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On deformation of 3-dimensional certain minimal Legendrian submanifolds

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Abstract. A minimal Legendrian submanifold in a Sasakian manifold is by definition a Legendrian submanifold in a Sasakian manifold which is a minimal submanifold in the sense of vanishing mean curvature vector field. The *minimal Legendrian deformation* means a smooth family of minimal Legendrian submanifolds.

In this note we discuss minimal Legendrian deformations of certain 3-dimensional compact minimal Legendrian submanifolds embedded in the 7-dimensional standard Einstein Sasakian manifolds, 7-dimensional unit sphere $S^7(1)$ and Stiefel manifold $V_2(\mathbf{R}^5)$. We prove that all non-trivial minimal Legendrian deformations of a certain non-totally geodesic minimal Legendrian orbit of $SU(2)$ in $S^7(1)$ are given by the 7-dimensional family of minimal Legendrian submanifolds which is constructed by the group action of $Sp(2,\mathbf{C})$. Moreover we show that a 3-dimensional compact minimal Legendrian submanifold $SO(3)/(\mathbf{Z}_2+\mathbf{Z}_2)$ in $V_2(\mathbf{R}^5)$ with constant positive sectional curvature has no nontrivial minimal Legendrian deformation.

Introduction

A smooth immersion $\psi : L \to M$ of a smooth manifold L into a contact manifold (M, η) is called *Legendrian immersion* if $\dim L = m$ and $\psi^*\eta = 0$. A *Legendrian deformation* of ψ is defined as a smooth family $\{\psi_t\}$ of Legendrian immersions $\psi_t : L \to M$ with $\psi_0 = \psi$. Let $(M^{2m+1}, \eta, g, \xi, \varphi)$ be a Sasakian manifold with the Sasakian structure (g, η, ξ, φ). A *minimal Legendrian submanifold* of a Sasakian manifold is a Legendrian submanifold relative to its contact structure which is a minimal submanifold with respect to the Riemannian metric of the Sasakian structure in the sense of vanishing mean curvature vector field, or equivalently extremal volume under any compactly supported smooth variations.

It is a natural question whether a given compact minimal Legendrian submanifold in a specfic Sasakian manifold can be deformed into a family of compact minimal Legendrian submanifolds or not. The *minimal Legendrian deformation* means a one-parameter smooth family of compact minimal Legendrian submanifolds. A

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minimal Legendrian deformation is said to be *trivial* if the minimal Legendrian deformation is induced by the automorphisms of the ambient Sasakian manifold.

Question. Determine all minimal Legendrian deformations of a given compact minimal Legendrian submanifold L in a Sasakian manifold.

The theory of minimal Legendrian deformations of compact minimal Legendrian submanifolds works well in the case when the ambient Sasakian manifold is an η-Einstein Sasakian manifold (see Section 3). It is known the standard construction of η-Einstein and Einstein Sasakian manifolds from Einstein-Kähler manifolds with positive Einstein constant, and Einstein Sasakian manifolds provide Ricci-flat Kähler cone metrics (cf. Section 2). In the construction minimal Legendrian submanifolds corresponds to both minimal Lagrangian submanifolds in Einstein-Kähler manifolds with positive Einstein constant and special Lagrangian subcones in Ricci-flat Kähler cone.

The purpose of this note is to discuss minimal Legendrian deformations of 3-dimensional certain compact minimal Lagrangian submanifolds in the 7-dimensional standard Einstein-Sasakian manifolds such as the 7-dimensional unit standard sphere $S^7(1)$ and the 7-dimensional Stiefel manifold $V_2(\mathbf{R})$ of orthonormal 2-frames in $\mathbf{R}^5$. Three examples will be treated. The simplest example should be a 3-dimensional totally geodesic Legendrian submanifold $S^3(1)$ embedded in $S^7(1)$ and we show that it has no non-trivial minimal Legendrian deformation (see Proposition 4.1).

Let (V_3, ρ_3) be the irreducible unitary representation of $SU(2)$ of degree 3. As the first non-trivial example, we know a non-totally geodesic minimal Legendrian orbit $L^3 := \rho_3(SU(2))(w)$ of $SU(2)$ in $S^7(1) \subset V_3 \cong \mathbf{C}^4$ (see Subsection 4.2, cf. [14]). One of main results in this note is as follows (see Theorem 4.1) :

Theorem. *All* non-trivial *minimal Legendrian deformations of* $L^3 = \rho_3(SU(2))(w) \subset S^7(1)$ *are given by the* 7*-dimensional family of minimal Legendrian submanifolds which is constructed by the group action of* $Sp(2, \mathbf{C})$*.*

Let $N^3 = SO(3)/(\mathbf{Z}_2 + \mathbf{Z}_2) \subset S^4(1)$ be a 3-dimensional isoparametric hypersurface embedded in $S^4(1)$ with 3 distinct principal curvatures, which is one of so called *Cartan hypersurfaces*. The second example is its Legendrian lift to $V_2(\mathbf{R}^5)$ which is a compact embedded minimal Legendrian submanifold $L^3 = SO(3)/(\mathbf{Z}_2 + \mathbf{Z}_2) \subset V_2(\mathbf{R}^5)$ whose induced metric from the Einstein Sasakian metric of $V_2(\mathbf{R}^5)$ is of constant positive sectional curvature. Our another result is as follows (see Theorem 4.2) :

Theorem. $L^3 = SO(3)/(\mathbf{Z}_2 + \mathbf{Z}_2) \subset V_2(\mathbf{R}^5)$ *has no non-trivial minimal Legendrian deformation.*

In Section 1 we shall prepare fundamental properties and formulas for Legendrian submanifolds in a contact manifold, the notion of Legendrian deformations and a Banach manifold structure of the space of Legendrian submanifolds. In Section 2 we shall describe differential geometry of Legendrian submanifolds in Sasakian manifolds and the notion of minimal Legendrian deformations. In Section 3 we shall describe a general theory of minimal Legendrian deformations for minimal Legendrian submanifolds in η-Einstein Sasakian manifolds. Section 4 we shall discuss minimal Legendrian deformation problem for three examples of 3-dimensional compact minimal Legendrian submanifolds in the 7-dimensional unit standard sphere $S^7(1)$ and the 7-dimensional Stiefel manifold $V_2(\mathbf{R}^5)$.

