{2}}\\ &\quad\times\int_{a\sqrt{\frac{4}{nr}}}^{b\sqrt{\frac{4}{nr}}}\left(2m-\frac{2}{r}y^2\right)^r y^{(2m-1)r}e^{-y^2}\phi\left(\sqrt{\frac{4}{rn}}y\right)dy.\end{aligned}$$

Since

$$\left|\left((\delta'_n)^r(x),\phi(x)\right)\right| \le M_3Q\left(\frac{4}{r}\right)^{\frac{(2m-1)r+1}{2}}n^{r-\frac{1}{2}}\int_{a\sqrt{\frac{4}{nr}}}^{b\sqrt{\frac{4}{nr}}}\left(2m-\frac{2}{r}y^2\right)^r y^{(2m-1)r}e^{-y^2}\,dy \to 0,$$

as $n\to\infty$,where $M_3 = \sup_{x\in\mathbb{R}}|\phi(x)|$, we deduce that

$$(\delta')^r(x) = 0 \quad \text{for all } 0 < r < \frac{1}{2}.$$

Case 4. For $r \ge \frac{1}{2}$,

$$\begin{aligned}(\delta')^r(x),\phi(x)) &= N\text{-}\lim_{n\to\infty}(\delta'_n)^r(x),\phi(x)) = N\text{-}\lim_{n\to\infty}\int_{-\infty}^{+\infty}(\delta'_n)^r(x)\phi(x)\,dx\\ &= N\text{-}\lim_{n\to\infty}\left(\int_0^{+\infty}(\delta'_n)^r(x)\phi(x)\,dx+\int_{-\infty}^{0}(\delta'_n)^r(x)\phi(x)\,dx\right)\\ &= N\text{-}\lim_{n\to\infty}\left(\int_0^{+\infty}(\delta'_n)^r(x)\phi(x)\,dx+\int_0^{+\infty}(\delta'_n)^r(-x)\phi(-x)\,dx\right)\\ &= N\text{-}\lim_{n\to\infty}(J_1+J_2),\end{aligned} \tag{31}$$

where

$$\begin{aligned}J_1 &= \int_0^{+\infty}(\delta'_n)^r(x)\phi(x)\,dx\\ &= \int_0^{+\infty}\left(Qn^{m+1/2}\right)^r\left(2m-\frac{n}{2}x^2\right)^r x^{(2m-1)r}e^{-nrx^2/4}\phi(x)\,dx.\end{aligned}$$

By the generalized Taylor's formula from (11), we obtain

$$\begin{aligned}J_1 &= \int_0^{+\infty}(\delta')^r(x)\left[\sum_{k=0}^{s-1}c_kx^{k\alpha}+c_sx^{s\alpha}+\frac{(T_0^\alpha\phi)^{(s+1)}(\zeta)}{\alpha^{(s+1)}(s+1)!}x^{(s+1)\alpha}\right]dx\\ &:= J_{11}+J_{12}+J_{13}.\end{aligned} \tag{32}$$

Setting $x = \sqrt{\frac{4}{rn}}y$ again, we get

$$J_{11} = \sum_{k=0}^{s-1} c_k \left(Qn^{m+1/2}\right)^r \left(\frac{4}{rn}\right)^{\frac{(2m-1)r+k\alpha+1}{2}} \int_0^{+\infty} \left(2m - \frac{2}{r}y^2\right)^r y^{(2m-1)r+k\alpha} e^{-y^2}\, dy, \tag{33}$$

$$J_{12} = c_s \left(Qn^{m+1/2}\right)^r \left(\frac{4}{rn}\right)^{\frac{(2m-1)r+s\alpha+1}{2}} \int_0^{\infty} \left(2m - \frac{2}{r}y^2\right)^r y^{(2m-1)r+s\alpha} e^{-y^2}\, dy, \tag{34}$$

$$J_{13} = \left(Qn^{m+1/2}\right)^r \left(\frac{4}{rn}\right)^{\frac{\beta+1}{2}} \int_0^{\infty} \left(2m - \frac{2}{r}y^2\right)^r \frac{(T_0^{\alpha}\phi)^{(s+1)}(\zeta)}{\alpha^{(s+1)}(s+1)!} y^{\beta} e^{-y^2}\, dy, \tag{35}$$

