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On dense ideals of C^*-algebras and generalizations of the Gelfand–Naimark Theorem

by

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Abstract. We develop the theory of Segal algebras of commutative C^*-algebras, with an emphasis on the functional representation. Our main results extend the Gelfand–Naimark Theorem. As an application, we describe faithful principal ideals of C^*-algebras. A key ingredient in our approach is the use of Nachbin algebras to generalize the Gelfand representation theory.

1. Introduction. The concept of a Segal algebra originated in the work of Reiter on subalgebras of the L_1-algebra of a locally compact group; cf. [30]. It was generalized to arbitrary Banach algebras by Burnham in [10]. A C^*-*Segal algebra* is a Banach algebra which is a dense ideal in a C^*-algebra; it need not be self-adjoint. The multiplier and the bidual algebras of self-adjoint C^*-Segal algebras were described in [1, 16, 18] and, in the presence of an approximate identity, the form of the closed ideals of C^*-Segal algebras was given in [7]. Otherwise, however, not much is known about the general structure of C^*-Segal algebras. Without a doubt, a major reason for this is the lack of Gelfand–Naimark type theorems for C^*-Segal algebras.

In this paper, we develop the theory of commutative C^*-Segal algebras, with an emphasis on the functional representation. Our main results extend the Gelfand–Naimark Theorem from C^*-algebras to a large class of C^*-Segal algebras. A key ingredient in our approach is the use of Nachbin algebras to generalize the Gelfand representation theory. Although such algebras have attracted much attention since being introduced by Nachbin in the 60s (to cite a few, see [25, 9, 29]), this appears to be the first paper where they are employed in the context of C^*-algebras. We consider Nachbin algebras in Section 2 and Gelfand representation in Section 4.

A fundamental difference between C^*-algebras and their Segal algebras is that a C^*-Segal algebra need not contain an approximate identity (bounded

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or unbounded). This necessitates developing new approaches, since most results on Segal algebras have been obtained under the assumption of an approximate identity. To this end, we will introduce a notion of "approximate ideal" which, together with the theory of multiplier modules, provides an efficient tool for the study of Segal algebras without an approximate identity. We consider the structure of Segal algebras of arbitrary Banach algebras in Section 3.

Section 5 contains our main results. In Theorems 5.7 and 5.21, we establish a module- and an order-theoretic generalization of the Gelfand–Naimark Theorem. As an application, we describe faithful principal ideals of C^*-algebras. Here, an important role is played by the notion of an order unit. However, in contrast to the C^*-algebra case, an order unit of a C^*-Segal algebra cannot serve as a multiplicative identity for the algebra. In fact, it emerges that a C^*-Segal algebra with an order unit cannot even have an approximate identity. It should be noted that in the order theory of rings and algebras, it has been customary to assume that the notions of multiplicative identity and order unit coincide; see [17], for example.

All algebras considered in this paper are assumed to be commutative and over the field $\mathbb{C}$ of complex numbers.

2. On proper ideals of $C_b(X)$ and $C_0(X)$. In this section, we recall some basic facts about Nachbin algebras. For a more detailed discussion of these algebras, including full proofs, we refer to [4, 5].

2.1. Notation. Throughout this paper, X is a locally compact Hausdorff space. We denote by $C(X)$ the set of continuous complex-valued functions on X, by $C_b(X)$ the set of bounded functions in $C(X)$, and by $C_0(X)$ the set of vanishing-at-infinity functions in $C(X)$. We view $C_b(X)$ and $C_0(X)$ as C^*-algebras in the usual way.

For a subset A of $C(X)$, we denote by $Z(A)$ its *zero set* and by A_{sa} its *self-adjoint part*, that is,

$$Z(A) := \{t \in X : f(t) = 0 \text{ for all } f \in A\},$$
$$A_{\mathrm{sa}} := \{f \in A : \overline{f} \in A\},$$

where the bar denotes the complex conjugate. Given a non-empty subset Y of X, we say that A *separates the points of* Y if, for every pair s, t of distinct points of Y, there exists $f \in A$ such that $f(s) \neq f(t)$. Furthermore, we say that A *strongly separates the points of* Y if it separates the points of Y and Y is contained in $X \setminus Z(A)$. Finally, A is called *self-adjoint* if $A_{\mathrm{sa}} = A$.

The next result will be useful. For the proof, see [4, Lemma 2.2].

Lemma 2.1. *Let A be a self-adjoint subalgebra of $C(X)$ which separates the points of $X \setminus Z(A)$. Then, for every ideal I of A, one has:*

(i) *I_{sa} is a self-adjoint ideal of A;*
(ii) *$Z(I_{\mathrm{sa}}) = Z(I)$;*
(iii) *I_{sa} and I strongly separate the points of $X \setminus Z(I)$.*

2.2. Weighted function algebras. Let v be an upper semicontinuous real-valued function on X such that $\inf_{t\in X} v(t) > 0$. We define

$$C_b^v(X) := \{f \in C(X) : vf \text{ is bounded on } X\},$$
$$C_0^v(X) := \{f \in C(X) : vf \text{ vanishes at infinity on } X\}.$$

Clearly, these sets are self-adjoint and strongly separate the points of X. When equipped with the pointwise operations, they become algebras. If v is unbounded, then $C_b^v(X)$ is a proper ideal of $C_b(X)$, and $C_0^v(X)$ is a proper ideal of $C_0(X)$. Moreover, if $v(t_\alpha) \to \infty$ whenever (t_α) is a net in X converging to the point at infinity of X, then also $C_b^v(X)$ is a proper ideal of $C_0(X)$. On the other hand, if v is bounded, then $C_b^v(X) = C_b(X)$ and $C_0^v(X) = C_0(X)$. Therefore, $C_b(X)$ and $C_0(X)$ are special cases of $C_b^v(X)$ and $C_0^v(X)$. From now on, we shall refer to v as a *weight function* on X.

Remark 2.2. It is easy to see that $C_b^v(X)$ and $C_0^v(X)$ need not be closed under the pointwise product without the condition $\inf_{t\in X} v(t) > 0$.

We will frequently use the following lemma; the proof is trivial.

Lemma 2.3. *For every $f \in C_b^v(X)$ and $g \in C_0(X)$, one has $fg \in C_0^v(X)$.*

In order to make $C_b^v(X)$ and $C_0^v(X)$ into C^*-Segal algebras, we endow them with the *weighted supremum norm*, defined by

$$\|f\|_v := \sup_{t\in X} v(t)|f(t)| \qquad (f \in C_b^v(X)).$$

In fact, although being a complete norm on both algebras, $\|\cdot\|_v$ is an algebra norm if and only if $v(t) \geq 1$ for all $t \in X$. Nevertheless, by multiplying $\|\cdot\|_v$ with a suitable constant, $C_b^v(X)$ and $C_0^v(X)$ can be regarded as Banach algebras.

2.3. Stone–Weierstrass property. It is well known that the proof of the Gelfand–Naimark Theorem for commutative C^*-algebras relies on the Stone–Weierstrass Theorem. Therefore, in order to extend the Gelfand–Naimark Theorem from C^*-algebras to a larger class of Banach algebras, it is natural to consider the Stone–Weierstrass property from a more general point of view. With this in mind, we make the following definition, introduced in [2].

Definition 2.4. Let A be a subalgebra of $C(X)$ which separates the points of $X \setminus Z(A)$, and let T be a topology on A making it into a topological algebra with jointly continuous multiplication. We call (A, T) a *Stone–Weierstrass algebra* if every self-adjoint subalgebra of A which strongly separates the points of $X \setminus Z(A)$ is T-dense in A.

We can now state the main result on the structure of $C_0^v(X)$. For the proof, see [4, Theorem 4.1].

Proposition 2.5. *Equipped with the weighted supremum norm topology, $C_0^v(X)$ is a Stone–Weierstrass algebra.*

Remark 2.6. Clearly, $C_b^v(X)$ with the weighted supremum norm topology is a Stone–Weierstrass algebra if and only if it coincides with $C_0^v(X)$.

2.4. Closed ideals and quotient algebras. Let A be a closed subalgebra of $C_b^v(X)$ which strongly separates the points of X. For a closed subset E of X, we put

$$I_A(E) := \{f \in A : f(t) = 0 \text{ for all } t \in E\}.$$

It is easy to see that $I_A(E)$ is a closed ideal of A.

2.4.1. *On the ideal structure of $C_0^v(X)$.* For a closed ideal I of $C_0^v(X)$, put $E := Z(I)$ and $w := v|_{X\setminus E}$. Then w is a weight function on the locally compact Hausdorff space $X \setminus E$. Moreover, by [4, Lemma 3.3], the mapping $f \mapsto f|_{X\setminus E}$ is an isometric conjugate-preserving isomorphism from $I_{C_0^v(X)}(E)$ onto $C_0^w(X \setminus E)$. As a result, $I_{C_0^v(X)}(E)$ is a Stone–Weierstrass algebra. Together with Lemma 2.1 and the inclusions $I_{\mathrm{sa}} \subseteq I \subseteq I_{C_0^v(X)}(E)$, this yields the following.