In the forthcoming paper we shall discuss these problems, results and their generalizations in detail.

1 Legendrian submanifolds and Legendrian deformations

Let (M^{2m+1}, η) be a $(2m+1)$-dimensional contact manifold with contact 1-form η and $\psi : L \longrightarrow M^{2m+1}$ be a smooth immersion a connected smooth manifold L into M^{2m+1}.

Definition 1.1. ψ is called a *Legendrian immersion* if

1. $\psi^*\eta = 0$,

2. $\dim L = m$.

For any $V \in C^\infty(\psi^{-1}TM)$, we define a 1-form $\alpha_V \in \Omega^1(L)$ on L by

$$\alpha_V(X) := -\frac{1}{2} d\eta(V, \psi_*(X)).$$

for each $X \in TL$. If ψ is a Legendrian immersion, then we have the canonical linear isomorphism

$$\chi : \varphi^{-1}TM/\varphi_*TL \ni v \longmapsto (\eta(v), \alpha_v) \in \mathbf{R} \oplus T^*L\,.$$

Let $\psi_t : L \longrightarrow M^{2m+1}$ be a smooth family of immersions with a Legendrian immersion $\psi_0 = \psi$. Set $V_t := \dfrac{\partial \psi_t}{\partial t} \in \mathcal{C}^\infty({\psi_t^{-1}}^*TM)$, which is the variational vector field of $\psi_t : L \longrightarrow M^{2m+1}$.

Definition 1.2. $\{\psi_t\}$ is called a *Legendrian deformation* of ψ if ψ_t is a Legendrian immersion for each t.

Proposition 1.1. *$\{\psi_t\}$ is a Legendrian deformation if and only if*

$$\alpha_{V_t} = \frac{1}{2} d(\eta(V_t))$$

for each t.

There were two notions of Hamiltonian deformations for Lagrangian deformations in Lagrangian Geometry. In contrast there is only a notion of Legendrian deformations in Legendrian Geometry.

The (suitably completed) space of all Lagrangian immersions of compact L into M is a Banach manifold modeled on the vector space of (suitable) functions on L in the following way (cf. [12]). Let $\varphi : L \longrightarrow M$ be a Legendrian immersion of an m-dimensional compact smooth manifold L into a $(2m+1)$-dimensional contact manifold (M, η). We may choose an almost contact metric structure (ξ, g) on M compatible with the contact structure η. Let $\mathcal{W}$ be a sufficiently small neighborhood of O in $C^\infty(\varphi^{-1}TM/\varphi_*TL)$. For each $V \in \mathcal{W} \subset C^\infty(\varphi^{-1}TM/\varphi_*TL)$, define a smooth map

$$\exp_\varphi V : L \ni x \longmapsto \exp_{\varphi(x)}(V_x).$$

We have a homeomorphism

$$C^\infty(\varphi^{-1}TM/\varphi_*TL) \supset \mathcal{W} \ni V \longmapsto \exp_\varphi V \in \bar{\mathcal{W}} \subset C^\infty(L, M)$$

and $\exp_\varphi \mathrm{O} = \varphi$. We define a function

$$\mathcal{F} : C^\infty(\varphi^{-1}TM/\varphi_*TL) \supset \mathcal{W} \ni V \longmapsto (\exp_\varphi V)^*\eta \in \Omega^1(L).$$

For each $V \in C^\infty(\varphi^{-1}TM/\varphi_*TL)$,

$$(d\mathcal{F})_\mathrm{O}(V) = d(\eta(V)) + \iota_V(d\eta) \in \Omega^1(L).$$

Since $\iota_V(d\eta)$, $V \in C^\infty(\varphi^{-1}TM/\varphi_*TL)$, can take all elements of $\Omega^1(L)$, the differential of $\mathcal{F}$ at O

$$(d\mathcal{F})_\mathrm{O} : C^\infty(\varphi^{-1}TM/\varphi_*TL) \longrightarrow \Omega^1(L)$$

is surjective. Hence *(the suitable completion of) a space of Legendrian immersions which are C^1-close to φ is a Banach manifold modeled on the vector space of (suitable) functions on L* ([12]).

2 Legendrian submanifolds in Sasakian manifolds

Let $(M^{2m+1}, \eta, g, \xi, \varphi)$ be a $(2m+1)$-dimensional Sasakian manifold with Sasakian structure (η, g, ξ, φ). Here

η : the contact 1-form of M

g : a Riemannian metric,

ξ : a Killing vector field,

ϕ : a tensor field of type $(1,1)$ on M

satisfying the following equations :

$$\begin{aligned}&\eta(\xi) = 1,\\ &\phi^2 = -\mathrm{Id} + \eta \otimes \xi,\\ &g(\phi(X), \phi(Y)) = g(X, Y) - \eta(X)\eta(Y),\\ &(d\eta)(X, Y) = 2g(X, \phi(Y)),\\ &[\phi, \phi](X, Y) + (d\eta)(X, Y)\xi = 0,\end{aligned}$$

where

$$[\phi, \phi](X, Y) := \phi^2[X, Y] + [\phi(X), \phi(Y)] - \phi[\phi(X), Y] - \phi[X, \phi(Y)].$$

A $(2m + 1)$-dimensional Sasakian manifold $(M^{2m+1}, \eta, g, \xi, \varphi)$ is called *η-Einstein* with η-Ricci constant A if its Ricci tensor field Ric_g satisfies

$$\mathrm{Ric}_g(X, Y) = A\, g + (2m - A)\, \eta \otimes \eta\, .$$

Note that an η-Einstein Sasakian manifold $(M^{2m+1}, \eta, g, \xi, \varphi)$ is Einstein-Sasakian if and only if $A = 2m$.

