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ON DERIVATIVES OF p-ADIC L-SERIES AT $s = 0$

DAVID BURNS

ABSTRACT. We use techniques of non-commutative Iwasawa theory to investigate the values at zero of higher derivatives of p-adic Artin L-series.

1. INTRODUCTION

Let p be an odd prime. In this article we use techniques and results from non-commutative Iwasawa theory to investigate detailed arithmetic properties of the values at zero of the higher derivatives of p-adic Artin L-series.

As concrete applications of our approach we shall (unconditionally) extend the main results of Federer and Gross in [11] from linear $\mathbb{Q}_p$-valued characters to general finite dimensional p-adic characters, define a canonical refinement of Gross's p-adic regulator map that gives a natural description of the image of Gross's map and also encodes p-adic valuations of the values at zero of higher derivatives of p-adic Artin L-series, and describe an explicit non-abelian generalisation of Gross's p-adic analytic approach to Hilbert's twelfth problem. At the same time we prove natural p-adic analogues of the central conjecture of Chinburg in [9] and of a natural non-abelian generalisation of the annihilation results proved by Rubin in [37].

In addition, our methods suggest the formulation of a natural conjectural 'refined p-adic class number formula' for $\mathbb{G}_m$ and allow us to prove this conjecture modulo Iwasawa's conjecture on the vanishing of cyclotomic μ-invariants, and even in some interesting cases unconditionally (see Remark 3.7).

We also deduce several concrete consequences of this refined p-adic class number formula, including explicit formulas for the (non-commutative) Fitting invariants of Selmer groups that are naturally associated to $\mathbb{G}_m$ over number fields.

In particular, in this way we are able to show that the validity of Gross's conjecture on the order of vanishing at zero of p-adic Artin L-series implies, in general modulo the above μ-invariant hypothesis, the p-component of the 'minus part' of the equivariant Tamagawa number conjecture for untwisted Tate motives over CM Galois extensions of totally real fields as well as the non-abelian extensions of Brumer's Conjecture and of the Brumer-Stark Conjecture that were formulated independently by Nickel in [31] and by the present author in [2].

The latter results are of interest because they make no assumptions concerning the decomposition behaviour of p-adic places and, ever since Wiles' seminal work in [45], it has been clear

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that trivial zeroes have a particularly important (if hitherto unclear) role to play in relation to Brumer's Conjecture and its variants.

In addition, and as first pointed out by Greither, there is a gap in the argument given in [45] and, building on ideas of Greither in [14], all previous attempts to breach that gap have either restricted to special classes of fields where trivial zeroes can be skilfully avoided or effectively ignored the problem by dealing with imprimitive L-series.

By contrast, our approach now both makes clear that the difficulties caused by trivial zeroes in relation to Brumer's Conjecture (and the equivariant Tamagawa number conjecture) are concerned solely with verifying that the order of vanishing at zero of all relevant p-adic Artin L-series agrees with an explicit formula predicted by Gross and also provides a concrete strategy for obtaining new results concerning these conjectures.

For example, in a subsequent article [5], jointly authored with Sano, we will show that the approach described here can be used to obtain the first unconditional verifications of both the minus part of the equivariant Tamagawa number conjecture for untwisted Tate motives and of the (non-abelian) Brumer-Stark Conjecture in the technically most difficult case of non-abelian Galois extensions that have degree divisible by p and characters for which the associated p-adic L-series possess trivial zeroes.

Finally, we remark that our approach also leads to new evidence for the 'refined class number formula for $\mathbb{G}_m$' that was conjectured independently by Mazur and Rubin in [27] and by Sano in [39]. A brief discussion of this aspect is given in §3.5 and full details can be found in [5].

In a little more detail, the main contents of this article is as follows. In §2 we review the p-adic Gross-Stark conjecture and then, in §3, we state the main results of this article and formulate a refined p-adic class number formula for $\mathbb{G}_m$. In §4 we construct a canonical family of perfect complexes that will play a key role in our arguments and in §5 we prove preliminary results in (non-commutative) Iwasawa-theory, define a canonical refinement of Gross's p-adic regulator map in terms of natural Bockstein homomorphisms and interpret the order of vanishing at zero of p-adic Artin L-series in terms of the semisimplicity of natural Iwasawa modules. In §6 we prove all of our main results concerning orders of vanishing, valuations and the p-adic analytic approach to Hilbert's twelfth problem and then, finally, in §7 we prove that our refined p-adic class number formula for $\mathbb{G}_m$ is valid modulo the vanishing of cyclotomic μ-invariants and deduce several concrete consequences of this result.

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2. The p-adic Gross-Stark Conjecture

In this section we quickly review certain conjectures of Gross concerning the derivatives at zero of p-adic Artin L-series.

To do this we fix a finite CM Galois extension F of a totally real number field k with group G. We write $\mathrm{Ir}(G)$ and $\mathrm{Ir}_p(G)$ for the set of irreducible characters of G over $\mathbb{C}$ and $\mathbb{C}_p$ and for each such character ψ we fix a representation V_χ over $\mathbb{C}$, respectively over $\mathbb{C}_p$, of character χ. For each such χ we also write $\check{\chi}$ for its contragredient.

We write F^+ for the maximal totally real subfield of F and τ for the (unique) non-trivial element of G_{F/F^+} and obtain central idempotents of $\mathbb{Q}[G]$ by setting $e_\pm = (1\pm\tau)/2$. We write $\mathrm{Ir}_p^\pm(G)$ and $\mathrm{Ir}^\pm(G)$ for the subsets of $\mathrm{Ir}_p(G)$ and $\mathrm{Ir}(G)$ comprising characters with $\chi(\tau) = \pm\chi(1)$. For any G-module M we also write $M^\pm$ for the G-submodule $\{m \in M : \tau(m) = \pm m\}$.

We fix a finite set of places Σ of k which contains both the set S_k^∞ of all archimedean places and the set S_k^p of all p-adic places. For any extension E of k we write Σ_E for the set of places of E above those in Σ.

For a Galois extension of fields F/E we set $G_{F/E} := \mathrm{Gal}(F/E)$. Unless explicitly specified otherwise, modules are regarded as left modules. For an abelian group A and homomorphism ϕ we write A_p and ϕ_p in place of $\mathbb{Z}_p \otimes_\mathbb{Z} A$ and $\mathrm{id}_{\mathbb{Z}_p} \otimes_\mathbb{Z} \phi$.

2.1. We first recall (from, for example, [43, Chap. I, Prop. 3.4]) that if a character χ in $\mathrm{Ir}(G)$ is not trivial, then the algebraic order $r_\Sigma(\chi)$ at $z = 0$ of the Σ-truncated Artin L-series $L_\Sigma(\chi, z)$ of χ (as defined in [43, §1]) can be computed via the explicit formula

$$r_{\Sigma,\chi} = \sum_{v\in\Sigma} \dim_\mathbb{C}(H^0(G_w, V_\chi)) = \dim_\mathbb{C}(\mathrm{Hom}_{\mathbb{C}[G]}(V_{\check{\chi}}, \mathbb{C}\cdot Y_{F,\Sigma})), \tag{1}$$

where G_w denotes the decomposition subgroup in G of any fixed place w of F above v and $Y_{F,\Sigma}$ the free abelian group on the set Σ_F, regarded as a $\mathbb{Z}[G]$-module via the natural action of G on Σ_F.

We observe this formula implies that $r_{\Sigma,\chi} = r_{\Sigma,\chi^\alpha}$ for all automorphisms α of $\mathbb{C}$ and so for any character ψ in $\mathrm{Ir}_p(G)$ we may unambiguously set

$$r_{\Sigma,\psi} := r_{\Sigma,\chi}$$

where χ is any character in $\mathrm{Ir}(G)$ with $\chi^j = \psi$ for some field isomorphism $j : \mathbb{C} \cong \mathbb{C}_p$.

For each character ψ in $\mathrm{Ir}_p^+(G)$ we write $L_{p,\Sigma}(\psi, s)$ for the Σ-truncated Deligne-Ribet p-adic Artin L-series of ψ (as discussed by Greenberg in [13]). We also write ω_k for the Teichmüller character $G_k \to \mathbb{Z}_p^\times$.

We can now state Gross's 'Order of Vanishing Conjecture' for p-adic Artin L-series (taken from [17, Conj. 2.12a)]).

Conjecture 2.1. *For each character ψ in $\mathrm{Ir}_p^-(G)$ the algebraic order at $s = 0$ of the series $L_{p,\Sigma}(\check{\psi}\omega_k, s)$ is equal to $r_{\Sigma,\psi}$.*

Remark 2.2. By using Brauer's Induction Theorem (as in the proof of Proposition 2.6 below) one shows easily that Conjecture 2.1 is valid if and only if it is valid for all L-series of the form $L_{p,\Sigma_K}(\check{\phi}\omega_K, s)$ where K is a totally real intermediate field of F/k and ϕ is a linear character in $\mathrm{Ir}_p^-(G_{F/K})$.

2.2. In the sequel we write $\mathcal{O}_{F,\Sigma}$ for the subring of F comprising elements that are integral at all places outside Σ_F. We also write

$$\phi_{F,\Sigma} : \mathcal{O}_{F,\Sigma}^{\times,-} \to Y_{F,\Sigma}^-$$

for the homomorphism of G-modules that sends each ϵ to $\sum_w \mathrm{ord}_w(\epsilon)\cdot w$, where in the sum w runs over all non-archimedean places in Σ_F and ord_w denotes the normalised additive valuation at w. We further write

$$\mathrm{R}_{F,\Sigma} : \mathbb{R}\cdot\mathcal{O}_{F,\Sigma}^{\times,-} \to \mathbb{R}\cdot Y_{F,\Sigma}^-$$

for the isomorphism of $\mathbb{R}[G]$-modules that sends each u in $\mathcal{O}_{F,\Sigma}^{\times,-}$ to $\sum_{\pi\in\Sigma_F}\log|u|_\pi\cdot\pi$, where $|\cdot|_\pi$ is the normalised absolute value at π.

The scalar extension $\mathbb{C}\otimes_{\mathbb{Z}}\phi_{F,\Sigma}$ is bijective and so, for each χ in $\mathrm{Ir}^-(G)$, we may define a non-zero 'regulator' element in $\mathbb{C}$ by setting

$$R_\Sigma(\chi) := \det_{\mathbb{C}}((\mathbb{C}\otimes_{\mathbb{R}} \mathrm{R}_{F,\Sigma})\circ(\mathbb{C}\otimes_{\mathbb{Z}}\phi_{F,\Sigma})^{-1} \mid \mathrm{Hom}_{\mathbb{C}[G]}(V_{\check{\chi}}, \mathbb{C}\cdot Y_{F,\Sigma}^-)) \in \mathbb{C}.$$

By using an observation of Tate from [42, 2.6], it is shown in [17, Prop. 2.11] that $R_\Sigma(\chi)$ differs from the coefficient $L_\Sigma^{r_{\Sigma,\chi}}(\check{\chi},0)$ of $z^{r_{\Sigma,\chi}}$ in the Taylor expansion at $z=0$ of $L_\Sigma(\check{\chi},z)$ by multiplication by an element of $\mathbb{Q}^{c,\times}$ and further that for each all α in $G_{\mathbb{Q}^c/\mathbb{Q}}$ one has

$$L_\Sigma^{r_{\Sigma,\chi^\alpha}}(\check{\chi}^\alpha,0)/R_\Sigma(\chi^\alpha) = (L_\Sigma^{r_{\Sigma,\chi}}(\check{\chi},0)/R_\Sigma(\chi))^\alpha. \tag{2}$$

2.3. For each place w of F Gross defines in [17, §1] a local p-adic absolute value $||\cdot||_{w,p}$ on $F_w^\times$ by means of the commutative diagram

$$\begin{tikzcd} F_w^\times \arrow[rr, "||\cdot||_{w,p}"] \arrow[dr, "r_w"'] & & \mathbb{Z}_p^\times \\ & G_{F_w^{\mathrm{ab}}/F_w} \arrow[ur, "\chi_{F_w}^{-1}"'] & \end{tikzcd} \tag{3}$$

where F_w^{ab} denotes the maximal abelian extension of F_w in F_w^c, r_w the reciprocity map of class field theory and χ_{F_w} the p-adic cyclotomic character.

We write

$$\lambda_{F,\Sigma,p} : \mathcal{O}_{F,\Sigma,p}^{\times,-} \to Y_{F,\Sigma,p}^-$$

for the homomorphism of $\mathbb{Z}_p[G]$-modules sending each u in $\mathcal{O}_{F,\Sigma}^{\times,-}$ to $\sum_{w\in\Sigma_F}\log_p||u||_{w,p}\cdot w$. For each ψ in $\mathrm{Ir}_p^-(G)$ we then define a canonical '$\mathscr{L}$-invariant' (or, if one prefers, 'p-adic regulator') in $\mathbb{C}_p$ by setting

$$\mathscr{L}_\Sigma(\psi) := \det_{\mathbb{C}_p}((\mathbb{C}_p\otimes_{\mathbb{Z}_p}\lambda_{F,\Sigma,p})\circ(\mathbb{C}_p\otimes_{\mathbb{Z}}\phi_{F,\Sigma})^{-1} \mid \mathrm{Hom}_{\mathbb{C}_p[G]}(V_{\check{\psi}}, \mathbb{C}_p\cdot Y_{F,\Sigma}^-)).$$

We then also define an associated idempotent of $\mathbb{Q}_p^c[G]$ by setting

$$e_{\mathrm{ss}} := \sum_\psi e_\psi \tag{4}$$

where the sum is over all ψ in $\mathrm{Ir}_p^-(G)$ for which $\mathscr{L}_\Sigma(\psi)$ is non-zero. It is straightforward to check that e_{ss} is independent of the set Σ and belongs to the centre of $\mathbb{Q}_p[G]$.

Remark 2.3.
(i) In [17, Conj. 1.15] Gross conjectures $\mathscr{L}_\Sigma(\psi)\neq 0$ for all ψ in $\mathrm{Ir}_p^-(G)$, and hence that $e_{\mathrm{ss}} = e_-$.
(ii) For ψ in $\mathrm{Ir}_p^-(G)$ set $r_\psi := r_{S_k^\infty\cup S_k^p,\psi}$. If $r_\psi = 0$, then it is straightforward to verify directly that $\mathscr{L}_\Sigma(\psi)\neq 0$. Excluding this case, however, the non-vanishing of $\mathscr{L}_\Sigma(\psi)$ has only been verified (by Gross in [17, Prop. 2.13]) in the case $r_\psi = 1$ when it follows as a consequence of Brumer's p-adic version of Baker's theorem.

For each integer m we write $L^m_{p,\Sigma}(\psi,0)$ for the coefficient of s^m in the power series expansion of $L_{p,\Sigma}(\psi,s)$ at $s=0$.

We can now state the 'Weak p-adic Gross-Stark Conjecture' as formulated by Gross in [17, Conj. 2.12b)].

Conjecture 2.4. *For every finite set Σ of places of k containing $S_k^\infty \cup S_k^p$, every character ψ in $\mathrm{Ir}_p^-(G)$ and every field isomorphism $j:\mathbb{C}_p\cong\mathbb{C}$ one has*

$$L^{r_{\Sigma,\psi}}_{p,\Sigma}(\check{\psi}\omega_k,0)=\mathscr{L}_\Sigma(\psi)\cdot j^{-1}(L^{r_{\Sigma,\psi}}_\Sigma(\check{\psi}^j,0)/R_\Sigma(\psi^j)).$$

Remark 2.5.
(i) If ψ validates Conjecture 2.1, then $L^{r_{\Sigma,\psi}}_{p,\Sigma}(\check{\psi}\omega_k,0)\neq 0$ and so Conjecture 2.4 implies $\mathscr{L}_\Sigma(\psi)\neq 0$ (as is consistent with Remark 2.3(i)). One can in fact prove directly that a character ψ validates Conjecture 2.1 if and only if $\mathscr{L}_\Sigma(\psi)$ does not vanish (see Theorem 3.1(i) and (iii) below).
(ii) Taken together, Conjectures 2.1 and 2.4 constitute the 'p-adic Gross-Stark Conjecture'.

The strongest evidence that one has in support of Conjecture 2.4 was obtained recently by Dasgupta, Kakde and Ventullo in [10] and is recorded in the following result.

Theorem 2.6. *Conjecture 2.4 is valid for all linear characters ψ. In general, the validity of Conjecture 2.4 is implied by the validity of Conjecture 2.1 for all linear characters ψ.*

Proof. One checks easily that for any ψ in $\mathrm{Ir}_p^-(G)$ both sides of the equality in Conjecture 2.4 are unchanged if one replaces F/k by $F^{\ker(\psi)}/k$. Given this observation, the first claim coincides with the main result of [10].

To prove the second claim we use a refined version of Brauer's Induction Theorem (due to Serre and proved, for example, in [43, Chap. III, Lem. 1.3]). This result guarantees that for any character ψ in $\mathrm{Ir}_p^-(G)$ there exists a (finite) set of totally real intermediate fields $\{F_i=F_{\psi,i}\}_{i\in I}$ of F/k and for each index i a linear character $\phi_{\psi,i}$ in $\mathrm{Ir}_p^-(G_{F/F_i})$ and an integer $n_i=n_{\psi,i}$ such that $\check{\psi}=\sum_{i\in I}n_i\cdot\mathrm{Ind}^G_{G_{F/F_i}}(\check{\phi}_{\psi,i})$.

This equality combines with the functoriality properties of p-adic Artin L-series to imply $L_{p,\Sigma}(\check{\psi}\omega_k,s)=\prod_{i\in I}L_{p,\Sigma_i}(\check{\phi}_{\psi,i}\omega_{F_i},s)^{n_i}$ with $\Sigma_i:=\Sigma_{F_i}$, and with the formula (1) to imply $r_{\Sigma,\psi}=\sum_{i\in I}n_i\cdot r_i$ with $r_i:=r_{\Sigma_i,\phi_{\psi,i}}$.

In particular, if the equality of Conjecture 2.1 is valid for each pair $(\Sigma_i,\phi_{\psi,i})$, then each term $L^{r_i}_{p,\Sigma_i}(\check{\phi}_{\psi,i}\omega_{F_i},0)$ is non-zero and one has

$$L^{r_{\Sigma,\psi}}_{p,\Sigma}(\check{\psi}\omega_k,0)=\prod_{i\in I}L^{r_i}_{p,\Sigma_i}(\check{\phi}_{\psi,i}\omega_{F_i},0)^{n_i}.$$

In addition, since the equality in Conjecture 2.4 is known to be valid for each pair $(\Sigma_i,\phi_{\psi,i})$, each $\mathscr{L}$-invariant $\mathscr{L}_{\Sigma_i}(\phi_{\psi,i})$ is non-zero and the above product is equal to

$$\prod_{i\in I}\mathscr{L}_{\Sigma_i}(\phi_{\psi,i})^{n_i}\cdot\prod_{i\in I}j^{-1}(L^{r_{\psi,i}}_{\Sigma_i}(\check{\phi}^j_{\psi,i},0)/R_{\Sigma_i}(\phi^j_{\psi,i}))^{n_i}.$$

To deduce the equality of Conjecture 2.4 it thus suffices to note that the standard functoriality properties of Artin L-series imply $\prod_{i\in I}L^{r_{\psi,i}}_{\Sigma_i}(\check{\phi}^j_{\psi,i},0)^{n_i}=L^{r_{\Sigma,\psi}}_\Sigma(\check{\psi}^j,0)$ and that easy computations show $\prod_{i\in I}\mathscr{L}_{\Sigma_i}(\phi_{\psi,i})^{n_i}=\mathscr{L}_\Sigma(\psi)$ and $\prod_{i\in I}R_{\Sigma_i}(\phi^j_{\psi,i})^{n_i}=R_\Sigma(\psi^j)$. $\square$

3. Statement of the main results

3.1. For an extension of number fields L/K and finite disjoint sets of places Σ and Σ' of K with $S_K^\infty \subseteq \Sigma$ we write $\mathrm{Cl}_\Sigma^{\Sigma'}(L)$ for the quotient of the group of fractional ideals of $\mathcal{O}_{L,\Sigma}$ that are coprime to all places in Σ'_L by the subgroup of principal ideals with a generator congruent to 1 modulo all of the places in Σ'_L. If $\Sigma = S_K^\infty$ or Σ' is empty, then we abbreviate $\mathrm{Cl}_\Sigma^{\Sigma'}(L)$ to $\mathrm{Cl}^{\Sigma'}(L)$ and $\mathrm{Cl}_\Sigma(L)$ respectively.

We recall that the 'Σ-relative Σ'-trivialized integral Selmer group' $\mathrm{Sel}_\Sigma^{\Sigma'}(L)$ for the multiplicative group $\mathbb{G}_m$ over L is defined to be the cokernel of a canonical homomorphism

$$\prod_w \mathbb{Z} \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(L^\times_{\Sigma'}, \mathbb{Z})$$

(see [4, Def. 2.1] where the notation $\mathcal{S}_{\Sigma,\Sigma'}(\mathbb{G}_{m/L})$ is used). Here in the product w runs over all places of L outside $\Sigma_L \cup \Sigma'_L$, $L^\times_{\Sigma'}$ is the subgroup of $L^\times$ comprising elements u for which $u-1$ has a strictly positive valuation at each place in Σ'_L and the unlabeled arrow sends each element $(x_w)_w$ to the map $(u \mapsto \sum \mathrm{ord}_w(u)x_w)$.

This group is a natural analogue for $\mathbb{G}_m$ of the integral Selmer groups of abelian varieties that are defined by Mazur and Tate in [28] and, in particular, lies in a canonical exact sequence of the form

$$0 \to \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Cl}_\Sigma^{\Sigma'}(L), \mathbb{Q}/\mathbb{Z}) \to \mathrm{Sel}_\Sigma^{\Sigma'}(L) \to \mathrm{Hom}_{\mathbb{Z}}(\mathcal{O}^\times_{L,\Sigma,\Sigma'}, \mathbb{Z}) \to 0 \tag{5}$$

(see [4, Prop. 2.2]), where $\mathcal{O}^\times_{L,\Sigma,\Sigma'}$ denotes the (finite index) subgroup $L^\times_{\Sigma'} \cap \mathcal{O}^\times_{L,\Sigma}$ of $\mathcal{O}^\times_{L,\Sigma}$ and, in the case that L/K is Galois, both duals are endowed with the contragredient action of $G_{L/K}$.

We further recall that $\mathrm{Sel}_\Sigma^{\Sigma'}(L)$ has a canonical transpose $\mathrm{Sel}_\Sigma^{\Sigma'}(L)^{\mathrm{tr}}$, in the sense of Jannsen's homotopy theory of modules [20], which itself lies in a canonical exact sequence

$$0 \longrightarrow \mathrm{Cl}_\Sigma^{\Sigma'}(L) \longrightarrow \mathrm{Sel}_\Sigma^{\Sigma'}(L)^{\mathrm{tr}} \longrightarrow X_{L,\Sigma} \longrightarrow 0, \tag{6}$$

where $X_{L,\Sigma}$ denotes the kernel of the homomorphism $Y_{L,\Sigma} \to \mathbb{Z}$ induced by sending each place in Σ_L to 1.

For a finite group Δ and a character ψ in $\mathrm{Ir}_p(\Delta)$ we fix a finite extension $\mathcal{O}_\psi$ of $\mathbb{Z}_p$ for which there exists a finitely generated $\mathcal{O}_\psi$-lattice T_ψ and a representation $\Delta \to \mathrm{Aut}_{\mathcal{O}_\psi}(T_\psi)$ of character ψ. For each homomorphism $\epsilon : M \to N$ of $\mathbb{Z}_p[\Delta]$-modules we consider the composite homomorphism of $\mathcal{O}_\psi$-modules

$$\epsilon^{\langle\psi\rangle} : H^0(\Delta, T_\psi \otimes_{\mathbb{Z}_p} M) \to H^0(\Delta, T_\psi \otimes_{\mathbb{Z}_p} N) \subseteq T_\psi \otimes_{\mathbb{Z}_p} N \to H_0(\Delta, T_\psi \otimes_{\mathbb{Z}_p} N)$$

where Δ acts diagonally on each of the tensor products, the first arrow is the map induced by $\mathrm{id}_{T_\psi} \otimes_{\mathbb{Z}_p} \epsilon$ and the second is the tautological map.

We also write $\zeta(A)$ for the centre of a ring A.

3.2. We now return to consider the extension of number fields F/k that was fixed in §2. In the sequel we shall also fix a finite non-empty set S of places of k containing both S_k^p and the set $S^{\mathrm{ram}}_{F/k}$ of places that ramify in F/k and a finite non-empty set of places T of k that is disjoint from S and such that $\mathcal{O}^\times_{F,S,T}$ is torsion-free (it is easy to see that, for a given set S, such a set T always exists).

For each character ψ in $\mathrm{Ir}_p^-(G)$ we then set

$$L_{p,S,T}(\psi\omega_k, s) := L_{p,S}(\psi\omega_k, s) \cdot \prod_{v\in T} \det_{\mathbb{C}_p}(1 - \mathrm{N}v \cdot \mathrm{Fr}_w^{-1} \mid V_{\check{\psi}}) \tag{7}$$

where $\mathrm{N}v$ is the cardinality of the residue field of v and Fr_w the Frobenius automorphism in G of any given place w of F above v.