where

$$\beta = (2m-1)r + (s+1)\alpha.$$

When $s\alpha = 2r-1$, we define the function

$$\mathcal{J}(r,m) = \int_0^{\infty} \left(2m - \frac{2}{r}y^2\right)^r y^{(2m+1)r-1} e^{-y^2}\, dy = 2^{3+r} e^{-mr} m^r \frac{\nu_1 + \nu_2 + \nu_3}{\nu_0}, \tag{36}$$

where

$$\begin{aligned}
\nu_1 &= r\left(-\frac{1}{mr}\right)^{\frac{1}{2}+r} \Gamma(-r)\Gamma\big((m+3/2)r\big)\left(\left(\frac{3}{4}+m\right)r - m - \frac{1}{2}\right)\big(1 + (m+3/2)\big)h_1, \\
h_1 &= \text{hypergeom}\big(\big[1-(m+1/2)r\big], \big[3-(m+3/2)r\big], mr\big),
\end{aligned}$$

and

$$\nu_2 = \frac{1}{4}\left(-\frac{1}{mr}\right)^{\frac{1}{2}-(2m+1)r} \Gamma\big(-(m+3/2)r\big)\big(\big((3+2m)r\big)^2 - 6(3+2m)r + 8\big)[h_{21} + h_{22}],$$

where

$$\begin{aligned}
h_{21} &= r(m+1)\Gamma\big((m+1/2)r\big)\,\text{hypergeom}\big([1+r], \big[2+(m+3/2)r\big], mr\big), \\
h_{22} &= \big(1 + (m+1/2)r\big)\Gamma\big((m+1/2)r\big)\,\text{hypergeom}\big([r], \big[2+(m+3/2)r\big], mr\big),
\end{aligned}$$

and

$$\begin{aligned}
\nu_3 &= \frac{3}{2}\left(-\frac{1}{mr}\right)^{\frac{1}{2}+r} \Gamma(-r)\Gamma\big((m+3/2)r\big)\left(r - \frac{2}{3}\right)\big(1 + (m+3/2)r\big)(r-2)h_3, \\
h_3 &= \text{hypergeom}\big(\big[-(m+1/2)r\big], \big[3-(m+3/2)r\big], mr\big),
\end{aligned}$$

and

$$\nu_0 = \left(-\frac{1}{mr}\right)^{\frac{1}{2}} \Gamma(-r)\big(2\big((3+2m)r\big)^3 - 8\big((3+2m)r\big)^2 - 8(3+2m)r + 32\big).$$

From (33), we have

$$J_{11} = \sum_{k=0}^{s-1} c_k Q^r n^{r-\frac{k\alpha+1}{2}} \left(\frac{4}{r}\right)^{\frac{(2m-1)r+k\alpha+1}{2}} \int_0^\infty \left(2m - \frac{2}{r}y^2\right)^r y^{(2m-1)r+k\alpha} e^{-y^2}\, dy. \tag{37}$$

Using the neutrix limit, we have

$$N\text{-}\lim_{n\to\infty} J_{11} = 0, \tag{38}$$

when $\mu = 1$ and $\lambda = 2r - k\alpha - 1 > 0$. Since $\phi(x) \in \mathcal{D}(\mathbb{R})$, there exists a positive real M_4 such that $\sup_{x\in\mathbb{R}^+} |(T_0^\alpha \phi)^{(s+1)}(x)| \le M_4$. Note that if y is fixed and n tends to infinity, we have

$$\begin{aligned} |J_{13}| &\le \frac{M_4}{\alpha^{(s+1)}(s+1)!} \big(Qn^{m+1/2}\big)^r \left(\frac{4}{rn}\right)^{\frac{\beta+1}{2}} \int_0^\infty \left(2m - \frac{2}{r}y^2\right)^r y^\beta e^{-y^2}\, dy \\ &= \frac{M_4 Q^r (\frac{4}{r})^{\frac{\beta+1}{2}}}{\alpha^{(s+1)}(s+1)!} n^{r-\frac{(s+1)\alpha+1}{2}} \int_0^\infty \left(2m - \frac{2}{r}y^2\right)^r y^\beta e^{-y^2}\, dy \to 0, \end{aligned}$$