We shall recall the standard construction of a Sasakian manifold $(M^{2m+1}, \eta, g, \xi, \phi)$ from a given Kähler manifold $(\bar{M}^{2m}, \omega, J, \bar{g})$ ([15, p331], cf. [2], [7]) : Suppose that there is a non-zero constant γ such that $\frac{1}{\gamma}[\omega] \in H^2(\bar{M}^{2m}, \mathbf{R})$ is an integral class. Then there is a principal $U(1)$-bundle $\pi_\gamma : P_\gamma \to \bar{M}^{2m}$ and a connection form θ_γ on P_γ whose curvature form coincides with $\frac{2\pi}{\gamma}\sqrt{-1}\pi^*\omega$. The standard Sasakian structure on $M^{2m+1} = P_\gamma$ induced from the Kähler structure of $\bar{M}^{2m}$ such that $\pi : (M^{2m+1}, g_\gamma) \to (\bar{M}^{2m}, \bar{g})$ is a Riemannian submersion with totally geodesic fibers can be defined as follows : $\eta_\gamma = \frac{\gamma}{\pi\sqrt{-1}}\theta_\gamma$, $g_\gamma = \pi_\gamma^*\bar{g} + \eta_\gamma \otimes \eta_\gamma$, $i_{\xi_\gamma} g_\gamma = \eta_\gamma$ and

$$\phi_\gamma(X) = \begin{cases} (J(\pi_* X))^* & \text{if } X \in \operatorname{Ker}\eta,\\ 0 & \text{if } X \in \mathbf{R}\xi,\end{cases}$$

where $(\cdot)^*$ denotes the horizontal lift with respect to the connection θ_γ. If $(\bar{M}^{2m}, \omega, J, \bar{g})$ is a Einstein-Kähler manifold, with Ricci form $\bar{\rho} = \kappa\omega$, then the Ricci tensor field Ric_{g_γ} satisfies

$$\mathrm{Ric}_{g_\gamma} = (\kappa - 2)g_\gamma + 2m\, \eta_\gamma \otimes \eta_\gamma,$$

that is, $(M^{2m+1}, \eta_\gamma, g_\gamma, \xi_\gamma, \phi_\gamma)$ is an η-Einstein-Sasakian manifold with η-Ricci constant $\kappa - 2$. In particular $\kappa = 2m+2$ if and only if g_γ is an Einstein-Sasakian metric. If $(\bar{M}^{2m}, \omega, J, \bar{g})$ is an Einstein-Kähler manifold with Einstein constant $\kappa = 2m+2$, then for each integer $l \in \mathbf{Z}$ by choosing $\gamma = \dfrac{2\pi}{(2m+2)l} = \dfrac{\pi}{(m+1)l}$ we obtain an Einstein-Sasakian manifold $(M^{2m+1} = P_\gamma, g_\gamma, \eta_\gamma, \xi_\gamma, \phi_\gamma)$.

Example 2.1. $\bar{M}^{2m} = \mathbf{C}P^m = SU(m+1)/S(U(1) \times U(m))$ is a complex projective space equipped with the Fubini-Study metric $\bar{g}$. Then $M^{2m+1} = S^{2m+1}(1)$ is the

$(2m+1)$-dimensional unit standard sphere, $\pi : S^{2m+1}(1) \to \mathbf{C}P^m$ is the Hopf fibration.

Example 2.2. $\bar{M}^{2m} = Q_m(\mathbf{C}) = \tilde{\mathrm{Gr}}_2(\mathbf{R}^{m+2}) = SO(m+2)/SO(2) \times SO(m)$ is the complex hyperquadric of $\mathbf{C}P^{m+1}$, which is compact Hermitian symmetric space of rank 2. $Q_m(\mathbf{C})$ is canonically isometric to the real Grassmannian manifold $\tilde{\mathrm{Gr}}_2(\mathbf{R}^{m+2})$ of oriented 2-dimensional vector subspaces of $\mathbf{R}^{m+2}$. Then $M^{2m+1} = V_2(\mathbf{R}^{m+2}) = SO(m+2)/SO(m)$ is the Stiefel manifold of orthonormal 2-frames in $\mathbf{R}^{m+2}$:

$$V_2(\mathbf{R}^{m+2}) := \{(\mathbf{a},\mathbf{b}) \mid \mathbf{a},\mathbf{b} \in \mathbf{R}^{m+2}, \|\mathbf{a}\| = \|\mathbf{b}\| = 1, \langle \mathbf{a},\mathbf{b}\rangle = 0\}$$

and

$$\pi : V_2(\mathbf{R}^{m+2}) \ni (\mathbf{a},\mathbf{b}) \longmapsto \mathbf{a} \wedge \mathbf{b} \in Q_m(\mathbf{C}) = \tilde{\mathrm{Gr}}_2(\mathbf{R}^{m+2}).$$

It is known that the cone metric $CM^{2m+1} \cong (0,\infty) \times_g M^{2m+1}$ over a Sasakian manifold M^{2m+1} is a Kähler metric and the converse holds :

$$\begin{aligned} \text{Kähler manifold } \bar{M}^{2m} &\Longrightarrow \text{Sasakian manifold } M^{2m+1} \\ &\Longleftrightarrow \text{Kähler cone } CM^{2m+1}. \end{aligned}$$

Moreover it is known that the Kähler cone metric $CM^{2m+1} \cong (0,\infty) \times_g M^{2m+1}$ over an Einstein-Sasakian manifold M^{2m+1} is Ricci-flat and the converse holds :

$$\begin{aligned} &\bar{M}^{2m} \text{ has an Einstein-Kähler metric} \\ &\Longrightarrow M^{2m+1} \text{ has an Einstein-Sasakian metric} \\ &\Longleftrightarrow CM^{2m+1} \text{ has a Ricci-flat Kähler cone metric .} \end{aligned}$$