Our first result extends the main result of Gross and Federer in [11]. Before stating this result we note that our techniques will construct (in Theorem 5.8(ii)) a canonical refinement $\lambda_{F,S,T,p}$ of Gross's regulator map $\lambda_{F,S,p}$ of the form

$$\begin{tikzcd} & \mathrm{Sel}_S^T(F)_p^{\mathrm{tr},-} \arrow[dr, two heads] & \\ \mathcal{O}_{F,S,T,p}^{\times,-} \arrow[ur, "\lambda_{F,S,T,p}"] \arrow[rr, "\lambda_{F,S,p}"] & & X_{F,S,p}^-, \end{tikzcd} \tag{8}$$

where the unlabeled arrow is the surjective homomorphism induced by the exact sequence (6).

In the sequel we write $\mathcal{D}_{L/K}$ for the different of a finite extension L/K of either local fields or number fields.

In claim (iii) of the following result we also write F^{cl} for the Galois closure of F over $\mathbb{Q}$, fix a primitive p-th root of unity ζ_p in $\mathbb{C}_p$ and write $\mathrm{val}_p(-)$ for the canonical valuation on $\mathbb{C}_p$.

Theorem 3.1. *For each character ψ in $\mathrm{Ir}_p^-(G)$ the following claims are valid.*

(i) $L_{p,S,T}(\check{\psi}\omega_k, z)$ *vanishes to order at least* $r_{S,\psi}$ *at* $z = 0$.

(ii) $L_{p,S,T}^{r_{S,\psi}}(\check{\psi}\omega_k, 0) \cdot \mathcal{O}_\psi = |G|^{-r_{S,\psi}} \mathrm{Fit}_{\mathcal{O}_\psi}(\mathrm{cok}(\lambda_{F,S,T,p}^{\langle\psi\rangle}))$.

(iii) $L_{p,S}^{r_{S,\psi}}(\check{\psi}\omega_k, 0) \neq 0$ *if and only if* $\mathscr{L}_S(\psi) \neq 0$. *In addition, if* $\mathscr{L}_S(\psi) \neq 0$ *and either* ψ *is* $\mathbb{Q}$*-valued,* $F^{\mathrm{cl}} \nsubseteq (F^{\mathrm{cl}})^+(\zeta_p)$ *or no* p*-adic place of* F^+ *splits in* F*, then for each field isomorphism* $j : \mathbb{C}_p \cong \mathbb{C}$ *one has*

$$\mathrm{val}_p(L_{p,S}^{r_{S,\psi}}(\check{\psi}\omega_k, 0)) = \mathrm{val}_p(\mathscr{L}_S(\psi) \cdot j^{-1}(L_S^{r_{S,\psi}}(\check{\psi}^j, 0)/R_S(\psi^j))).$$

(iv) *If* $\mathscr{L}_S(\psi) \neq 0$ *and* a_ψ *is any element of* $\mathcal{D}_{\mathbb{Q}_p(\psi)/\mathbb{Q}_p}^{-1} \cdot \bigcap_{v\in S} \mathrm{Fit}_{\mathcal{O}_\psi}(\hat{H}^0(G_w, T_\psi))$*, then the sum*

$$|G|^{r_{S,\psi}} \sum_{g\in G} \mathrm{Tr}_{\mathbb{Q}_p(\psi)/\mathbb{Q}_p}(a_\psi \check{\psi}(g) L_{p,S,T}^{r_{S,\psi}}(\check{\psi}\omega_k, 0) \mathscr{L}_S(\psi)^{-1}) g$$

belongs to $\mathbb{Z}_p[G]$ *and annihilates the module* $\mathrm{Cl}^T(F)_p$.

Remark 3.2.

(i) For linear characters ψ the assertion of Theorem 3.1(i) was already known to follow as a consequence of Wiles's proof of the main conjecture for totally real fields. For such characters it has also recently been proved directly by Spiess [40] by using Shintani cocycles.

(ii) The equality in claim (iii) constitutes a natural weakening of Conjecture 2.4. The main result of Gross and Federer in [11] is equivalent to the result of this claim in the case that ψ is both linear and $\mathbb{Q}_p$-valued. The condition $F^{\mathrm{cl}} \nsubseteq (F^{\mathrm{cl}})^+(\zeta_p)$ is automatically satisfied if, for example, p is unramified in $F/\mathbb{Q}$.

We next record a consequence of Theorem 3.1 that combines with Conjecture 2.4 (and Remark 2.3(ii)) to predict a precise p-adic analytic construction of families of algebraic p-units that both generate non-abelian Galois extensions of totally real fields and also encode structural information about ideal class groups. (In particular, see Remark 3.4(iv) below).

In claim (iii)(b) of this result we use the first derivative of the T-modified Σ-truncated Artin L-series of a character ψ in $\mathrm{Ir}(G)$ that is defined (just as in (7)) for each set of places Σ of k with $S_k^\infty \subseteq \Sigma \subseteq S$ by setting

$$L_{\Sigma,T}(\psi,s) := L_\Sigma(\psi,s)\cdot\prod_{v\in T}\det_{\mathbb{C}}(1-\mathrm{N}v\cdot\mathrm{Fr}_w^{-1}\mid V_{\check{\psi}}). \tag{9}$$

Corollary 3.3. *Let v_1 be a p-adic place in S that has absolute degree one and ψ a character in $\mathrm{Ir}^-(G)$ which satisfies*

$$L_S(\psi,0)=0 \quad \text{and} \quad L_{S\setminus\{v_1\}}(\psi,0)\cdot L'_S(\psi,0)\neq 0. \tag{10}$$

Fix a place w_1 of F above v_1 and set $G_1 := G_{w_1}$. Fix an embedding $j:\mathbb{Q}^c\to\mathbb{Q}_p^c$ whose restriction to F corresponds to w_1 and use it to identify $F_1 := F^{G_1}$ as a subfield of $\mathbb{Q}_p$, ψ as an element of $\mathrm{Ir}_p^-(G)$ and $G_{\mathbb{Q}_p(\psi)/\mathbb{Q}_p}$ as a subgroup of $G_{\mathbb{Q}(\psi)/\mathbb{Q}}$. Fix an element d in $\mathcal{D}^{-1}_{\mathbb{Q}(\psi)/\mathbb{Q}}$.

Then $\mathscr{L}_S(\psi^\gamma)\neq 0$ for every γ in $G_{\mathbb{Q}_p(\psi)/\mathbb{Q}_p}$ and for every g in G the sum

$$a_{\psi,d,g,p} := \sum_{g'\in G_1}\frac{(p-1)}{|G_1|}\sum_{\gamma\in G_{\mathbb{Q}_p(\psi)/\mathbb{Q}_p}} L'_{p,S,T}(\check{\psi}^\gamma\omega_k,0)\mathscr{L}_S(\psi^\gamma)^{-1}\check{\psi}^\gamma(gg')d^\gamma$$

belongs to $\mathbb{Z}_p$. Further, the element

$$\epsilon_{\psi,d,p} := p^{a_{\psi,d,\mathrm{id},p}}\cdot\exp_p\Big(\frac{(p-1)}{|G_1|}\sum_{g'\in G_1}\sum_{\gamma\in G_{\mathbb{Q}_p(\psi)/\mathbb{Q}_p}} L'_{p,S,T}(\check{\psi}^\gamma\omega_k,0)\check{\psi}^\gamma(g')d^\gamma\Big)$$

belongs to $\mathcal{O}^{\times,-}_{F_1,\{v_1\},p}$ and has all of the following properties.

(i) *For every g in G one has*

$$g(\epsilon_{\psi,d,p}) = p^{a_{\psi,d,g,p}}\cdot\exp_p\Big(\frac{(p-1)}{|G_1|}\sum_{g'\in G_1}\sum_{\gamma\in G_{\mathbb{Q}_p(\psi)/\mathbb{Q}_p}} L'_{p,S,T}(\check{\psi}^\gamma\omega_k,0)\check{\psi}^\gamma(gg')d^\gamma\Big).$$

(ii) *If d is an algebraic integer, then for every θ in $\mathrm{Hom}_{\mathbb{Z}_p[G]}(\mathcal{O}^{\times,-}_{F,\{v_1\},p},\mathbb{Z}_p[G])$ the product $|G|^2\cdot\theta(\epsilon_{\psi,d,p})$ belongs to $\mathbb{Z}_p[G]$ and annihilates $\mathrm{Cl}^T_{\{v_1\}}(F)_p$.*

(iii) *Assume Conjecture 2.4 is valid for the character ψ^α for all α in $G_{\mathbb{Q}^c/\mathbb{Q}}$. Fix a set of representatives $\{\psi_i : i\in I\}$ for the orbits of $G_{\mathbb{Q}_p(\psi)/\mathbb{Q}_p}$ on $\{\psi^\alpha:\alpha\in G_{\mathbb{Q}^c/\mathbb{Q}}\}$ and set*

$$\epsilon_{\psi,d} := \prod_{i\in I}\epsilon_{\psi_i,d,p}\in\mathcal{O}^{\times,-}_{F_1,\{v_1\},p}.$$

Then there exists an integer m that is prime to p and such that both of the following claims are valid.

(a) *If $d\neq 0$, then the element $\epsilon^m_{\psi,d}$ is a $k[G]$-generator of the field $F^{\ker(\psi)}$.*

(b) *For every g in G the sum*

$$y_{\psi,d,g,m} := \sum_{g'\in G_1} \frac{m(p-1)}{|G_1|} \sum_{\alpha\in G_{\mathbb{Q}(\psi)/\mathbb{Q}}} L'_{S,T}(\check{\psi}^\alpha, 0) R_S(\psi^\alpha)^{-1}\check{\psi}^\alpha(gg')d^\alpha$$

is a rational integer. Further, if $\mathfrak{B}$ is the prime ideal of F that corresponds to w_1, then $\epsilon^m_{\psi,d}$ generates the ideal $\prod_g g(\mathfrak{B})^{y_{\psi,d,g,m}}$ where in the product g runs over a set of coset representatives for G_1 in G.

Remark 3.4.
(i) In [5, Rem. 13.5] Sano and the present author describe explicit families of examples in which all of the hypotheses of Corollary 3.3 are satisfied for characters ψ that are both faithful and of arbitrarily large degree.
(ii) The result of Corollary 3.3(ii) is analogous to the main annihilation result proved (for cyclotomic fields) by Rubin in [37]. A close analysis of our argument will also show that the factor $|G|^2$ in Corollary 3.3(ii) is not always best possible.
(iii) Let $\{\psi_j : j \in J\}$ be a finite set of characters in $\mathrm{Ir}^-(G)$ that satisfy the hypotheses of Corollary 3.3(iii) and for each j fix an element d_j of $\mathcal{D}^{-1}_{\mathbb{Q}(\psi_j)/\mathbb{Q}}$. Our proof of Corollary 3.3 will show that if the $G_{\mathbb{Q}^c/\mathbb{Q}}$-orbits of the characters ψ_j are distinct, then there exists an integer m that is prime to p and such that the product $(\prod_{j\in J}\epsilon_{\psi_j,d_j})^m$ is a $k[G]$-generator of the compositum of the fields $F^{\ker(\psi_j)}$ as j runs over J.
(iv) In the spirit of Gross's conjecture [17, Conj. 3.13] it is natural to predict that the claims in Corollary 3.3(iii)(a) and (b) should both be valid with $m = 1$. In this way Corollary 3.3 both extends and refines an observation (concerning the case that ψ is linear) made by Gross in [17, Prop. 3.14] and also provides a natural p-adic analogue of the question considered by Stark in [41] and the conjecture formulated by Chinburg in [9] for characters ψ of degree two.

3.3. In this subsection we formulate a refined p-adic class number formula for $\mathbb{G}_m$ and state several results related to it.

In the sequel, for any noetherian ring R we write $D(R)$ for the derived category of left R-modules and $D^{\mathrm{p}}(R)$ for the full triangulated subcategory of $D(R)$ comprising complexes that are isomorphic to a bounded complex of finitely generated projective R-modules.

We recall that for any homomorphism $R \to R'$ of associative unital noetherian rings, any object C of $D^{\mathrm{p}}(R)$ and any exact sequence of R'-modules

$$\epsilon:\ 0 \to \cdots \to R'\otimes_R H^i(C) \to R'\otimes_R H^{i+1}(C) \to R'\otimes_R H^{i+2}(C) \to \cdots \to 0 \tag{11}$$

one can define a canonical element $\chi_{R,R'}(C,\epsilon)$ in the relative algebraic K_0-group $K_0(R,R')$.

If, in addition, R' is a semisimple $\mathbb{Q}_p$-algebra, then the associated reduced norm map $K_1(R') \to \zeta(R')^\times$ is bijective and so there exists a composite homomorphism

$$\delta_{R,R'} : \zeta(R')^\times \to K_1(R') \to K_0(R,R')$$

where the first arrow is the inverse of the reduced norm map and the second is the canonical connecting homomorphism of relative K-theory (normalised as in [7, §1.2]).

In the case that R' is the total quotient ring of R we abbreviate $\chi_{R,R'}(-,-)$ and $\delta_{R,R'}$ to $\chi_R(-,-)$ and δ_R respectively.

In the sequel we set
$$\Lambda[G]^{\rm ss} := \Lambda[G]e_{\rm ss}$$
for any subring Λ of $\mathbb{C}_p$, where $e_{\rm ss}$ is the idempotent of $\zeta(\mathbb{Q}_p[G])$ defined in (4), and for any object C of $D^{\rm p}(\Lambda[G])$ we write $C^{\rm ss}$ for the associated object $\Lambda[G]^{\rm ss}\otimes^{\mathbb{L}}_{\Lambda[G]} C$ of $D^{\rm p}(\Lambda[G]^{\rm ss})$. We note that if $\Lambda[G]$ is semisimple, then in each degree i there is a natural identification $H^i(C^{\rm ss}) = e_{\rm ss}\cdot H^i(C)$.

We fix a set of places T of k as in §3.2 and note that in Lemma 4.1 (and Remark 4.2) below we construct a canonical object $R\Gamma_{\text{ét},T}(\mathcal{O}_{F,S},\mathbb{Z}_p(1))$ of $D^{\rm p}(\mathbb{Z}_p[G])$ which is acyclic outside degrees one and two and is such that there are canonical identifications

$$H^i(R\Gamma_{\text{ét},T}(\mathcal{O}_{F,S},\mathbb{Z}_p(1)))^- \cong \begin{cases} \mathcal{O}^{\times,-}_{F,S,T,p}, & \text{if } i=1\\ \mathrm{Sel}^T_S(F)^{\mathrm{tr},-}_p, & \text{if } i=2\\ 0, & \text{otherwise}\end{cases} \tag{12}$$

and hence also a canonical exact sequence of $\mathbb{Q}_p[G]$-modules
$$\lambda^{\rm ss}_{F/k,S,p}: 0\to \mathbb{Q}_p\cdot H^1(R\Gamma_{\text{ét},T}(\mathcal{O}_{F,S},\mathbb{Z}_p(1))^{\rm ss}) \xrightarrow{\lambda_{F,S,p}} \mathbb{Q}_p\cdot H^2(R\Gamma_{\text{ét},T}(\mathcal{O}_{F,S},\mathbb{Z}_p(1))^{\rm ss})\to 0.$$

Finally we define a $\zeta(\mathbb{C}_p[G])$-valued meromorphic function of a p-adic variable z by setting
$$\theta_{p,F/k,S,T}(z) := \sum_{\psi\in \mathrm{Ir}^-_p(G)} e_\psi L_{p,S,T}(\check{\psi}\omega_k, z)$$
and we note that the leading term $\theta^*_{p,F/k,S,T}(0)$ in the Taylor expansion of this function at $z=0$ belongs to $\zeta(\mathbb{Q}_p[G])^\times$.

We can now state our conjectural refined p-adic class number formula for $\mathbb{G}_m$.

Conjecture 3.5. *In $K_0(\mathbb{Z}_p[G]^{\rm ss},\mathbb{Q}_p[G]^{\rm ss})$ one has*
$$\delta_{\mathbb{Z}_p[G]^{\rm ss}}(\theta^*_{p,F/k,S,T}(0)e_{\rm ss}) = -\chi_{\mathbb{Z}_p[G]^{\rm ss}}(R\Gamma_{\text{ét},T}(\mathcal{O}_{F,S},\mathbb{Z}_p(1))^{\rm ss},\lambda^{\rm ss}_{F/k,S,p})$$

The main evidence that we can currently provide in support of this conjecture is provided by the next result.

In the sequel we write $\mu_p(E)$ for the p-adic μ-invariant of the cyclotomic $\mathbb{Z}_p$-extension of a number field E. We recall Iwasawa has conjectured in [19] that $\mu_p(E)$ should always vanish.

Theorem 3.6. *If $\mu_p(F)$ vanishes or p does not divide $[F:k]$, then Conjecture 3.5 is valid.*

Remark 3.7. In [22] Johnston and Nickel identify families of extensions F/k for which one can prove the main conjecture of non-commutative Iwasawa theory for $F^{\rm cyc}/k$ without assuming that either $\mu_p(F)$ vanishes or p does not divide $[F:k]$. In all such cases our method shows that the equality of Theorem 3.6, and all of its consequences described below in Corollaries 3.8, 3.10 and 3.11 are valid without any assumption on the odd prime p.

By replacing the role played by p-adic L-functions in the equality of Theorem 3.6 by Artin L-functions we will directly obtain the following result.

Corollary 3.8. *If either $\mu_p(F)$ vanishes or p does not divide $[F:k]$, then the following claims are valid.*

(i) *If the Weak p-adic Gross-Stark Conjecture (Conjecture 2.4) is valid for all ψ in* $\mathrm{Ir}_p^-(G)$, *then the equivariant Tamagawa number conjecture for* $(h^0(\mathrm{Spec}(F)), \mathbb{Z}_p[G]^-)$ *is valid modulo the subgroup* $\delta_{\mathbb{Z}_p[G]^-,\mathbb{C}_p[G]^-}(\zeta(\mathbb{C}_p[G](e_- - e_{\mathrm{ss}}))^\times)$.

(ii) *If Gross's Order of Vanishing Conjecture (Conjecture 2.1) is valid for all ψ in* $\mathrm{Ir}_p^-(G)$, *then the equivariant Tamagawa number conjecture for* $(h^0(\mathrm{Spec}(F)), \mathbb{Z}_p[G]^-)$ *is valid.*

Remark 3.9.
(i) If F/k is abelian and either $\mu_p(F)$ vanishes or p does not divide $[F : k]$, then Corollary 3.8(i) combines with the theorem of Dasgupta, Kakde and Ventullo recalled in Theorem 2.6 to imply the validity modulo $\delta_{\mathbb{Z}_p[G]^-,\mathbb{C}_p[G]^-}(\zeta(\mathbb{C}_p[G](e_- - e_{\mathrm{ss}}))^\times)$ of the equivariant Tamagawa number conjecture for $(h^0(\mathrm{Spec}(F)), \mathbb{Z}_p[G]^-)$.
(ii) In [5, Cor. 10.6 and Exam. 10.7] it is shown by Sano and the present author that Corollary 3.8 can also be used to give the first verifications of the equivariant Tamagawa number conjecture for $(h^0(\mathrm{Spec}(F)), \mathbb{Z}_p[G]^-)$ in the technically most difficult case that F/k is non-abelian of degree divisible by p and the relevant p-adic L-series possess trivial zeroes.

In the rest of this section we record several concrete results that are obtained by deriving consequences of Conjecture 3.5 and then applying Theorem 3.6.

3.4. For each non-negative integer r we now define an 'r-th order p-adic Stickelberger series' for the data $F/k, S$ and T by setting

$$\theta^{(r)}_{p,F/k,S,T}(z) := \theta_{p,F/k,S,T}(z) \cdot \sum_{\psi \in \mathrm{Ir}_p^-(G)} e_\psi z^{-r\psi(1)} = \sum_{\psi \in \mathrm{Ir}_p^-(G)} e_\psi z^{-r\psi(1)} L_{p,S,T}(\check{\psi}\omega_k, z).$$

We also define a $\zeta(\mathbb{Q}_p[G])$-valued $\mathscr{L}$-invariant for F/k by setting

$$\mathscr{L}_S(F/k) := \sum_{\psi \in \mathrm{Ir}_p^-(G)} e_\psi \mathscr{L}_S(\psi).$$

For a finite group Δ we write $\xi(\mathbb{Z}_p[\Delta])$ for the $\mathbb{Z}_p$-order in $\zeta(\mathbb{Q}_p[\Delta])$ that is (additively) generated over $\mathbb{Z}_p$ by the reduced norms over the semisimple algebra $\mathbb{Q}_p[\Delta]$ of all finite square matrices with entries in $\mathbb{Z}_p[\Delta]$. If Δ is abelian, then it is clear that $\xi(\mathbb{Z}_p[\Delta]) = \mathbb{Z}_p[\Delta]$ but in general one finds that $\xi(\mathbb{Z}_p[\Delta])$ is neither contained in nor contains $\zeta(\mathbb{Z}_p[\Delta])$

A $\mathbb{Z}_p[\Delta]$-module N is said to have a 'quadratic presentation' if there exists a natural number d and an exact sequence of $\mathbb{Z}_p[\Delta]$-modules of the form $\mathbb{Z}_p[\Delta]^d \to \mathbb{Z}_p[\Delta]^d \to N \to 0$.

We recall that for such modules N there exists a canonical $\xi(\mathbb{Z}_p[\Delta])$-submodule $\mathrm{Fit}_{\mathbb{Z}_p[\Delta]}(N)$ of $\zeta(\mathbb{Q}_p[\Delta])$ that constitutes a natural generalization of the classical notion of zeroth Fitting ideal (for more details of this construction see Parker [33] and Nickel [30]).

Corollary 3.10. *Write V for the subset of S comprising all (non-archimedean) places that split completely in F/k and set $r := |V|$. Then the following claims are valid.*

(i) $\theta^{(r)}_{p,F/k,S,T}(z)$ *is holomorphic at $z = 0$.*

(ii) $\mathrm{Sel}^T_{S\setminus V}(F)_p$ *and* $\mathrm{Sel}^T_{S\setminus V}(F)^{\mathrm{tr}}_p$ *have quadratic $\mathbb{Z}_p[G]$-module presentations.*

(iii) *If $\mu_p(F)$ vanishes or p does not divide $[F : k]$, then one has both*

$$\xi(\mathbb{Z}_p[G]) \cdot \theta^{(r)}_{p,F/k,S,T}(0) = \mathscr{L}_S(F/k) \cdot \mathrm{Fit}_{\mathbb{Z}_p[G]}(\mathrm{Sel}^T_{S\setminus V}(F)^{\mathrm{tr}}_p)$$

and

$$\xi(\mathbb{Z}_p[G])\cdot\theta^{(r)}_{p,F/k,S,T}(0)^{\#} = \mathscr{L}_S(F/k)^{\#}\cdot \mathrm{Fit}_{\mathbb{Z}_p[G]}(\mathrm{Sel}^T_{S\setminus V}(F)_p).$$

Before stating the next result we recall that for every natural number m and matrix M in $\mathrm{M}_m(\mathbb{Z}[G])$ there is a unique matrix M^* in $\mathrm{M}_m(\mathbb{Q}[G])$ with $MM^* = M^*M = \mathrm{Nrd}_{\mathbb{Q}[G]}(M)\cdot I_m$ and such that for every primitive central idempotent e of $\mathbb{Q}[G]$ the matrix M^*e is invertible if and only if $\mathrm{Nrd}_{\mathbb{Q}[G]}(M)e$ is non-zero: we then obtain an ideal of $\zeta(\mathbb{Z}[G])$ by setting

$$\mathcal{A}(G) := \{x\in\zeta(\mathbb{Q}[G]) : \text{ if } d>0 \text{ and } M\in \mathrm{M}_d(\mathbb{Z}[G]) \text{ then } xM^*\in \mathrm{M}_d(\mathbb{Z}[G])\}.$$

This ideal was introduced in [30] and is studied extensively by Johnston and Nickel in [21].

In the next result we state several consequences of Corollaries 3.8 and 3.10 that are in the spirit of (stronger versions of) Brumer's Conjecture.

In this result we continue to use the notation of Corollary 3.10. In claims (ii) and (iii) we also use the $\zeta(\mathbb{Q}[G])$-valued 'Stickelberger elements' that are defined for sets of places Σ of k with $S_k^\infty\subseteq\Sigma\subseteq S$ by setting

$$\theta_{F/k,\Sigma,T}(0) = \sum_{\psi\in\mathrm{Ir}(G)} e_\psi L_{\Sigma,T}(\check{\psi},0).$$

Corollary 3.11. *If either $\mu_p(F)$ vanishes or p does not divide $[F:k]$, then the following claims are valid.*

(i) *For each a in $\mathcal{A}(G)$ the product $a\cdot\theta^{(r)}_{p,F/k,S,T}(0)$ belongs to $\mathscr{L}_S(F/k)\cdot\mathrm{Ann}_{\mathbb{Z}[G]}(\mathrm{Cl}^T(F))_p$.*

(ii) *Assume the Weak p-adic Gross-Stark Conjecture (Conjecture 2.4) to be valid for all characters in $\mathrm{Ir}_p^-(G)$. Then for each a in $\mathcal{A}(G)\cap\mathbb{Q}_p[G]^{\mathrm{ss}}$ the elements $a\cdot\theta_{F/k,S\setminus V,T}(0)^{\#}$ and $a\cdot\theta_{F/k,S\setminus V,T}(0)$ belong to $\mathbb{Z}_p[G]$ and respectively annihilate $\mathrm{Sel}^T_{S\setminus V}(F)_p$ and both $\mathrm{Sel}^T_{S\setminus V}(F)_p^{\mathrm{tr}}$ and $\mathrm{Cl}^T(F)_p$.*

(iii) *Assume Gross's Order of Vanishing Conjecture (Conjecture 2.1) to be valid for all characters in $\mathrm{Ir}_p^-(G)$. Write Σ for the set $S^{\mathrm{ram}}_{F/k}$ of places of k that ramify in F (and note that $S^k_\infty\subseteq\Sigma$). Then $\mathrm{Sel}^T_\Sigma(F)_p$ has a quadratic $\mathbb{Z}_p[G]$-module presentation and*

$$\xi(\mathbb{Z}_p[G])^-\cdot\theta_{F/k,\Sigma,T}(0)^{\#} = \mathrm{Fit}_{\mathbb{Z}_p[G]^-}(\mathrm{Sel}^T_\Sigma(F)_p^-).$$

In particular, for a in $\mathcal{A}(G)^-$ the elements $a\cdot\theta_{F/k,\Sigma,T}(0)^{\#}$ and $a\cdot\theta_{F/k,\Sigma,T}(0)$ belong to $\mathbb{Z}_p[G]$ and respectively annihilate $\mathrm{Sel}^T_\Sigma(F)_p$ and both $\mathrm{Sel}^T_\Sigma(F)_p^{\mathrm{tr}}$ and $\mathrm{Cl}^T(F)_p$.