Proposition 2.7. *The mapping $E \mapsto I_{C_0^v(X)}(E)$ is a bijection between the closed subsets of X and the closed ideals of $C_0^v(X)$. In particular, for every closed ideal I of $C_0^v(X)$, one has $I = I_{C_0^v(X)}(E)$, where $E = Z(I)$.*

Corollary 2.8. *The mapping $t \mapsto I_{C_0^v(X)}(\{t\})$ is a bijection between the points of X and the closed maximal ideals of $C_0^v(X)$. In particular, for every closed maximal ideal M of $C_0^v(X)$, one has $M = I_{C_0^v(X)}(\{t\})$, where $\{t\} = Z(M)$.*

For a closed ideal I of $C_0^v(X)$, put $E := Z(I)$, $w := v|_E$, and $B := \{f|_E : f \in C_0^v(X)\}$. Then E is a locally compact Hausdorff space, w is a weight function on E, and B is a self-adjoint subalgebra of $C_0^w(E)$ which strongly separates the points of E. Since $C_0^w(E)$ is a Stone–Weierstrass algebra, B is dense in it. On the other hand, by [4, Theorem 4.5], the mapping $f + I \mapsto f|_E$ is an isometric conjugate-preserving homomorphism from $C_0^v(X)/I$ into $C_0^w(E)$. Taken together, these observations yield the following.

PROPOSITION 2.9. *Let I be a closed ideal of $C_0^v(X)$. Then $C_0^v(X)/I$ is isometrically conjugate isomorphic to $C_0^w(E)$, where $E = Z(I)$ and $w = v|_E$. In particular, $C_0^v(X)|_E = C_0^w(E)$.*

We end the study of the ideal structure of $C_0^v(X)$ with a result on its sum ideals. For the proof, see [4, Theorem 4.7].

PROPOSITION 2.10. *Let I and J be closed ideals of $C_0^v(X)$. Then $I + J$ is a closed ideal of $C_0^v(X)$.*

2.4.2. *On the ideal structure of $C_b^v(X)$.* Our knowledge of the ideal structure of $C_b^v(X)$ is restricted to the proposition below.

PROPOSITION 2.11. *Let A be a closed subalgebra of $C_b^v(X)$ such that $C_0^v(X) \subseteq A \subseteq C_0(X)$. Then, for every closed regular ideal I of A, one has $I = I_A(E)$, where $E = Z(I)$.*

Proof. Put $E := Z(I)$ and $J := C_0^v(X) \cap I$. Using Lemma 2.3 and the regularity of I, it is not hard to show that

$$I_A(E) = I_{C_0^v(X)}(E) + I.$$

On the other hand, since J is a closed ideal of $C_0^v(X)$ with zero set E, one has $J = I_{C_0^v(X)}(E)$ by Proposition 2.7. Consequently, $I_A(E) = J + I = I$. ∎

COROLLARY 2.12. *Let A be a closed subalgebra of $C_b^v(X)$ such that $C_0^v(X) \subseteq A \subseteq C_0(X)$. Then, for every maximal regular ideal M of A, one has $M = I_A(\{t\})$, where $\{t\} = Z(M)$.*

REMARK 2.13. The size of the class of non-regular ideals of $C_b^v(X)$ can be immense. For instance, every subspace L of $C_b^v(X)$ satisfying the inclusions $C_0^v(X) \subseteq L \subseteq C_0(X)$ is an ideal of $C_b^v(X)$, by Lemma 2.3.

We finish this section with a weighted version of the Banach–Stone Theorem.

PROPOSITION 2.14. *Let Y be a locally compact Hausdorff space, and let w be a weight function on Y. The following conditions are equivalent:*

(a) *$C_0^v(X)$ and $C_0^w(Y)$ are isometrically conjugate isomorphic;*
(b) *there exists a homeomorphism $\phi\colon X \to Y$ such that $v = w \circ \phi$.*

3. Irregularity of Banach algebras. In this section, we discuss and analyse Banach algebras which are (possibly non-closed) ideals in a Banach algebra.

3.1. Notation and basic definitions. Throughout the remainder of this paper, let A be a Banach algebra with norm $\|\cdot\|$. For an ideal I of A, we denote by $\mathrm{ann}_A(I)$ its *annihilator* in A, that is,

$$\mathrm{ann}_A(I) := \{a \in A : ax = 0 \text{ for all } x \in I\}.$$

The ideal I is called *faithful* if $\mathrm{ann}_A(I) = \{0\}$. Similarly, the Banach algebra A is called faithful if $\mathrm{ann}_A(A) = \{0\}$.

The basic notion of this paper is that of the *multiplier seminorm*, defined on A by

$$\|a\|_M := \sup_{\|b\| \le 1} \|ab\| \quad (a, b \in A).$$

It is not difficult to verify that $\|\cdot\|_M$ is an algebra seminorm on A whose kernel coincides with $\mathrm{ann}_A(A)$. If A has an identity element (denoted by e), then each $a \in A$ satisfies

$$\|a\|_M \le \|a\| \le \|e\| \, \|a\|_M.$$

However, in the non-unital case, the interrelations between $\|\cdot\|$ and $\|\cdot\|_M$ become more involved. The left-hand inequality remains true for every $a \in A$, but even if $\|\cdot\|_M$ is a norm, it need not be equivalent to $\|\cdot\|$.

Example 3.1. Let G be an infinite compact abelian group, and let λ be a Haar measure on it which is normalized so that $\lambda(G) = 1$. For $1 \le p < \infty$, denote by $L_p(G)$ the Banach space of (equivalence classes of) complex-valued functions f on G such that

$$\|f\|_p := \Big(\int_G |f(t)|^p \, d\lambda(t)\Big)^{1/p} < \infty.$$

With convolution as multiplication, $L_p(G)$ is a commutative Banach algebra. Since any $f, g \in L_p(G)$ satisfy $\|f * g\|_p \le \|f\|_1 \, \|g\|_p$, it follows that, for $1 < p < \infty$, the multiplier norm on $L_p(G)$ is not equivalent to $\|\cdot\|_p$. On the other hand, it is well known that the two norms coincide on $L_1(G)$.

In order to simplify the discussion, we introduce some terminology, following [6, 8].

Definition 3.2. The Banach algebra A is called

(i) *norm regular* if $\|\cdot\|$ and $\|\cdot\|_M$ coincide on A;
(ii) *weakly norm regular* if $\|\cdot\|$ and $\|\cdot\|_M$ are equivalent on A;
(iii) *norm irregular* if $\|\cdot\|$ is strictly stronger than $\|\cdot\|_M$ on A.

Example 3.3. It is easy to see that the multiplier norm on $C_0^v(X)$ and $C_b^v(X)$ coincides with the supremum norm. Therefore, $C_0^v(X)$ and $C_b^v(X)$ are norm regular if and only if v is identically 1, weakly norm regular if and only if v is bounded, and norm irregular if and only if v is unbounded.

Besides the multiplier seminorm, the following family of algebra norms will play a role in our work. The terminology will be justified shortly.

Definition 3.4. Let $|\cdot|$ be an algebra norm on A. We call it a *Segal norm* if there exist strictly positive constants k and l such that

$$k\|a\|_M \le |a| \le l\|a\| \quad \text{for all } a \in A.$$

3.2. Segal algebras. Given a Banach algebra B with norm $\|\cdot\|_B$, recall that A is said to be a *Segal algebra in* B if it is a dense ideal of B and there exists a constant $l > 0$ such that $\|a\|_B \leq l\|a\|$ for all $a \in A$. Of course, the latter condition is superfluous whenever B is semisimple. For future reference, we record the following standard result of Barnes [7, Theorem 2.3]. For other basic properties of Segal algebras, the reader may consult [23, 30].

LEMMA 3.5. *Let B be a Banach algebra in which A is a Segal algebra. Then A is a Banach B-module, i.e., there exists a constant $l > 0$ such that*

$$\|ax\| \leq l\|a\|\,\|x\|_B \quad \textit{for all } a \in A \textit{ and } x \in B.$$

In our context, it is natural to reverse the notion of a Segal algebra as follows.

DEFINITION 3.6. By a *Segal extension* of A we mean a Banach algebra in which A is a Segal algebra.

The proposition below establishes a useful relation between the Segal extensions of A and the Segal norms on A. (Here and in the sequel, we identify a normed algebra with its canonical image in its completion.)

PROPOSITION 3.7. *The following conditions are equivalent for a Banach algebra B:*

(a) *B is a Segal extension of A;*
(b) *B is the completion of A with respect to a Segal norm on A.*

Proof. (a)⇒(b). It follows from Lemma 3.5 that the multiplier seminorm on A is majorized by the norm on B. Together with the definition of a Segal extension, this means that the restriction of $\|\cdot\|_B$ to A is the desired Segal norm on A.

(b)⇒(a). It is enough to prove that A is an ideal of B. Let $a \in A$ and $x \in B$. Then there is a sequence (a_n) in A such that $\|a_n - x\|_B \to 0$. Noting that each $b \in A$ satisfies $\|ab\| \leq \|a\|\,\|b\|_M$, it is easy to deduce from the definition of a Segal norm that (aa_n) is a Cauchy sequence in A. Thus, for some $b \in A$, one has $\|b - aa_n\| \to 0$. Now

$$\begin{aligned}\|b - ax\|_B &\leq \|b - aa_n\|_B + \|aa_n - ax\|_B \\ &\leq l\|b - aa_n\| + \|a\|_B\,\|a_n - x\|_B \to 0,\end{aligned}$$

where $l > 0$ is a constant. Therefore, $ax = b$ and the assertion follows. ∎

In particular, this result shows that norm irregular Banach algebras provide the natural framework for our investigation.

Corollary 3.8. *Let A be a faithful Banach algebra. The following conditions are equivalent:*

(a) *A is norm irregular;*
(b) *A is a Segal algebra.*

Furthermore, the completion of A under the multiplier norm is a Segal extension of A with the property that any Segal extension of A can be embedded as a dense subalgebra.

Remark 3.9. The assumption that A is faithful is not needed in proving that (b) implies (a).

For the remainder of this paper, we shall assume that A is faithful. The normed algebra $(A, \|\cdot\|_M)$ will be denoted by A_M and its completion by $\widetilde{A}_M$.