Then there are bijective correspondences among minimal Lagrangian submanifolds in $\bar{M}^{2m}$, minimal Legendrian submanifolds in M^{2m+1} and special Lagrangian sub-cones in CM^{2m+1} :

$$\begin{array}{ccll} CL^m & \xrightarrow{\text{SL}} & CM^{2m+1} & : \text{Ricci flat E-K. cone} \\ \cup & & \cup & \\ L^m & \xrightarrow[\text{min. Leg.}]{} & M^{2m+1} & : \text{Einstein-Sasakian mfd.} \\ \downarrow & & \pi \downarrow U(1) = S^1 & \\ \bar{L}^m & \xrightarrow[\text{min. Lag.}]{} & \bar{M}^{2m} & : \text{Einstein-Kähler mfd.} \end{array}$$

Let $(M^{2m+1}, \eta, g, \xi, \phi)$ be a Sasakian manifold and $\psi : L \longrightarrow M$ be a Legendrian immersion. Let B denote the second fundamental form of L in (M, g) and H denote the mean curvature vector field of ψ defined by

$$H = \sum_{i=1}^{m} B(e_i, e_i)$$

where $\{e_i\}$ is an orthonormal basis of T_xL relative to the induced metric on L. The 1-form α_H on L corresponding to the mean curvature vector field H is called the *mean curvature form* of ψ. The mean curvature form α_H of ψ satisfies the identity

$$(d\alpha_H)(X, Y) = -\mathrm{Ric}^M(\psi_* X, \phi\psi_*(Y)))$$

for each $X, Y \in TL$. This identity follows from the Coddazi equation. Hence if M^{2m+1} is η-Einstein, then the mean curvature form α_H of any Legendrian immersion ψ is always a closed 1-form on L.

Suppose that L is compact without boundary. A Legendrian immersion ψ is *Legendrian minimal* (or shortly *L-minimal*) if for every Legendrian deformation $\psi_t : L \longrightarrow M^{2m+1}$ with $\psi_0 = \varphi$,

$$\frac{d}{dt}\mathrm{Vol}\ (L, \varphi_t^* g)|_{t=0} = 0.$$

Its Euler-Lagrange equation is $\delta\alpha_H = 0$ and thus a Legendrian immersion ψ into η-Einstein manifold M^{2m+1} is Legendrian minimal if and only if the mean curvature form α_H of ψ is a harmonic 1-form on L.

A *minimal* Legendrian immersion ψ is by definition a Legendrian immersion whose mean curvature vector field (or equivalently, mean curvature form) identically vanishes. The Legendrian stability of minimal Legendrian submanifolds were studied in [15], [10].

Definition 2.1. A one-parameter smooth family $\psi_t : L \longrightarrow M$ is called a *minimal Legendrian deformation* if $\psi_t : L \longrightarrow M$ is a Legendrian deformation such that ψ_t is a minimal immersion (i.e. its mean curvature vector field $H = 0$) for each t.

A minimal Legendrian deformation $\psi_t : L \longrightarrow M$ is called *trivial* if the minimal Legendrian deformation ψ_t is induced by the one-parameter family of automorphisms of the ambient Sasakian manifold $(M^{2m+1}, \eta, g, \xi, \varphi)$. The Lie algebra of the automorphism group $\mathrm{Aut}(M^{2m+1}, g, \eta, \xi, \varphi)$ of the Sasakian manifold $(M^{2m+1}, \eta, g, \xi, \varphi)$ consists of Sasakian Killing vector fields on M^{2m+1}, namely Killing vector fields preserving the Sasakian structure of M^{2m+1}. Let X be a Sasakian Killing vector field on M^{2m+1}. Then

$$0 = \mathcal{L}_X d\phi = (d \circ \iota_X + \iota_X \circ d)d\phi = d(\iota_X d\phi).$$

Suppose that M^{2m+1} is simply connected, more generally the first Betti number of M^{2m+1} is zero. Then $\iota_X d\phi$ is exact, that is, $\iota_X d\phi = df$ for some $f \in C^\infty(M^{2m+1})$.

Setting $V = X\circ\phi$, we have $\alpha_V = -\frac{1}{2}\psi^*(\iota_V d\eta) = -\frac{1}{2}d(f\circ\psi)$ and thus each Sasakian Killing vector field generates a Legendrian deformation. For a minimal Legendrian immersion $\psi : L \to M$, we define the *Sasakian Killing nullity* of ψ by

$$n_{sk}(\psi) := \dim\{X^\perp \mid X \in \mathrm{Lie}(\mathrm{Aut}(M^{2m+1}, g, \eta, \xi, \varphi))\},$$

where $X^\perp$ denotes the component of $X\circ\psi$ normal to ψ_*TL for each $X \in \mathrm{Lie}(\mathrm{Aut}(M^{2m+1}, g, \eta, \xi, \varphi))$. Then the dimension of all *trivial* infinitesimal minimal Legendrian deformations of ψ is equal to the Sasakian Killing nullity $n_{sk}(\psi)$.

3 Minimal Legendrian deformations in η-Einstein Sasakian manifolds

3.1 Infinitesimal minimal Legendrian deformations

Suppose that $(M^{2m+1}, \eta, g, \xi, \phi)$ is an η-Einstein Sasakian manifold with η-Einstein constant A. Let L^m be a compact m-dimensional smooth manifold without boundary and $\psi : L^m \longrightarrow M^{2m+1}$ be a minimal Legendrian immersion.

Lemma 3.1. *The vector space of all infinitesimal minimal Legendrian deformations of ψ can be identified with*

$$E_\psi := \mathbf{R} \oplus \{f \in C^\infty(L) \mid \Delta^0_\psi f = (A+2)f\}.$$

where Δ^0_ψ denotes the Hodge-de Rham-Laplace operator of L acting on $\Omega^0(L) = C^\infty(L)$ relative to the induced metric by ψ.

Under the canonical linear isomorphism $\chi : NL \cong \psi^*TM/\psi_*TL \to C^\infty(L) \oplus \Omega^1(L)$, the vector space of all infinitesimal Legendrian deformations of ψ is given by

$$\{(f,\alpha) \in C^\infty(L) \oplus \Omega^1(L) \mid \alpha = \frac{1}{2}df\} \cong C^\infty(L)\ .$$

Let $\nabla^\perp$ denote the normal connection in the normal bundle of ψ.