Remark 3.12.
(i) In the context of Corollary 3.11(ii) recall that Conjecture 2.4 is known to be valid whenever F/k is abelian (see Theorem 2.6).
(ii) The 'non-abelian Brumer-Stark conjecture' (as formulated independently by Nickel in [31] and the present author in [2]) asserts that the element $a\cdot\theta_{F/k,\Sigma,T}(0)$ occurring in Corollary 3.11(iii) belongs to $\mathbb{Z}[G]$ and annihilates $\mathrm{Cl}^T(F)$. In the case that G is abelian, this prediction recovers the classical Brumer-Stark Conjecture, as formulated by Tate in [43, Chap. IV, §6]. Previous investigations of these questions (such as in the recent work of Greither and Popescu [15] and Johnston and Nickel [23]) study a weaker version of the conjectures in which $\theta_{F/k,\Sigma,T}(0)$ is replaced by the 'imprimitive' Stickelberger element $\theta_{F/k,\Sigma\cup S_k^p,T}(0)$ since that allows one to

avoid technical difficulties arising from the existence of trivial zeros in the relevant p-adic L-series. Corollary 3.11(iii) now provides a concrete strategy for proving the non-abelian Brumer-Stark conjecture in the presence of trivial zeroes and is used by Sano and the present author in [5] to give the first unconditional verifications of the conjecture in this setting.

3.5. We now assume $G = G_{F/k}$ is abelian. We also fix an ordering of S and for each v in S a place w_v of F lying over v. For each CM extension L of k in F we write $w_{v,L}$ for the restriction of w_v to L and V_L for the subset of S comprising places that split completely in L/k (so that, in the notation of Corollary 3.10, one has $V_F = V$) and we set $r_L := |V_L|$.

Then Theorem 3.1(iii) implies $\theta^{(r_L)}_{p,L/k,S,T}(0)$ is stable under multiplication by $e_{\rm ss}$ and hence that, with $H = G_{F/L}$, there exists a unique element $\epsilon^p_{L/k,S,T}$ of $\bigwedge^{r_L}_{\mathbb{Q}_p[G/H]} e_{\rm ss}(\mathbb{Q}_p \cdot \mathcal{O}^\times_{L,S})$ with

$$(\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \bigwedge\nolimits^{r_L}_{\mathbb{Z}_p[G/H]} \lambda_{L,S,p})(\epsilon^p_{L/k,S,T}) = \theta^{(r_L)}_{p,L/k,S,T}(0) \cdot \bigwedge_{v \in V_L} (w_{v,L} - \overline{w_{v,L}}).$$

By using the approach of Kurihara, Sano and the present author in [4], one can show Conjecture 3.5 implies that the family of elements $\epsilon^p_{L/k,S,T}$ has the same arithmetic properties that are conjectured for Rubin-Stark elements in loc. cit., including validating refinements of the 'refined class number formula for $\mathbb{G}_m$' that is formulated independently by Mazur and Rubin in [27] and Sano in [39].

On the other hand, if Gross's Order of Vanishing Conjecture (Conjecture 2.1) is valid for all characters in $\mathrm{Ir}^-_p(G)$, then Theorem 2.6 implies that the p-adic Gross-Stark Conjecture is valid for L/k for every CM-extension L of k in F and, in this case, a direct comparison of definitions shows that $\epsilon^p_{L/k,S,T} = e_-(\epsilon^{V_L}_{L/k,S,T})$ with $\epsilon^{V_L}_{L/k,S,T}$ the Rubin-Stark element for the data $(L/k, S, T, V_L)$.

In this way one obtains concrete new evidence for the conjectures formulated in [4]. This aspect of the theory is discussed by Sano and the present author in [5], where it is also extended naturally to the setting of arbitrary Galois CM extensions of totally real fields.

4. Canonical complexes

In this section we explicitly describe the complex $R\Gamma_{\text{ét},T}(\mathcal{O}_{F,S}, \mathbb{Z}_p(1))$ that occurs in Conjecture 3.5 and, in particular, describe its connection to the 'T-modified compactly supported Weil-étale cohomology complexes' $R\Gamma_{c,T}((\mathcal{O}_{F,S})_{\mathcal{W}}, \mathbb{Z})$ that are defined in [4, Prop. 2.4].

For each finite Galois extension L of k in $F^{\rm cyc}$ we write κ_w for the residue field of each place w in T_L.

Lemma 4.1. *Let L be a finite Galois extension of k in $F^{\rm cyc}$ and set $G' := G_{L/k}$. Then there exists a canonical object $R\Gamma_{\text{ét},T}(\mathcal{O}_{L,S}, \mathbb{Z}_p(1))$ in $D^{\rm p}(\mathbb{Z}_p[G'])$ with all of the following properties.*

We write $H^i_T(\mathcal{O}_{L,S}, \mathbb{Z}_p(1))$ for the cohomology of $R\Gamma_{\text{ét},T}(\mathcal{O}_{L,S}, \mathbb{Z}_p(1))$ in each degree i.

(i) *There is a canonical exact triangle in $D^{\rm p}(\mathbb{Z}_p[G'])$ of the form*

$$R\Gamma_{\text{ét},T}(\mathcal{O}_{L,S}, \mathbb{Z}_p(1)) \to R\Gamma_{\text{ét}}(\mathcal{O}_{L,S}, \mathbb{Z}_p(1)) \xrightarrow{\theta_{L,S,T}} \bigoplus_{w \in T_L} R\Gamma_{\text{ét}}(\kappa_w, \mathbb{Z}_p(1)) \to$$

in which $H^1(\theta_{L,S,T})$ is induced by the natural projection maps $\mathcal{O}^\times_{L,S} \to \kappa^\times_w$.

(ii) *$R\Gamma_{\text{ét},T}(\mathcal{O}_{L,S},\mathbb{Z}_p(1))$ is acyclic outside degrees one and two, $H^1_T(\mathcal{O}_{L,S},\mathbb{Z}_p(1)))$ identifies with $\mathcal{O}^\times_{L,S,T,p}$ and there exists a canonical short exact sequence of $\mathbb{Z}_p[G']$-modules*

$$0 \to H^2_T(\mathcal{O}_{L,S},\mathbb{Z}_p(1)) \to \mathrm{Sel}^T_S(L)^{\mathrm{tr}}_p \to Y_{L,S^\infty_k,p} \to 0$$

where the last arrow denotes the composite of the scalar extensions of the homomorphism $\mathrm{Sel}^T_S(L)^{\mathrm{tr}} \to X_{L,S}$ in (6) and the natural projection $X_{L,S} \to Y_{L,S^\infty_k}$.

(iii) *There is a natural exact triangle in $D^{\mathrm{p}}(\mathbb{Z}_p[G'])$ of the form*

$$R\Gamma_T(\mathcal{O}_{L,S},\mathbb{Z}_p(1)) \to \mathbb{Z}_p \otimes_{\mathbb{Z}} R\,\mathrm{Hom}_{\mathbb{Z}}(R\Gamma_{c,T}((\mathcal{O}_{L,S})_{\mathcal{W}},\mathbb{Z}),\mathbb{Z})[-3] \to Y_{L,S^\infty_k,p}[-2] \to,$$

where $R\,\mathrm{Hom}_{\mathbb{Z}}(R\Gamma_{c,T}((\mathcal{O}_{L,S})_{\mathcal{W}},\mathbb{Z}),\mathbb{Z})$ is endowed with contragredient G'-action.

(iv) *Let ϵ be any exact sequence of the form (11) with $C = R\Gamma_{\text{ét},T}(\mathcal{O}_{L,S},\mathbb{Z}_p(1))^-, R = \mathbb{Z}_p[G]^-$ and $R' = \mathbb{C}_p[G]^-$. Then in $K_0(R,R')$ one has*

$$\chi_{R,R'}(C,\epsilon) = \chi_{R,R'}(R\Gamma_{\text{ét}}(\mathcal{O}_{L,S},\mathbb{Z}_p(1))^-,\epsilon) - \delta_{R,R'}(\prod_{v\in T}\mathrm{Nrd}_{\mathbb{Q}_p[G]}(1-\mathrm{N}v\cdot \mathrm{Fr}^{-1}_{w_v})e_-)$$

where each w_v is a choice of place of L above v.

Proof. For each w in T_L we write ι_w for the closed immersion $\mathrm{Spec}(\kappa_w) \to \mathrm{Spec}(\mathcal{O}_{L,S})$. We then write $\theta_{L,S,T}$ for the morphism $R\Gamma_{\text{ét}}(\mathcal{O}_{L,S},\mathbb{Z}_p(1)) \to \bigoplus_{w\in T_L} R\Gamma_{\text{ét}}(\kappa_w,\mathbb{Z}_p(1))$ induced by the inverse limits over n of the natural morphisms $\mu_{p^n} \to \bigoplus_{w\in T_L}\iota_{w,*}(\mu_{p^n})$ on $\mathrm{Spec}(\mathcal{O}_{L,S})$.

With respect to the natural identifications $H^1_{\text{ét}}(\mathcal{O}_{L,S},\mathbb{Z}_p(1)) = \mathcal{O}^\times_{L,S,p}$ and $H^1_{\text{ét}}(\kappa_w,\mathbb{Z}_p(1)) = \kappa^\times_{w,p}$ the map $H^1(\theta_{L,S,T})$ is the natural map $\mathcal{O}^\times_{L,S,p} \to \bigoplus_{w\in T_L}\kappa^\times_{w,p}$. In particular, by defining $R\Gamma_{\text{ét},T}(\mathcal{O}_{L,S},\mathbb{Z}_p(1))$ to be the mapping fibre of $\theta_{L,T}$ one directly obtains an exact triangle of the form stated in claim (i). (We are grateful to the referee for pointing out that an alternative description of the complex $R\Gamma_{\text{ét},T}(\mathcal{O}_{L,S},\mathbb{Z}_p(1))$ is given by the approach of Greither and Popescu in [16, Lem 5.6].)

Next we note that, as p is odd, the complex $R\Gamma_{\text{ét}}(\mathcal{O}_{L,S},\mathbb{Z}_p(1))$ belongs to $D^{\mathrm{p}}(\mathbb{Z}_p[G'])$ and is acyclic outside degrees one and two. In addition, $\bigoplus_{w\in T_L} R\Gamma_{\text{ét}}(\kappa_w,\mathbb{Z}_p(1))$ belongs to $D^{\mathrm{p}}(\mathbb{Z}_p[G'])$ and is acyclic outside degree one. These facts combine with the triangle in claim (i) (and its associated long exact sequence of cohomology) to imply $R\Gamma_{\text{ét},T}(\mathcal{O}_{L,S},\mathbb{Z}_p(1))$ belongs to $D^{\mathrm{p}}(\mathbb{Z}_p[G'])$, is acyclic outside degrees one and two and has cohomology in degree one that identifies with the kernel $\mathcal{O}^\times_{L,S,T,p}$ of $H^1(\theta_{L,S,T})$.

The triangle in claim (i) fits into a commutative diagram of exact triangles in $D(\mathbb{Z}_p[G'])$

$$\begin{tikzcd}
 & {} & & \\
 & Y_{L,S^\infty_k,p}[-2] \arrow[u] & & \\
\mathbb{Z}_p\otimes_{\mathbb{Z}} R\,\mathrm{Hom}_{\mathbb{Z}}(C,\mathbb{Z})[-3] \arrow[r] & R\Gamma_c(\mathcal{O}_{L,S},\mathbb{Z}_p)^*[-3] \arrow[u] \arrow[r] & \bigoplus_{w\in T_L}\kappa^\times_{w,p}[-1] \arrow[r] & {} \\
R\Gamma_T(\mathcal{O}_{L,S},\mathbb{Z}_p(1)) \arrow[r] & R\Gamma(\mathcal{O}_{L,S},\mathbb{Z}_p(1)) \arrow[u] \arrow[r, "\theta_{L,S,T}"] & \bigoplus_{w\in T_L} R\Gamma_{\text{ét}}(\kappa_w,\mathbb{Z}_p(1)) \arrow[u, "\cong"'] \arrow[r] & {}.
\end{tikzcd} \tag{13}$$

Here we write C for $R\Gamma_{c,T}((\mathcal{O}_{L,S})_{\mathcal{W}},\mathbb{Z})$ (as defined in [4, Prop. 2.4]) and set $R\Gamma_c(\mathcal{O}_{L,S},\mathbb{Z}_p)^* := R\operatorname{Hom}_{\mathbb{Z}_p}(R\Gamma_c(\mathcal{O}_{L,S},\mathbb{Z}_p),\mathbb{Z}_p)$. The existence of the vertical triangle follows from the Artin-Verdier Duality theorem, the upper horizontal triangle is obtained by applying the exact functor $\mathbb{Z}_p\otimes_{\mathbb{Z}} R\operatorname{Hom}_{\mathbb{Z}}(-,\mathbb{Z})[-1]$ to the right hand vertical triangle in the diagram [4, (6)] and the right hand vertical map is the natural isomorphism resulting from the fact that each complex $R\Gamma_{\text{ét}}(\kappa_w,\mathbb{Z}_p(1))$ is acyclic outside degree one.

Recalling that $\mathrm{Sel}_S^T(L)^{\rm tr}$ is defined in [4] to be equal to $H^{-1}(R\operatorname{Hom}_{\mathbb{Z}}(R\Gamma_{c,T}((\mathcal{O}_{L,S})_{\mathcal{W}},\mathbb{Z}),\mathbb{Z}))$, it is then straightforward to check that this exact diagram gives rise to an exact sequence of the form stated in claim (ii).

The exact triangle in claim (iii) is obtained directly upon completing the diagram (13) to a morphism of exact triangles.

To deduce the equality in claim (iv) as a consequence of the exact triangle in claim (i) (and the standard additivity properties of Euler characteristics in relative K-theory) it suffices to prove that for each v in T one has

$$\chi_{R,R'}(\bigoplus_{w|v} R\Gamma_{\text{ét}}(\kappa_w,\mathbb{Z}_p(1))^-,0) = \delta_{R,R'}(\mathrm{Nrd}_{\mathbb{Q}_p[G]}(1-\mathrm{N}v\cdot\mathrm{Fr}_{w_v}^{-1})e_-).$$

This equality follows directly from the fact that $\bigoplus_{w|v} R\Gamma_{\text{ét}}(\kappa_w,\mathbb{Z}_p(1))^-$ is naturally isomorphic to a complex $R\to R$, where the first term is placed in degree zero and the differential is right multiplication by $(1-\mathrm{N}v\cdot\mathrm{Fr}_{w_v}^{-1})e_-$. □

Remark 4.2. If the field L in Lemma 4.1 is CM, then $Y^-_{L,S_k^\infty,p}$ vanishes. This shows that the identifications in (12) follow directly from Lemma 4.1(ii).

5. Iwasawa theoretic preliminaries

In this section we shall define the refined regulator map $\lambda_{F,S,T,p}$ that occurs in diagram (8) and also prove several Iwasawa-theoretic results that will be used in the proofs of Theorem 3.1 and Corollary 3.3.

5.1. In this subsection we discuss some preliminary algebraic constructions and introduce convenient notation.

5.1.1. Let Δ be a finite group. For each ψ in $\mathrm{Ir}_p(\Delta)$ we fix a subfield E_ψ of $\mathbb{Q}_p^c$ which is both Galois and of finite degree over $\mathbb{Q}_p$ and over which ψ can be realised. We also fix an indecomposable idempotent f_ψ of $E_\psi[\Delta]e_\psi$, write $\mathcal{O}_\psi$ for the valuation ring of E_ψ, choose a maximal $\mathcal{O}_\psi$-order $\mathcal{M}_\psi$ in $E_\psi[\Delta]$ which contains f_ψ and define an $\mathcal{O}_\psi$-free right $\mathcal{O}_\psi[\Delta]$-module $T_\psi := f_\psi\mathcal{M}_\psi$. The associated right $E_\psi[\Delta]$-module $V_\psi := E_\psi\otimes_{\mathcal{O}_\psi} T_\psi$ has character ψ.

For any (left) $\mathbb{Z}_p[\Delta]$-module M we set $M[\psi] := T_\psi\otimes_{\mathbb{Z}_p} M$, upon which Δ acts on the left by $t\otimes m\mapsto t\delta^{-1}\otimes\delta(m)$ for each $t\in T_\psi, m\in M$ and $\delta\in\Delta$.

For any $\mathbb{Z}_p[\Delta]$-module M and subgroup Υ of Δ we also write $\hat H^i(\Upsilon,M)$ for the Tate cohomology group in degree i and M^Υ, resp. M_Υ, for the maximal submodule, resp. maximal quotient module, of M upon which Υ acts trivially. We thereby obtain left, respectively right, exact functors $M\mapsto M^{(\psi)}$ and $M\mapsto M_{(\psi)}$, from left $\mathbb{Z}_p[\Delta]$-modules to the category of $\mathcal{O}_\psi$-modules by setting

$$M^{(\psi)} := M[\psi]^\Delta \quad\text{and}\quad M_{(\psi)} := M[\psi]_\Delta \cong T_\psi\otimes_{\mathbb{Z}_p[\Delta]} M.$$

The action on M of the element $\sum_{\delta\in\Delta}\delta$ then gives rise to a natural exact sequence of $\mathcal{O}_\psi$-modules

$$0 \to \hat{H}^{-1}(\Delta, M[\psi]) \to M_{(\psi)} \xrightarrow{\mathrm{Tr}^{\psi}_{\Delta,M}} M^{(\psi)} \to \hat{H}^{0}(\Delta, M[\psi]) \to 0. \tag{14}$$

In particular, if M, and hence also $M[\psi]$, is a cohomologically-trivial Δ-module, then $\mathrm{Tr}^{\psi}_{\Delta,M}$ is bijective and so the functors $M_{(\psi)}$ and $M^{(\psi)}$ extend to give naturally isomorphic (exact) functors from the derived category of bounded complexes of cohomologically-trivial $\mathbb{Z}_p[\Delta]$-modules to the derived category of bounded complexes of $\mathcal{O}_\psi$-modules.

The following technical result will be useful in subsequent sections.

In claims (ii) and (iii) of this result we use the construction $\epsilon \mapsto \epsilon^{\langle\psi\rangle}$ introduced just prior to the statement of Theorem 3.1.

Lemma 5.1. *Fix ψ in $\mathrm{Ir}_p(\Delta)$ and set $\mathcal{O} := \mathcal{O}_\psi$.*

(i) *Let C be a complex of cohomologically-trivial $\mathbb{Z}_p[\Delta]$-modules that is acyclic outside degrees a and $a+1$ for an integer a. Then $C_{(\psi)}$ is acyclic outside degrees a and $a+1$ and there are natural isomorphisms $H^a(C_{(\psi)}) \cong H^a(C)^{(\psi)}$ and $H^{a+1}(C_{(\psi)}) \cong H^{a+1}(C)_{(\psi)}$.*

(ii) *Let $\lambda : M \to N$ be an injective homomorphism of finitely generated $\mathbb{Z}_p[\Delta]$-modules which induces an isomorphism $\mathbb{Q}_pM \cong \mathbb{Q}_pN$. Then*

$$\mathrm{Fit}_{\mathcal{O}}(\mathrm{cok}(\lambda^{(\psi)})) = |\Delta|^{-r}\mathrm{Fit}_{\mathcal{O}}(\mathrm{cok}(\lambda^{\langle\psi\rangle}))\cdot \mathrm{Fit}_{\mathcal{O}}(\hat{H}^0(\Delta, N[\psi]))\cdot \mathrm{Fit}_{\mathcal{O}}(\hat{H}^{-1}(\Delta, N[\psi]))^{-1}$$

with $r = \dim_{\mathbb{Q}_p^c}(\mathbb{Q}_p^c\cdot N^{(\psi)})$.

(iii) *Let λ and $\tilde{\lambda}$ be injective homomorphisms $M \to N$ as in claim (ii). Then if $M^{(\psi)}$ is torsion-free one has*

$$\begin{aligned}\mathrm{Fit}_{\mathcal{O}}(\mathrm{cok}(\lambda^{\langle\psi\rangle}))\mathrm{Fit}_{\mathcal{O}}(\mathrm{cok}(\tilde{\lambda}^{\langle\psi\rangle}))^{-1} &= \mathrm{Fit}_{\mathcal{O}}(\mathrm{cok}(\lambda^{(\psi)}))\mathrm{Fit}_{\mathcal{O}}(\mathrm{cok}(\tilde{\lambda}^{(\psi)}))^{-1}\\ &= \mathrm{det}_{\mathbb{Q}_p^c}((\mathbb{Q}_p^c\otimes_{\mathbb{Z}_p}\lambda)\circ(\mathbb{Q}_p^c\otimes_{\mathbb{Z}_p}\tilde{\lambda})^{-1}\mid \mathbb{Q}_p^c\cdot N^{(\psi)})\cdot\mathcal{O}.\end{aligned}$$

Proof. Fix a complex $M^a \xrightarrow{d} M^{a+1}$ of cohomologically-trivial $\mathbb{Z}_p[\Delta]$-modules that is isomorphic to C in $D(\mathbb{Z}_p[\Delta])$. Then there is an exact commutative diagram of $\mathcal{O}$-modules

$$\begin{tikzcd}
 & M^a_{(\psi)} \arrow[r, "d_{(\psi)}"] \arrow[d, "\mathrm{Tr}^{\psi}_{\Delta,M^a}"] & M^{a+1}_{(\psi)} \arrow[r, two heads] \arrow[d, "\mathrm{Tr}^{\psi}_{\Delta,M^{a+1}}"] & H^{a+1}(C)_{(\psi)} \\
H^a(C)^{(\psi)} \arrow[r, hook] & M^{a,(\psi)} \arrow[r, "d^{(\psi)}"] & M^{a+1,(\psi)} &
\end{tikzcd}$$

with bijective vertical arrows. Claim (i) follows immediately from this diagram.

To prove claim (ii) we use the exact commutative diagram

$$\begin{tikzcd}
 & \ker(\tau) \arrow[r, "\mathrm{id}"] \arrow[d, hook] & \ker(\tau) \arrow[d, hook] \\
M^{(\psi)} \arrow[r, "\lambda^{(\psi)}"] \arrow[d, "\mathrm{id}"] & N^{(\psi)} \arrow[r, two heads] \arrow[d, "\tau"] & \mathrm{cok}(\lambda^{(\psi)}) \arrow[d] \\
M^{(\psi)} \arrow[r, "\lambda^{\langle\psi\rangle}"] & N_{(\psi)} \arrow[r, two heads] \arrow[d, two heads] & \mathrm{cok}(\lambda^{\langle\psi\rangle}) \arrow[d, two heads] \\
 & \mathrm{cok}(\tau) \arrow[r, "\mathrm{id}"] & \mathrm{cok}(\tau)
\end{tikzcd}$$

where τ is the tautological map and the left hand square commutes by definition of $\lambda^{\langle\psi\rangle}$. All modules in the last column of this diagram are finite and so one has

$$\mathrm{Fit}_{\mathcal{O}}(\mathrm{cok}(\lambda^{(\psi)})) = \mathrm{Fit}_{\mathcal{O}}(\mathrm{cok}(\lambda^{\langle\psi\rangle}))\mathrm{Fit}_{\mathcal{O}}(\ker(\tau))\mathrm{Fit}_{\mathcal{O}}(\mathrm{cok}(\tau))^{-1}.$$

In addition, the composite $\mathrm{Tr}^{\psi}_{\Delta,N} \circ \tau$ is equal to the endomorphism $\mu_{|\Delta|}$ of $N^{(\psi)}$ induced by multiplication by $|\Delta|$ and so the kernel-cokernel sequence of this composite implies that

$$\begin{aligned}
&\mathrm{Fit}_{\mathcal{O}}(\ker(\tau))\mathrm{Fit}_{\mathcal{O}}(\mathrm{cok}(\tau))^{-1} \\
&\qquad = \mathrm{Fit}_{\mathcal{O}}(\hat{H}^0(\Delta, N[\psi]))\mathrm{Fit}_{\mathcal{O}}(\hat{H}^{-1}(\Delta, N[\psi]))^{-1}\mathrm{Fit}_{\mathcal{O}}(\ker(\mu_{|\Delta|}))\mathrm{Fit}_{\mathcal{O}}(\mathrm{cok}(\mu_{|\Delta|}))^{-1}.
\end{aligned}$$

Claim (ii) now follows because an application of the Snake lemma to the exact commutative diagram

$$\begin{tikzcd}
N^{(\psi)}_{\mathrm{tor}} \arrow[r, hook] \arrow[d, "\mu_{|\Delta|}"] & N^{(\psi)} \arrow[r, two heads] \arrow[d, "\mu_{|\Delta|}"] & N^{(\psi)}_{\mathrm{tf}} \arrow[d, hook, "\mu_{|\Delta|}"] \\
N^{(\psi)}_{\mathrm{tor}} \arrow[r, hook] & N^{(\psi)} \arrow[r, two heads] & N^{(\psi)}_{\mathrm{tf}}
\end{tikzcd}$$

implies $\mathrm{Fit}_{\mathcal{O}}(\ker(\mu_{|\Delta|}))\mathrm{Fit}_{\mathcal{O}}(\mathrm{cok}(\mu_{|\Delta|}))^{-1} = \det_{\mathcal{O}}(\mu_{|\Delta|} \mid N^{(\psi)}_{\mathrm{tf}})^{-1}\mathcal{O} = |\Delta|^{-r}\mathcal{O}$.