3.3. Approximate identities of norm irregular Banach algebras. Given a normed algebra B with norm $\|\cdot\|_B$, recall that an *approximate identity* for B is a net $(e_\alpha)_{\alpha\in\Omega}$ in B such that $\|xe_\alpha - x\|_B \to 0$ for every $x \in B$. It is said to be *bounded* if there exists a constant $l > 0$ such that $\|e_\alpha\|_B \le l$ for all $\alpha \in \Omega$. Moreover, it is said to be *minimal* if $\|e_\alpha\|_B \le 1$ for all $\alpha \in \Omega$. In case $\Omega = \mathbb{N}$, it is said to be *sequential.*

One of the drawbacks of norm irregular Banach algebras is that they cannot have a bounded approximate identity. Indeed, it is easy to see that if A has a bounded approximate identity $(e_\alpha)_{\alpha\in\Omega}$, then each $a \in A$ satisfies $\|a\| \le l\|a\|_M$, where $l = \sup_{\alpha\in\Omega} \|e_\alpha\|$. Thus, in the context of norm irregular Banach algebras, the crucial point turns out to be the existence of a bounded approximate identity with respect to the multiplier norm. In order to make this precise, we first need a simple lemma.

Lemma 3.10. *The following conditions are equivalent for the Banach algebra A:*

(a) *A_M has a bounded approximate identity;*
(b) *$\widetilde{A}_M$ has a bounded approximate identity;*
(c) *A has a Segal extension with a bounded approximate identity.*

The proof is immediate from Proposition 3.7 and the fact that a normed algebra has a bounded approximate identity if and only if its completion has a bounded approximate identity (see, e.g., [13, Lemma 2.1]).

Example 3.11. Since a C^*-algebra always has a minimal approximate identity, it follows that every C^*-Segal algebra contains a bounded approximate identity under the multiplier norm. The same is true for the group algebras $L_p(G)$, as they are Segal algebras in $L_1(G)$, which has a minimal approximate identity.

REMARK 3.12. In general, the existence of a bounded approximate identity for A_M does not guarantee the existence of even an unbounded approximate identity for A. Example 3.15 below illustrates this well.

Now, consider the set

$$A\widetilde{A}_M := \{ax : a \in A \text{ and } x \in \widetilde{A}_M\}.$$

As A is a Banach $\widetilde{A}_M$-module, we conclude from the Cohen–Hewitt Factorization Theorem [15, Theorem B.7.1] and part (b) of Lemma 3.10 that $A\widetilde{A}_M$ is a closed faithful ideal of A whenever A_M has a bounded approximate identity. The importance of $A\widetilde{A}_M$ lies in the fact that it is the largest closed ideal of A with an approximate identity (necessarily unbounded in the norm irregular case).

PROPOSITION 3.13. *Let A be a Banach algebra such that A_M has a bounded approximate identity $(e_\alpha)_{\alpha\in\Omega}$. Then:*

(i) $A\widetilde{A}_M = \{a \in A : \|ae_\alpha - a\| \to 0\}$;
(ii) $A\widetilde{A}_M$ *has an approximate identity;*
(iii) *every closed ideal of A with an approximate identity is contained in* $A\widetilde{A}_M$.

Proof. (i) The inclusion "$\supseteq$" is evident from the closedness of $A\widetilde{A}_M$ in A, and "$\subseteq$" follows easily from Lemmas 3.5 and 3.10(b).

(ii) Noting that the net $(e_\alpha^2)_{\alpha\in\Omega}$ is also a bounded approximate identity for A_M, contained in $A\widetilde{A}_M$, the assertion follows from (i) by replacing $(e_\alpha)_{\alpha\in\Omega}$ with $(e_\alpha^2)_{\alpha\in\Omega}$.

(iii) Given a closed ideal I of A with an approximate identity, the set I^2 is dense in it, and so the statement follows. ■

Motivated by this result, we make the following definition.

DEFINITION 3.14. Let A be a Banach algebra such that A_M has a bounded approximate identity. We put $E_A := A\widetilde{A}_M$ and call it the *approximate ideal* of A.

EXAMPLE 3.15. Let B be a closed subalgebra of $C_b^v(X)$ such that $C_0^v(X) \subseteq B \subseteq C_0(X)$. It follows immediately from Lemma 2.3 and the density of $C_0^v(X)$ in $C_0(X)$ that B is a C^*-Segal algebra with approximate ideal of the form $E_B = BC_0(X)$. Clearly, the zero set of E_B is empty, whence $E_B = C_0^v(X)$ by Lemma 2.3 and Proposition 2.7.

REMARK 3.16. Our approach is particularly well suited to the study of Banach algebras having an unbounded approximate identity. That is, it is not easy to give an example of a Banach algebra with an approximate identity not bounded in the multiplier norm; Willis constructed such an algebra in [33, Example 5]. Moreover, an application of the Uniform Boundedness

Principle shows that if a Banach algebra has a sequential approximate identity, then it is automatically bounded with respect to the multiplier norm; see, for instance, [14, p. 191].

As a consequence of the above discussion, we have the following factorization results.

COROLLARY 3.17. *Let A be a Banach algebra with an approximate identity bounded under $\|\cdot\|_M$. Then $E_A = A$.*

COROLLARY 3.18. *Let A be a Banach algebra with a sequential approximate identity. Then $E_A = A$.*

EXAMPLE 3.19. For $1 \leq p < \infty$, denote by ℓ_p the Banach space of complex-valued sequences $x = (x_n)$ such that $\|x\|_p := (\sum_n |x_n|^p)^{1/p} < \infty$. Under pointwise multiplication, ℓ_p is a commutative Banach algebra with a sequential approximate identity (e.g., the sequence (e_n), where $e_n(k) = 1$ for every $1 \leq k \leq n$, and $e_n(k) = 0$ for every $k > n$). Furthermore, it is not hard to see that the multiplier norm on ℓ_p coincides with the supremum norm. Together with the previous corollary and the fact that ℓ_p is dense in the C^*-algebra c_0 of complex-valued sequences converging to zero, this yields the well-known factorization property $\ell_p = \ell_p c_0$.

We finish this subsection with some useful observations on the approximate ideal.

LEMMA 3.20. *Let A be a Banach algebra such that A_M has a bounded approximate identity, and let B be a Segal extension of A with a bounded approximate identity. Then:*

(i) *A^2 is dense in E_A;*
(ii) *$E_A = AB := \{ax : a \in A \text{ and } x \in B\}$;*
(iii) *B is a Segal extension of E_A.*

Proof. (i) This is a direct consequence of the density of A in $\widetilde{A}_M$ and Lemma 3.5.

(ii) One can use the Cohen–Hewitt Factorization Theorem again to deduce that AB is a closed ideal of A. The identity now follows from (i) and the inclusions $A^2 \subseteq AB \subseteq E_A$.

(iii) It is enough to prove that E_A is dense in B. But this is immediate from (ii) together with the facts that A is dense in B and that $B = B^2$. ∎

3.4. Multipliers of norm irregular Banach algebras. Multiplier modules will play a central role in this paper, as they allow us to reduce the study of certain properties of A to those of E_A. For general information on multiplier modules, we refer to [31, 12].

DEFINITION 3.21. Let A be a Banach algebra such that A_M has a bounded approximate identity, and let B be a Segal extension of A. By a *B-multiplier* of A we mean a mapping $m\colon B \to A$ such that

$$m(xy) = m(x)y \quad (x, y \in B).$$

Each $a \in A$ determines a B-multiplier l_a of A given by $l_a(x) := ax$ for $x \in B$. We write $M_B(A)$ for the set of B-multipliers of A. Clearly, it is a closed commutative subalgebra of the Banach algebra $\mathcal{L}(B, A)$ of bounded linear mappings from B into A. (The fact that $\mathcal{L}(B, A)$ is an algebra is a direct consequence of A being a Segal algebra in B.) In addition, $M_B(A)$ carries a natural B-module structure defined by

$$m \cdot x := l_{m(x)} \quad (m \in M_B(A),\ x \in B).$$

There exists a continuous injective algebra and B-module homomorphism $\varphi\colon A \to M_B(A)$ given by $\varphi(a) := l_a$ for $a \in A$. In case B has a bounded approximate identity, the image of E_A under φ is a closed faithful ideal of $M_B(A)$.

REMARK 3.22. If A and B coincide, then $M_B(A)$ is the usual *multiplier algebra* $M(A)$ of A. As mentioned in the Introduction, multiplier algebras of Segal algebras have received some attention; see also, e.g., [21, 32]. However, the drawback in the norm irregular case is that, although they can be considered as faithful ideals of $M(A)$, neither E_A nor A is closed in it.

EXAMPLE 3.23. It can be shown that, up to an isometric isomorphism, the identities $M(C_0^v(X)) = C_b(X)$ and $M_{C_0(X)}(C_0^v(X)) = C_b^v(X)$ hold.

To end this section, we describe a universal property of the multiplier module. Recall that a B-module V is said to be *faithful* if, for every non-zero $x \in B$, there exists $v \in V$ such that $v \cdot x \neq 0$.

PROPOSITION 3.24. *Let A be a Banach algebra such that A_M has a bounded approximate identity. Then, for every Segal extension B of A with a bounded approximate identity, $(M_B(A), \varphi)$ satisfies the following conditions:*

(i) *$M_B(A)$ is a faithful B-module;*
(ii) *$\varphi(E_A) = M_B(A) \cdot B$;*
(iii) *if V is a faithful B-module and ϕ is an injective B-module homomorphism from A into V such that $\phi(E_A) = V \cdot B$, then there exists a unique injective B-module homomorphism ψ of V into $M_B(A)$ such that $\varphi = \psi \circ \phi$.*

Proof. (i) This is a straightforward calculation.