In minimal submanifold theory, the equation of infinitesimal minimal deformations of ψ is known as the Jacobi equation :

$$\mathcal{J}_\psi(V) = -\Delta^\perp V + \bar{\mathcal{R}}(V) - \tilde{\mathcal{A}}(V) = 0$$

for $V \in C^\infty(NL)$, where the Jacobi differential operator $\mathcal{J}_\psi = -\Delta^\perp + \bar{\mathcal{R}} - \tilde{\mathcal{A}} : C^\infty(NL) \to C^\infty(NL)$ is defined as

$$\begin{aligned}
\Delta^\perp(V) &:= \sum_{i=1}^m (\nabla^\perp_{e_i}\nabla^\perp_{e_i}V - \nabla^\perp_{\nabla^L_{e_i}e_i}V),\\
g(\bar{\mathcal{R}}(V), V) &= \sum_{i=1}^m g(R(e_i, V)e_i, V),\\
g(\tilde{\mathcal{A}}(V), V) &= \sum_{i,j=1}^m g(B(e_i, e_j), V)^2 = \mathrm{tr}(A_V \circ A_V)\,.
\end{aligned}$$

For each $V \in C^\infty(NL)$ with $\chi(V) = (f, \alpha) \in C^\infty(L) \oplus \Omega^1(L)$,

$$\begin{aligned}\chi(\mathcal{J}_\psi(V)) = &\big(\Delta^0_L f - 2\delta\alpha, -2df + \Delta^1\alpha - (A-2)\alpha\big)\\ &\in C^\infty(L) \oplus \Omega^1(L)\,.\end{aligned}$$

Suppose that V is an infinitesimal Legendrian deformation of ψ, i.e. $\alpha = \frac{1}{2}df$. Then

$$\chi(\mathcal{J}_\psi(V)) = \left(0,\, \frac{1}{2}(\Delta^1_L df - (A+2)df)\right) \in C^\infty(L) \oplus \Omega^1(L).$$

Now we set a vector subspace

$$\Gamma := \{(f, \frac{1}{2}df) \mid f \in C^\infty(L)\} \subset C^\infty(L) \oplus \Omega^1(L)$$

and we define a linear differential operator

$$\begin{aligned}\mathcal{J}^\chi_\psi : \Gamma \ni (f, \frac{1}{2}df) \longmapsto &\left(0,\, \frac{1}{2}(\Delta^1_L - (A+2)\mathrm{Id})df\right)\\ = &\left(0,\, \frac{1}{2}d(\Delta^0_L f - (A+2)f)\right) \in \Gamma,\end{aligned}$$

which can be considered as a linearized operator at ψ of the *minimal Legendrian submanifold equation* on the space of Legendrian immersions of L into M^{2m+1}. Then $\mathcal{J}^\chi_\psi$ is self-adjoint, i.e. $(\mathcal{J}^\chi_\psi)^* = \mathcal{J}^\chi_\psi$ and thus $\mathrm{Ker}(\mathcal{J}^\chi_\psi) = \mathrm{Ker}(\mathcal{J}^\chi_\psi)^* = E_\psi$.

Hence the vector space of all infinitesimal minimal Legendrian deformations of ψ corresponds to a vector space

$$\begin{aligned}\mathrm{Ker}(\mathcal{J}^\chi_\psi) =&\{\, (f, df) \mid \Delta^1_\psi df = (A+2)df\,\}\\ \cong&\,\mathbf{R} \oplus \{\, f \in C^\infty(L) \mid \Delta^0_\psi f = (A+2)f\,\} = E_L\,.\end{aligned}$$

3.2 Kuranishi type deformation theory

We can apply the Kuranishi type deformation theory to our problem. See also [12].

Let $\mathcal{M}(L)$ be the space of minimal Legendrian immersions near ψ from compact L^m into an η-Einstein Sasakian manifold M^{2m+1}. Then there exist a neighborhood U of 0 in a vector space $\mathrm{Ker}\mathcal{J}^\chi_\psi$ and a nonlinear map, so called *Kuranishi map*,

$$\Phi : \mathrm{Ker}(\mathcal{J}^\chi_\psi) = E_\psi \supset U \longrightarrow \mathrm{Ker}(\mathcal{J}^\chi_\psi)^* = E_\psi$$

such that $\Phi(0) = 0$ and

$$\begin{aligned}&[\text{a nbd. of } \Phi^{-1}(0) \text{ around } 0] \cong [\text{a nbd. of } \mathcal{M}(L) \text{ around } \psi]\\ &\qquad\qquad\qquad\qquad\qquad\qquad\text{(homeomorphic)}.\end{aligned}$$

Here note that if M^{2m+1} is a real analytic η-Einstein Sasakian manifold, then the Kuranishi map Φ is real analytic. Hence we know that *if every infinitesimal minimal Legendrian deformation of ψ is integrable, that is, generates a minimal Legendrian deformation of ψ, then there is a neighborhood in $\mathcal{M}(L)$ around psi which is a smooth manifold of dimension equal to* $\dim(E_\psi)$.

4 Minimal Legendrian deformations of 3-dimensional certain minimal Legendrian submanifolds

We shall give our attention to the case when $m = 3$ and

$$\psi : L^3 \longrightarrow M^7$$

is a 3-dimensional compact minimal Legendrian submanifold embedded in the 7-dimensional standard (η-)Einstein Sasakian manifolds

4.1 The simplest example

Let $M^5 = S^7(1) = U(4)/U(3)$ be the 5-dimensional standard unit sphere and $L^3 = S^3(1) = SO(4)/SO(3)$ be a totally geodesic Legendrian submanifold embedded in $S^7(1)$. The Hopf fibration $\pi : S^7(1) \to \mathbf{C}P^3$ induces the double covering

$$\pi : S^7(1) \supset S^3(1) \longrightarrow \mathbf{R}P^3 \subset \mathbf{C}P^3.$$