Regarding claim (iii), the first equality is a direct consequence of the equality in claim (ii) and so it suffices to prove the second equality.

The assumption that $M^{(\psi)}$ is torsion-free (and λ and $\tilde{\lambda}$ are injective) also implies that if κ denotes either λ or $\tilde{\lambda}$, then $\mathrm{Fit}_{\mathcal{O}}(\mathrm{cok}(\kappa^{(\psi)}) = \mathrm{Fit}_{\mathcal{O}}(N^{(\psi)}_{\mathrm{tor}}) \cdot \mathrm{Fit}_{\mathcal{O}}(\mathrm{cok}(\kappa^{(\psi)}_{\mathrm{tf}}))$ with $\kappa^{(\psi)}_{\mathrm{tf}}$ denoting the composite of $\kappa^{(\psi)}$ and the tautological projection $N^{(\psi)} \to N^{(\psi)}_{\mathrm{tf}}$ and so it is actually enough to prove the second equality with $\lambda^{(\psi)}$ and $\tilde{\lambda}^{(\psi)}$ replaced by $\lambda^{(\psi)}_{\mathrm{tf}}$ and $\tilde{\lambda}^{(\psi)}_{\mathrm{tf}}$.

In this case, the required equality is true because, by definition, the ideal $\mathrm{Fit}_{\mathcal{O}}(\mathrm{cok}(\kappa^{(\psi)}_{\mathrm{tf}}))$ is generated over $\mathcal{O}$ by the determinant of the matrix of $\mathbb{Q}_p^c \otimes_{\mathcal{O}} \kappa^{(\psi)}$ with respect to any choice of $\mathcal{O}$-bases of $M^{(\psi)}$ and $N^{(\psi)}_{\mathrm{tf}}$. □

5.1.2. We next recall some necessary background material concerning Bockstein homomorphisms in Iwasawa theory.

To do this we assume to be given any compact p-adic Lie group $\mathcal{G}$ that contains a closed normal subgroup $\mathcal{H}$ such that the quotient group $\Gamma := \mathcal{G}/\mathcal{H}$ is topologically isomorphic to $\mathbb{Z}_p$. We fix a topological generator γ of Γ.

We also fix a finitely generated $\mathbb{Z}_p$-algebra R and a continuous homomorphism

$$\rho : \mathcal{G} \to \mathrm{Aut}_R(T_\rho)$$

with T_ρ a finitely generated free left R-module. We set $\Lambda_R(\Gamma) := R \otimes_{\mathbb{Z}_p} \Lambda(\Gamma)$ and consider the tensor product $\Lambda_R(\Gamma) \otimes_R T_\rho$ as an $(\Lambda_R(\Gamma), \Lambda(\mathcal{G}))$-bimodule where $\Lambda_R(\Gamma)$ acts by multiplication on the left and $\Lambda(\mathcal{G})$ acts on the right via the rule $((r \otimes_{\mathbb{Z}_p} \lambda) \otimes_R t)g := (r \otimes_{\mathbb{Z}_p} \lambda\bar{g}) \otimes_R g^{-1}(t)$ for each r in R, $\lambda \in \Lambda(\Gamma)$, $t \in T_\rho$ and $g \in \mathcal{G}$ with image $\bar{g}$ in Γ.

Then for each bounded complex of finitely generated projective $\Lambda(\mathcal{G})$-modules C we obtain a bounded complex of finitely generated projective $\Lambda_R(\Gamma)$-modules by setting

$$C_\rho := (\Lambda_R(\Gamma) \otimes_R T_\rho) \otimes_{\Lambda(\mathcal{G})} C.$$

For each open normal subgroup U of $\mathcal{G}$ we set $C_U := \mathbb{Z}_p[\mathcal{G}/U] \otimes_{\Lambda(\mathcal{G})} C$. In particular, if $U \subseteq \ker(\rho)$, then there are natural isomorphisms of (left) R-modules

$$\mathbb{Z}_p \otimes_{\Lambda(\Gamma)} C_\rho \cong T_\rho \otimes_{\Lambda(\mathcal{G})} C \cong T_\rho \otimes_{\mathbb{Z}_p[\mathcal{G}/U]} C_U.$$

Thus, if for each such U we extend the notation of §5.1.1 by setting $C_{U,(\rho)} := T_\rho \otimes_{\mathbb{Z}_p[\mathcal{G}/U]} C_U$, then one has a natural exact triangle in $D(R)$

$$C_\rho \xrightarrow{\gamma-1} C_\rho \to C_{U,(\rho)} \to C_\rho[1]. \tag{15}$$

In each degree i this triangle then induces a short exact sequence of R-modules

$$0 \to H^i(C_\rho)_\Gamma \xrightarrow{\alpha^i} H^i(C_{U,(\rho)}) \xrightarrow{\tilde{\alpha}^i} H^{i+1}(C_\rho)^\Gamma \to 0, \tag{16}$$

a composite homomorphism

$$\beta^i_{C,\rho,\gamma} : H^i(C_{U,(\rho)}) \xrightarrow{\tilde{\alpha}^i} H^{i+1}(C_\rho)^\Gamma \xrightarrow{\tau^{i+1}_{C_\rho}} H^{i+1}(C_\rho)_\Gamma \xrightarrow{\alpha^{i+1}} H^{i+1}(C_{U,(\rho)})$$

where $\tau^{i+1}_{C_\rho}$ denotes the tautological map, and a bounded complex of R-modules

$$\Delta_{C_\rho,\gamma} : \ \cdots \xrightarrow{\beta^{i-1}_{C,\rho,\gamma}} H^i(C_{U,(\rho)}) \xrightarrow{\beta^i_{C,\rho,\gamma}} H^{i+1}(C_{U,(\rho)}) \xrightarrow{\beta^{i+1}_{C,\rho,\gamma}} \cdots$$

where the term $H^i(C_{U,(\rho)})$ occurs in degree i.

We refer to $\beta^i_{C,\rho,\gamma}$ as the 'Bockstein homomorphism in degree i of the data (C, ρ, γ)' and abbreviate this notation to $\beta^i_{C,\gamma}$ in the case that ρ is the trivial representation of $\mathcal{G}$.

We also say that C is 'semisimple at ρ' if the complex $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \Delta_{C_\rho,\gamma}$ is acyclic for any, and therefore every, choice of topological generator γ of Γ.

5.2. In this subsection we shall state the main result that is to be proved in §5.

To do this we fix in the sequel data F/k, G, S and T as in Theorem 3.1 and set $G_k := G_{k^c/k}$. We also set $\mathcal{G} := G_{F^{\rm cyc}/k}$, $\mathcal{H} := G_{F^{\rm cyc}/k^{\rm cyc}}$ and $\Gamma := \mathcal{G}/\mathcal{H} \cong G_{k^{\rm cyc}/k}$ and note that the triple $(\mathcal{G}, \mathcal{H}, \Gamma)$ satisfies the conditions of §5.1.2.

If E is a field which is a finite extension of either $\mathbb{Q}$ or $\mathbb{Q}_\ell$ for some prime ℓ, then we write χ_E for the p-adic cyclotomic character $G_E \to \mathbb{Z}_p^\times$.

We write $\Lambda(\mathcal{G})$ for the p-adic Iwasawa algebra of $\mathcal{G}$ and for each integer a let $\Lambda(\mathcal{G})^\#(a)$ denote the (left) $\Lambda(\mathcal{G})$-module $\Lambda(\mathcal{G})$ endowed with the action of G_k whereby each element σ acts as right multiplication by $\bar{\sigma}^{-1}\chi_k(\sigma)^a$ where $\bar{\sigma}$ is the image of σ in $\mathcal{G}$.

For any finite set of places Σ of k containing $S_k^p \cup S_{F/k}^{\rm ram}$ we regard $\Lambda(\mathcal{G})^\#(a)$ as an étale sheaf of $\Lambda(\mathcal{G})$-modules on $\mathcal{O}_{k,\Sigma}$ in the natural way and then (since p is odd) we obtain following the approach of Nekovář in [29] a canonical object $R\Gamma_{\text{ét}}(\mathcal{O}_{k,\Sigma}, \Lambda(\mathcal{G})^\#(a))$ of $D^{\rm p}(\Lambda(\mathcal{G}))$.

We note that claim (iv) of the following result refers to the Strong Stark Conjecture formulated by Chinburg in [8, Conj. 2.2].

Theorem 5.2. *Fix a character ψ in $\mathrm{Ir}_p^-(G)$, recall the integer $r_{S,\psi}$ defined in §2.1 and write $r_{p,S,\psi}$ for the order of vanishing of $L_{p,S}(\check{\psi}\omega_k, s)$ at $s = 0$. Then all of the following assertions are valid.*

- (i) $r_{p,S,\psi} \geq r_{S,\psi}$.
- (ii) *The following conditions are equivalent.*
  - (a) $r_{p,S,\psi} = r_{S,\psi}$.
  - (b) *The complex $R\Gamma_{\text{ét}}(\mathcal{O}_{k,S}, \Lambda(\mathcal{G})^\#(1))$ is semisimple at ψ.*
  - (c) $\mathscr{L}_S(\psi) \neq 0$.
- (iii) *There exists a canonical homomorphism $\lambda_{F,S,T,p} : \mathcal{O}_{F,S,T,p}^{\times,-} \to \mathrm{Sel}_S^T(F)_p^{\mathrm{tr},-}$ of $\mathbb{Z}_p[G]$-modules that lies in a commutative diagram of the form (8) and for which the equality in Theorem 3.1(ii) is valid.*
- (iv) *Assume that the conditions in claim (ii) are satisfied and that the character ψ^j validates the Strong-Stark Conjecture at p for any isomorphism of fields $j : \mathbb{C}_p \cong \mathbb{C}$. Then $L_{p,S}^{r_{S,\psi}}(\check{\psi}\omega_k, 0)$ is non-zero and satisfies the displayed equality in Theorem 3.1(iii).*

After proving certain preliminary results in §5.3 and §5.4 we shall prove Theorem 5.2 in §5.5. In the process we shall also show (in Theorem 5.8(ii)) that the map $\lambda_{F,S,T,p}$ constructed in Theorem 5.2(iii) naturally gives rise to an explicit description of the image of Gross's regulator map $\lambda_{F,S,p}$.

Remark 5.3. In the special case that F/k has degree two and ψ is the unique non-trivial homomorphism $G \to \mathbb{Q}_p^\times$, the equivalence of the conditions in Theorem 5.2(ii) can also be directly derived from the observations of Sinnott given in [11, (6.3), (6.4)]. We are very grateful to the referee for pointing this out to us.

5.3. With $\mathcal{G}, \mathcal{H}$ and Γ as specified above, in this subsection we define an object of $D^{\rm p}(\Lambda(\mathcal{G}))$ that will be central to the proof of Theorem 5.2 and also establish some of its basic properties.

We write τ for the (unique) complex conjugation in $\mathcal{G}$ and, noting that τ is central in $\mathcal{G}$, we obtain idempotents $e_\pm$ in $\zeta(\Lambda(\mathcal{G}))$ by setting $e_\pm := (1 \pm \tau)/2$. In particular, each $\Lambda(\mathcal{G})$-module M decomposes as a direct sum $M^+ \oplus M^-$ with $M^\pm := e_\pm \cdot M$ and for objects C of $D^{\rm p}(\Lambda(\mathcal{G}))$ one can similarly define subcomplexes C^+ and C^-.

We write $A(\mathcal{G})$ for the set of continuous representations

$$\rho : \mathcal{G} \to \mathrm{GL}_n(\mathcal{O})$$

where $\mathcal{O}$ is a finite extension of $\mathbb{Z}_p$ and $\ker(\rho)$ is open in $\mathcal{G}$. We also write $A^+(\mathcal{G})$ and $A^-(\mathcal{G})$ for the subsets of $A(\mathcal{G})$ comprising representations that are respectively totally even (that is, for which $\rho(\tau) = I_n$) and totally odd (so $\rho(\tau) = -I_n$).

5.3.1. For each open normal subgroup V of $\mathcal{G}$, with $F_V := (F^{\mathrm{cyc}})^V$, there are natural isomorphisms in $D^{\mathrm{p}}(\mathbb{Z}_p[\mathcal{G}/V])$ of the form

$$\mathbb{Z}_p[\mathcal{G}/V] \otimes^{\mathbb{L}}_{\Lambda(\mathcal{G})} R\Gamma_{\text{ét}}(\mathcal{O}_{k,S}, \Lambda(\mathcal{G})^{\#}(1)) \cong R\Gamma_{\text{ét}}(\mathcal{O}_{F_V,S}, \mathbb{Z}_p(1))$$

and for each place v of k also

$$\mathbb{Z}_p[\mathcal{G}/V] \otimes^{\mathbb{L}}_{\Lambda(\mathcal{G})} R\Gamma_{\text{ét}}(\kappa_v, \Lambda(\mathcal{G})^{\#}(1)) \cong \bigoplus_w R\Gamma_{\text{ét}}(\kappa_w, \mathbb{Z}_p(1))$$

where in the direct sum w runs over all places of F_V above v.

By passing to the inverse limit of the exact triangles in Lemma 4.1(i) over all finite extensions L of F in F^{cyc} one can thus construct an exact triangle in $D^{\mathrm{p}}(\Lambda(\mathcal{G}))$ of the form

$$R\Gamma_{\text{ét},T}(\mathcal{O}_{k,S}, \Lambda(\mathcal{G})^{\#}(1)) \to R\Gamma_{\text{ét}}(\mathcal{O}_{k,S}, \Lambda(\mathcal{G})^{\#}(1)) \to \bigoplus_{v \in T} R\Gamma_{\text{ét}}(\kappa_v, \Lambda(\mathcal{G})^{\#}(1)) \to \tag{17}$$

in which, for each open normal subgroup V of $\mathcal{G}$, there is a natural isomorphism in $D^{\mathrm{p}}(\mathbb{Z}_p[\mathcal{G}/V])$

$$\mathbb{Z}_p[\mathcal{G}/V] \otimes^{\mathbb{L}}_{\Lambda(\mathcal{G})} R\Gamma_{\text{ét},T}(\mathcal{O}_{k,S}, \Lambda(\mathcal{G})^{\#}(1)) \cong R\Gamma_{\text{ét},T}(\mathcal{O}_{F_V,S}, \mathbb{Z}_p(1)). \tag{18}$$

The complex $R\Gamma_{\text{ét},T}(\mathcal{O}_{k,S}, \Lambda(\mathcal{G})^{\#}(1))^-$ will play a key role in the proof of Theorem 5.2.

5.3.2. To investigate $R\Gamma_{\text{ét},T}(\mathcal{O}_{k,S}, \Lambda(\mathcal{G})^{\#}(1))^-$ it is convenient to use the following construction from non-commutative Iwasawa theory.

Write $Q(\mathcal{G})$ for the total quotient ring of $\Lambda(\mathcal{G})$ and note each homomorphism $\rho : \mathcal{G} \to \mathrm{GL}_n(\mathcal{O})$ in $A(\mathcal{G})$ gives rise to a ring homomorphism

$$Q(\mathcal{G}) \to \mathrm{M}_n(\mathcal{O}) \otimes_{\mathbb{Z}_p} \mathrm{Q}(\Gamma) \cong \mathrm{M}_n(\mathrm{Q}_{\mathcal{O}}(\Gamma))$$

that sends g in $\mathcal{G}$ to $\rho(g) \otimes \tilde{g} \in \mathrm{M}_n(\mathrm{Q}_{\mathcal{O}}(\Gamma))$, where $\tilde{g}$ denotes the image of g in Γ and $\mathrm{Q}_{\mathcal{O}}(\Gamma)$ the total quotient ring of $\mathcal{O} \otimes_{\mathbb{Z}_p} \Lambda(\Gamma)$.

We then fix a topological generator γ of Γ and note that the above ring homomorphism induces a group homomorphism

$$\Phi_\rho : K_1(Q(\mathcal{G})) \to K_1(\mathrm{M}_n(\mathrm{Q}_{\mathcal{O}}(\Gamma))) \cong K_1(\mathrm{Q}_{\mathcal{O}}(\Gamma)) \cong \mathrm{Q}_{\mathcal{O}}(\Gamma)^\times \cong \mathrm{Q}(\mathcal{O}[[u]])^\times$$

where $\mathcal{O}[[u]]$ denotes the ring of power series over $\mathcal{O}$ in the formal variable u, the first isomorphism is induced by Morita equivalence, the second by taking determinants over $\mathrm{Q}_{\mathcal{O}}(\Gamma)$ and the last by sending $\gamma - 1$ to u.

The relevant properties of this homomorphism are recorded in the next result. Before stating this result we recall (from the beginning of §3.3) that any C in $D^{\mathrm{p}}(\Lambda(\mathcal{G}))$ for which $Q(\mathcal{G}) \otimes_{\Lambda(\mathcal{G})} C$ is acyclic gives a canonical element $\chi_{\Lambda(\mathcal{G})}(C, 0)$ of $K_0(\Lambda(\mathcal{G}), Q(\mathcal{G}))$. In any such case we say that an element ξ of $K_1(Q(\mathcal{G}))$ is a 'characteristic element for C' if one has

$$\partial_{\Lambda(\mathcal{G})}(\xi) = -\chi_{\Lambda(\mathcal{G})}(C, 0). \tag{19}$$

Here $\partial_{\Lambda(\mathcal{G})}$ denotes the natural connecting homomorphism $K_1(Q(\mathcal{G})) \to K_0(\Lambda(\mathcal{G}), Q(\mathcal{G}))$ (normalised as in §3.3) and the sign occurs in the displayed equality to ensure consistency with the conventions fixed in [7, §1.4].

In the sequel we write $x \mapsto x^*$ and $x \mapsto \mathrm{tw}_a(\xi)$ for each integer a for the $\mathbb{Q}_p^c$-linear anti-involution and ring automorphism of $\mathbb{Q}_p^c \otimes_{\mathbb{Z}_p} \Lambda(\mathcal{G})$ that respectively send each g in $\mathcal{G}$ to g^{-1} and to $\chi_k(g)^a g$. In claims (iv) and (v) of the following result we also use the same notation to denote the induced group automorphisms of $K_1(Q(\mathcal{G}))$.

Lemma 5.4. *For all elements ξ of $K_1(Q(\mathcal{G}))$ and all representations $\rho : \mathcal{G} \to \mathrm{GL}_n(\mathcal{O})$ and ρ' in $A(\mathcal{G})$ the following properties are valid.*

- (i) $\Phi_{\rho\oplus\rho'}(\xi) = \Phi_\rho(\xi)\Phi_{\rho'}(\xi)$;
- (ii) $\Phi_{\rho^\alpha}(\xi) = \iota_\alpha(\Phi_\rho(\xi))$ *for all* $\alpha \in G_{\mathbb{Q}_p}$, *where* ι_α *is the automorphism of the* $\mathbb{Q}_p$*-algebra* $\mathbb{Q}_p^c \otimes_{\mathcal{O}} Q(\mathcal{O}[[u]])$ *obtained by letting* α *act on the coefficient of each power of* u;
- (iii) $\Phi_{\rho\otimes\psi}(\xi) = \iota_\psi(\Phi_\rho(\xi))$ *for all* $\psi \in A(\Gamma)$, *where* ι_ψ *is the automorphism of* $Q(\mathcal{O}[[u]])$ *that sends* u *to* $\psi(\gamma)(1+u) - 1$;
- (iv) $\Phi_\rho(\xi^*) = \iota_*(\Phi_{\check\rho}(\xi))$ *where* ι_* *is the automorphism of* $Q(\mathcal{O}[[u]])$ *sending* u *to* $(1+u)^{-1}-1$;
- (v) $\Phi_\rho(\mathrm{tw}_a(\xi)) = \iota_a(\Phi_{\rho\omega_k^a}(\xi))$ *for all integers* a, *where* ι_a *is the automorphism of* $Q(\mathcal{O}[[u]])$ *that sends* u *to* $\kappa_k(\gamma)^a(1+u) - 1$;
- (vi) *For each open subgroup* $\mathcal{U}$ *of* $\mathcal{G}$, *finite normal subgroup* $\mathcal{V}$ *of* $\mathcal{U}$ *and* ψ *in* $A(\mathcal{U}/\mathcal{V})$ *one has* $\Phi_{\mathrm{Ind}_{\mathcal{U}}^{\mathcal{G}}(\mathrm{Inf}_{\mathcal{U}/\mathcal{V}}^{\mathcal{U}}\psi)}(\xi) = \Phi_\psi(\xi_{\mathcal{U}/\mathcal{V}})$ *where* $\xi_{\mathcal{U}/\mathcal{V}}$ *is the image of* ξ *under the natural composite map* $K_1(Q(\mathcal{G})) \to K_1(Q(\mathcal{U})) \to K_1(Q(\mathcal{U}/\mathcal{V}))$ *and* $\Phi_\psi(-)$ *is computed with respect to the unique topological generator of* $G_{(F^{\mathrm{cyc},\mathcal{U}})^{\mathrm{cyc},\mathcal{V}}/F^{\mathrm{cyc},\mathcal{U}}}$ *whose restriction to* $G_{k^{\mathrm{cyc},\mathcal{V}}/k}$ *coincides with the restriction of an element of the form* γ^{p^n} *with* $n > 0$.
- (vii) *If* ξ *is a characteristic element for* C *in* $D^{\mathrm{p}}(\Lambda(\mathcal{G}))$, *then* $\Phi_\rho(\xi)$ *is a characteristic element for* C_ρ *in* $D^{\mathrm{p}}(\mathcal{O}[[u]])$.

Proof. Claims (i)-(iv) are obvious, (v) and (vi) are proved in [3, Lem. 9.5 and Lem. 3.6(i)] and (vii) follows easily from the naturality of connecting homomorphisms in relative K-theory. □

5.3.3. In this section we prove the main result concerning the complex $R\Gamma_{\acute{e}t,T}(\mathcal{O}_{k,S}, \Lambda(\mathcal{G})^\#(1))^-$ that will be used in the proof of Theorem 5.2.

Before stating this result we note that it suffices to prove Theorem 5.2 with F/k replaced by $F(\zeta_p)/k$ for any primitive p-th root of unity ζ_p in k^c. In particular, if necessary after replacing F by $F(\zeta_p)$, we may and will assume in the sequel that F^{cyc} contains all p-power roots of unity in $\mathbb{Q}^c$.

In particular, in the cases we study the cyclotomic character χ_k will always factor through the projection $G_k \to \mathcal{G}$. In addition, the homomorphism $\kappa_k := \chi_k \cdot \omega_k^{-1}$ factors through the restriction map $G_k \to \Gamma$ and we recall (from, for example, the discussion of Greenberg in [13]) that for any fixed topological generator γ of Γ and any representation ψ in $A^+(\mathcal{G})$ Deligne and Ribet have shown that there exists a unique element $f_{S,\psi}(u)$ in the total quotient ring of $\mathbb{Q}_p^c \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[[u]]$ for which one has

$$L_{p,S}(1-s,\psi) = f_{S,\psi}(\kappa_k(\gamma)^s - 1). \tag{20}$$

Finally, for each v in T we fix a place w_v of $F^{\rm cyc}$ above v, write ${\rm Fr}_{w_v}$ for the Frobenius automorphism of w_v in $\mathcal{G}$ and note that the product

$$\kappa_T := \prod_{v\in T}(1-{\rm N}v\cdot {\rm Fr}_{w_v}^{-1})$$

belongs to $\Lambda(\mathcal{G})\cap Q(\mathcal{G})^\times$. We write κ_T' for the image of κ_T in $K_1(Q(\mathcal{G}))$ and then for each representation ρ in $A(\mathcal{G})$ set

$$\kappa_{T,\rho}(u) := \Phi_\rho(\kappa_T').$$

Proposition 5.5. *Assume F contains ζ_p and set $C := R\Gamma_{\text{ét},T}(\mathcal{O}_{k,S},\Lambda(\mathcal{G})^\#(1))^-$.*

(i) *Then C is acyclic outside degree two and the $\Lambda(\mathcal{G})$-module $H^2(C)$ is finitely generated, torsion and of projective dimension at most one.*

(ii) *For each ρ in $A^-(\mathcal{G})$ the complex C_ρ is acyclic outside degree two, the $\mathcal{O}_\rho[[u]]$-module $H^2(C_\rho)$ is finitely generated, torsion and of projective dimension at most one and its characteristic ideal is generated by the series*

$$f_{S,T,\rho}(u) := \kappa_{T,\rho}(u)\cdot f_{S,\check\rho\omega_k}(\kappa_k(\gamma)(1+u)^{-1}-1).$$

Proof. The isomorphisms (18) imply each group $H^i(C)$ can be computed as the inverse limit (with respect to the natural corestriction maps) of the groups $H^i_T(\mathcal{O}_{L,S},\mathbb{Z}_p(1)))^-$ as L varies over all finite extensions of F in $F^{\rm cyc}$.