(ii) By Lemma 3.20(ii), it is sufficient to show that $M_B(A) \cdot B$ is contained in $\varphi(E_A)$. Let $m \in M_B(A)$ and $x \in B$. Then there exist $y, z \in B$ such that

$x = yz$. It follows that $m \cdot x = m \cdot yz = l_{m(yz)} = l_{m(y)z} \in \varphi(AB) = \varphi(E_A)$, as wanted.

(iii) The desired map ψ is given by $\psi(v) := \phi_v$ for $v \in V$, where $\phi_v(x) := \phi^{-1}(v \cdot x)$ for each $x \in B$. ∎

4. Gelfand representation of norm irregular Banach algebras. In this section, we consider functional representation of norm irregular Banach algebras.

4.1. Notation. Our notation is standard and generally follows that of [22, 27]. For the Banach algebra A, we denote by $\Delta(A)$ its *Gelfand space*. Unless otherwise explicitly stated, we assume that it is non-empty. For $a \in A$, we denote by $\widehat{a}$ its *Gelfand transform*. By the *Gelfand transformation* on A we mean the mapping $a \mapsto \widehat{a}$ from A onto the *Gelfand transform algebra* of A, denoted by $\widehat{A}$. Recall that $\Delta(A)$ is a locally compact Hausdorff space and that $\widehat{A}$ is a subalgebra of $C_0(\Delta(A))$ which strongly separates the points of $\Delta(A)$.

4.2. The weight function. Following [11, 3], we now describe a natural weight function on the Gelfand space of A.

Definition 4.1. We define a mapping $\widehat{v}_A \colon \Delta(A) \to \mathbb{R}$ by setting

$$\widehat{v}_A(\tau) := 1/\|\tau\| \quad (\tau \in \Delta(A)),$$

where $\|\tau\|$ denotes the dual norm of τ. Whenever convenient, we shall abbreviate it by $\widehat{v}$.

Lemma 4.2. *$\widehat{v}$ is upper semicontinuous on $\Delta(A)$, and $\widehat{v}(\tau) \geq 1$ for all $\tau \in \Delta(A)$.*

Proof. It is sufficient to establish the upper semicontinuity of $\widehat{v}$. For $\alpha > 0$, put $U := \{\tau \in \Delta(A) : \widehat{v}(\tau) < \alpha\}$. To show that U is open in $\Delta(A)$, suppose that it is non-empty, and let $\tau \in U$. Then there exists $a \in A$ satisfying $\|a\| \leq 1$ and $|\widehat{a}(\tau)| > 1/\alpha$. Since $\widehat{a}$ is continuous on $\Delta(A)$, τ has a neighbourhood V such that $|\widehat{a}(\omega)| > 1/\alpha$ for all $\omega \in V$. This implies that V is contained in U, and so the proof is complete. ∎

Our choice for the weight function is motivated by the example below.

Example 4.3. Let B be a closed subalgebra of $C_b^v(X)$ such that $C_0^v(X) \subseteq B \subseteq C_0(X)$. It follows from Corollary 2.12 that $\Delta(B) = \{\tau_t : t \in X\}$, where τ_t denotes the point evaluation at t. Moreover, by [26, Lemma 1], one has $\widehat{v}(\tau_t) = v(t)$ for all $t \in X$.

In order to give an idea of how the interrelations between $\|\cdot\|$ and $\|\cdot\|_M$ affect the behaviour of $\widehat{v}$, we first note that in the unital case, each $\tau \in \Delta(A)$

satisfies $\widehat{v}(\tau) \leq \|e\|$. More generally, if A has a bounded approximate identity $(e_\alpha)_{\alpha\in\Omega}$, then

$$|\tau(e_\alpha)\tau(a) - \tau(a)| = |\tau(e_\alpha a - a)| \leq \|\tau\|\,\|e_\alpha a - a\| \to 0$$

for all $a \in A$, which implies that

$$1 = \lim_{\alpha\in\Omega} |\tau(e_\alpha)| \leq \|\tau\| \lim_{\alpha\in\Omega} \|e_\alpha\| \leq \|\tau\| \sup_{\alpha\in\Omega} \|e_\alpha\|.$$

We thus obtain the following estimate.

Lemma 4.4. *Let A be a Banach algebra with a bounded approximate identity $(e_\alpha)_{\alpha\in\Omega}$. Then*

$$\widehat{v}(\tau) \leq \sup_{\alpha\in\Omega} \|e_\alpha\| \quad \textit{for all } \tau \in \Delta(A).$$

These considerations raise the question of whether $\widehat{v}$ is bounded whenever $\|\cdot\|$ and $\|\cdot\|_M$ are equivalent. In general, however, (weak) norm regularity is neither a sufficient nor a necessary condition for the boundedness of the weight function.

Example 4.5. Let $\mathbb{D} := \{z \in \mathbb{C} : |z| \leq 1\}$, and let $A(\mathbb{D})$ be the *disc algebra*, that is, the subalgebra of $C(\mathbb{D})$ of functions which are analytic in the interior of $\mathbb{D}$. Let M_0 denote the maximal ideal of $A(\mathbb{D})$ of functions vanishing at zero. It is a Banach algebra with the supremum norm, and its Gelfand space consists of the point evaluations τ_t, where t runs through the elements of $\mathbb{D} \setminus \{0\}$. Of course, M_0 is norm regular, but as each $t \in \mathbb{D} \setminus \{0\}$ satisfies $\|\tau_t\|_\infty = |t|$, it follows that $\widehat{v}$ is unbounded.

Example 4.6. For $1 \leq p < \infty$, consider the C^*-Segal algebra ℓ_p of Example 3.19. Then $\Delta(\ell_p)$ consists of the point evaluations τ_n, where n runs through the natural numbers. Clearly, $\|\tau_n\|_p = 1$ for all $n \in \mathbb{N}$, and so $\widehat{v}$ is identically 1.

We end this subsection with two useful results concerning the Gelfand space and the weight function of Segal algebras. The first lemma is well known (see [10, Theorem 2.1]), but we include a proof for completeness.

Lemma 4.7. *Let B be a Segal extension of A. Then $\Delta(A)$ and $\Delta(B)$ are homeomorphic. In particular, every multiplicative functional on A has a unique extension to a multiplicative functional on B.*

Proof. Let $\tau \in \Delta(A)$ and pick $u \in A$ such that $\tau(u) = 1$. Given a sequence (a_n) in A such that $\|a_n\|_B \to 0$, it is immediate from Lemma 3.5 that $\|ua_n\| \to 0$. Consequently, $\tau(a_n) = \tau(u)\tau(a_n) = \tau(ua_n) \to 0$ so that τ is continuous under $\|\cdot\|_B$. By the density of A in B, it follows that τ has a unique extension $\widetilde{\tau}$ to a multiplicative functional on B. Obviously, the mapping $\tau \mapsto \widetilde{\tau}$ is a homeomorphism from $\Delta(A)$ onto $\Delta(B)$. ∎

Lemma 4.8. *Let A be a Banach algebra such that A_M has a minimal approximate identity $(e_\alpha)_{\alpha\in\Omega}$. Then:*

(i) *the mapping $\tau \mapsto \tau|_{E_A}$ is a homeomorphism from $\Delta(A)$ onto $\Delta(E_A)$;*
(ii) *$\widehat{v}_A(\tau) = \widehat{v}_{E_A}(\tau|_{E_A})$ for all $\tau \in \Delta(A)$.*

Proof. (i) By Lemma 3.20(iii), $\widetilde{A}_M$ is a Segal extension of both E_A and A, whence the claim follows from the previous lemma.

(ii) Let $\tau \in \Delta(A)$. Since it is continuous under $\|\cdot\|_M$, one can deduce, as in the proof of Lemma 4.4, that $|\tau(e_\alpha)| \le 1$ for all $\alpha \in \Omega$ and that $\lim_{\alpha\in\Omega} |\tau(e_\alpha)| = 1$. Together with the fact that any $a \in A$ and $\alpha \in \Omega$ satisfy $\|ae_\alpha\| \le \|a\|\,\|e_\alpha\|_M \le \|a\|$, this yields

$$\|\tau\| \ge \|\tau|_{E_A}\| = \sup_{a\in A\setminus\{0\},\, x\in\widetilde{A}_M\setminus\{0\}} \frac{|\tau(ax)|}{\|ax\|} \ge \sup_{a\in A\setminus\{0\},\,\alpha\in\Omega} \frac{|\tau(ae_\alpha)|}{\|ae_\alpha\|}$$

$$\ge \sup_{a\in A\setminus\{0\},\,\alpha\in\Omega} \frac{|\tau(a)|\,|\tau(e_\alpha)|}{\|a\|} = \sup_{a\in A\setminus\{0\}} \frac{|\tau(a)|}{\|a\|} = \|\tau\|,$$

proving the assertion. ∎

Remark 4.9. It is easy to see that Lemma 4.8 also holds true with the assumption that $\widetilde{A}_M$ has a minimal approximate identity.

4.3. Weighted uniform algebras. We are now in a position to represent A by means of weighted function algebras. Clearly, for every $\tau \in \Delta(A)$, one has

$$\widehat{v}(\tau) = \sup\{l > 0 : l|\widehat{a}(\tau)| \le \|a\| \text{ for all } a \in A\}.$$

Therefore, besides being a subalgebra of $C_0(\Delta(A))$, $\widehat{A}$ is also a subalgebra of $C_b^{\widehat{v}}(\Delta(A))$. Moreover, since each $a \in A$ satisfies

$$\|\widehat{a}\|_\infty \le \|\widehat{a}\|_{\widehat{v}} \le \|a\|, \tag{4.1}$$

it follows that the Gelfand transformation on A is a contractive homomorphism into $C_b^{\widehat{v}}(\Delta(A))$. If A has an approximate identity $(e_\alpha)_{\alpha\in\Omega}$, then

$$\|\widehat{a}\widehat{e}_\alpha - \widehat{a}\|_{\widehat{v}} \le \|ae_\alpha - a\| \to 0$$

for all $a \in A$, implying, by Lemma 2.3, that $\widehat{A}$ is contained in $C_0^{\widehat{v}}(\Delta(A))$. This observation will be useful later.