Since the multiplicity of the second eigenvalue $2m + 2 = 8$ of $\Delta^0_{S^3(1)}$ is equal to 9, we have $\dim(E_{S^3(1)}) = 1 + 9 = 10$. On the other hand $n_{sk}(S^3(1)) = \dim(U(4)) - \dim(SO(4)) = 16 - 6 = 10$. Therefore we obtain

Proposition 4.1. *The 3-dimensional compact totally geodesic Legendrian submanifold $S^3(1)$ embedded in $S^7(1)$ has only trivial minimal Legendrian deformations. Its deformation space is $U(4)/O(4)$.*

4.2 The first example

Let (V_3, ρ_3) be the irreducible unitary representation of $SU(2)$ of degree 3, where

$$V_3 := \{f(z_1, z_2) \mid \text{ complex homogeneous polynomials with two variable } z_1, z_2 \text{ of degree } 3\}.$$

V_3 is a 4-dimensional complex vector space equipped with the standard Hermitian inner product such that

$$\{\frac{1}{\sqrt{3!}} z_1^3, \frac{1}{\sqrt{2!}} z_1^2 z_2, \frac{1}{\sqrt{2!}} z_1 z_2^2, \frac{1}{\sqrt{3!}} z_2^3\}$$

is a unitary basis of V_3. We shall consider the $SU(2)$-orbit on $S^7(1)$:

$$L := \rho_3(SU(2))(w) \subset S^7(1)$$

through the point

$$w := \frac{1}{\sqrt{2}}(\frac{1}{\sqrt{3!}} z_1^3 + \frac{1}{\sqrt{3!}} z_2^3).$$

Then we have

Proposition 4.2. *The orbit L is a non-totally geodesic 3-dimensional compact minimal Legendrian submanifold embedded in $S^7(1)$. Moreover its fundamental group is $\pi_1(L) \cong \mathbf{Z}_3$ a finite cyclic group of order 3 and thus $L \cong SU(2)/\mathbf{Z}_3 \cong S^3/\mathbf{Z}_3$.*

Remark. The induced metric on L is never of constant sectional curvatures. This compact minimal Legendrian submanifold was also treated in [14]. For higher dimensional examples of compact minimal Legendrian orbits, see also [3], [14].

We denote by $\psi_0 : L \to S^7(1)$ the minimal Legendrian embedding of $L = \rho_3(SU(2))w$ into $S^7(1)$. Moreover

Lemma 4.1 ([14], Theorem 3.1)**.** *The multiplicity of the eigenvalue $2m+2 = 8$ of $\Delta^0_{\psi_0}$ is equal to 19.*

Thus we have $\dim(E_{\psi_0}) = 1 + 19 = 20$. On the other hand $n_{sk}(\psi_0) = \dim(U(4)) - \dim(SU(2)) = 16 - 3 = 13$.

Hence we see that L can have at most 7-dimensional family of non-trivial minimal Legendrian deformations. In fact, we obtain the following result

Theorem 4.1. *All non-trivial minimal Legendrian deformations of ψ_0 are given by the 7-dimensional family of minimal Legendrian embeddings which is induced by the group action of $Sp(2, \mathbf{C})$.*

Such deformations can be explained in the following diagram :

$$
\begin{array}{ccccc}
 & & \mathbf{H}^2 & \cong & \mathbf{C}^4 \\
 & & \cup & & \cup \\
L & \xrightarrow{\psi_0} & S^7(1) & = & S^7(1) \\
\downarrow S^1 & & p_2 \downarrow S^1 & & p_1 \downarrow S^1 \\
S^2 & \xrightarrow{h_0} & \mathbf{C}P^3 & & \mathbf{C}P^3 \supset p_1(\psi_0(L)) \\
\downarrow & & \downarrow & & \\
\mathbf{R}P^2 \subset S^4 & = & \mathbf{H}P^1 & &
\end{array}
$$

Remark. (1) $p_1(\psi_0(L)) \subset \mathbf{C}P^3$ is a 3-dimensional compact *strictly Hamiltonian stable* minimal Lagrangian embedded in $\mathbf{C}P^3$ with non-parallel second fundamental form ([4], [14]).

(2) The embedding $\mathbf{R}P^2 \subset S^4$ is the *Veronese surface*, which is a real projective plane with constant positive Gaussian curvature minimally embedded in the standard 4-sphere by the first eigenfunctions of the Laplacian of $\mathbf{R}P^2$.

(3) $h_0 : S^2 \to \mathbf{C}P^3$ is its horizontal holomorphic lift into the twistor space $\mathbf{C}P^3$ over S^4.

Let $\langle\ ,\ \rangle$ be the standard inner product of $\mathbf{R}^8$. Let I, J, $IJ = K$ be the standard quaternionic structure of $\mathbf{R}^8$. For each $\mathbf{x} \in S^7(1) \subset \mathbf{R}^8$,

$$\mathbf{R}^8 = \mathbf{R}\mathbf{x} \oplus \mathbf{R}I\mathbf{x} \oplus \mathbf{R}J\mathbf{x} \oplus \mathbf{R}K\mathbf{x} \oplus \mathcal{H}_{\mathbf{x}}.$$

Relative to I, we have an identification

$$\mathbf{R}^8 \cong \mathbf{H}^2 \cong \mathbf{C}^4.$$

and the standard fibrations

$$S^7(1) \longrightarrow \mathbf{C}P^3 \longrightarrow \mathbf{H}P^1 = S^4.$$

Then $\mathbf{C}P^3$ has the standard complex contact structure and the holomorphic contact 1-form η on $\mathbf{C}P^3$ defined by

$$\tilde{\eta}_{\mathbf{x}}(X) := \langle X, J\mathbf{x}\rangle + \sqrt{-1}\langle X, K\mathbf{x}\rangle = \langle X, J\mathbf{x}\rangle + \sqrt{-1}\langle X, IJ\mathbf{x}\rangle.$$

for each $X \in \mathbf{R}J\mathbf{x} \oplus \mathbf{R}K\mathbf{x} \oplus \mathcal{H}_{\mathbf{x}}$.