with $s\alpha = 2r - 1$, $s \in Z^+$, and $0 < \alpha \le 1$. Thus, we conclude that

$$N\text{-}\lim_{n\to\infty} J_{13} = 0. \tag{39}$$

For J_{12}, we use formula (36) to imply that

$$J_{12} = c_s \big(Qn^{m+1/2}\big)^r \left(\frac{4}{rn}\right)^{\frac{(2m-1)r+s\alpha+1}{2}} \mathcal{J}(s\alpha+1, m). \tag{40}$$

Thus,

$$J_{12} = \left(\frac{1}{2^{2m+1}\Gamma(m+1/2)}\right)^r \left(\frac{4}{r}\right)^{\frac{(2m-1)r+s\alpha+1}{2}} \left(\frac{(T_0^\alpha\phi)^{[\frac{2r-1}{\alpha}]}(0)}{\alpha^{[\frac{2r-1}{\alpha}]}[\frac{2r-1}{\alpha}]!}\right) \mathcal{J}(s\alpha+1, m). \tag{41}$$

It follows from (32), (38), (39), and (41) that

$$N\text{-}\lim_{n\to\infty} J_1 = \left(\frac{1}{2^{2m+1}\Gamma(m+1/2)}\right)^r \left(\frac{4}{r}\right)^{\frac{(2m-1)r+2r}{2}} \left(\frac{(T_0^\alpha\phi)^{[\frac{2r-1}{\alpha}]}(0)}{\alpha^{[\frac{2r-1}{\alpha}]}[\frac{2r-1}{\alpha}]!}\right) \mathcal{J}(r, m). \tag{42}$$

A similar treatment for

$$J_2 = \int_0^{+\infty} (\delta')^r(-x) \left[\sum_{k=0}^{s-1} c_k(-x)^{k\alpha} + c_s^{s\alpha}(-x)^s + \frac{(T_0^\alpha\phi)^{(s+1)}(\zeta)}{\alpha^{(s+1)}(s+1)!}(-x)^{(s+1)\alpha}\right] dx. \tag{43}$$

The proof of (43) is similar to the one of (42).

$$\begin{aligned} N\text{-}\lim_{n\to\infty} J_2 &= (-1)^{s\alpha} \left(\frac{1}{2^{2m+1}\Gamma(m+1/2)}\right)^r \left(\frac{4}{r}\right)^{\frac{(2m-1)r+s\alpha+1}{2}} \\ &\quad \times \left(\frac{(T_0^\alpha\phi)^{[\frac{2r-1}{\alpha}]}(0)}{\alpha^{[\frac{2r-1}{\alpha}]}[\frac{2r-1}{\alpha}]!}\right) \mathcal{J}(s\alpha+1, m). \end{aligned} \tag{44}$$

Finally, from (42) and (44), we have

$$\begin{aligned}\left((\delta')^r(x),\phi(x)\right) &= N\text{-}\lim_{n\to\infty}\left((\delta'_n)^r(x),\phi(x)\right)\\ &= \frac{(1+(-1)^{3r-1})}{(2^{2m+1}\Gamma(m+1/2))^r}\left(\frac{4}{r}\right)^{\frac{(2m+1)r}{2}}\left(\frac{(T_0^\alpha\phi)^{[\frac{2r-1}{\alpha}]}(0)}{\alpha^{[\frac{2r-1}{\alpha}]}[\frac{2r-1}{\alpha}]!}\right)\mathcal{J}(r,m). \qquad (45)\end{aligned}$$