The following proposition summarizes the above discussion.

Proposition 4.10. *The Gelfand transformation $a \mapsto \widehat{a}$ is a contractive homomorphism from A into $C_b^{\widehat{v}}(\Delta(A))$. If A has an approximate identity, then its image is contained in $C_0^{\widehat{v}}(\Delta(A))$.*

Recall that A is said to be a *uniform algebra* if the Gelfand transformation on A is an isometry into $C_0(\Delta(A))$. The next definition provides a natural generalization of this fundamental concept.

DEFINITION 4.11. We call A a *weighted uniform algebra* if

$$\|a\| = \|\widehat{a}\|_{\widehat{v}} \quad \text{for all } a \in A.$$

As a consequence of (4.1), we see that every uniform algebra is also a weighted uniform algebra. More generally, if A is a weighted uniform algebra, then it is easy to verify that each $a \in A$ satisfies

$$\|a\|_M = \|\widehat{a}\|_\infty. \tag{4.2}$$

Note that, by Example 4.5, a weighted uniform algebra can be norm regular even though $\widehat{v}$ would be unbounded. It is immediate from Lemma 4.4 that the existence of a bounded approximate identity in A_M excludes this possibility.

We close this subsection by an example which shows that the converse of Definition 4.11 holds true, too.

EXAMPLE 4.12. Let B be a closed subalgebra of $C_b^v(X)$ which strongly separates the points of X. Then the set $\{\tau_t : t \in X\}$ is contained in $\Delta(B)$. Clearly, for each $t \in X$, one has $\widehat{v}(\tau_t) \geq v(t)$, so that $\|f\|_v \leq \|\widehat{f}\|_{\widehat{v}}$ for all $f \in B$. Since the reverse inequality is always true, it follows that B is a weighted uniform algebra.

4.4. Almost self-adjoint weighted uniform algebras. Recall that A is said to be *self-adjoint* if its Gelfand transform algebra is self-adjoint. In the context of weighted uniform algebras, it is useful to generalize this notion as follows.

DEFINITION 4.13. We call A *almost self-adjoint* if the self-adjoint part of $\widehat{A}$ strongly separates the points of $\Delta(A)$.

It is readily deduced from the Stone–Weierstrass Theorem that $\widehat{A} = C_0(\Delta(A))$ whenever A is an almost self-adjoint uniform algebra. Therefore, the two notions coincide in a uniform algebra. The next example shows that this is not generally true for weighted uniform algebras. First, we record a simple lemma; the proof is evident from Example 4.3.

LEMMA 4.14. *Let B be a closed subalgebra of $C_b^v(X)$ such that $C_0^v(X) \subseteq B \subseteq C_0(X)$. Then B is almost self-adjoint.*

EXAMPLE 4.15. Let c be the C^*-algebra of complex-valued convergent sequences. Put $B := \{ux : x \in c\}$, where $u \colon \mathbb{N} \to \mathbb{C}$ is given by $u(n) := 1/n$ for odd n, and $u(n) := i/n$ for even n. Define a weight function $v \colon \mathbb{N} \to \mathbb{R}$ by setting $v(n) := n$ for $n \in \mathbb{N}$. It is easy to see that B is a closed subalgebra of $C_b^v(\mathbb{N})$ satisfying the above inclusions. Therefore, B is an almost self-adjoint weighted uniform algebra. However, it fails to be self-adjoint: one has $u \in B$, but if $\overline{u} \in B$, then $\overline{u} = ux$ for some $x \in c$. This implies that $x(n) = 1$ for odd n, and $x(n) = -1$ for even n, a contradiction.

Combining Lemma 4.14 with the following proposition, we obtain a characterization of almost self-adjointness in a weighted uniform algebra.

Proposition 4.16. *Let A be an almost self-adjoint weighted uniform algebra. Then $\widehat{A}$ is a closed subalgebra of $C_b^{\widehat{v}}(\Delta(A))$ such that $C_0^{\widehat{v}}(\Delta(A)) \subseteq \widehat{A} \subseteq C_0(\Delta(A))$.*

Proof. Since $\widehat{A}_{\mathrm{sa}}$ is a self-adjoint subalgebra of $C_0(\Delta(A))$ which strongly separates the points of $\Delta(A)$, one can deduce from the Stone–Weierstrass Theorem that $\widehat{A}$ is dense in $C_0(\Delta(A))$. Together with (4.2), this implies that $\widehat{A}$ is a Segal algebra in $C_0(\Delta(A))$ with approximate ideal of the form $E_{\widehat{A}} = \widehat{A}C_0(\Delta(A))$. Clearly, the zero set of $E_{\widehat{A}}$ is empty, whence $E_{\widehat{A}} = C_0^{\widehat{v}}(\Delta(A))$ by Lemma 2.3 and Proposition 2.7. This proves the claim. ∎

The importance of almost self-adjoint weighted uniform algebras in the study of Banach algebras without a bounded approximate identity is explained by the classification result below. The proof is immediate from Lemma 4.4 and Propositions 4.10 and 4.16.

Theorem 4.17. *Let A be an almost self-adjoint weighted uniform algebra. Then:*

(i) *A has a bounded approximate identity if and only if $\widehat{A} = C_0(\Delta(A))$;*
(ii) *A has an approximate identity if and only if $\widehat{A} = C_0^{\widehat{v}}(\Delta(A))$;*
(iii) *A does not have an approximate identity if and only if $\widehat{A}$ is a closed subalgebra of $C_b^{\widehat{v}}(\Delta(A))$ such that $C_0^{\widehat{v}}(\Delta(A)) \subseteq \widehat{A} \subseteq C_0(\Delta(A))$, both inclusions being strict.*

By means of Proposition 2.14 and Lemmas 4.7 and 4.8, this result can be reformulated as follows.

Corollary 4.18. *Let A be an almost self-adjoint weighted uniform algebra. Then:*

(i) *$\widetilde{A}_M$ is isometrically isomorphic to $C_0(\Delta(A))$;*
(ii) *E_A is isometrically isomorphic to $C_0^{\widehat{v}}(\Delta(A))$;*
(iii) *A is isometrically isomorphic to a closed almost self-adjoint subalgebra of $C_b^{\widehat{v}}(\Delta(A))$ which strongly separates the points of $\Delta(A)$.*

Remark 4.19. As a consequence of Theorem 4.17, we see that an almost self-adjoint weighted uniform algebra with an approximate identity is automatically self-adjoint.

We end this subsection with a discussion of the ideal structure of self-adjoint weighted uniform algebras having an approximate identity. For a closed ideal I of A and a closed subset E of $\Delta(A)$, we denote by $h(I)$ the *hull* of I and by $k(E)$ the *kernel* of E, i.e.,

$$h(I) := \{\tau \in \Delta(A) : \widehat{x}(\tau) = 0 \text{ for all } x \in I\},$$
$$k(E) := \{a \in A : \widehat{a}(\tau) = 0 \text{ for all } \tau \in E\}.$$

Clearly, $h(I)$ is a closed subset of $\Delta(A)$ and $k(E)$ is a closed ideal of A.

PROPOSITION 4.20. *Let A be a self-adjoint weighted uniform algebra with an approximate identity. Then, for every closed ideal I of A, one has:*

(i) $I = k(h(I))$;

(ii) *I is a self-adjoint weighted uniform algebra with an approximate identity;*

(iii) *A/I is a self-adjoint weighted uniform algebra with an approximate identity;*

(iv) *$I + J$ is a closed ideal of A for any closed ideal J of A.*

Proof. Parts (i) and (ii) follow from Proposition 2.7, and parts (iii) and (iv) from Propositions 2.9 and 2.10. ∎

4.5. Multipliers of weighted uniform algebras. It is well known that if A is semisimple, then each $m \in M(A)$ determines a unique $\widehat{m} \in C_b(\Delta(A))$ with the properties that $m(a)^\wedge = \widehat{m}\widehat{a}$ for all $a \in A$ and $\|\widehat{m}\|_\infty \le \|m\|$. Furthermore, in the particular case where A is a uniform algebra, the supremum norm of $\widehat{m}$ coincides with the operator norm of m. We shall now apply the weighted Gelfand representation to prove an analogous result for the multiplier module $M_{\widetilde{A}_M}(A)$.

LEMMA 4.21. *Let A be a semisimple Banach algebra such that A_M has a minimal approximate identity. Then, for every $m \in M_{\widetilde{A}_M}(A)$, there exists a unique $\widehat{m} \in C_b^{\widehat{v}}(\Delta(A))$ such that:*

(i) $m(a)^\wedge = \widehat{m}\widehat{a}$ *for all* $a \in A$;

(ii) $\|\widehat{m}\|_{\widehat{v}} \le \|m\|$.