Suppose that $h : \Sigma \to \mathbf{C}P^3$ is a horizontal holomorphic map, that is, a holomorphic contact curve, which is a holomorphic map satisfying $h^*\eta = 0$.

$$\begin{array}{ccccccc}
 & & \mathbf{H}^2 & \cong & \mathbf{C}^4 & & \\
 & & \cup & & \cup & & \\
L = h^{-1}(S^7(1)) & \xrightarrow{\ \psi\ } & S^7(1) & = & S^7(1) & & \\
\downarrow{\scriptstyle S^1} & & {\scriptstyle p_2}\downarrow{\scriptstyle S^1} & & {\scriptstyle p_1}\downarrow{\scriptstyle S^1} & & \\
\Sigma & \xrightarrow{\ h\ } & \mathbf{C}P^3 & & \mathbf{C}P^3 \supset p_1(\psi(L)) & & \\
\downarrow{\scriptstyle F} & & \downarrow & & & & \\
F(\Sigma) \subset & & S^4 = \mathbf{H}P^1 & & & &
\end{array}$$

If $W \subset \mathbf{R}J\mathbf{x} \oplus \mathbf{R}K\mathbf{x} \oplus \mathcal{H}_{\mathbf{x}}$ is a vector subspace of $\dim W = 2$, $I(W) = W$ and $\tilde{\eta}(W) = 0$, then we have an orthogonal direct sum as

$$\mathbf{R}^8 = \mathbf{R}\mathbf{x} \oplus \mathbf{R}I\mathbf{x} \oplus W \oplus \mathbf{R}J\mathbf{x} \oplus \mathbf{R}K\mathbf{x} \oplus JW.$$

Indeed, we express W as

$$W = \mathbf{R}w \oplus \mathbf{R}I(w).$$

Then we have $JW = \mathbf{R}Jw \oplus \mathbf{R}Kw = KW$. Since $Jw \perp w$, $Jw \perp Iw$, $JIw \perp w$, $JIw \perp Iw$, we have $W \perp JW = KW$. Since

$$\tilde{\eta}(W) = 0 \quad \Leftrightarrow \quad J\mathbf{x} \perp W,\ K\mathbf{x} \perp W,$$

we have

$$\mathbf{x} \perp W,\ I\mathbf{x} \perp W,\ J\mathbf{x} \perp W,\ K\mathbf{x} \perp W$$

and thus

$$\mathbf{x} \perp JW,\ I\mathbf{x} \perp JW,\ J\mathbf{x} \perp JW,\ K\mathbf{x} \perp JW.$$

Hence we obtain

$$\mathbf{R}^8 = \mathbf{R}\mathbf{x} \oplus \mathbf{R}J\mathbf{x} \oplus (\mathbf{R}I\mathbf{x} \oplus W) \oplus (\mathbf{R}K\mathbf{x} \oplus JW)$$

and

$$J(\mathbf{R}I\mathbf{x} \oplus W) = \mathbf{R}K\mathbf{x} \oplus JW.$$

Therefore if we take another identification relative to J :

$$\mathbf{R}^8 \cong \mathbf{H}^2 \cong \mathbf{C}^4.$$

and the standard fibration

$$p_1 : S^7(1) \longrightarrow \mathbf{C}P^3,$$

then the induced map

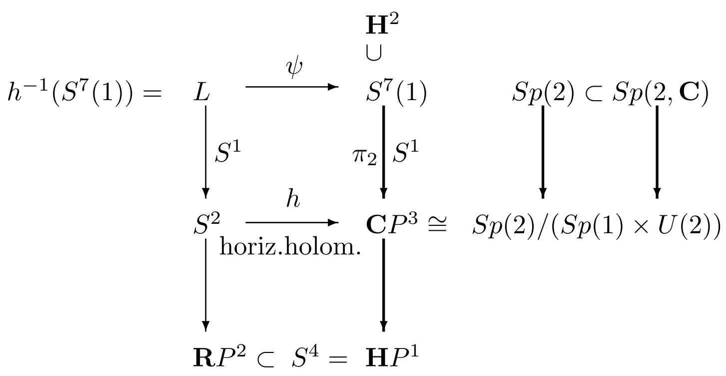
$$\psi = \tilde{h} : L = h^{-1}(S^7(1)) \longrightarrow S^7(1)$$

is a minimal Legendrian immersion relative to J and thus

$$p_1 \circ \tilde{h} : L = h^{-1}(S^7(1)) \longrightarrow \mathbf{C}P^3$$

is a minimal Lagrangian immersion relative to J

The complex Lie group $Sp(2, \mathbf{C})$ acts holomorphically on $\mathbf{C}P^3$ preserving the horizontal distribution with respect to the Penrose twistor fibration $\mathbf{C}P^3 \to \mathbf{H}P^1 \cong S^4$ and transforms a horizontal holomorphic curve to another horizontal holomorphic curve in $\mathbf{C}P^3$.



This complex group action induces horizontal holomorphic deformations of $h_0 : S^2 \to \mathbf{C}P^3$ and hence minimal Legendrian deformations of $\psi_0 : L = h_0^{-1}(S^7(1)) \to S^7(1)$. The dimension of the so obtained non-trivial family of minimal Legendrian immersions can be calculated as follows :

$$\begin{aligned} &\dim(Sp(2, \mathbf{C})) - \dim(Sp(2)) - (\dim(\mathrm{Hol}(S^2)) - \dim(\mathrm{Isom}(S^2))) \\ =&20 - 10 - (6 - 3) = 7\,. \end{aligned}$$

Remark. Compare this construction with [9], [1], [8]. This family are also very related to Lagrangian submanifolds attaining the equality in the B. Y. Chen's inequality on curvatures (see [5]).