In particular, for $r = 1$ and $\alpha = 1$, we have

$$\begin{aligned}\left((\delta')(x),\phi(x)\right) &= \left(\frac{(1+(-1)^{3-1})}{2^{2m+1}\Gamma(m+1/2)}\right)\left(4^{\frac{(2m+1)}{2}}\right)\left(-\frac{1}{2}\Gamma(m+1/2)\right)\phi'(0)\\ &= -\phi'(0).\end{aligned}$$

It follows that

$$(\delta')^{2k}(x) = 0 \quad \text{for } k = 1,2,\ldots.$$

When $\alpha = 1$, we have

$$(\delta')^{2k+1}(x) = \frac{((-1)^{6k+2}+1)}{(4k+1)!(\Gamma(m+1/2))^{2k+1}}\left(\frac{1}{2k+1}\right)^{(m+1/2)(2k+1)}\left(\delta^{(4k+1)}(x)\right)\mathcal{J}(2k+1,m)$$

for all $k = 0,1,2,\ldots$. This completes the proof. □

Remark 20 We should note that Theorem 19 is the generalization of Theorem 11 in [9], where the case for $r \in \mathbb{R}$ is mainly considered. In both theorems, the even powers of $\delta'(x)$ turn out to be zero, while the odd powers are expressible as a constant multiple of a derivative of $\delta(x)$.

We end this section by the following example.

Example 21 The choice of $\alpha \in (0,1]$, r, and m in Theorem 19, we have

$$(\delta')^3(x) = \frac{2}{(\Gamma(m+1/2))^3}\left(\frac{1}{3}\right)^{3(m+1/2)}\left(\frac{(T_0^\alpha\phi)^{[\frac{5}{\alpha}]}(0)}{\alpha^{[\frac{5}{\alpha}]}[\frac{5}{\alpha}]!}\right)\mathcal{J}(3,m).$$

Using formula (29) with $z = 1$, we have

$$\delta')^3(x) = -\frac{m(52m+35)}{18\pi\sqrt{3}}\left(\frac{(T_0^\alpha\phi)^{[\frac{5}{\alpha}]}(0)}{\alpha^{[\frac{5}{\alpha}]}[\frac{5}{\alpha}]!}\right).$$

When $\alpha = 1$,

$$(\delta')^3(x) = -\frac{m(52m+35)}{2160\pi\sqrt{3}}\delta^{(5)}(x).$$

5 Defining $\delta^r(x^\lambda)$ for all $r \in \mathbb{R}$, $\lambda \in \mathbb{R}_+^*$

Theorem 22 (Generalization of Theorem 10) *For any positive integer m and $\lambda r - \rho \geq 0$:*

$$\left(\delta^r\left(x^\lambda\right), \phi\left(x^\rho\right)\right) = \frac{\rho^2(1+(-1)^{\frac{\lambda r-\rho}{\rho}})}{2\lambda 2^{(2m+1)r}} \frac{\Gamma((m+1/2)r)}{(\Gamma(m+1/2))^r} \left(\frac{4}{r}\right)^{(m+1/2)r} \frac{(T_0^\alpha \phi)^{[\frac{\lambda r-\rho}{\alpha\rho}]}(0)}{(\alpha\rho)^{[\frac{\lambda r-\rho}{\alpha\rho}]}[\frac{\lambda r-\rho}{\alpha\rho}]!},$$

where $\lambda, \rho \in \mathbb{R}_+^*$, $\alpha \in (0,1]$ and T_0^α is the conformable derivative at $a = 0$ and $[r]$ is the smallest integer greater than or equal to r. In particular, for $r = 1$, $\rho = 1$, $\alpha = 1$, we have when $\lambda = 2k$, $\delta(x^{2k}) = 0$ for $k = 1, 2, \ldots$ and when $\lambda = 2k+1$, for $k = 0, 1, 2, \ldots,$

$$\delta\left(x^{2k+1}\right) = \frac{1}{(2k+1)(2k)!}\phi^{(2k)}(0).$$

Proof For all $r, \lambda, \rho \in \mathbb{R}_+^*$,

$$\begin{aligned}\left(\delta^r\left(x^\lambda\right), \phi\left(x^\rho\right)\right) &= N\text{-}\lim_{n\to\infty}\left(\delta_n^r\left(m, x^\lambda\right), \phi\left(x^\rho\right)\right) \\ &= N\text{-}\lim_{n\to\infty}\int_{-\infty}^{+\infty}\left(\frac{n^{m+1/2}}{2^{2m+1}\Gamma(m+1/2)}x^{2\lambda m}e^{-nx^{2\lambda}/4}\right)^r \phi\left(x^\rho\right)dx. \end{aligned} \tag{46}$$