In particular, by combining the explicit descriptions of $H^i_T(\mathcal{O}_{L,S},\mathbb{Z}_p(1))$ given in (12) with the exact sequences (6) one finds that C is acyclic outside degrees one and two, that $H^1(C)$ identifies with $\varprojlim_L \mathcal{O}^{\times,-}_{L,S,T,p}$ and that there is a canonical exact sequence

$$0\longrightarrow \varprojlim_L {\rm Cl}^T_S(L)^-_p\longrightarrow H^2(C)\longrightarrow \varprojlim_L Y^-_{L,S,p}\longrightarrow 0,$$

where in each limit L runs over finite extensions of F in $F^{\rm cyc}$. Since each non-archimedean place in S has open decomposition subgroup in $\mathcal{G}$, it is then straightforward to deduce the groups $H^1(C)$ and $H^2(C)$ are finitely generated torsion $\Lambda(\mathcal{G})$-modules.

To complete the proof of claim (i) it is thus enough to prove C is represented by a complex of finitely generated projective $\Lambda(\mathcal{G})$-modules of the form $P^1\to P^2$ (since then $H^1(C)$ is a torsion submodule of P^1, and hence zero, and so $P^1\to P^2$ is a projective resolution of $H^2(C)$).

To do this it is turn enough to fix a bounded complex of finitely generated projective $\Lambda(\mathcal{G})$-modules $\cdots\to Q^0\xrightarrow{d^0} Q^1\xrightarrow{d^1} Q^2$ that is isomorphic in $D^{\rm p}(\Lambda(\mathcal{G}))$ to C and to prove that ${\rm cok}(d^0)$ is a projective $\Lambda(\mathcal{G})$-module.

To analyse ${\rm cok}(d^0)$ we note first that for each open normal subgroup V of $\mathcal{G}$ the isomorphism (18) implies that the complex of V-coinvariants ${\rm cok}(d^0_V)\xrightarrow{d^1_V} Q^2_V$ is isomorphic in $D(\mathbb{Z}_p[\mathcal{G}/V])$ to $R\Gamma_{\text{ét},T}(\mathcal{O}_{F_V,S},\mathbb{Z}_p(1))^-$. Since the $\mathbb{Z}_p[\mathcal{G}/V]$-module Q^2_V is projective, this isomorphism implies that ${\rm cok}(d^0_V)$ is $\mathbb{Z}_p$-free (as both $\ker(d^1_V)\cong H^1_T(\mathcal{O}_{F_V,S},\mathbb{Z}_p(1))^-=\mathcal{O}^{\times,-}_{F_V,S,T,p}$ and ${\rm im}(d^1_V)\subseteq Q^2_V$ are $\mathbb{Z}_p$-free) and also of finite projective dimension as a $\mathbb{Z}_p[\mathcal{G}/V]$-module (since $R\Gamma_{\text{ét},T}(\mathcal{O}_{F_V,S},\mathbb{Z}_p(1))^-$ belongs to $D^{\rm p}(\mathbb{Z}_p[\mathcal{G}/V])$). These facts then combine with [1, Th. 8] to imply that ${\rm cok}(d^0_V)$ is a projective $\mathbb{Z}_p[\mathcal{G}/V]$-module.

Upon passing to the limit over V (and noting that all involved modules are compact) one then deduces that $\mathrm{cok}(d^0) \cong \varprojlim_V \mathrm{cok}(d^0)_V = \varprojlim_V \mathrm{cok}(d^0_V)$ is a projective $\Lambda(\mathcal{G})$-module, as required to prove claim (i).

Turning to claim (ii), the result of Lemma 5.6 below implies this claim is true if for any given characteristic element ξ of $R\Gamma_{c,\text{ét}}(\mathcal{O}_{k,S}, \Lambda(\mathcal{G})^{\#}(1))^+$ one has for each representation ρ in $A^+(\mathcal{G})$ a containment of the form

$$f_{S,\rho}(u)/\Phi_\rho(\xi) \in \mathcal{O}[[u]]^\times, \tag{21}$$

with $\mathcal{O}$ the valuation ring of any sufficiently large finite Galois extension of $\mathbb{Q}_p$ in $\mathbb{Q}_p^c$.

Modulo a change of approach and notation, such containments were first deduced from Wiles' proof of the main conjecture for totally real fields by Ritter and Weiss in the course of the proof of [35, Th. 16]. However, for completeness, we now quickly indicate how the above containments can also be more directly deduced from Wiles' result by using Lemma 5.4.

We recall first that the series $f_{S,\rho}(u)$ are known to satisfy precise analogues of the properties given in Lemma 5.4(i)-(vi) (this is shown by Greenberg in [13]) and we shall often use this point without explicit comment. In particular, since the naturality of connecting homomorphisms in relative algebraic K-theory implies, in the notation of Lemma 5.4(vi), that $\xi_{\mathcal{U}/\mathcal{V}}$ is a characteristic element for the complex

$$\Lambda(\mathcal{U}/\mathcal{V}) \otimes^{\mathbb{L}}_{\Lambda(\mathcal{U})} R\Gamma_{c,\text{ét}}(U_{k,S}, \Lambda(\mathcal{G})^{\#}(1)) \cong R\Gamma_{c,\text{ét}}(U_{F^{\mathrm{cyc},\mathcal{U}},S}, \Lambda(\mathcal{U}/\mathcal{V})^{\#}(1)),$$

a standard Brauer induction argument allows one to assume in the sequel that the extension F/k is cyclic.

In this case we can choose a finite abelian CM extension F' of k for which $F' \cap k^{\mathrm{cyc}} = k$ and $F^{\mathrm{cyc}} = (F')^{\mathrm{cyc}}$. The group $\mathcal{G}$ is then canonically isomorphic to the direct product $G_{F'/k} \times \Gamma$ and, since each automorphism ι_ψ in Lemma 5.4(iii) preserves the group $\mathcal{O}[[u]]^\times$, after replacing F by F' and ρ by a suitable twist of the form $\rho \otimes \psi$ with ψ in $A(\Gamma)$ we can, and will, assume in the sequel that Γ is contained in the kernel of ρ and hence that ρ identifies with a linear character of the finite (cyclic) group $H := G_{F^{\mathrm{cyc}}/k^{\mathrm{cyc}}} \cong G_{F^{\ker(\rho)}/k}$.

Write $D(\rho)$ for the complex $R\Gamma_{c,\text{ét}}(\mathcal{O}_{k,S}, \Lambda(\mathcal{G})^{\#}(1))_\rho$. Then, under the present hypotheses, Lemma 5.1(i) implies $D(\rho)$ is acyclic outside degrees two and three and gives identifications

$$H^2(D(\rho)) = (T_\rho \otimes_{\mathbb{Z}_p} M_S^{\mathrm{cyc}})^H, \quad H^3(D(\rho)) = (T_\rho)_H$$

where M_S^{cyc} denotes the maximal abelian pro-p extension of F^{cyc} unramified outside S.

Thus, since Lemma 5.4(vii) implies $\Phi_\rho(\xi)$ is a characteristic element for $D(\rho)$, to prove (21) it is enough to show that the series

$$G_{S,\rho}(u) := f_{S,\rho}(u) \cdot \mathrm{ch}_{\mathcal{O}[[u]]}((T_\rho)_H)$$

differs from $\mathrm{ch}_{\mathcal{O}[[u]]}((T_\rho \otimes_{\mathbb{Z}_p} M_S^{\mathrm{cyc}})^H)$ by a unit of $\mathcal{O}[[u]]$, where we write $\mathrm{ch}_{\mathcal{O}[[u]]}(N)$ for the characteristic polynomial of any finitely generated torsion $\mathcal{O}[[u]]$-module N.

Now, the $\mathcal{O}[[u]]$-module $(T_\rho)_H$ is isomorphic to $\mathbb{Z}_p$ if $\rho = \mathbf{1}_{\mathcal{G}}$ and is otherwise finite and so $\mathrm{ch}_{\mathcal{O}[[u]]}((T_\rho)_H)$ is equal to u if $\rho = \mathbf{1}_{\mathcal{G}}$ and to 1 otherwise. Given this, the result of Wiles [44, Th. 1.3] asserts $\mathrm{ch}_{\mathcal{O}[[u]]}((T_\rho \otimes_{\mathbb{Z}_p} M_S^{\mathrm{cyc}})^H)$ is equal to the distinguished polynomial part of $G_{S,\rho}(u)$. It thus suffices to show the powers μ_ρ and μ'_ρ of the uniformising parameter of $\mathcal{O}$ that occur in the Weierstrass decompositions of $f_{S,\rho}(u)$ and $\mathrm{ch}_{\mathcal{O}[[u]]}((T_\rho\otimes_{\mathbb{Z}_p} M_S^{\mathrm{cyc}})^H)\mathrm{ch}_{\mathcal{O}[[u]]}((T_\rho)_H)^{-1}$, or equivalently of $f_{S,\rho}(u)$ and $\Phi_\rho(\xi)$, coincide.

To do this we decompose H as $H_p\times H'_p$ with H_p its Sylow p-subgroup and accordingly write ρ as a product $\rho_p\times\rho'_p$. Then there exists a finite set Υ of elements α of $G_{\mathbb{Q}_p}$ for which

$$\sum_{\alpha\in\Upsilon}\rho^\alpha = \mathrm{Ind}^H_{J_1\times H'_p}(\mathbf{1}_{J_1}\times\rho'_p) - \mathrm{Ind}^H_{J_2\times H'_p}(\mathbf{1}_{J_2}\times\rho'_p)$$

for suitable subgroups J_1 and J_2 of H_p. Thus, since Lemma 5.4(ii) implies $\mu_{\rho^\alpha}=\mu_\rho$ and $\mu'_{\rho^\alpha}=\mu'_\rho$ for α in $G_{\mathbb{Q}_p}$, the properties of Lemma 5.4(i) and (vi) combine to imply it is enough to consider the case that ρ has order prime to p and in this case the equality $\mu_\rho=\mu'_\rho$ is proved directly by Wiles in [44, Th. 1.4].

This completes the proof of Proposition 5.5. □

Lemma 5.6. *Assume F contains ζ_p and set $C' := R\Gamma_{c,\text{ét}}(\mathcal{O}_{k,S},\Lambda(\mathcal{G})^{\#}(1))^+$.*

(i) *Then C' belongs to $D^{\mathrm{p}}(\Lambda(\mathcal{G}))$ and $Q(\mathcal{G})\otimes_{\Lambda(\mathcal{G})}C'$ is acyclic.*

(ii) *Proposition 5.5(ii) is true provided that for any characteristic element ξ of C' and each representation ψ in $A^+(\mathcal{G})$ the quotient $f_{S,\psi}(u)/\Phi_\psi(\xi)$ belongs to $\mathcal{O}[[u]]^\times$ with $\mathcal{O}$ the valuation ring of any sufficiently large finite Galois extension of $\mathbb{Q}_p$ in $\mathbb{Q}_p^c$.*

Proof. We set $R:=\Lambda(\mathcal{G})$ and $Q:=Q(\mathcal{G})$ and for each R-module M we regard $M^*:=\mathrm{Hom}_R(M,R)$ as a (left) R-module by setting $x(\theta)(m):=\theta(m)x^*$ for all $x\in R$, $\theta\in M^*$ and $m\in M$. We observe that this induces an exact functor $E\mapsto E^*:=R\,\mathrm{Hom}_R(E,R[0])$ from $D^{\mathrm{p}}(R)$ to $D^{\mathrm{p}}(R)$ that is self-inverse.

Then, with this notation, the key point is that Artin-Verdier duality gives an isomorphism $R\Gamma_{\text{ét}}(\mathcal{O}_{k,S},\Lambda(\mathcal{G})^{\#}(1))[2]\cong R\Gamma_{c,\text{ét}}(\mathcal{O}_{k,S},\Lambda(\mathcal{G})^{\#}(1))^*\otimes_{\mathbb{Z}_p}\mathbb{Z}_p(1)$ in $D^{\mathrm{p}}(R)$ (where $\mathcal{G}$ acts diagonally on the tensor product) which in turn restricts to give an isomorphism

$$R\Gamma_{\text{ét}}(\mathcal{O}_{k,S},\Lambda(\mathcal{G})^{\#}(1))^-[2]\cong (C')^*\otimes_{\mathbb{Z}_p}\mathbb{Z}_p(1). \tag{22}$$

Claim (i) follows directly from this isomorphism and the fact that the left hand complex belongs to $D^{\mathrm{p}}(\Lambda(\mathcal{G}))$ and becomes acyclic after tensoring with $Q(\mathcal{G})$ (by the same argument as used in the proof of Proposition 5.5(i)).

To prove claim (ii) we use the complex C defined in Proposition 5.5. In particular, if we represent C by the complex $P^1\xrightarrow{d}P^2$ constructed in the proof of Proposition 5.5(i), then C_ρ is represented by $(T_\rho\otimes_{\mathbb{Z}_p}P^1)^{\mathcal{H}}\xrightarrow{d'}(T_\rho\otimes_{\mathbb{Z}_p}P^2)^{\mathcal{H}}$ with $\mathcal{H}$ the kernel of the projection map $\mathcal{G}\to\Gamma$ and the map $d':=(\mathrm{id}_{T_\rho}\otimes d)^{\mathcal{H}}$ is injective since d' is.

This shows that all but the final assertion of Proposition 5.5(ii) follow directly from Proposition 5.5(i). To complete the proof of claim (ii) it thus suffices to show that the validity of the containment (21) for each ρ in $A^+(\mathcal{G})$ implies that for each ρ in $A^-(\mathcal{G})$ the characteristic ideal of the torsion $\mathcal{O}[[u]]$-module $H^2(C_\rho)$ is generated by the element $f_{S,T,\rho}$ defined in claim (ii).

As a first step we note that Lemma 5.7 below combines with the results of Lemma 5.4(iv)-(vi) to imply that for each ρ in $A^-(\mathcal{G})$ the natural connecting homomorphism $Q(\mathcal{O}[[u]])^\times = K_1(Q(\mathcal{O}[[u]]))^\times\to K_0(\mathcal{O}[[u]],Q(\mathcal{O}[[u]]))$ sends the element

$$\Phi_\rho(\kappa_T\cdot\mathrm{tw}_{-1}(\xi^*))=\Phi_\rho(\kappa_T)\Phi_\rho(\mathrm{tw}_{-1}(\xi^*))=\kappa_{T,\rho}(u)\cdot\iota_{-1}\iota_*(\Phi_{\check\rho\omega_k}(\xi))$$

to $-\chi_{\mathcal{O}[[u]]}(C_\rho,0)$.

Taken in conjunction with the assumed containment (21) this then implies that for each such ρ the same is true of the product

$$\kappa_{T,\rho}(u)\cdot\iota_{-1}\iota_*(f_{S,\check\rho\omega_k}(u)) = \kappa_{T,\rho}(u)\cdot f_{S,\check\rho\omega_k}(\kappa_k(\gamma)(1+u)^{-1}-1) = f_{S,T,\rho}(u),$$

and, since C_ρ is acyclic outside degree two, our chosen normalisation (19) of characteristic element implies this last statement is equivalent to the final claim of Proposition 5.5(ii). $\square$

Lemma 5.7. *If ξ is any element as in Lemma 5.6(ii), then the product $\kappa_T\cdot\mathrm{tw}_{-1}(\xi^*)$ is a characteristic element for the complex C in Proposition 5.5.*

Proof. We use the module $R\otimes_{R,\mathrm{tw}_{-1}} R^\#(0)$ where the tensor product indicates that the first term R is regarded a right R-module via the homomorphism tw_{-1}. This module is endowed with commuting left actions of R (via left multiplication on the first factor) and G_k (via the given action on the second factor) and, with respect to these actions, the map $1\otimes g\mapsto\chi_k(g)g$ induces an isomorphism of $R\times\mathbb{Z}_p[G_k]$-modules $R\otimes_{R,\mathrm{tw}_{-1}} R^\#(0)\cong R^\#(1)$. This isomorphism in turn induces for the complex C' in Lemma 5.6 an isomorphism in $D^{\mathrm{p}}(R)$ of the form

$$R\otimes_{R,\mathrm{tw}_{-1}}(C')^*\cong(C')^*\otimes_{\mathbb{Z}_p}\mathbb{Z}_p(1).$$

Noting that the normalisation (19) implies ξ satisfies $-\partial_R(\xi^*) = \chi_R((C')^*,0)$ (since if ξ is represented by an automorphism α of Q, then ξ^* is represented by the automorphism of $Q\otimes_R R^*$ that is induced by α), the displayed isomorphism combines with the naturality of connecting homomorphisms in relative K-theory to imply

$$-\partial_R(\mathrm{tw}_{-1}(\xi^*))=\chi_R(R\otimes_{R,\mathrm{tw}_{-1}}(C')^*,0) = \chi_R((C')^*\otimes_{\mathbb{Z}_p}\mathbb{Z}_p(1),0)=\chi_R(R\Gamma_{\text{ét}}(\mathcal{O}_{k,S},\Lambda(\mathcal{G})^\#(1))^-,0)$$

where the last equality is a consequence of the isomorphism (22).

From the exact triangle (17) one also has an equality

$$\chi_R(C,0) = \chi_R(R\Gamma_{\text{ét}}(\mathcal{O}_{k,S},\Lambda(\mathcal{G})^\#(1))^-,0) - \sum_{v\in T}\chi_R(R\Gamma_{\text{ét}}(\kappa_v,\Lambda(\mathcal{G})^\#(1))^-,0).$$

We also note that, as each place v in T is prime to p, the complex $R\Gamma_{\text{ét}}(\kappa_v,\Lambda(\mathcal{G})^\#(1))$ is naturally isomorphic to a complex $R\to R$, where the first term is placed in degree zero and the differential is induced by right multiplication by the element $1-\mathrm{N}v\cdot\mathrm{Fr}_{w_v}^{-1}$, and hence that

$$\partial_R(e_-\cdot\kappa_T) = \sum_{v\in T}\chi_R(R\Gamma_{\text{ét}}(\kappa_v,\Lambda(\mathcal{G})^\#(1))^-,0).$$

Since $\partial_R(\kappa_T\cdot\mathrm{tw}_{-1}(\xi^*)) = \partial_R((e_-\cdot\kappa_T)\cdot\mathrm{tw}_{-1}(\xi^*)) = \partial_R(e_-\cdot\kappa_T)+\partial_R(\mathrm{tw}_{-1}(\xi^*))$ the claimed result is now a direct consequence of the last three displayed equalities. $\square$

5.4. In this section we introduce a canonical refinement of Gross's p-adic regulator map that fits into a commutative diagram of the form (8) and deduce an explicit description of the image of Gross's map.

At the outset we fix groups $\mathcal{G},\mathcal{H}$ and Γ as in §5.2. For any finite Galois extension L of k in F^{cyc} we write $\rho_{L/k}$ for the representation of $\mathcal{G}$ afforded by regarding $T:=\mathbb{Z}_p[G_{L/k}]$ as a left $\mathbb{Z}_p[G_{L/k}]\otimes_{\mathbb{Z}_p}\Lambda(\mathcal{G})$-module via the action $(x\otimes_{\mathbb{Z}_p} g)(t) = xtg_L^{-1}$ for each x and t in $\mathbb{Z}_p[G_{L/k}]$ and each g in $\mathcal{G}$ with image g_L in $G_{L/k}$.

Then for a bounded complex of finitely generated projective $\Lambda(\mathcal{G})$-modules C and an open normal subgroup U of $\mathcal{G}$ with $U \subseteq G_{F^{\rm cyc}/L}$ the complex $C_{U,(\rho_{L/k})}$ described in §5.1.2 identifies naturally with $\mathbb{Z}_p[G_{L/k}] \otimes_{\Lambda(\mathcal{G})} C$ and in each degree i we write

$$\beta^i_{C,L/k,\gamma} : H^i(\mathbb{Z}_p[G_{L/k}] \otimes_{\Lambda(\mathcal{G})} C) \to H^{i+1}(\mathbb{Z}_p[G_{L/k}] \otimes_{\Lambda(\mathcal{G})} C)$$

for the corresponding Bockstein homomorphism $\beta^i_{C,\rho_{L/k},\gamma}$ of $\mathbb{Z}_p[G_{L/k}]$-modules.

In this section we regard the sets S and T as fixed and for each extension E of F in $F^{\rm cyc}$ write C_E for the complex $R\Gamma_{\text{ét},T}(\mathcal{O}_{k,S}, \Lambda(G_{E/k})^{\#}(1))^-$ in $D^{\rm p}(\Lambda(G_{E/k}))$. For simplicity we also set $C_F^{\rm cyc} := C_{F^{\rm cyc}}$.

We can now state the main result that will be proved in this section.

Theorem 5.8. *Fix a topological generator γ of Γ and set $\beta^1_{F/k,\gamma} := \beta^1_{C_F^{\rm cyc},F/k,\gamma}$ and, for each character ρ in $\mathrm{Ir}_p^-(G)$, also $\beta^1_{\rho,\gamma} := \beta^1_{C_F^{\rm cyc},\rho,\gamma}$. Then each of the following assertions is valid.*

(i) *There is a commutative diagram of $\mathbb{Z}_p[G]$-modules*

$$\begin{CD} H^1(C_F) @>{-\log_p(\chi_k(\gamma))\cdot\beta^1_{F/k,\gamma}}>> H^2(C_F) \\ @V{\alpha^1_F}VV @VV{\alpha^2_F}V \\ \mathcal{O}^{\times,-}_{F,S,T,p} @>{\lambda_{F,S,p}}>> Y^-_{F,S,p}, \end{CD}$$

in which α^1_F and α^2_F are respectively the canonical identification and surjection that are induced by Lemma 4.1(ii) and the exact sequence (6).

(ii) *For any fixed topological generator γ_F of $\Delta := G_{F^{\rm cyc}/F}$ one has*

$$\lambda_{F,S,p}(\mathcal{O}^{\times,-}_{F,S,T,p}) = \log_p(\chi_F(\gamma_F)) \cdot \pi_F(H^2_T(\mathcal{O}_{F,S}, \Lambda(\Gamma_F)^{\#}(1))^{\Delta})^-,$$

with π_F the natural composite projection

$$H^2_T(\mathcal{O}_{F,S}, \Lambda(\Gamma_F)^{\#}(1)) \to H^2_T(\mathcal{O}_{F,S}, \mathbb{Z}_p(1)) \to Y_{F,S,p}.$$

(iii) *For each ψ in $\mathrm{Ir}_p^-(G)$ there is a commutative diagram*

$$\begin{CD} H^1(C_F)^{(\psi)} @>{\sim}>> H^1(C_{F,(\psi)}) @>{-\log_p(\chi_k(\gamma))\cdot\beta^1_{\psi,\gamma}}>> H^2(C_{F,(\psi)}) @>{\sim}>> H^2(C_F)_{(\psi)} \\ @V{(\alpha^1_F)^{(\psi)}}VV @. @. @VV{|G|\cdot(\alpha^2_F)_{(\psi)}}V \\ (\mathcal{O}^{\times,-}_{F,S,T,p})^{(\psi)} @>{\lambda^{\langle\psi\rangle}_{F,S,p}}>> @. @. (Y^-_{F,S,p})_{(\psi)} \end{CD}$$

in which the unlabeled arrows are the isomorphisms induced by Lemma 5.1(i) (with $C = C_F$ and $\Delta = G$).

The proof of this result occupies the next two subsections.

5.4.1. We first prove commutativity of the diagram in Theorem 5.8(i).

We set $d := [F : F\cap k^{\rm cyc}]$ and use the restriction map to identify $\Delta := G_{F^{\rm cyc}/F}$ with the subgroup of Γ of index d. Then $C^{\rm cyc}_{F,\rho_{F/k}}$ is isomorphic in $D^{\rm p}(\Lambda(\Gamma))$ to $\Lambda(\Gamma)\otimes_{\Lambda(\Delta)} C^{\rm cyc}_F$ and so Lemma 5.9 below implies an equality of homomorphisms

$$\log_p(\chi_k(\gamma))\cdot\beta^1_{F/k,\gamma} = \log_p(\chi_k(\gamma))\cdot(d\times\beta^1_{F,\gamma^d}) = \log_p(\chi_F(\gamma^d))\cdot\beta^1_{F,\gamma^d}.$$

It is therefore enough to prove the commutativity of the diagram in claim (i) after replacing the upper map by $\log_p(\chi_F(\gamma^d))\cdot\beta^1_{F,\gamma^d}$.