Proof. Clearly, the restriction of m to A is a multiplier of A; thus there exists a unique $\widehat{m} \in C_b(\Delta(A))$ such that $m(a)^\wedge = \widehat{m}\widehat{a}$ for all $a \in A$. To see that $\widehat{m} \in C_b^{\widehat{v}}(\Delta(A))$ and that $\|\widehat{m}\|_{\widehat{v}} \le \|m\|$, let $\tau \in \Delta(A)$. Then each $a \in A$ satisfies

$$\widehat{v}(\tau)|\widehat{m}(\tau)|\,|\widehat{a}(\tau)| = \widehat{v}(\tau)|m(a)^\wedge(\tau)| \le \|m(a)\| \le \|m\|\,\|a\|_M,$$

so that

$$\widehat{v}(\tau)|\widehat{m}(\tau)| \le \inf_{\|a\|_M=1} \frac{\|m\|}{|\widehat{a}(\tau)|} = \frac{\|m\|}{\sup_{\|a\|_M=1}|\tau(a)|} = \|m\|. \ ∎$$

For semisimple A, put

$$\mathcal{M}^{\widehat{v}}(A) := \{f \in C_b^{\widehat{v}}(\Delta(A)) : f\widehat{a} \in \widehat{A} \text{ for all } a \in A\}.$$

Then $\mathcal{M}^{\widehat{v}}(A)$ is a subalgebra of $C_b^{\widehat{v}}(\Delta(A))$ and the mapping $m \mapsto \widehat{m}$ is a contractive injective homomorphism from $M_{\widetilde{A}_M}(A)$ into $\mathcal{M}^{\widehat{v}}(A)$ whenever A_M has a minimal approximate identity. In the special case of a weighted uniform algebra, we have:

PROPOSITION 4.22. *Let A be a weighted uniform algebra such that A_M has a minimal approximate identity. Then $M_{\widetilde{A}_M}(A)$ is isometrically isomorphic to $\mathcal{M}^{\widehat{v}}(A)$.*

Proof. Since each $m \in M_{\widetilde{A}_M}(A)$ satisfies

$$\begin{aligned}\|m\| &= \sup_{\|a\|_M \le 1} \|m(a)\| = \sup_{\|\widehat{a}\|_\infty \le 1} \|m(a)^\wedge\|_{\widehat{v}} = \sup_{\|\widehat{a}\|_\infty \le 1} \|\widehat{m}\widehat{a}\|_{\widehat{v}} \\ &\le \sup_{\|\widehat{a}\|_\infty \le 1} \|\widehat{m}\|_{\widehat{v}} \|\widehat{a}\|_\infty = \|\widehat{m}\|_{\widehat{v}}, \quad a \in A,\end{aligned}$$

the mapping $m \mapsto \widehat{m}$ is an isometry. To show that it is a surjection, let $f \in \mathcal{M}^{\hat{v}}(A)$ and define $m\colon \widetilde{A}_M \to A$ by the formula $m(x)^\wedge = f\widehat{x}$ for $x \in \widetilde{A}_M$. One easily checks that m belongs to $M_{\widetilde{A}_M}(A)$ and that it satisfies $\widehat{m} = f$. ∎

REMARK 4.23. It is easy to see that the above results also hold true with the assumption that $\widetilde{A}_M$ has a minimal approximate identity.

By Lemma 2.3 and Theorem 4.17, $\widehat{A}$ is an ideal of $C_b^{\widehat{v}}(\Delta(A))$ whenever A is an almost self-adjoint weighted uniform algebra. We thus obtain the following corollary.

COROLLARY 4.24. *Let A be an almost self-adjoint weighted uniform algebra. Then $M_{\widetilde{A}_M}(A)$ is isometrically isomorphic to $C_b^{\widehat{v}}(\Delta(A))$.*

5. C^*-Segal algebras. This section contains our main results. In Theorems 5.7 and 5.21, we establish a module- and an order-theoretic generalization of the Gelfand–Naimark Theorem. As an application, we describe faithful principal ideals of C^*-algebras.

5.1. General properties of C^*-Segal algebras

DEFINITION 5.1. We call A a *C^*-Segal algebra* if it has a Segal extension C, where C is a C^*-algebra.

We leave it to the reader to verify the following basic properties of C^*-Segal algebras.

LEMMA 5.2. *Let A be a C^*-Segal algebra. Then:*

(i) *C is unique up to an isometric *-isomorphism;*
(ii) *$\|\cdot\|_M$ and $\|\cdot\|_C$ are equivalent norms on A;*
(iii) *every closed ideal of A is a C^*-Segal algebra;*
(iv) *E_A and $M_C(A)$ are C^*-Segal algebras;*
(v) *A is semisimple;*
(vi) *A is self-adjoint if and only if it is closed under the involution of C.*

Remark 5.3. In general, the quotient of a C^*-Segal algebra by a closed ideal is not a C^*-Segal algebra. As an example, the annihilator of $C_b^v(X)/C_0^v(X)$ is the whole algebra whenever $C_b^v(X)$ is contained in $C_0(X)$.

Concerning the self-adjointness of C^*-Segal algebras, we have the following result.

Lemma 5.4. *Let A be a C^*-Segal algebra. Then A is almost self-adjoint. Moreover, the following conditions are equivalent:*

(a) *E_A is self-adjoint;*
(b) *$M_C(A)$ is self-adjoint.*

Finally, E_A is self-adjoint whenever A is self-adjoint.

Proof. The almost self-adjointness of A follows from Lemmas 2.1(iii) and 4.7. The implication (b)⇒(a) and the last claim follow from Proposition 3.24(ii) and Lemma 3.20(ii). The implication (a)⇒(b) follows from the easily verified fact that the image of every multiplier in $M_C(A)$ is contained in E_A. ■

There is an interesting relation between approximate identities, closed ideals, and the Stone–Weierstrass property of C^*-Segal algebras, as described by the proposition below. Recall that A is said to have

(i) the *property of spectral synthesis* if $I = k(h(I))$ for every closed ideal I of A;
(ii) the *Stone–Weierstrass property* if every self-adjoint subalgebra of A which strongly separates the points of $\Delta(A)$ is dense in A.

In the following, we let $l > 0$ designate a constant as in Lemma 3.5.

Proposition 5.5. *Let A be a C^*-Segal algebra. The following conditions are equivalent:*

(a) *A has an approximate identity $(e_\alpha)_{\alpha\in\Omega}$;*
(b) *A has the property of spectral synthesis;*
(c) *A has the Stone–Weierstrass property.*

Proof. (a)⇒(b). Let I be a closed ideal of A. Since the hulls of I and $k(h(I))$ agree on $\Delta(A)$, Lemmas 4.7 and 5.2(iii) together with the spectral synthesis property of C imply that I and $k(h(I))$ have the same closure in C. Given $x \in k(h(I))$ and $\epsilon > 0$, one can thus find $\alpha \in \Omega$ and $y \in I$ with $\|x - xe_\alpha\| < \epsilon$ and $\|x - y\|_C < \epsilon l^{-1}\|e_\alpha\|^{-1}$. As $ye_\alpha \in I$ and

$$\|x - ye_\alpha\| \le \|x - xe_\alpha\| + \|xe_\alpha - ye_\alpha\| \le \|x - xe_\alpha\| + l\|e_\alpha\|\,\|x - y\|_C < 2\epsilon,$$

it follows that I is dense in $k(h(I))$, whence $I = k(h(I))$ as wanted.

(b)⇒(c). Let B be a self-adjoint subalgebra of A strongly separating the points of $\Delta(A)$, and let J denote the closure of B in A. Then, by the Stone–Weierstrass property of C and Lemma 4.7, B is dense in C. Given

$a \in A$, $x \in J$, and $\epsilon > 0$, one can thus find $y, z \in B$ with $\|x - y\| < \epsilon \|a\|^{-1}$ and $\|a - z\|_C < \epsilon l^{-1} \|y\|^{-1}$. As $yz \in B$ and

$$\|ax - yz\| \le \|ax - ay\| + \|ay - yz\| \le \|a\| \, \|x - y\| + l\|y\| \, \|a - z\|_C < 2\epsilon,$$

it follows that J is an ideal of A, whence $J = A$ as required.

(c)⇒(a). Put $B := \{x \in E_A : x^* \in E_A\}$. Then Lemmas 5.2(iv) and 5.4 imply that B is a self-adjoint subalgebra of E_A strongly separating the points of $\Delta(E_A)$. Thus, by Lemma 4.8(i) and the Stone–Weierstrass property of A, it follows that B is dense in A, whence $E_A = A$ as desired. ∎

Remark 5.6. The concept of a Stone–Weierstrass property was introduced by Katznelson and Rudin [19] for semisimple Banach algebras. In view of that paper, where it was demonstrated that the connection between the Stone–Weierstrass property and the ideal structure of semisimple Banach algebras is not generally a very close one, the above result is slightly surprising.

5.2. Weighted C^*-algebras. We now come to the module-theoretic generalization of the Gelfand–Naimark Theorem.

Theorem 5.7. *Let A be a C^*-Segal algebra. The following conditions are equivalent:*

(a) *A is an almost self-adjoint weighted uniform algebra with $\widehat{v}$ continuous on $\Delta(A)$;*
(b) *there exists a positive isometric C-module homomorphism $\pi : A \to M(C)$.*

Remark 5.8. (i) By *positivity* of π we mean that the spectrum of $\pi(a)$ in $M(C)$ is non-negative whenever the spectrum of a in A is non-negative.

(ii) Letting ϕ be the canonical embedding of C into $M(C)$, it is easy to see that the C-module action on $M(C)$ satisfies $\pi(a) \cdot x = \pi(a)\phi(x)$ for all $a \in A$ and $x \in C$. In what follows, we shall identify C with its image in $M(C)$ under ϕ.

Proof of Theorem 5.7. (a)⇒(b) The desired mapping π is given by $\pi(a) := m_a$ for $a \in A$, where m_a is defined by the formula $\widehat{m}_a(\widetilde{\tau}) = \widehat{v}(\tau)\widehat{a}(\tau)$ for $\tau \in \Delta(A)$.