Setting

$$y = x^\rho,$$

we obtain

$$\left(\delta^r\left(x^\lambda\right), \phi\left(x^\rho\right)\right) = N\text{-}\lim_{n\to\infty}\int_{-\infty}^{+\infty}\rho\left(\frac{n^{m+1/2}}{2^{2m+1}\Gamma(m+1/2)}\right)^r y^{\frac{2\lambda mr+s\alpha\rho+\rho-1}{\rho}} e^{-\frac{nr}{4}y^{\frac{2\lambda}{\rho}}})\phi(y)\,dy.$$

Making the change of variables

$$\frac{nr}{4}y^{\frac{2\lambda}{\rho}} = z^{\frac{2\lambda}{\rho}} \quad\Longrightarrow\quad y = \left(\frac{4}{nr}\right)^{\frac{\rho}{2\lambda}} z,$$

we get

$$\left(\delta^r\left(x^\lambda\right), \phi\left(x^\rho\right)\right) = N\text{-}\lim_{n\to\infty}\int_{-\infty}^{+\infty}\rho\left(\frac{n^{m+1/2}}{2^{2m+1}\Gamma(m+1/2)}\right)^r z^{\frac{2\lambda mr+s\alpha\rho+\rho-1}{\rho}} e^{-z^{\frac{2\lambda}{\rho}}})\phi(z)\,dz. \tag{47}$$

Setting

$$s\alpha = \frac{\lambda r - 2\rho + 1}{\rho},$$

we obtain

$$\int_0^{+\infty} z^{\frac{2m\lambda r+s\alpha\rho+\rho-1}{\rho}} e^{-z^{2\lambda}}\,dz = \frac{\rho}{2\lambda}\Gamma\left((m+1/2)r\right). \tag{48}$$

Using the arguments similar to those in the proofs of the previous theorems, we reach

$$\left(\delta^r\left(x^\lambda\right),\phi\left(x^\rho\right)\right)=\frac{\rho^2(1+(-1)^{\frac{\lambda r-\rho}{\rho}})}{2\lambda 2^{(2m+1)r}}\frac{\Gamma((m+1/2)r)}{(\Gamma(m+1/2))^r}\left(\frac{4}{r}\right)^{(m+1/2)r}\frac{(T_0^\alpha\phi)^{[\frac{\lambda r-\rho}{\alpha\rho}]}(0)}{(\alpha\rho)^{[\frac{\lambda r-\rho}{\alpha\rho}]}[\frac{\lambda r-\rho}{\alpha\rho}]!}.$$

In particular, for $r=1$, $\lambda=1$, $\alpha=1$, $\rho=1$, we have

$$\left(\delta^1(x),\phi(x)\right)=\phi(0),$$

for $\lambda=2k$, it reads

$$\delta^r\left(x^{2k}\right)=0,$$

and for $\lambda=2k+1$,

$$\begin{aligned}\left(\delta^r\left(x^{2k+1}\right),\phi(x)\right)&=\frac{\rho^2(1+(-1)^{\frac{(2k+1)r-\rho}{\rho}})}{2(2k+1)2^{(2m+1)r}}\frac{\Gamma((m+1/2)r)}{(\Gamma(m+1/2))^r}\left(\frac{4}{r}\right)^{(m+1/2)r}\\&\quad\times\frac{(T_0^\alpha\phi)^{[\frac{(2k+1)r-\rho}{\alpha\rho}]}(0)}{(\alpha\rho)^{[\frac{(2k+1)r-\rho}{\alpha\rho}]}[\frac{(2k+1)r-\rho}{\alpha\rho}]!}\end{aligned}$$