For each finite extension E of F in $F^{\rm cyc}$ and each v in $S\setminus S_k^\infty$ we set $E_v := \prod_w E_w$ where w runs over the set of places of E above v and we use the composite morphism

$$\lambda^v_E : C_E \to R\Gamma_{\text{ét}}(\mathcal{O}_{E,S},\mathbb{Z}_p(1)) \to R\Gamma_{\text{ét}}(E_v,\mathbb{Z}_p(1)),$$

where the first morphism is induced by the exact triangle in Lemma 4.1(i) and the second is the natural localisation morphism. These morphisms are compatible with change of E and so, by passing to the inverse limit over subfields E of $F^{\rm cyc}$ that are of finite degree over F, one obtains a morphism of exact triangles of $\mathbb{Z}_p$-modules

$$\begin{array}{ccccccc} C^{\rm cyc}_F & \xrightarrow{\gamma^d-1} & C^{\rm cyc}_F & \to & C_F & \to & C^{\rm cyc}_F[1] \\ \lambda^{\rm cyc}_{F_v}\big\downarrow & & \lambda^{\rm cyc}_{F_v}\big\downarrow & & \lambda^v_F\big\downarrow & & \lambda^{\rm cyc}_{F_v}[1]\big\downarrow \\ C^{\rm cyc}_{F_v} & \xrightarrow{\gamma^d-1} & C^{\rm cyc}_{F_v} & \to & R\Gamma_{\text{ét}}(F_v,\mathbb{Z}_p(1)) & \to & C^{\rm cyc}_{F_v}[1]. \end{array} \tag{23}$$

Here we fix a place w of F above v and set $C^{\rm cyc}_{F_v} := \Lambda(\Delta)\otimes_{\Lambda(\Delta_w)} C^{\rm cyc}_{F_w}$ with Δ_w denoting the decomposition subgroup of w in Δ and $C^{\rm cyc}_{F_w}$ the complex $R\Gamma_{\text{ét}}(F^{\rm cyc}_w,\mathbb{Z}_p(1))$ where $F^{\rm cyc}_w$ is the p-cyclotomic extension of F_w in F^c_w.

In addition, setting $d_w := d\cdot[\Delta:\Delta_w]$, the result of Lemma 5.9 below applies (with Γ,Δ,C and γ replaced by $\Delta,\Delta_w,C^{\rm cyc}_{F_w}$ and γ^d) to give an equality of homomorphisms

$$\log_p(\chi_F(\gamma^d))\cdot\beta^1_{C^{\rm cyc}_{F_v},\gamma^d} = \log_p(\chi_F(\gamma^d))\cdot([\Delta:\Delta_w]\times\beta^1_{C^{\rm cyc}_{F_w},\gamma^{d\cdot[\Delta:\Delta_w]}}) = \log_p(\chi_{F_w}(\gamma^{d_w}))\cdot\beta^1_{C^{\rm cyc}_{F_w},\gamma^{d_w}}.$$

The morphism of triangles (23) thus induces a commutative diagram of $\mathbb{Z}_p$-modules

$$\begin{array}{ccccc} H^1(C_F) & \xrightarrow{H^1(\lambda^v_F)} & H^1_{\text{ét}}(F_v,\mathbb{Z}_p(1)) & \xrightarrow{\cong} & \prod_{w\mid v} H^1_{\text{ét}}(F_w,\mathbb{Z}_p(1)) \\ \big\downarrow{\scriptstyle \log_p(\chi_F(\gamma^d))\cdot\beta^1_{F,\gamma^d}} & & \big\downarrow{\scriptstyle \log_p(\chi_F(\gamma^d))\cdot\beta^1_{C^{\rm cyc}_{F_v},\gamma^d}} & & \big\downarrow{\scriptstyle (\log_p(\chi_{F_w}(\gamma^{d_w}))\cdot\beta^1_{C^{\rm cyc}_{F_w},\gamma^{d_w}})_{w\mid v}} \\ H^2(C_F) & \xrightarrow{H^2(\lambda^v_F)} & H^2_{\text{ét}}(F_v,\mathbb{Z}_p(1)) & \xrightarrow{\cong} & \prod_{w\mid v} H^2_{\text{ét}}(F_w,\mathbb{Z}_p(1)) \end{array} \tag{24}$$

in which both products are over all places w of F above v and, with respect to the identification $H^1(C_F)\cong\mathcal{O}^{\times,-}_{F,S,T,p}$ and canonical surjection $H^2(C_F)\to Y^-_{F,S,p}$ induced by Lemma 4.1(ii) and (6) and the natural identifications of $H^1_{\text{ét}}(F_w,\mathbb{Z}_p(1))$ and $H^2_{\text{ét}}(F_w,\mathbb{Z}_p(1))$ with the pro-p-completion $\mathbb{Z}_p\hat{\otimes}F_w^\times$ of $F_w^\times$ and $\mathbb{Z}_p$ respectively, the upper composite homomorphism is induced by the natural localisation map $\mathcal{O}^\times_{F,S,T,p}\subset F^\times\to\prod_{w\mid v}F_w^\times$ and the lower composite homomorphism by the map $Y_{F,S,p}\to\prod_{w\mid v}\mathbb{Z}_p$ which sends each element $(n_{w'})_{w'\in S_F}$ to $(n_w)_{w\mid v}$.

To describe each map $\beta^1_w := \beta^1_{C^{\rm cyc}_{F_w},\gamma^{d_w}}$ explicitly we recall the argument of Rapoport and Zink in [34, 1.2] (see also [6, §3.2.1]) implies this map is equal to cup product with the element of $H^1_{\text{ét}}(F_w,\mathbb{Z}_p) = \mathrm{Hom}_{\rm cont}(G_{F^c_w/F_w},\mathbb{Z}_p)$ obtained by composing the projection $G_{F^c_w/F_w} \twoheadrightarrow G_{F^{\rm cyc}_w/F_w}$ together with the continuous homomorphism $G_{F^{\rm cyc}_w/F_w} \to \mathbb{Z}_p$ that sends γ^{d_w} to 1. In particular, since cup products commute with corestriction, one has a commutative diagram

$$\begin{array}{ccc} H^1_{\text{ét}}(F_w,\mathbb{Q}_p(1)) & \xrightarrow{\kappa^1} & H^1_{\text{ét}}(\mathbb{Q}_{\ell(w)},\mathbb{Q}_p(1)) \\ {\scriptstyle\beta^1_w}\big\downarrow & & \big\downarrow{\scriptstyle e(\gamma_w)^{-1}\cdot\beta^1_{\gamma_w}} \\ H^2_{\text{ét}}(F_w,\mathbb{Q}_p(1)) & \xrightarrow{\kappa^2} & H^2_{\text{ét}}(\mathbb{Q}_{\ell(w)},\mathbb{Q}_p(1)) \end{array} \tag{25}$$

in which we write $\ell(w)$ for the residue characteristic of w, κ^1 and κ^2 for the natural corestriction maps, $e(\gamma_w)$ for the p-adic integer defined by the condition that γ^{d_w} acts on $\mathbb{Q}^{\rm cyc}_{\ell(w)}$ as the $e(\gamma_w)$-th power of a fixed topological generator γ_w of $G_{\mathbb{Q}^{\rm cyc}_{\ell(w)}/\mathbb{Q}_{\ell(w)}}$ and $\beta^1_{\gamma_w}$ for the Bockstein homomorphism in degree one of the data $(R\Gamma_{\text{ét}}(\mathbb{Q}^{\rm cyc}_{\ell(w)},\mathbb{Z}_p(1)),\gamma_w)$.

Now, with respect to the identifications $H^1_{\text{ét}}(F_w,\mathbb{Q}_p(1)) \cong \mathbb{Q}_p\hat{\otimes}F^\times_w$ and $H^2_{\text{ét}}(F_w,\mathbb{Q}_p(1)) \cong \mathbb{Q}_p$ (and similarly with F_w replaced by $\mathbb{Q}_{\ell(w)}$), the maps κ^1 and κ^2 in (25) are induced by $\mathrm{N}_{F_w/\mathbb{Q}_{\ell(w)}}$ and the identity map respectively whilst an explicit exercise in local class field theory (as in Kato [25, Chap. II, Lem. 1.4.5]) shows $\beta^1_{\gamma_w}$ is induced by the map $\log_p(\chi_{\mathbb{Q}_{\ell(w)}}(\gamma_w))^{-1}\cdot\log_p$ if $\ell(w)\neq p$ and by -1 times this map if $\ell(w)=p$. The diagram (25) therefore implies that β^1_w is equal to the composite

$$\begin{aligned} &-(-1)^{a(w)}e(\gamma_w)^{-1}\log_p(\chi_{\mathbb{Q}_{\ell(w)}}(\gamma_w))^{-1}\cdot\log_p\circ\mathrm{N}_{F_w/\mathbb{Q}_{\ell(w)}} \\ &\qquad\qquad = -\log_p(\chi_{F_w}(\gamma^{d_w}))^{-1}\cdot((-1)^{a(w)}\times\log_p\circ\mathrm{N}_{F_w/\mathbb{Q}_{\ell(w)}}) \end{aligned}$$

with $a(w)=0$ if $\ell(w)=p$ and $a(w)=1$ if $\ell(w)\neq p$. In addition, for x in $\mathbb{Z}_p\hat{\otimes}F^\times_w$ one has

$$(-1)^{a(w)}\times\log_p(\mathrm{N}_{F_w/\mathbb{Q}_p}(x)) = \log_p((\mathrm{N}w)^{-\mathrm{val}_w(x)}\cdot\mathrm{N}_{F_w/\mathbb{Q}_p}(x))$$

if $\ell(w)=p$ (since then $\log_p(\mathrm{N}w)=0$), and

$$(-1)^{a(w)}\times\log_p(\mathrm{N}_{F_w/\mathbb{Q}_{\ell(w)}}(x)) = \log_p(\mathrm{N}_{F_w/\mathbb{Q}_{\ell(w)}}(\pi_w^{-\mathrm{val}_w(x)})) = \log_p((\mathrm{N}w)^{-\mathrm{val}_w(x)})$$

if $\ell(w)\neq p$ and π_w is any choice of uniformising parameter of F_w.

Given this explicit description of $\beta^1_w = \beta^1_{C^{\rm cyc}_{F_w},\gamma^{d_w}}$, the commutativity of the diagram in Theorem 5.8(i) follows directly from the commutativity of (24), the definition of $\lambda_{F,S,p}$ and the fact that (as observed by Gross in [17, (1.8)]) an explicit description of the local reciprocity map combines with the commutative diagram (3) to imply that for each x in $F^\times_w$ one has

$$||x||_{w,p} = \begin{cases} (\mathrm{N}w)^{-\mathrm{val}_w(x)}\cdot\mathrm{N}_{F_w/\mathbb{Q}_p}(x), & \text{if } \ell(w)=p \\ (\mathrm{N}w)^{-\mathrm{val}_w(x)}, & \text{otherwise.} \end{cases}$$

Lemma 5.9. *Let C be an object of $D^{\rm p}(\Lambda(\Gamma))$. Then for each open subgroup Δ of Γ there is a natural exact triangle in $D^{\rm p}(\Lambda(\Gamma))$*

$$\Lambda(\Gamma)\otimes_{\Lambda(\Delta)}C \xrightarrow{1-\gamma} \Lambda(\Gamma)\otimes_{\Lambda(\Delta)}C \xrightarrow{\pi} C_\Delta \to \Lambda(\Gamma)\otimes_{\Lambda(\Delta)}C[1],$$

in which γ acts as multiplication on the first factor in each tensor product. In each degree i the bockstein homomorphism $H^i(C_\Delta) \to H^{i+1}(C_\Delta)$ of this triangle coincides with $[\Gamma:\Delta]\times\beta^i_{C,\gamma^{[\Gamma:\Delta]}}$.

Proof. The existence of the displayed exact triangle is clear.

It is also clear that, setting $d := [\Gamma:\Delta]$, there is a morphism of exact triangles of $\mathbb{Z}_p$-modules

$$\begin{tikzcd}
\Lambda(\Gamma)\otimes_{\Lambda(\Delta)} C \arrow[r, "1-\gamma"] & \Lambda(\Gamma)\otimes_{\Lambda(\Delta)} C \arrow[r, "\pi"] & C_\Delta \arrow[r] & \Lambda(\Gamma)\otimes_{\Lambda(\Delta)} C[1] \\
C \arrow[u, "t_d"] \arrow[r, "1-\gamma^d"] & C \arrow[u, "1\otimes\mathrm{id}"] \arrow[r, "\pi'"] & C_\Delta \arrow[u, "\mathrm{id}"] \arrow[r] & C[1] \arrow[u, "t_d[1]"]
\end{tikzcd}$$

in which, in each degree j, the map t^j_d sends x in C^j to $\sum_{i=0}^{i=d-1}\gamma^i(1\otimes x) \in \Lambda(\Gamma)\otimes_{\Lambda(\Delta)} C^j$, π^j sends $x\otimes y^j$ in $\Lambda(\Gamma)\otimes_{\Lambda(\Delta)} C^j$ to $x(y^j_\Delta)\in C^j_\Delta$ with y^j_Δ the image of y^j in C^j_Δ and π' is the natural projection.

Thus, by passing to cohomology this morphism of triangles gives a commutative diagram

$$\begin{tikzcd}
H^i(C_\Delta) \arrow[r] & H^{i+1}(\Lambda(\Gamma)\otimes_{\Lambda(\Delta)} C)^\Gamma \arrow[r, "H^{i+1}(\pi)"] & H^{i+1}(C_\Delta) \\
H^i(C_\Delta) \arrow[u, equal] \arrow[r] & H^{i+1}(C)^\Delta \arrow[u, "H^{i+1}(t_d)"] \arrow[r, "H^{i+1}(\pi')"] & H^{i+1}(C_\Delta) \arrow[u, "\times d"']
\end{tikzcd}$$

and this implies the claimed result since the lower composite homomorphism in this diagram is equal to β^i_{C,γ^d}. □

5.4.2. We now complete the proof of Theorem 5.8 by deriving claims (ii) and (iii) as consequences of claim (i).

Regarding claim (ii) the commutativity of the diagram in claim (i) combines with the surjectivity of the map $\tilde\alpha^i$ in (16) to give an equality

$$\lambda_{F,S,p}(\mathcal{O}^{\times,-}_{F,S,T,p}) = \alpha^2_F(\mathrm{im}(\log_p(\chi_F(\gamma_F))\cdot\beta^1_{F/k,\gamma})) = \log_p(\chi_F(\gamma_F))\cdot\pi_{F/k}(H^2(C^{\mathrm{cyc}}_{F,\rho_{F/k}})^\Gamma),$$

with $\pi_{F/k}$ the natural composite homomorphism

$$H^2(C^{\mathrm{cyc}}_{F,\rho_{F/k}})^\Gamma \subseteq H^2(C^{\mathrm{cyc}}_{F,\rho_{F/k}}) \to H^2(\mathbb{Z}_p\otimes^{\mathbb{L}}_{\Lambda(\Gamma)} C^{\mathrm{cyc}}_{F,\rho_{F/k}}) \cong H^2_T(\mathcal{O}_{F,S},\mathbb{Z}_p(1)) \to Y_{F,S,p}.$$

The equality in claim (ii) is the special case $F=k$ of this equality.

To prove claim (iii) we use the subgroups $\mathcal{U} := G_{F^{\mathrm{cyc}}/F}$ and $\mathcal{H} := G_{F^{\mathrm{cyc}}/k^{\mathrm{cyc}}}$ of $\mathcal{G} = G_{F^{\mathrm{cyc}}/k}$ and the representative $P^1 \xrightarrow{d} P^2$ of C^{cyc}_F constructed in the proof of Proposition 5.5(i).

We note in particular that if $\rho : \mathcal{G} \to \mathrm{Aut}_R(T_\rho)$ is any homomorphism as in §5.1.2 which factors through the surjection $\mathcal{G}\to G$, then the map $\beta^1_{C^{\mathrm{cyc}}_F,\rho,\gamma} : H^1(C_{F,(\rho)}) \to H^2(C_{F,(\rho)})$ can be computed as $\pi_\rho\circ\kappa^1_\rho$ with κ^1_ρ the connecting homomorphism $H^1(C_{F,(\rho)}) \to (T_\rho\otimes_{\mathbb{Z}_p} H^2(C^{\mathrm{cyc}}_F))_{\mathcal{H}}$ in the exact commutative diagram of R-modules

$$\begin{tikzcd}
& (T_\rho \otimes_{\mathbb{Z}_p} P^1)_{\mathcal{H}} \arrow[r, hook, "(\mathrm{id}_{T_\rho}\otimes_{\mathbb{Z}_p} d)_{\mathcal{H}}"] \arrow[d, hook, "1-\gamma"] & (T_\rho \otimes_{\mathbb{Z}_p} P^2)_{\mathcal{H}} \arrow[r, two heads] \arrow[d, hook, "1-\gamma"] & (T_\rho \otimes_{\mathbb{Z}_p} H^2(C_F^{\mathrm{cyc}}))_{\mathcal{H}} \arrow[d, "1-\gamma"] \\
& (T_\rho \otimes_{\mathbb{Z}_p} P^1)_{\mathcal{H}} \arrow[r, hook, "(\mathrm{id}_{T_\rho}\otimes_{\mathbb{Z}_p} d)_{\mathcal{H}}"] \arrow[d, two heads] & (T_\rho \otimes_{\mathbb{Z}_p} P^2)_{\mathcal{H}} \arrow[r, two heads] \arrow[d, two heads] & (T_\rho \otimes_{\mathbb{Z}_p} H^2(C_F^{\mathrm{cyc}}))_{\mathcal{H}} \arrow[d, two heads, "\pi_\rho"] \\
H^1(C_{F,(\rho)}) \arrow[r, hook, "\iota_\rho"] & T_\rho \otimes_{\mathbb{Z}_p[G]} P^1_{\mathcal{U}} \arrow[r, "\mathrm{id}_{T_\rho}\otimes_{\mathbb{Z}_p[G]} d_{\mathcal{U}}"] & T_\rho \otimes_{\mathbb{Z}_p[G]} P^2_{\mathcal{U}} \arrow[r, two heads] & H^1(C_{F,(\rho)})
\end{tikzcd}$$

where the maps ι_ρ and π_ρ are defined via the identifications $H^1(C_{F,(\rho)}) = \ker(\mathrm{id}_{T_\rho} \otimes_{\mathbb{Z}_p[G]} d_{\mathcal{U}})$ and $H^2(C_{F,(\rho)}) \cong H^2(C_F)_{(\rho)} \cong (T_\rho \otimes_{\mathbb{Z}_p} H^2(C_F^{\mathrm{cyc}}))_{\mathcal{G}}$.

Note that if $\rho = \rho_{F/k}$ (so $R = T_\rho = \mathbb{Z}_p[G]$), then $\beta^1_{C_F^{\mathrm{cyc}},\rho,\gamma} = \beta^1_{F/k,\gamma}$ and there is a natural identification of the $\mathbb{Z}_p[G]$-module $H^1(C_{F,(\rho)})$ with $H^1(C_F)$.

On the other hand, the above description means that for any ψ in $\mathrm{Ir}_p^-(G)$ one can compute the upper row of the diagram in claim (iii) by first identifying the sublattice

$$H^1(C_F)^{(\psi)} = (T_\psi \otimes_{\mathbb{Z}_p} \ker(d_{\mathcal{U}}))^G = \ker((\mathrm{id}_{T_\psi} \otimes_{\mathbb{Z}_p} d_{\mathcal{U}})^G) \subseteq (T_\psi \otimes_{\mathbb{Z}_p} P^1_{\mathcal{U}})^G = \mathrm{Tr}_G(T_\psi \otimes_{\mathbb{Z}_p} P^1_{\mathcal{U}})$$

of $V_\psi \otimes_{\mathbb{Z}_p[G]} H^1(C_F) \subseteq \mathbb{Q}_p^c \cdot H^1(C_F)$ with $H^1(C_{F,(\psi)})$ via the map sending each element $\mathrm{Tr}_G(x)$ of $\ker((\mathrm{id}_{T_\psi} \otimes_{\mathbb{Z}_p} d_{\mathcal{U}})^G)$ to the image $\overline{x}$ of x in $\ker(\mathrm{id}_{T_\psi} \otimes_{\mathbb{Z}_p[G]} d_{\mathcal{U}}) \subseteq (T_\psi \otimes_{\mathbb{Z}_p} P^1_{\mathcal{U}})_G = T_\psi \otimes_{\mathbb{Z}_p[G]} P^1_{\mathcal{U}}$ and then applying $\pi_\psi \circ \kappa^1_\psi$ to $\overline{x}$.

In particular, since the image of $\mathrm{Tr}_G(x)$ in $T_\psi \otimes_{\mathbb{Z}_p[G]} P^1_{\mathcal{U}}$ is equal to $|G| \cdot \overline{x}$ this gives a commutative diagram of $\mathcal{O}_\psi$-modules

$$\begin{tikzcd}
H^1(C_F)^{(\psi)} \arrow[r, "\sim"] \arrow[d, "\mathrm{id}"'] & H^1(C_{F,(\psi)}) \arrow[r, "\beta^1_{\psi,\gamma}"] & H^2(C_{F,(\psi)}) \arrow[r, "\sim"] & H^2(C_F)_{(\psi)} \arrow[d, "\times|G|"] \\
H^1(C_F)^{(\psi)} \arrow[rrr, "(\beta^1_{F/k,\gamma})^{\langle\psi\rangle}"] & & & H^2(C_F)_{(\psi)}
\end{tikzcd}$$

and the commutativity of the diagram in claim (iii) follows directly by combining this diagram with that in claim (i).

This completes the proof of Theorem 5.8.

5.5. In this subsection we combine the results obtained above in order to prove Theorem 5.2.

To do this we set $C := R\Gamma_{\text{ét},T}(\mathcal{O}_{k,S}, \Lambda(\mathcal{G})^{\#}(1))^-$ and note that (by a standard Shapiro Lemma argument) we may, and will, identify the complex C_F defined just prior to Theorem 5.8 with $R\Gamma_{\text{ét},T}(\mathcal{O}_{F,S}, \mathbb{Z}_p(1))^-$.

We also fix a character ψ in $\mathrm{Ir}_p^-(G)$ and a corresponding representation $\mathcal{G} \to \mathrm{Aut}_{\mathcal{O}}(T_\psi)$ in $A^-(\mathcal{G})$ which we continue to denote by ψ and we write E for the fraction field of $\mathcal{O}$.

Then, since C_ψ is acyclic outside degree two (by Proposition 5.5(ii)), the exact sequence (16) induces an isomorphism of $\mathcal{O}$-modules $H^1(C_F)_{(\psi)} \cong H^2(C_\psi)^\Gamma$.

Hence, as u annihilates $H^2(C_\psi)^\Gamma$, the tautological short exact sequence

$$0 \to H^2(C_\psi)^\Gamma \to H^2(C_\psi) \to H^2(C_\psi)/H^2(C_\psi)^\Gamma \to 0$$

implies that the maximal power $r^{\mathrm{alg}}_{p,S,\psi}$ of u that divides $\mathrm{ch}_{\mathcal{O}[[u]]}(H^2(C_\psi))$ satisfies

$$r^{\mathrm{alg}}_{p,S,\psi} \geq \dim_E(E \cdot H^2(C_\psi)^\Gamma) = \dim_E(E \cdot H^1(C_F)_{(\psi)}),$$

with equality if and only if u acts injectively on the quotient $H^2(C_\psi)/H^2(C_\psi)^\Gamma$. In particular, since the $\mathbb{Q}_p[G]$-modules $\mathbb{Q}_p \cdot H^1(C_F)$ and $\mathbb{Q}_p \cdot H^2(C_F)$ are isomorphic one therefore has

$$r^{\mathrm{alg}}_{p,S,\psi} \geq \dim_E(E \cdot H^1(C_F)_{(\psi)}) = \dim_E(E \cdot H^2(C_F)_{(\psi)}) = \dim_E(E \cdot Y_{F,S,(\psi)}) = r_{S,\psi}$$

where the last equality is a direct consequence of (1).

Claim (i) now follows from the last displayed inequality because Proposition 5.5(ii) combines with the interpolation property (20) to imply that $r^{\mathrm{alg}}_{p,S,\psi}$ coincides with $r_{p,S,\psi}$.

Next we note that, since C_F is acyclic outside degrees one and two (by Lemma 4.1(ii)), the complex C is semisimple at ψ if and only if the map $\beta^1_{C,\psi,\gamma}$ is bijective. In addition, as the $\mathbb{Q}_p[G]$-modules $\mathbb{Q}_p \cdot H^1(C_F)$ and $\mathbb{Q}_p \cdot H^2(C_F)$ are isomorphic, the map $\beta^1_{C,\psi,\gamma}$ is bijective if and only if it is injective. Given this, the equivalence of (b) and (c) in claim (ii) follows directly by combining the explicit description of Theorem 5.8(ii) with the obvious fact that $\mathscr{L}_S(\psi) \neq 0$ if and only if the map $\lambda^{\langle\psi\rangle}_{F,S,p}$ is injective.

The discussion in §5.1.2 also shows that the complex C is semisimple at ψ if and only if the tautological map $E \cdot H^1(C_\psi)^\Gamma \to E \cdot H^1(C_\psi)_\Gamma$ is bijective, or equivalently the element $u = \gamma - 1$ acts invertibly on the quotient module $E \cdot (H^1(C_\psi)/H^2(C_\rho)^\Gamma)$. This implies that the equivalence of the conditions (a) and (b) in claim (ii) also follows directly from the discussion above.

To proceed we now set

$$\lambda_{F,S,T,p} := -\log_p(\chi_k(\gamma)) \cdot \beta^1_{F/k,\gamma}$$

and note that Theorem 5.8(i) implies this homomorphism lies in a commutative diagram of the form (8). To prove claim (iii) it is thus enough to show that this choice of $\lambda_{F,S,T,p}$ also satisfies the equality in Theorem 3.1(ii) for the character ψ.