(b)⇒(a). The proof is divided into seven steps.

Step 1: *$\|a\|_M = \|a\|_C$ for all $a \in A$.* Let $a \in A$. Then

$$\|a\|_M = \sup_{b \in A \setminus \{0\}} \frac{\|ab\|}{\|b\|} = \sup_{b \in A \setminus \{0\}} \frac{\|\pi(ab)\|_C}{\|\pi(b)\|_C} = \sup_{b \in A \setminus \{0\}} \frac{\|a\pi(b)\|_C}{\|\pi(b)\|_C} \le \|a\|_C;$$

the reverse inequality follows from the minimality property of the C^*-norm; see [27, Theorem 3.4.22], for instance.

Step 2: *$\pi(E_A)$ is a closed ideal of C.* This is immediate from the assumptions on π.

Step 3: *There is a unique non-negative $f_\pi \in C(\Delta(A))$ such that $\pi(x)^\wedge(\widetilde{\tau}) = f_\pi(\tau)\tau(x)$ for all $x \in E_A$ and $\tau \in \Delta(A)$.* Let $\tau \in \Delta(A)$ and $x, y \in E_A$ be such that $\tau(x)$ and $\tau(y)$ are non-zero. Since $\pi(x)y = \pi(xy) = x\pi(y)$ and $\pi(E_A) \subseteq C$, one has

$$\frac{\widetilde{\tau}(\pi(x))}{\tau(x)} = \frac{\widetilde{\tau}(\pi(y))}{\tau(y)}.$$

In view of this and Lemma 4.8(i), we can define a mapping $f_\pi \colon \Delta(A) \to \mathbb{C}$ by setting

$$f_\pi(\tau) := \frac{\widetilde{\tau}(\pi(x))}{\tau(x)},$$

where $x \in E_A$ is such that $\tau(x) \neq 0$. It is easy to verify that f_π is continuous on $\Delta(A)$. Moreover, if $\tau \in \Delta(A)$ and $x \in E_A$ are such that $\tau(x) = 0$, then each $y \in E_A$ satisfies $\widetilde{\tau}(\pi(x))\tau(y) = \tau(x)\widetilde{\tau}(\pi(y)) = 0$. As a result, $\widetilde{\tau}(\pi(x)) = 0$ so that $\pi(x)^\wedge(\widetilde{\tau}) = f_\pi(\tau)\tau(x)$ for all $x \in E_A$ and $\tau \in \Delta(A)$. Next, to establish the non-negativity of f_π, let $\tau \in \Delta(A)$ and pick $x \in E_A$ such that $\tau(x) \neq 0$. Since the spectrum of x^*x is non-negative and $\tau(x^*x)$ is a strictly positive real number, one has

$$f_\pi(\tau) = \frac{\widetilde{\tau}(\pi(x^*x))}{\tau(x^*x)} \geq 0$$

as desired. The uniqueness of f_π is trivial.

Step 4: *$\Delta(A) \setminus Z(f_\pi)$ is dense in $\Delta(A)$.* Suppose $\Delta(A) \setminus Z(f_\pi)$ is not dense in $\Delta(A)$. Then there is a non-empty open subset U of $\Delta(A)$ contained in $Z(f_\pi)$. Clearly, one can find a non-zero $x \in E_A$ such that $\Delta(A) \setminus U$ is contained in $Z(\widehat{x})$. Together with Step 3, this implies $\widetilde{\tau}(\pi(x)) = 0$ for all $\tau \in \Delta(A)$. Thus, by Lemma 4.7 and the semisimplicity of C, one has $\pi(x) = 0$, contradicting the injectivity of π.

Step 5: *$f_\pi(\tau) = \widehat{v}(\tau)$ for all $\tau \in \Delta(A)$.* Let $\tau \in \Delta(A) \setminus Z(f_\pi)$. It follows from Step 3 that each $x \in E_A$ with $\tau(x) \neq 0$ satisfies $\widetilde{\tau}(\pi(x)) \neq 0$. Thus, as $\pi(E_A)$ is a closed ideal of C and $\widetilde{\tau}|_{\pi(E_A)}$ is a non-zero multiplicative functional on it, one has $\|\widetilde{\tau}|_{\pi(E_A)}\|_C = 1$. Together with Lemma 4.8(ii) and Steps 1 and 3, this yields

$$\|\tau\| = \|\tau|_{E_A}\| = \sup_{x \in E_A \setminus \{0\}} \frac{|\tau(x)|}{\|x\|} = \frac{1}{f_\pi(\tau)} \sup_{x \in E_A \setminus \{0\}} \frac{|\widetilde{\tau}(\pi(x))|}{\|\pi(x)\|_C} = \frac{1}{f_\pi(\tau)},$$

and therefore, to complete the proof, it suffices to show that $Z(f_\pi)$ is empty. But this is immediate from the density of $\Delta(A) \setminus Z(f_\pi)$ in $\Delta(A)$ and the fact that f_π is continuous on $\Delta(A)$ with $f_\pi(\tau) \geq 1$ for all $\tau \in \Delta(A) \setminus Z(f_\pi)$.

Step 6: $\|x\| = \|\widehat{x}\|_{\widehat{v}}$ *for all* $x \in E_A$. Let $x \in E_A$. Then Steps 3 and 5 together with Lemma 4.7 imply

$$\begin{aligned}\|x\| = \|\pi(x)\|_C &= \sup_{\tau\in\Delta(A)} |\pi(x)^\wedge(\widetilde{\tau})| = \sup_{\tau\in\Delta(A)} |f_\pi(\tau)\tau(x)| \\ &= \sup_{\tau\in\Delta(A)} \widehat{v}(\tau)|\widehat{x}(\tau)| = \|\widehat{x}\|_{\widehat{v}},\end{aligned}$$

as claimed.

Step 7: *A is an almost self-adjoint weighted uniform algebra with $\widehat{v}$ continuous.* By Lemma 5.4 and Step 5, it is enough to prove that A is a weighted uniform algebra. Let $(e_\alpha)_{\alpha\in\Omega}$ be a minimal approximate identity for C. Then each $a \in A$ satisfies

$$\begin{aligned}\|\widehat{a}\|_{\widehat{v}} &\overset{(*)}{\geq} \sup_{\|x\|_C\leq 1} \|ax\| = \sup_{\|x\|_C\leq 1} \|\pi(ax)\|_C \geq \lim_{\alpha\in\Omega} \|\pi(ae_\alpha)\|_C \\ &= \lim_{\alpha\in\Omega} \|\pi(a)e_\alpha\|_C \overset{(**)}{=} \|\pi(a)\|_C = \|a\|, \qquad x \in C,\end{aligned}$$

where $(*)$ is given by Step 6 and $(**)$ by [24, Theorem 7.3.1], for example. The reverse inequality is always true, and so the result follows. ∎

Remark 5.9. (i) It is easy to deduce from Steps 3 and 5 together with the faithfulness of C in $M(C)$ and Lemma 4.7 that the mapping π is unique, i.e., if $\pi'\colon A \to M(C)$ is a positive isometric C-module homomorphism, then $\pi = \pi'$.

(ii) It is not difficult to see that Theorem 5.7 also holds true without the positivity assumption on π; however, such a mapping is not unique.

(iii) It follows from Steps 3 and 5 together with Lemma 4.7 that the hull of $\pi(E_A)$ in C is empty, whence $\pi(E_A) = C$ by the spectral synthesis property of C.

Motivated by Theorem 5.7, we make the following definition.

Definition 5.10. By a *weighted C^*-algebra* we mean a pair (A, π), where

(i) A is a C^*-Segal algebra;
(ii) $\pi\colon A \to M(C)$ is a positive isometric C-module homomorphism.

Among the most important examples of weighted C^*-algebras are the faithful principal ideals of C^*-algebras, as shown by the proposition below.

Proposition 5.11. *Let B be a C^*-algebra, and let $u \in B$ be such that uB is faithful in B. Then there exists a norm on uB making it into a weighted C^*-algebra, and uB has an approximate identity if and only if it is dense in B.*

Proof. One can assume that $\|u\|_B = 1$. Put $I := uB$ and define a mapping $|\cdot|: I \to \mathbb{R}$ by setting $|ux| := \|x\|_B$ for $x \in B$. Clearly, $|\cdot|$ is a norm on I making it into a C^*-Segal algebra. Since J, the closure of I in B, is an ideal of B, the multiplication operator l_x belongs to $M(J)$ for every $x \in B$. It is easy to verify that the mapping $ux \mapsto l_x$ from I into $M(J)$ is an isometric J-module homomorphism. Consequently, I is a weighted C^*-algebra. The second assertion follows from the identities $E_I = IJ = uBJ = uJ$. ∎

REMARK 5.12. (i) Clearly, $\widehat{v}_I(\tau) = 1/|\tau(u)|$ for $\tau \in \Delta(I) = \Delta(B) \setminus Z(\widehat{u})$.

(ii) Another equivalent condition for the density of I in B is the *strict positivity* of u, that is, $\widehat{u}(\tau) > 0$ for all $\tau \in \Delta(B)$.

It is not difficult to see that Proposition 5.11 admits the following generalization.