for all $k=0,1,2,\ldots$. □

Example 23 Using formula (29) with $z=1$,we have

$$\left(\delta^r\left(x^\lambda\right),\phi\left(x^\rho\right)\right)=\frac{\rho^2(1+(-1)^{\frac{\lambda r-\rho}{\rho}})}{2\lambda 2^{(2m+1)r}}\frac{\Gamma((m+1/2)r)}{(\Gamma(m+1/2))^r}\left(\frac{4}{r}\right)^{(m+1/2)r}\frac{(T_0^\alpha\phi)^{[\frac{\lambda r-\rho}{\alpha\rho}]}(0)}{(\alpha\rho)^{[\frac{\lambda r-\rho}{\alpha\rho}]}[\frac{\lambda r-\rho}{\alpha\rho}]!}.$$

If $r=1$, $\lambda=1$, $\rho=1$,

$$\left(\delta(x),\phi(x)\right)=\phi(0).$$

If $r=1$, $\lambda=2$, $\rho=1$,

$$\left(\delta\left(x^2\right),\phi(x)\right)=0. \tag{49}$$

If $r=1$, $\lambda=3$, $\rho=1$, $\alpha=1$,

$$\left(\delta\left(x^3\right),\phi(x)\right)=\frac{1}{3}\frac{(T_0^\alpha\phi)^{[\frac{\lambda r-\rho}{\alpha\rho}]}(0)}{(\alpha\rho)^{[\frac{\lambda r-\rho}{\alpha\rho}]}[\frac{\lambda r-\rho}{\alpha\rho}]!}=\frac{1}{6}\phi''(0). \tag{50}$$

If $r=1$, $\lambda=3$, $\rho=3$, $\alpha=1$,

$$\left(\delta\left(x^3\right),\phi\left(x^3\right)\right)=3\phi(0).$$

Remark 24 (49) was found in [8] and [22], and (50) was found in [22].

Remark 25 We would like to point out that Theorem 22 is the generalization of Theorem 10 obtained in [9], where the case for $r \in \mathbb{R}$ is mainly discussed.

We would like to mention that one can also find $\ln(\delta(x))$ using the δ-sequence and the neutrix limit to derive the following identity:

$$\ln\delta(x) = \ln\left(\frac{1}{2^{2m+1}\Gamma(m+1/2)}\right) + \ln x^{2m}. \tag{51}$$

Indeed,

$$\begin{aligned}
&\bigl(\ln\delta(x), \phi(x)\bigr) \\
&\quad = N\text{-}\lim_{n\to\infty}\bigl(\ln\delta_n(x), \phi(x)\bigr) \\
&\quad = N\text{-}\lim_{n\to\infty}\left(\ln\left(\frac{1}{2^{2m+1}\Gamma(m+1/2)}n^{m+1/2}x^{2m}e^{-nx^2/4}\right), \phi(x)\right) \\
&\quad = N\text{-}\lim_{n\to\infty}\left(\left(\ln\left(\frac{1}{2^{2m+1}\Gamma(m+1/2)}\right) + \ln n^{m+1/2} + \ln x^{2m} + \ln e^{-nx^2/4}\right), \phi(x)\right) \\
&\quad = N\text{-}\lim_{n\to\infty}\left(\ln\left(\frac{1}{2^{2m+1}\Gamma(m+1/2)}\right) + \ln x^{2m}, \phi(x)\right) \\
&\quad = \left(\ln\left(\frac{1}{2^{2m+1}\Gamma(m+1/2)}\right) + \ln x^{2m}, \phi(x)\right).
\end{aligned}$$

6 Conclusion

The powers of the delta function and its derivatives have potential applications in elementary physics and quantum mechanics. In addition, finding such powers is a difficult task. In this manuscript we used a certain δ-sequence, neutrix limits, and the conformable derivatives to define powers of the δ function and its derivatives. Some results obtained in the manuscript are generalizations of some results found in the literature. However, it might be very interesting if the recently introduced fractional derivatives with nonsingular kernels were used to define the powers of δ-function.

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Authors' contributions

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