To do this we assume first that $\beta^1_{C,\psi,\gamma}$ is not injective. In this case the map $\lambda^{\langle\psi\rangle}_{F,S,T,p}$ is not injective, and hence the $\mathcal{O}$-module $\mathrm{cok}(\lambda^{\langle\psi\rangle}_{F,S,p})$ is not torsion, whilst the above proof of claim (ii) implies $r_{p,S,\psi} \neq r_{S,\psi}$ so that (by claim (i)) $L^{r_{S,\psi}}_{p,S,T}(\check{\psi}\omega_k, 0) = 0$. In this case, therefore, the equality of Theorem 3.1(ii) is satisfied trivially since both sides are zero.

Assuming now $\beta^1_{C,\psi,\gamma}$ is injective, we derive the claimed equality by combining Proposition 5.5(ii) with the computation of generalised Euler characteristics given in [6, Prop. 3.19]. More precisely, since $C^{(\psi)}_F$ is acyclic outside degrees one and two and $\beta^1_{C,\psi,\gamma}$ is injective one finds in this way a formula

$$\mathrm{Fit}_{\mathcal{O}}(\mathrm{cok}(\beta^1_{C,\psi,\gamma})) = f^*_{S,T,\psi}(0) \cdot \mathcal{O} \tag{26}$$

where $f^*_{S,T,\psi}(0)$ is the leading term at $u = 0$ of the series $f_{S,T,\psi}(u)$.

In addition, in this case the proof of claim (ii) and the formula (1) together imply that $r_{p,S,\psi} = \dim_E(E\cdot H^1(C_F^{(\psi)}))$ and so the explicit description of Theorem 5.8(ii) combines with Lemma 5.1(iii) to give a formula

$$\mathrm{Fit}_{\mathcal{O}}(\mathrm{cok}(\beta^1_{C,\psi,\gamma})) = \log_p(\kappa_k(\gamma))^{-r_{p,S,\psi}}|G|^{-r_{p,S,\psi}}\mathrm{Fit}_{\mathcal{O}}(\mathrm{cok}(\lambda^{\langle\psi\rangle}_{F,S,T,p})).$$

In this case the equality of Theorem 3.1(ii) is now obtained by comparing the last equality to (26) and then using the following explicit computation of the leading term.

Lemma 5.10. $f^*_{S,T,\psi}(0) = \log_p(\kappa_k(\gamma))^{-r_{p,S,\psi}} L^*_{p,S,T}(\check{\psi}\omega_k, 0)$.

Proof. We set $\tilde{f}_{S,\psi}(u) := f_{S,\check{\psi}\omega_k}(\kappa_k(\gamma)(1+u)^{-1}-1)$ and $r_p := r_{p,S,\psi}$. Then the interpolation property (20) implies $\tilde{f}_{S,\psi}(u)$ vanishes to order r_p at $u=0$ and hence that

$$\begin{aligned}
\tilde{f}^*_{S,\psi}(0) &= \lim_{u\to 0} u^{-r_p} f_{S,\check{\psi}\omega_k}(\kappa_k(\gamma)(1+u)^{-1}-1)\\
&= (-1)^{r_p}\lim_{u\to 0} u^{-r_p} f_{S,\check{\psi}\omega_k}(\kappa_k(\gamma)(1+u)-1)\\
&= (-1)^{r_p}\lim_{s\to 0}(\kappa_k(\gamma)^s-1)^{-r_p} f_{S,\check{\psi}\omega_k}(\kappa_k(\gamma)^{1+s}-1)\\
&= (-1)^{r_p}\log_p(\kappa_k(\gamma))^{-r_p}\lim_{s\to 0} s^{-r_p} L_{p,S}(\check{\psi}\omega_k,-s)\\
&= \log_p(\kappa_k(\gamma))^{-r_p}\lim_{s\to 0} s^{-r_p} L_{p,S}(\check{\psi}\omega_k,s)\\
&= \log_p(\kappa_k(\gamma))^{-r_p} L^*_{p,S}(\check{\psi}\omega_k,0).
\end{aligned}$$

In addition, an explicit computation shows that the value of $\kappa_{T,\psi}(u) := \Phi_\psi(\kappa_T)$ at $u=0$ is equal to $\prod_{v\in T}\det(1-\mathrm{N}t\cdot\mathrm{Fr}_{w_v}^{-1}\mid V_\psi)\neq 0$ and so the definition of $L_{p,S,T}(\check{\psi}\omega_k,s)$ in (7) implies directly that $L^*_{p,S,T}(\check{\psi}\omega_k,0) = \kappa_{T,\psi}(0)L^*_{p,S}(\check{\psi}\omega_k,0)$.

Since $f_{S,T,\psi}(u)$ is defined to be $\kappa_{T,\psi}(u)\tilde{f}_{S,\psi}(u)$ one therefore has

$$\begin{aligned}
f^*_{S,T,\psi}(0) = \kappa_{T,\psi}(0)\tilde{f}^*_{S,\psi}(0) &= \log_p(\kappa_k(\gamma))^{-r_p}\kappa_{T,\psi}(0)L^*_{p,S}(\check{\psi}\omega_k,0)\\
&= \log_p(\kappa_k(\gamma))^{-r_p}L^*_{p,S,T}(\check{\psi}\omega_k,0),
\end{aligned}$$

as claimed. □

Turning to claim (iv) the isomorphism $R\Gamma_c(\mathcal{O}_{F,S},\mathbb{Z}_p)^*[-3]^-\cong R\Gamma(\mathcal{O}_{F,S},\mathbb{Z}_p(1))^-$ induced by the vertical exact triangle in (13) combines with Lemma 4.1(iv) to imply the equivariant Tamagawa number conjecture for $(h^0(\mathrm{Spec}(F),\mathbb{Z}_p[G]^-)$ is valid if and only if for any isomorphism of fields $j:\mathbb{C}\cong\mathbb{C}_p$ there is an equality in $K_0(\mathbb{Z}_p[G]^-,\mathbb{C}_p[G]^-)$

$$\delta_{\mathbb{Z}_p[G]^-,\mathbb{C}_p[G]^-}\Big(\sum_{\psi\in\mathrm{Ir}^-(G)} e_{\psi^j}L^*_{S,T}(\check{\psi},0)^j\Big) = -\chi_{\mathbb{Z}_p[G]^-,\mathbb{C}_p[G]^-}(C_F,R_{S,j}), \tag{27}$$

where $R_{S,j}$ denotes the exact sequence of $\mathbb{C}_p[G]$-modules

$$0\to\mathbb{C}_p\cdot H^1(C_F)\xrightarrow{\mathbb{C}_p\otimes_{\mathbb{R},j}R_{F,S}}\mathbb{C}_p\cdot H^2(C_F)\to 0. \tag{28}$$

Thus, if ψ validates the Strong Stark Conjecture, then [2, Remark 6.1.1(iii) and Lem. A.3] together imply that for each such j one has

$$L^*_{S,T}(\check{\psi},0)^j\cdot\wedge^{r_\psi}_{\mathcal{O}_j}H^2(C_F)_{(\psi^j),\mathrm{tf}} = |G|^{-r_{S,\psi}}\mathrm{Fit}_{\mathcal{O}}(H^2(C_F)_{(\psi^j),\mathrm{tor}})\cdot\wedge^{r_\psi}_{\mathcal{O}_j}(R_{S,j}(H^1(C_F))^{(\psi^j)})$$

where we set $r_\psi := r_{S,\psi}$, $\mathcal{O} := \mathcal{O}_\psi$ and $\mathcal{O}_j := \mathcal{O}_{\psi^j}$, or equivalently

$$(L^*_{S,T}(\check{\psi},0)/R_S(\psi))^j\cdot\wedge^{r_\psi}_{\mathcal{O}_j}H^2(C_F)_{(\psi^j),\mathrm{tf}} = |G|^{-r_\psi}\mathrm{Fit}_{\mathcal{O}_j}(H^2(C_F)_{(\psi^j),\mathrm{tor}})\cdot\wedge^{r_\psi}_{\mathcal{O}_j}(\phi_p(H^1(C_F))^{(\psi^j)})$$

with $\phi_p := \mathbb{Z}_p\otimes_{\mathbb{Z}}\phi_{F,S}$.

In addition, as $H^1(C_F)^{(\psi^j)}$ is torsion-free, there is an exact diagram of $\mathcal{O}_j$-modules

$$\begin{tikzcd}
 & H^2(C_F)_{(\psi^j),\mathrm{tor}} \arrow[d, hook] & \\
\phi_p(H^1(C_F))^{(\psi^j)} \arrow[r, hook] \arrow[d, "\mathrm{id}"] & H^2(C_F)_{(\psi^j)} \arrow[r, two heads] \arrow[d] & \mathrm{cok}(\phi_p^{\langle\psi^j\rangle}) \\
\phi_p(H^1(C_F))^{(\psi^j)} \arrow[r, hook] & H^2(C_F)_{(\psi^j),\mathrm{tf}} &
\end{tikzcd}$$

and hence an equality

$$\wedge^{r_\psi}_{\mathcal{O}_j}(\phi_p(H^1(C_F))^{(\psi^j)}) = \mathrm{Fit}_{\mathcal{O}_j}(\mathrm{cok}(\phi_p^{\langle\psi^j\rangle}))\mathrm{Fit}_{\mathcal{O}_j}(H^2(C_F)_{(\psi^j),\mathrm{tor}})^{-1}\cdot\wedge^{r_\psi}_{\mathcal{O}_j}H^2(C_F)_{(\psi^j),\mathrm{tf}}.$$

Comparing the last two displayed equalities one deduces that

$$(L^*_{S,T}(\check{\psi},0)/R_S(\psi))^j\mathcal{O}_j = |G|^{-r_\psi}\mathrm{Fit}_{\mathcal{O}_j}(\mathrm{cok}(\phi_p^{\langle\psi^j\rangle})). \tag{29}$$

On the other hand, since we are assuming $\mathscr{L}(\psi^j)\neq 0$, the equality of Theorem 3.1(ii) (with ψ replaced by ψ^j) combines with Lemma 5.1(iii) to imply

$$(L^*_{p,S,T}(\check{\psi}^j\omega_k,0)/\mathscr{L}(\psi^j))\cdot\mathcal{O}_j = |G|^{-r_\psi}\mathrm{Fit}_{\mathcal{O}_j}(\mathrm{cok}(\phi_p^{\langle\psi^j\rangle}))$$

and the equality of Theorem 3.1(iii) follows immediately by comparing this formula with (29).

This completes the proof of Theorem 5.2.

6. Orders of vanishing, valuations and Hilbert's twelfth problem

In this section we derive Theorem 3.1 and Corollary 3.3 as consequences of Theorem 5.2.

6.1. In this subsection we prove Theorem 3.1.

It is at first clear that Theorem 3.1(i) and (ii) follow directly from Theorem 5.2(i) and (iii) respectively and that the equivalence of (a) and (c) in Theorem 5.2(ii) implies the first assertion of Theorem 3.1(iii).

In addition, the remainder of Theorem 3.1(iii) follows from Theorem 5.2(iv) and the fact that if either ψ is rational valued or no p-adic place of F^+ splits in F whenever $F^{\mathrm{cl}}\nsubseteq(F^{\mathrm{cl}})^+(\zeta_p)$, then it validates the Strong Stark Conjecture at p by Tate [43, Ch. II, Th. 6.8] and by Nickel [32, Cor. 2] respectively.

It therefore only remains to prove Theorem 3.1(iv) and to do this we assume, as required, that $\mathscr{L}_S(\psi)\neq 0$.

Then to deduce the containment in claim (iv) from the result of Lemma 6.1 below one need only note $L^{r_{S,\psi}}_{p,S,T}(\check{\psi}\omega_k,0) = L^*_{p,S,T}(\check{\psi}\omega_k,0)$ (by Theorem 5.2(ii)) and that $\mathrm{Fit}(\hat{H}^0(G,Y^-_{S,p}[\psi]))$ contains $\mathrm{Fit}(\hat{H}^0(G,Y_{S,p}[\psi])) = \prod_{v\in S}\mathrm{Fit}(\hat{H}^0(G_v,T_\psi))$ and then apply a slight refinement of (the natural p-adic analogue of) the result of [2, Lem. 11.1.2(i) and (ii)].

To be precise, one directly applies the result of [2, Lem. 11.1.2(ii)] in this context. However, to obtain the containment stated in claim (iv) one must also note that, whilst the result of [2, Lem. 11.1.2(i)] asserts (in the notation of loc. cit.) that for any y_ψ in $\mathrm{Ann}_{\mathcal{O}}(M_\psi)$ one has $|G|y_\psi\mathrm{pr}_\psi \in \mathcal{O}\otimes\mathrm{Ann}_{\mathbb{Z}[G]}(M)$, if one uses the natural isomorphism $\mathrm{Hom}(M_\psi,\mathbb{Q}/\mathbb{Z}) \cong \mathrm{Hom}(M,\mathbb{Q}/\mathbb{Z})^{\check{\psi}}$ and equality $\mathrm{Ann}_{\mathbb{Z}[G]}(\mathrm{Hom}(M,\mathbb{Q}/\mathbb{Z})) = \mathrm{Ann}_{\mathbb{Z}[G]}(M)^{\#}$ it follows easily from the same argument as in loc. cit. that if y_ψ is any element of either $\mathrm{Ann}_{\mathcal{O}}(M_\psi)$ or $\mathrm{Ann}_{\mathcal{O}}(M^\psi)$, then the product $y_\psi\cdot\mathrm{pr}_\psi$ belongs to $\mathcal{O}\otimes\mathrm{Ann}_{\mathbb{Z}[G]}(M)$.

This completes the proof of Theorem 3.1.

Lemma 6.1. *If $\mathscr{L}_S(\psi)\neq 0$, then $(L^*_{p,S,T}(\check{\psi}\omega_k,0)/\mathscr{L}_S(\psi))\cdot\mathrm{Fit}_{\mathcal{O}}(\hat{H}^0(G,Y^-_{F,S,p}[\psi]))$ is contained in $\mathrm{Fit}_{\mathcal{O}}(\mathrm{Cl}^T(F)^{(\psi)}_p)$.*

Proof. We regard ψ as fixed and so abbreviate $\mathcal{O}_\psi$ and $\mathrm{Fit}_{\mathcal{O}_\psi}(-)$ to $\mathcal{O}$ and $\mathrm{Fit}(-)$ respectively.

By the argument of [2, Lem. 5.1.1] we can choose a finite set S'' of places of k that are not in S, are totally split in F/k and are such that $\mathrm{Cl}^T_{S_k^\infty\cup S''}(F)_p$ vanishes. Then, setting $S' := S\cup S''$, one has $L^*_{p,S,T}(\check{\psi}\omega_k,0)/\mathscr{L}_S(\psi) = L^*_{p,S',T}(\check{\psi}\omega_k,0)/\mathscr{L}_{S'}(\psi)$ (cf. [2, Lem. 5.1.3]).

To further analyse this quotient we write C'_F in place of $R\Gamma_{\text{ét},T}(\mathcal{O}_{F,S'},\mathbb{Z}_p(1))^-$ and note that the exact sequence (6) combines with Lemma 4.1(ii) to imply $H^2(C'_F)$ identifies with $\mathrm{Sel}^T_{S'}(F)^{\mathrm{tr},-}_p = Y^-_{F,S',p}$.

Hence, writing $\phi_{S',T,p}$ for the restriction of $\phi_{F,S',p}$ to $\mathcal{O}^\times_{F,S',T}$, one has

$$\begin{aligned}(L^*_{p,S,T}(\tilde{\psi}\omega_k,0)/\mathscr{L}_S(\psi))\cdot\mathcal{O} =&(L^*_{p,S',T}(\tilde{\psi}\omega_k,0)/\mathscr{L}_{S'}(\psi))\cdot\mathcal{O}\\ =&|G|^{-r_{p,S',\psi}}\mathscr{L}_{S'}(\psi)^{-1}\mathrm{Fit}(\mathrm{cok}(\lambda^{\langle\psi\rangle}_{F,S',T,p}))\\ =&|G|^{-r_{p,S',\psi}}\mathrm{Fit}(\mathrm{cok}(\phi^{\langle\psi\rangle}_{S',T,p}))\\ =&\mathrm{Fit}(\mathrm{cok}(\phi^{(\psi)}_{S',T,p}))\mathrm{Fit}(\hat{H}^{-1}(G,Y^-_{F,S',p}[\psi]))\mathrm{Fit}(\hat{H}^0(G,Y^-_{F,S',p}[\psi]))^{-1}\\ =&\mathrm{Fit}(\mathrm{cok}(\phi^{(\psi)}_{S',T,p}))\mathrm{Fit}(\hat{H}^{-1}(G,Y^-_{F,S,p}[\psi]))\mathrm{Fit}(\hat{H}^0(G,Y^-_{F,S,p}[\psi]))^{-1}.\end{aligned}$$

Here the second equality is by Theorem 3.1(ii) (with S replaced by S'), the third follows from Lemma 5.1(iii), the fourth from Lemma 5.1(ii) and the last from the fact that each place in S'' splits completely in F/k and so $Y^-_{F,S'',p}[\psi]$ is a cohomologically-trivial G-module.

We next note that the natural sequence $0\to\mathcal{O}^{\times,-}_{F,S',T,p}\xrightarrow{\phi_{S',T,p}} Y^-_{F,S',p}\to\mathrm{Cl}^T(F)^-_p\to 0$ is exact (as $\mathrm{Cl}^T_{S_k^\infty\cup S''}(F)$ vanishes) so that there is an induced exact sequence of $\mathcal{O}$-modules

$$0\to\mathrm{cok}(\phi^{(\psi)}_{S',T,p})\to\mathrm{Cl}^T(F)^{(\psi)}_p\to H^1(G,\mathcal{O}^{\times,-}_{F,S',T,p}[\psi])$$

and hence, by general properties of Fitting ideals over $\mathcal{O}$, an inclusion

$$\mathrm{Fit}(\mathrm{cok}(\phi^{(\psi)}_{S',T,p}))\cdot\mathrm{Fit}(H^1(G,\mathcal{O}^{\times,-}_{F,S',T,p}[\psi]))\subseteq\mathrm{Fit}(\mathrm{Cl}^T(F)^{(\psi)}_p).$$

In addition, since C'_F belongs to $D^{\rm p}(\mathbb{Z}_p[G])$ the descriptions in Lemma 4.1(ii) combine to imply that the $\mathcal{O}$-module $H^1(G,\mathcal{O}^{\times,-}_{F,S',T,p}[\psi])$ is isomorphic to $\hat{H}^{-1}(G,Y^-_{F,S',p}[\psi])\cong\hat{H}^{-1}(G,Y^-_{F,S,p}[\psi])$ and hence that $\mathrm{Fit}(H^1(G,\mathcal{O}^{\times,-}_{F,S',T,p}[\psi]))=\mathrm{Fit}(\hat{H}^{-1}(G,Y^-_{F,S,p}[\psi]))$.

The claimed inclusion therefore follows directly upon comparing the displayed equality and inclusion above. □

6.2. We now prove Corollary 3.3.

To do this we set $\mathcal{U}:=\mathcal{O}^{\times,-}_{F,\{v_1\},T,p}$ and $\mathcal{U}_1:=\mathcal{O}^{\times,-}_{F_1,\{v_1\},T,p}$ and for any set Σ of places of k we abbreviate $Y_{F,\Sigma,p}$ to Y_Σ. Regarding ψ as fixed, we also set $E:=\mathbb{Q}_p(\psi)$, $\mathcal{O}:=\mathbb{Z}_p[\psi]$, $\Gamma:=G_{E_\psi/\mathbb{Q}_p}$ and $n:=\psi(1)$ and abbreviate $\mathrm{Fit}_{\mathcal{O}}(-)$ to $\mathrm{Fit}(-)$ and $\hat{H}^i(G,-)$ to $\hat{H}^i(-)$.

Finally, we write $\lambda_{S,T}$ in place of $\lambda_{F,S,T,p}$ and $\tilde{\lambda}_{S,T}$ for the restriction of $\lambda_{F,S,p}$ to $\mathcal{O}^{\times,-}_{F,S,T,p}$ and $\pi_{S,T}$ for the surjective homomorphism $H^2(C_F)\to Y^-_S$ induced by (6).

6.2.1. At the outset we note the assumptions (10) combine with the formula (1), Theorem 5.2(ii) and the result of Gross recalled in Remark 2.3(ii) to imply $r_{S,\psi}=r_{p,S,\psi}=1$ and hence that the $\mathcal{O}$-modules $\mathcal{O}^{\times,(\psi)}_{F,S,T,p}=\mathcal{U}^{(\psi)}$ and $Y^{(\psi)}_S=Y^{(\psi)}_{\{v_1\}}$ are both free of rank one and that the map on $\mathcal{U}^{(\psi)}$ induced by $\tilde{\lambda}_{S,T}$ is injective.

In particular, since in this case the formula in Theorem 3.1(ii) combines with Lemma 5.1(ii) to imply that $\mathrm{Fit}(\mathrm{cok}(\tilde{\lambda}^{(\psi)}_{S,T}))$ is equal to

$$\begin{aligned}&|G|^{-1}\mathrm{Fit}(\mathrm{cok}(\tilde{\lambda}^{\langle\psi\rangle}_{S,T}))\mathrm{Fit}(\hat{H}^0(Y^-_S[\psi]))\mathrm{Fit}(\hat{H}^{-1}(Y^-_S[\psi]))^{-1}\\ =&|G|^{-1}\cdot\mathrm{Fit}(\mathrm{cok}(\lambda^{\langle\psi\rangle}_{S,T}))\mathrm{Fit}(\ker(\pi_{S,T,(\psi)}))^{-1}\mathrm{Fit}(\hat{H}^0(Y^-_S[\psi]))\mathrm{Fit}(\hat{H}^{-1}(Y^-_S[\psi]))^{-1}\\ =&L'_{p,S,T}(\check{\psi}\omega_k,0)\cdot\mathrm{Fit}(\hat{H}^0(Y^-_S[\psi]))\mathrm{Fit}(\hat{H}^{-1}(Y^-_S[\psi]))^{-1}\mathrm{Fit}(\ker(\pi_{S,T,(\psi)}))^{-1}\end{aligned}$$

one has

$$L'_{p,S,T}(\check{\psi}\omega_k,0)\mathrm{Fit}(\hat{H}^0(Y^-_S[\psi]))\mathrm{Fit}(\hat{H}^{-1}(Y^-_S[\psi]))^{-1}\cdot Y^{(\psi)}_{\{v_1\}}=\tilde{\lambda}^{(\psi)}_{S,T}(\mathrm{Fit}(\ker(\pi_{S,T,(\psi)}))\cdot\mathcal{U}^{(\psi)}).\tag{30}$$

To investigate this equality we decompose e_ψ as a sum of (non-zero) indecomposable idempotents $\sum_{m=1}^{m=n}f_{m,\psi}$ in $E[G]e_\psi$. For each index m we choose a maximal $\mathcal{O}$-order $\mathcal{M}_{m,\psi}$ in $E[G]$ which contains $f_{m,\psi}$ and then, as in §5.1.1, we set $T_{m,\psi}:=f_{m,\psi}\mathcal{M}_{m,\psi}$ and, for any $\mathbb{Z}_p[G]$-module M, write $M^{(\psi,m)}$ and $M_{(\psi,m)}$ for the $\mathcal{O}$-modules $H^0(G,T_{m,\psi}\otimes_{\mathbb{Z}_p}M)$ and $H_0(G,T_{m,\psi}\otimes_{\mathbb{Z}_p}M)$.

In particular, writing $f_{m,\psi}\cdot w_1$ for the image of $f_{m,\psi}\otimes_{\mathbb{Z}_p[G]}w_1$ in $Y_{S,(\psi,m),\mathrm{tf}}$, and Tr_G for the element $\sum_{g\in G}g$ of $\mathbb{Z}[G]$, one has

$$|G|f_{m,\psi}\cdot w_1=\mathrm{Tr}_G(f_{m,\psi}\cdot w_1)\in\mathrm{Tr}_G(Y_{S,(\psi,m),\mathrm{tf}})=\mathrm{Tr}_G(Y_{S,(\psi,m)})=\mathrm{Fit}(\hat{H}^0(Y^-_S[\psi]))\cdot Y^{(\psi,m)}_{\{v_1\}}$$

where the last equality is valid because $Y^{(\psi,m)}_S=Y^{(\psi,m)}_{\{v_1\}}$ is a free rank one $\mathcal{O}$-module.

Thus, from (30), we deduce the existence of elements $a_{m,\psi}$ of $\mathrm{Fit}(\ker(\pi_{S,T,(\psi)}))\mathrm{Fit}(\hat{H}^{-1}(Y^-_S[\psi]))$ and $u'_{m,\psi}$ of $\mathcal{U}^{(\psi,m)}$ with $L'_{p,S,T}(\check{\psi}\omega_k,0)|G|f_{m,\psi}\cdot w_1=\tilde{\lambda}_{S,T}(a_{m,\psi}u'_{m,\psi})$. In particular, for every d_ψ in $\mathcal{D}^{-1}_{E/\mathbb{Q}_p}\mathrm{Fit}(\hat{H}^{-1}(Y^-_S[\psi]))^{-1}$ the element

$$\eta(\psi,d_\psi,m):=d_\psi\cdot a_{m,\psi}u'_{m,\psi}\in\mathcal{D}^{-1}_{E/\mathbb{Q}_p}\mathrm{Fit}(\ker(\pi_{S,T,(\psi)}))\cdot\mathcal{U}^{(\psi,m)}$$

satisfies $d_\psi L'_{p,S,T}(\check\psi\omega_k,0)|G|f_{m,\psi}(w_1) = \tilde\lambda_{S,T}(\eta(\psi,d_\psi,m))$.