COROLLARY 5.13. *Let B be a C^*-algebra, and let m be an injective multiplier of B. Then there exists a norm on $m(B)$ making it into a weighted C^*-algebra, and $m(B)$ has an approximate identity if and only if it is dense in B.*

5.3. Order structure of C^*-Segal algebras. We now proceed to the order-theoretic generalization of the Gelfand–Naimark Theorem. Let A be a C^*-Segal algebra. The *positive cone* of A is defined by

$$A_+ := \{a \in A : \tau(a) \geq 0 \text{ for all } \tau \in \Delta(A)\}.$$

Let A_h denote the real vector space of hermitian elements of A. Then A_h becomes a partially ordered vector space when equipped with the relation

$$a \leq b \quad \text{if} \quad b - a \in A_+ \qquad (a, b \in A_h).$$

An element $u \in A_+$ is called an *order unit* for A if each $a \in A_h$ satisfies $a \leq lu$ for some constant $l > 0$. Clearly, if A has an order unit, then it is strictly positive.

EXAMPLE 5.14. Consider $C_b^v(X)$ with v continuous on X. Then the function $1/v$ is an order unit for $C_b^v(X)$.

For future reference, we record the following standard lemma.

LEMMA 5.15. *Let A be a self-adjoint C^*-Segal algebra with order unit u. Then*

$$A = A_h + iA_h \quad \text{and} \quad A_h = A_+ - A_+.$$

Moreover, for each $c \in C$, one has $uc = 0$ if and only if $c = 0$.

Our next result is a purely algebraic characterization of self-adjoint C^*-Segal algebras with an order unit. Here, 1 denotes the identity element of $M(C)$.

Proposition 5.16. *Let A be a self-adjoint C^*-Segal algebra, and let $u \in A_+$. The following conditions are equivalent:*

(a) *u is an order unit for A;*
(b) *$A = uD$ for some subspace D of $M(C)$ containing C and 1.*

Proof. It is enough to prove that (a) implies (b). Let w be the weight function on $\Delta(A)$ given by $w(\tau) := 1/\widehat{u}(\tau)$ for $\tau \in \Delta(A)$. By Proposition 5.11 and Remark 5.12 together with Lemma 4.7, the Gelfand transform algebra of uC can be identified with $C_0^w(\Delta(A))$. The remainder of the proof goes as follows.

Step 1: $\widehat{A} \subseteq C_b^w(\Delta(A))$. Let $a \in A$. Then there exist $b, c \in A_h$ such that $a = b + ic$. Since, for some positive constants k and l, one has $b \leq ku$ and $c \leq lu$, it follows that the Gelfand transforms of b and c belong to $C_b^w(\Delta(A))$. This proves the inclusion.

Step 2: $E_A = uC$. By Lemma 2.3 and Step 1 together with Lemma 4.7, the Gelfand transform algebra of E_A is contained in $C_0^w(\Delta(A))$, whence the identity follows from the above discussion and the semisimplicity of A.

Step 3: $M_C(A) = uM(C)$. Let $m \in M_C(A)$ and recall that its image is contained in E_A. By the last assertion of Lemma 5.15 and Step 2, we can thus define a mapping $n \colon C \to C$ by the formula $m(x) = un(x)$ for $x \in C$. To verify that n is a multiplier of C, let $x, y \in C$. Then $un(xy) = m(xy) = m(x)y = un(x)y$, so that $n(xy) = n(x)y$ by Lemma 5.15 again. The reverse inclusion is trivial, and so the identity follows.

Step 4: *$A = uD$ for some subspace D of $M(C)$ containing C and* 1. Putting $D := \{m \in M(C) : um \in A\}$, the assertion is immediate from Steps 2 and 3 together with the inclusions $E_A \subseteq A \subseteq M_C(A)$. ■

In view of the identity $E_A = M_C(A)C$ together with Lemmas 5.2(iv) and 5.4, it is not difficult to see that Proposition 5.16 admits the following generalization.

Corollary 5.17. *Let A be a C^*-Segal algebra such that E_A is self-adjoint, and let $u \in M_C(A)_+$. The following conditions are equivalent:*

(i) *u is an order unit for $M_C(A)$;*
(ii) *$A = uD$ for some subspace D of $M(C)$ containing C.*

Using a construction of Paulsen and Tomforde [28, Definition 4.4], we now describe a natural norm on a self-adjoint C^*-Segal algebra with an order unit. Call a positive functional ω on a C^*-Segal algebra B with order unit u a *state* if $\omega(u) = 1$.

Definition 5.18. Let A be a self-adjoint C^*-Segal algebra with order unit u. We define a mapping $\|\cdot\|_u\colon A \to \mathbb{R}$ by setting

$$\|a\|_u := \sup\{|\omega(a)| : \omega \text{ is a state on } A\} \qquad (a \in A)$$

and call it the *order norm* on A induced by u.

In [28], this map was defined in the setting of partially ordered *-vector spaces with an order unit, and it was called the *minimal order norm*. The terminology is explained by the following important result (see [28, Theorem 4.5 and Proposition 4.9]).

Lemma 5.19. *Let A be a self-adjoint C^*-Segal algebra with order unit u. Then $\|\cdot\|_u$ is a vector space norm on A satisfying the following conditions:*

(i) *$\|a^*\|_u = \|a\|_u$ for all $a \in A$;*
(ii) *$\|a\|_u = \inf\{l > 0 : -lu \le a \le lu\}$ for all $a \in A_h$;*
(iii) *if $|\cdot|$ is a vector space norm on A with the above two properties, then there exists a constant $l > 0$ such that $\|a\|_u \le |a| \le l\|a\|_u$ for all $a \in A$.*

Definition 5.20. By an *order unit C^*-Segal algebra* we mean a pair (A, u), where A is a self-adjoint C^*-Segal algebra and u is an order unit for A such that

$$\|a\| = \|a\|_u \quad \text{for all } a \in A.$$

We can now state and prove the order-theoretic generalization of the Gelfand–Naimark Theorem.

Theorem 5.21. *Let (A, u) be an order unit C^*-Segal algebra. Then:*

(i) *$\widehat{u}(\tau) = \|\tau\|$ for all $\tau \in \Delta(A)$;*
(ii) *$\Delta(A)$ is σ-compact;*
(iii) *A is a weighted C^*-algebra.*

Furthermore, A is isometrically *-isomorphic to $C_b^{\widehat{v}}(\Delta(A))$ if and only if it is an order ideal of C, i.e., if $0 \le x \le a$ for $a \in A$ and $x \in C$ then $x \in A$.

Proof. Let $|\cdot|$ be the *-algebra norm on A defined by $|a| := \|\widehat{a}\|_{\widehat{v}}$ for $a \in A$. By [28, Lemma 4.16], one has $\widehat{u}(\tau) = \|\tau\|$ for all $\tau \in \Delta(A)$. This yields the continuity of $\widehat{v}$ and the σ-compactness of $\Delta(A)$ because $\widehat{A}$ contains a strictly positive element. To establish the property (ii) of Lemma 5.19, let $a \in A_h$. Then

$$\begin{aligned}|a| &= \inf\{l > 0 : -l \le \widehat{v}(\tau)\,\widehat{a}(\tau) \le l \text{ for all } \tau \in \Delta(A)\}\\ &= \inf\{l > 0 : -l\widehat{u}(\tau) \le \widehat{a}(\tau) \le l\widehat{u}(\tau) \text{ for all } \tau \in \Delta(A)\}\\ &= \inf\{l > 0 : -lu \le a \le lu\},\end{aligned}$$

so that A is a weighted C^*-algebra by Theorem 5.7 and the minimality property of $\|\cdot\|_u$.

Next, suppose that A is an order ideal of C. To show that $\widehat{A} = C_b^{\widehat{v}}(\Delta(A))$, it suffices to prove that $A = M_C(A)$, by Corollary 4.24. Let D be a subspace of $M(C)$ satisfying $A = uD$ and containing C and 1. By Step 3 of the proof of Proposition 5.16, one has $A = M_C(A)$ if and only if $D = M(C)$. To verify the latter identity, note that the order ideal hypothesis together with Lemma 5.15 implies that A, and therefore D as well, is an ideal of $M(C)$. But since the identity element of $M(C)$ belongs to D, it follows that the two sets coincide. The other direction of the statement is trivial. ∎

In order to extend this result to weighted C^*-algebras without an order unit, we need to generalize the notion of a unitization of a C^*-algebra.

DEFINITION 5.22. Let A be a C^*-Segal algebra. By an *order unitization* of A we mean a pair (B, ι), where

(i) B is an order unit C^*-Segal algebra;
(ii) ι is a positive isometric homomorphism from A into B;
(iii) $\iota(A)$ is a faithful ideal of B.

In view of Corollary 4.24, it is evident that every weighted C^*-algebra has an order unitization. The result below shows that the converse holds too. First, we recall a basic fact about multipliers of C^*-algebras.

LEMMA 5.23. *Let I and J be closed faithful ideals in a C^*-algebra such that $I \subseteq J$. Then the mapping $m \mapsto m|_I$ is a positive isometric algebra and I-module homomorphism from $M(J)$ into $M(I)$.*

PROPOSITION 5.24. *Let A be a C^*-Segal algebra. The following conditions are equivalent:*

(a) *A has an order unitization;*
(b) *A is a weighted C^*-algebra.*

Proof. Let (B, ι) be an order unitization of A. Then there is a C^*-algebra C' containing B as a Segal algebra and a positive isometric C'-module homomorphism $\psi \colon B \to M(C')$. Without loss of generality, identify A with its image in B under ι. It is not hard to verify that the C^*-algebra C in which A is a Segal algebra is the closure of A in C'. Using the faithfulness of A in B together with the density of A in C and B in C', respectively, it is easy to see that C is a faithful ideal of C'. In view of this and Lemma 5.23, it is evident that the desired mapping $\pi \colon A \to M(C)$ is given by $\pi(a) := \psi(a)|_C$ for $a \in A$. ∎

REMARK 5.25. Theorems 5.7 and 5.21 suggest that the theory of Nachbin algebras can be extended to the non-commutative setting. In fact, first steps in this direction have already been taken in [20].

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