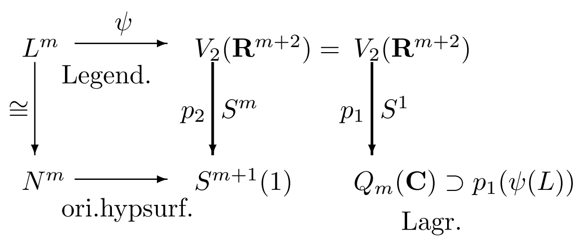
4.3 The second example

We shall consider the $(2m + 1)$-dimensional real Stiefel manifold of orthonormal 2-frames in $\mathbf{R}^{m+2}$:

$$V_2(\mathbf{R}^{m+2}) := \{(\mathbf{a}, \mathbf{b}) \mid \mathbf{a}, \mathbf{b} \in \mathbf{R}^{m+2} \text{ orthonormal } \} \cong SO(m + 2)/SO(m)$$

which is the standard Einstein-Sasakian manifold over a complex m-dimensional complex hyperquadric $Q_m(\mathbf{C}) \cong \widetilde{\mathrm{Gr}}_2(\mathbf{R}^{m+2})$. The natural projection $p_1 : V_2(\mathbf{R}^{n+2}) \to Q_m(\mathbf{C})$ is defined by $p_1(\mathbf{a}, \mathbf{b}) = [\mathbf{a} + \sqrt{-1}\mathbf{b}] = \mathbf{a} \wedge \mathbf{b}$. The natural projection $p_2 : V_2(\mathbf{R}^{n+2}) \to S^{m+}(1)$ is defined by $p_2(\mathbf{a}, \mathbf{b}) = \mathbf{a}$.

Let N^m be an oriented hypersurface in the $(m+1)$-dimensional the unit standard sphere $S^{m+1}(1) \subset \mathbf{R}^{m+2}$. We denote by $\mathbf{x}$ the position vector of a point of N^n and by $\mathbf{n}$ the unit normal vector field to N^m in $S^{m+1}(1)$.



Here the Legendrian life L^m of $N^m \subset S^{m+1}(1)$ to $V_2(\mathbf{R}^{m+2})$ is defined by $N^m \ni p \longmapsto (\mathbf{x}(p), \mathbf{n}(p)) \in V_2(\mathbf{R}^{m+2})$.

The *Gauss map* $\mathcal{G}$ of N^m is defined as a smooth map

$$\mathcal{G} : N^m \ni p \longmapsto \mathbf{x}(p) \wedge \mathbf{n}(p) \in Q_n(\mathbf{C}),$$

which we discussed in [13], and then the Gauss map $\mathcal{G}$ coincides with the composition map

$$p_1 \circ (p_2|_L)^{-1} : N^m \longrightarrow Q_m(\mathbf{C}).$$

We know that for any isoparametric hypersurface N^m in $S^{m+1}(1)$, the Gauss map $\mathcal{G} : N^m \to Q_n(\mathbf{C})$ is a minimal Lagrangian immersion and the Legendrian life L^m of $N^m \subset S^{m+1}(1)$ is a minimal Legendrian submanifold in $V_2(\mathbf{R}^{m+2})$.

Now we shall discuss the case of the 7-dimensional real Stiefel manifold of orthonormal 2-frames in $\mathbf{R}^5$ $(m = 3)$:

$$V_2(\mathbf{R}^5) := \{(\mathbf{a}, \mathbf{b}) \mid \mathbf{a}, \mathbf{b} \in \mathbf{R}^5 \text{ orthonormal } \} \cong SO(5)/SO(3)$$

which is the standard Einstein-Sasakian manifold over a 3-dimensional complex hyperquadric $Q_3(\mathbf{C})$

$$\begin{array}{ccccc} L^3 & \xrightarrow[\text{Legend.}]{\psi} & V_2(\mathbf{R}^5) & = & V_2(\mathbf{R}^5) \\ \cong \downarrow & & \pi_2 \downarrow S^3 & & \pi_1 \downarrow S^1 \\ N^3 & \xrightarrow[\text{ori.hypsurf.}]{} & S^4(1) & & Q_3(\mathbf{C}) \supset \pi_1(\psi(L)) \\ & & & & \text{Lagr.} \end{array}$$

Suppose that

$$N^3 = SO(3)/(\mathbf{Z}_2 + \mathbf{Z}_2) \subset S^4(1)$$

which is a compact 3-dimensional isoparametric hypersurface with 3 distinct principal curvatures embedded in $S^4(1)$, which is one of so called *Cartan hypersurfaces.* We choose an irreducible orthogonal representation of $SO(3)$ which acts by conjugation on the vector space $S^2_0(\mathbf{R}^3) \cong \mathbf{R}^5$ of all real symmetric matrices with trace 0 of degree 3. Then N^3 is a codimension 1 orbit of $SO(3)$ in the unit hypersphere $S^4(1)$ of $S^2_0(\mathbf{R}^3)$. Then the corresponding Legendrian submanifold

$$L^3 = SO(3)/(\mathbf{Z}_2 + \mathbf{Z}_2) \subset V_2(\mathbf{R}^5)$$

is a 3-dimensional compact minimal Legendrian submanifold embedded in $V_2(\mathbf{R}^5)$ and we denote by ψ_0 the minimal Legendrian embedding. Note that the induced metric is of constant positive sectional curvature. Since the right action of $SO(2)$ on $V_2(\mathbf{R}^5) = SO(5)/SO(3)$ induces the Killing vector field ξ, its Sasakian-Killing nullity is $n_{sk}(\varphi) = \dim(SO(5)) + \dim(SO(2)) - \dim(SO(3)) = 10 + 1 - 3 = 8$.

On the other hand, we have

Lemma 4.2 ([13], Lemma 5.3)**.** *The multiplicity of eigenvalue* $2m + 2 = 8$ *of* $\Delta^0_{\psi_0}$ *is equal to* 7.

Hence we have $\dim(E_{\psi_0}) = 1 + 7 = 8$. Therefore we obtain

Theorem 4.2. *The 3-dimensional compact minimal Legendrian submanifold* $L^3 = SO(3)/\mathbf{Z}_2 + \mathbf{Z}_2 \subset V_2(\mathbf{R}^5)$ *has only trivial minimal Legendrian deformations. Its deformation space is* $SO(5)/SO(3)$.

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