This in turn implies that the element

$$\eta(\psi,d_\psi) := n^{-1}\sum_{m=1}^{m=n}\eta(\psi,d_\psi,m) = n^{-1}\sum_{m=1}^{m=n} d_\psi\cdot a_{m,\psi}u'_{m,\psi} \tag{31}$$

belongs to the lattice $\Xi_{\psi,d_\psi} := n^{-1}\mathcal{D}^{-1}_{E/\mathbb{Q}_p}\mathrm{Fit}(\ker(\pi_{S,T,(\psi)}))\cdot\bigoplus_{m=1}^{m=n}\mathcal{U}^{(\psi,m)}$ and satisfies

$$\begin{aligned}\sum_{g\in G} L'_{p,S,T}(\check\psi\omega_k,0)\check\psi(g)d_\psi\cdot g(w_1) &= L'_{p,S,T}(\tilde\psi\omega_k,0)n^{-1}|G|d_\psi e_\psi(w_1)\\ &= \sum_{m=1}^{m=n} L'_{p,S,T}(\check\psi\omega_k,0)n^{-1}|G|d_\psi f_{m,\psi}(w_1)\\ &= \tilde\lambda_{S,T}(\eta(\psi,d_\psi)).\end{aligned}$$

Since $\tilde\lambda_{S,T}$ is injective on Ξ_{ψ,d_ψ} this equality has two important consequences. Firstly, it implies $\eta(\psi,d_\psi)$ is fixed by the natural action of G_1 on $E\cdot\Xi_{\psi,d_\psi} = e_\psi(E\cdot\mathcal{U})$. Secondly, since for every γ in Γ one has $L'_{p,S,T}(\check\psi\omega_k,0)^\gamma = L'_{p,S,T}(\check\psi^\gamma\omega_k,0)$ it implies $\eta(d_\psi^\gamma,\psi^\gamma) = \eta(\psi,d_\psi)^\gamma$ with respect to the natural $\mathcal{O}$-semi-linear action of Γ on Ξ_{ψ,d_ψ}.

The element $\epsilon'_{\psi,d_\psi} := \sum_{\gamma\in\Gamma}\eta(\psi,d_\psi)^\gamma$ therefore belongs to

$$H^0(G_1,\mathrm{Tr}_{E/\mathbb{Q}_p}(\Xi_{\psi,d_\psi}))\subseteq H^0(G_1,\mathcal{U}) = \mathcal{U}_1,$$

(where the first inclusion is true because both $n^{-1}\cdot\mathcal{U}_1^{(\psi,m)}\subseteq\mathcal{O}\cdot\mathcal{U}$ and $\mathrm{Tr}_{E/\mathbb{Q}_p}(\mathcal{D}^{-1}_{E/\mathbb{Q}_p})\subseteq\mathbb{Z}_p$) and satisfies

$$\sum_{g\in G}(\sum_{\gamma\in\Gamma} L'_{p,S,T}(\check\psi^\gamma\omega_k,0)\check\psi^\gamma(g)d_\psi^\gamma)g(w_1) = \tilde\lambda_{S,T}(\epsilon'_{\psi,d_\psi})$$

and hence for every g in G also

$$\log_p(p^{-\mathrm{val}_{w_1}(g(\epsilon'_{\psi,d_\psi}))}g(\epsilon'_{\psi,d_\psi})) = \frac{1}{|G_1|}\sum_{g'\in G_1}\sum_{\gamma\in\Gamma} L'_{p,S,T}(\check\psi^\gamma\omega_k,0)\check\psi^\gamma(gg')d_\psi^\gamma.$$

Finally we set

$$\epsilon_{\psi,d_\psi} := (\epsilon'_{\psi,d_\psi})^{p-1}.$$

Then for every g in G the element $g(\epsilon_{\psi,d_\psi})$ belongs to the subgroup $p^{\mathbb{Z}}(1+p\mathbb{Z}_p)$ of $\mathbb{Q}_p^\times$ and so the above equalities imply

$$g(\epsilon_{\psi,d_\psi}) = p^{\mathrm{val}_{w_1}(g(\epsilon_{\psi,d_\psi}))}\exp_p(\frac{(p-1)}{|G_1|}\sum_{g'\in G_1}\sum_{\gamma\in\Gamma} L'_{p,S,T}(\tilde\psi^\gamma\omega_k,0)\check\psi^\gamma(gg')d_\psi^\gamma).$$

In addition, one has

$$\begin{aligned}\phi_{F,S}(\epsilon'_{\psi,d_\psi}) = \sum_{\gamma\in\Gamma} e_{\psi^\gamma}\phi_{F,S}(\epsilon'_{\psi,d_\psi}) &= \sum_{\gamma\in\Gamma} e_{\psi^\gamma}\mathscr{L}_S(\psi^\gamma)^{-1}\tilde\lambda_{S,T}(\epsilon'_{\psi,d_\psi})\\ &= \sum_{\gamma\in\Gamma}\mathscr{L}_S(\psi^\gamma)^{-1}L'_{p,S,T}(\check\psi^\gamma\omega_k,0)n^{-1}|G|d_\psi^\gamma e_{\psi^\gamma}(w_1)\end{aligned} \tag{32}$$

and hence for every integer t also

$$\mathrm{val}_{w_1}(g(\epsilon^t_{\psi,d_\psi})) = \sum_{g'\in G_1} \frac{t(p-1)}{|G_1|} \sum_{\gamma\in\Gamma} \mathscr{L}_S(\psi^\gamma)^{-1} L'_{p,S,T}(\check{\psi}^\gamma\omega_k, 0)\check{\psi}^\gamma(gg')d^\gamma_\psi. \tag{33}$$

This verifies all of the assertions of Corollary 3.3 up to the end of claim (i).

6.2.2. To verify Corollary 3.3(ii) we regard each θ in $\mathrm{Hom}_{\mathbb{Z}_p[G]}(\mathcal{U}, \mathbb{Z}_p[G])$ as an element of $\mathrm{Hom}_{E[G]}(E\cdot\mathcal{U}, E[G])$ and note that $\theta(\mathcal{U}_1^{(\psi)}) \subseteq \mathbb{Z}_p[G]^{(\psi)} = |G|\cdot T_\psi \subseteq \mathcal{O}\cdot\mathrm{pr}_\psi$.

In addition, the $\mathcal{O}$-module $Y^-_{S,(\psi),\mathrm{tor}}$ is isomorphic to $\hat{H}^{-1}(Y^-_S[\psi])$ and so the exact sequence of G-modules (6) induces an exact sequence $\hat{H}^{-2}(Y^-_S[\psi]) \to \mathrm{Cl}^T_{S,p,(\psi)} \to \ker(\pi_{S,T,(\psi)})$ and hence an inclusion

$$\mathrm{Ann}_{\mathcal{O}}(\hat{H}^{-2}(Y^-_S[\psi]))\cdot\mathrm{Fit}(\ker(\pi_{S,T,(\psi)})) \subseteq \mathrm{Ann}_{\mathcal{O}}(\mathrm{Cl}^T_{S,p,(\psi)}).$$

Thus, if we fix any a in $\mathbb{Z}_p$ (such as $a = |G|$) that annihilates $\hat{H}^{-2}(Y^-_S[\psi])$, and assume d_ψ belongs to $\mathrm{Fit}(\hat{H}^{-1}(Y^-_S[\psi]))^{-1}$, then the explicit definition of ϵ'_{ψ,d_ψ} implies that

$$\begin{aligned} a\theta(\epsilon_{\psi,d_\psi}) = (p-1)a\theta(\epsilon'_{\psi,d_\psi}) &= (p-1)\sum_{\gamma\in\Gamma}\sum_{m=1}^{m=n} a(d^\gamma_\psi a_{m,\psi^\gamma})\theta(u_{m,\psi^\gamma}) \\ &\in \sum_{\gamma\in\Gamma} \mathrm{Ann}_{\mathcal{O}}(\mathrm{Cl}^T_S(F)_{p,(\psi)})\cdot\mathrm{pr}_\psi \subseteq \mathcal{O}\cdot\mathrm{Ann}_{\mathbb{Z}_p[G]}(\mathrm{Cl}^T_S(F)_p) \end{aligned}$$

where the inclusion follows from the observation made just before the statement of Lemma 6.1, and hence that $a\theta(\epsilon_{\psi,d_\psi})$ belongs to $\mathbb{Q}_p[G]\cap(\mathcal{O}\cdot\mathrm{Ann}_{\mathbb{Z}[G]}(\mathrm{Cl}^T_S(F))) = \mathrm{Ann}_{\mathbb{Z}[G]}(\mathrm{Cl}^T_S(F))_p$.

We next note that the idempotent $e_1 := \sum_{\gamma\in\Gamma} e_{\psi^\gamma}$ fixes ϵ_{ψ,d_ψ} and annihilates $\mathbb{Q}_p\cdot Y^-_{S\setminus\{v_1\}}$. By using the natural exact sequence $Y^-_{S\setminus\{v_1\}} \to \mathrm{Cl}^T_{\{v_1\}}(F)^-_p \to \mathrm{Cl}^T_S(F)^-_p \to 0$ we may therefore deduce that $|G|a\theta(\epsilon_{\psi,d_\psi}) = (|G|e_1)(a\theta(\epsilon_{\psi,d_\psi}))$ belongs to $\mathrm{Ann}_{\mathbb{Z}[G]}(\mathrm{Cl}^T_{\{v_1\}}(F))_p$, as required to complete the proof of claim (ii).

Turning to claim (iii) we note (32) combines with the assumed validity of Conjecture 2.4 (and a direct comparison of the definitions (7) and (9) of T-modified L-series) to imply that for any integer m the element $\epsilon_{\psi,d} := \prod_{i\in I}\epsilon_{\psi_i,d,p}$ satisfies

$$\phi_{F,S}(\epsilon^m_{\psi,d}) = m(p-1)\sum_{\alpha\in G_{\mathbb{Q}(\psi)/\mathbb{Q}}} L'_{S,T}(\check{\psi}^\alpha, 0)R_S(\psi^\alpha)^{-1}\psi^\alpha(1)^{-1}|G|d^\alpha e_{\psi^\alpha}(w_1), \tag{34}$$

whilst the expression on the right hand side of (33) implies $\mathrm{val}_{w_1}(g(\epsilon^m_{\psi,d}))$ is equal to the sum $y_{\psi,g,d,m}$ defined in claim 3.3(iii)(b).

Now the Galois invariance property (2) implies that the sum on the right hand side of (34) belongs to $\mathbb{Q}\cdot Y^-_{F,\{v_1\}} \subset \mathbb{Q}_p\cdot Y^-_{\{v_1\}}$ and hence that $\epsilon_{\psi,d}$ belongs to the localisation of $\mathcal{O}^{\times,-}_{F,\{v_1\},T}$ at p. For some integer m prime to p one therefore has $\epsilon^m_{\psi,d} \in \mathcal{O}^{\times,-}_{F,\{v_1\},T}$.

For any such integer m claim (iii)(b) then follows directly from the formula (33) whilst claim (iii)(a) is true because (34) combines with the injectivity of $\phi_{F,S}$ to imply that if $d \neq 0$, then the largest normal subgroup N of G that fixes $\epsilon^m_{\psi,d}$ is $\ker(\psi)$. (In a similar way, one can justify the claim in Remark 3.4(iii) by using a suitable sum of expressions of the form (34).)

This completes the proof of Corollary 3.3.

7. The refined p-adic class number formula and some consequences

In this section we prove Theorem 3.6 and then deduce Corollaries 3.8, 3.10 and 3.11.

7.1. To prove Theorem 3.6 we fix data F/k, G, S and T as at the beginning of §3.2, set $\mathcal{G} := G_{F^{\rm cyc}/k}$ and write $C_T^{\rm cyc}$ for the complex $R\Gamma_{\text{ét},T}(\mathcal{O}_{k,S}, \Lambda(\mathcal{G})^{\#}(1))$.

We also write Σ for the multiplicatively closed left and right Ore set of non-zero divisors of $\Lambda(\mathcal{G})$ comprising elements f such that $\Lambda(\mathcal{G})/\Lambda(\mathcal{G})f$ is a finitely generated $\mathbb{Z}_p$-module. We set

$$Q'(\mathcal{G}) := \begin{cases} \Lambda(\mathcal{G})_\Sigma, & \text{if } \mu_p(F) = 0,\\ Q(\mathcal{G}), & \text{if both } \mu_p(F) \neq 0 \text{ and } p \text{ does not divide } |G| \end{cases}$$

and write $\partial_{\Lambda(\mathcal{G}),Q'(\mathcal{G})}$ for the connecting homomorphism $K_1(Q'(\mathcal{G})) \to K_0(\Lambda(\mathcal{G}), Q'(\mathcal{G}))$.

The following result provides a convenient interpretation of the known validity of the main conjecture of non-commutative Iwasawa theory for totally real fields.

Proposition 7.1. *Assume either $\mu_p(F) = 0$ or both $\mu_p(F) \neq 0$ and p does not divide $|G|$.*

Then $Q'(\mathcal{G}) \otimes_{\Lambda(\mathcal{G})} C_T^{\rm cyc}$ is acyclic and there exists an element ξ of $K_1(Q'(\mathcal{G}))$ with

$$\partial_{\Lambda(\mathcal{G}),Q'(\mathcal{G})}(\xi) = -\chi_{\Lambda(\mathcal{G}),Q'(\mathcal{G})}(C_T^{\rm cyc}, 0)$$

and such that for all ψ in $A^-(\mathcal{G})$ one has

$$\Phi_\psi(\xi)^*(0) = \log_p(\kappa_k(\gamma))^{-r_{p,S,\psi}} L^*_{p,S,T}(\check{\psi}\omega_k, 0).$$

Proof. The complex $C' := R\Gamma_{c,\text{ét}}(\mathcal{O}_{k,S}, \Lambda(\mathcal{G})^{\#}(1))$ is acyclic outside degrees two and three and its cohomology identifies with $M_S^{\rm cyc}$ and $\mathbb{Z}_p$ in these respective degrees. Since these groups are finitely generated torsion $\Lambda(\mathcal{G})$-modules that are finitely generated over $\mathbb{Z}_p$ if $\mu_p(F)$ vanishes the given assumptions on p imply the acyclicity of $Q'(\mathcal{G}) \otimes_{\Lambda(\mathcal{G})} C'$.

We next claim that there exists an element ξ' of $K_1(Q'(\mathcal{G}))$ with both $\partial_{\Lambda(\mathcal{G}),Q'(\mathcal{G})}(\xi') = \chi_{\Lambda(\mathcal{G}),Q'(\mathcal{G})}(C', 0)$ and $\Phi_\rho(\xi') = f_{S,\rho}(u)$ for all ρ in $A^+(\mathcal{G})$.

If p does not divide $|G|$, then $\Lambda(\mathcal{G})$ is a maximal $\Lambda(G_{F^{\rm cyc}/F})$-order in $Q(\mathcal{G})$ and the existence of a suitable element ξ' in $K_1(Q(\mathcal{G}))$ can be directly deduced from the main result of Wiles in [44]. To do this one can use the same reduction arguments as in the proof of Proposition 5.5(ii) or simply note that, after interpreting the homomorphisms Φ_ρ in terms of the reduced norm of the semisimple algebra $Q(\mathcal{G})$ (as in [3, Lem. 3.1]), this result is equivalent to earlier results of Ritter and Weiss in [35, Th. 16 and Rem. (H)].

In a similar way, if one assumes only that $\mu_p(F)$ vanishes, then the existence of such an element in $K_1(\Lambda(\mathcal{G})_\Sigma)$ is equivalent to the validity of the main conjecture of non-commutative Iwasawa theory for totally real fields, as proved independently by Ritter and Weiss in [36] and by Kakde in [24].

These facts combine with the proof of Proposition 5.5 to imply the acyclicity of the complex $Q'(\mathcal{G}) \otimes_{\Lambda(\mathcal{G})} C_T^{\rm cyc}$ and the existence of an element ξ of $K_1(Q'(\mathcal{G}))$ with $\partial_{\Lambda(\mathcal{G}),Q'(\mathcal{G})}(\xi) = -\chi_{\Lambda(\mathcal{G}),Q'(\mathcal{G})}(C_T^{\rm cyc}, 0)$ and such that, for each ψ in $A^-(\mathcal{G})$, the image of ξ under Φ_ψ is equal to the series $f_{S,T,\psi}(u)$ that occurs in Proposition 5.5(ii).

The claimed formula for each leading term $\Phi_\psi(\xi)^*(0) = f^*_{S,T,\psi}(0)$ then follows directly from Lemma 5.10. □

Turning now to the proof of Theorem 3.6, the first assertion of Proposition 7.1 implies $C_T^{\rm cyc}$ belongs to the category $D^{\rm p}_{\Sigma}(\Lambda(\mathcal{G}))$ defined in [6, §1.4]. Further, if ψ is any element of $\mathrm{Ir}_p^-(G)$ that factors through the surjection $\mathbb{Q}_p[G]^- \to \mathbb{Q}_p[G]e_{\rm ss}$, then Theorem 5.2(ii) implies $C_T^{\rm cyc}$ is semisimple at ψ and Theorem 5.8 implies the Bockstein homomorphism in degree one of the data $(C_T^{\rm cyc}, \psi, \gamma)$ is induced by the map $-\log_p(\kappa_k(\gamma))^{-1}\cdot\lambda_{F,S,p}$.

The semisimplicity at ψ of $C_T^{\rm cyc}$ also combines with [7, Lem. 5.5(iv)] to imply that the term $(-1)^{r_G(C_T^{\rm cyc})(\psi)}$ that occurs in the descent formula proved by Venjakob and the present author in [7, Th. 2.2] (for the groups $G=\mathcal{G}$ and $\overline{G}=G$ and the complex $C=C_T^{\rm cyc}$) is equal to $(-1)^{-r_{S,\psi}} = \det_{\mathbb{Q}_p^c}(-1 \mid \mathbb{Q}_p^c\cdot H^1_{\text{ét},T}(\mathcal{O}_{F,S},\mathbb{Z}_p(1))^{(\psi)})$.

Given these facts, the first displayed equality in Proposition 7.1 combines with [7, Th. 2.2] to imply that

$$\delta_{\mathbb{Z}_p[G]^{\rm ss}}\Big(\sum_{\psi\in\mathrm{Ir}_p^{\rm ss}(G)} e_\psi\Phi_\psi(\xi)^*(0)\Big) = -\chi_{\mathbb{Z}_p[G]^{\rm ss}}(R\Gamma_{\text{ét},T}(\mathcal{O}_{F,S},\mathbb{Z}_p(1))^{\rm ss}, \hat\lambda^{\rm ss}_{F/k,S,p}). \tag{35}$$

where $\hat\lambda^{\rm ss}_{F/k,S,p}$ denotes the exact sequence of $\mathbb{Q}_p[G]$-modules

$$0\to \mathbb{Q}_p\cdot H^1(R\Gamma_{\text{ét},T}(\mathcal{O}_{F,S},\mathbb{Z}_p(1))^{\rm ss}) \xrightarrow{\log_p(\kappa_k(\gamma))^{-1}\cdot\lambda_{F,S,p}} \mathbb{Q}_p\cdot H^2(R\Gamma_{\text{ét},T}(\mathcal{O}_{F,S},\mathbb{Z}_p(1))^{\rm ss})\to 0.$$

In addition, as Theorem 5.2(ii) combines with (1) to imply that

$$r_{p,S,\psi} = r_{S,\psi} = \dim_{\mathbb{Q}_p^c}(\mathbb{Q}_p^c\cdot H^2_{\text{ét},T}(\mathcal{O}_{F,S},\mathbb{Z}_p(1))^{(\psi)})$$

for each ψ in $\mathrm{Ir}_p^{\rm ss}(G)$, the result of [2, Lem. A.1(ii)] implies

$$\begin{aligned}
&\chi_{\mathbb{Z}_p[G]^{\rm ss}}(R\Gamma_{\text{ét},T}(\mathcal{O}_{F,S},\mathbb{Z}_p(1))^{\rm ss},\hat\lambda^{\rm ss}_{F,S,p}) - \chi_{\mathbb{Z}_p[G]^{\rm ss}}(R\Gamma_{\text{ét},T}(\mathcal{O}_{F,S},\mathbb{Z}_p(1))^{\rm ss},\lambda^{\rm ss}_{F,S,p})\\
=&-\delta_{\mathbb{Z}_p[G]^{\rm ss}}(\mathrm{Nrd}_{\mathbb{Q}_p[G]^{\rm ss}}(\log_p(\kappa_k(\gamma))^{-1}\mid e_{\rm ss}(\mathbb{Q}_p\cdot H^2_T(\mathcal{O}_{F,S},\mathbb{Z}_p(1)))))\\
=&-\delta_{\mathbb{Z}_p[G]^{\rm ss}}\Big(\sum_{\psi\in\mathrm{Ir}_p^{\rm ss}(G)} e_\psi\log_p(\kappa_k(\gamma))^{-r_{p,S,\psi}}\Big).
\end{aligned}$$

From the final assertion of Proposition 7.1 we also know that for each such character ψ one has $e_\psi\Phi_\psi(\xi)^*(0) = e_\psi\log_p(\kappa_k(\gamma))^{-r_{p,S,\psi}}\cdot 2\theta^*_{p,F/k,S,T}(0)$ and hence that

$$\delta_{\mathbb{Z}_p[G]^{\rm ss}}\Big(\sum_{\psi\in\mathrm{Ir}_p^{\rm ss}(G)} e_\psi\Phi_\psi(\xi)^*(0)\Big) = \delta_{\mathbb{Z}_p[G]^{\rm ss}}(\theta^*_{p,F/k,S,T}(0)) + \delta_{\mathbb{Z}_p[G]^{\rm ss}}\Big(\sum_{\psi\in\mathrm{Ir}_p^{\rm ss}(G)} e_\psi\log_p(\kappa_k(\gamma))^{-r_{p,S,\psi}}\Big).$$

The assertion of Theorem 3.6 now follows directly upon substituting the last two displayed equalities into (35).

7.2. To prove Corollary 3.8 we write $\chi(-,-)$ in place of $\chi_{\mathbb{Z}_p[G]^{\rm ss},\mathbb{C}_p[G]^{\rm ss}}(-,-)$ and abbreviate both $\delta_{\mathbb{Z}_p[G]^{\rm ss},\mathbb{C}_p[G]^{\rm ss}}(-)$ and $\delta_{\mathbb{Z}_p[G]^-,\mathbb{C}_p[G]^-}(-)$ to $\delta(-)$ and $\delta^-(-)$ respectively. We also use the invertible element of $\zeta(\mathbb{C}[G])^\times$ obtained by setting

$$R_S(F/k) := \sum_{\chi\in\mathrm{Ir}(G)} R_S(\chi)\cdot e_\chi.$$

Then for each isomorphism of fields $j:\mathbb{C}_p\cong\mathbb{C}$ the validity of Conjecture 2.4 for F/k implies

$$j_*^{-1}(\theta^*_{F/k,S,T}(0))(\theta^*_{p,F/k,S,T}(0)e_{\rm ss})^{-1}=j_*^{-1}(R_S(F/k))(\mathscr{L}_S(F/k)e_{\rm ss})^{-1}, \tag{36}$$

with j_* the ring isomorphism $\mathbb{C}_p[G]\cong\mathbb{C}[G]$ induced by j.

This in turn combines with the result of Theorem 3.6 to imply that $\delta(j_*^{-1}(\theta^*_{F/k,S,T}(0))e_{\rm ss})$ is equal (under the hypotheses of Corollary 3.8) to

$$\begin{aligned}&\delta(\theta^*_{p,F/k,S,T}(0)e_{\rm ss})+\delta(j_*^{-1}(R_S(F/k))(\mathscr{L}_S(F/k)e_{\rm ss})^{-1})\\ =&\chi(R\Gamma_{\text{ét},T}(\mathcal{O}_{F,S},\mathbb{Z}_p(1))^{\rm ss},\lambda^{\rm ss}_{F,S,p})+\delta(\mathrm{Nrd}_{\mathbb{C}_p[G]^{\rm ss}}((\mathbb{C}_p\otimes_{\mathbb{R},j}\mathrm{R}_{F,S})\circ\lambda^{-1}_{F,S,p}\mid e_{\rm ss}(\mathbb{C}_p\cdot Y^-_{F,S,p}))\\ =&\chi(R\Gamma_{\text{ét},T}(\mathcal{O}_{F,S},\mathbb{Z}_p(1))^{\rm ss},R^{\rm ss}_{S,j}),\end{aligned}$$

where the last equality follows from [2, Lem. A.1(ii)].

This shows that the equality (27) is valid modulo the kernel of the natural homomorphism $\pi:K_0(\mathbb{Z}_p[G]^-,\mathbb{C}_p[G]^-)\to K_0(\mathbb{Z}_p[G]^{\rm ss},\mathbb{C}_p[G]^{\rm ss})$ and so to complete the proof of Corollary 3.8(i) it is enough to prove that $\ker(\pi)=\delta^-(\zeta(\mathbb{C}_p[G](e_--e_{\rm ss}))^\times)$.

To show this we use the exact commutative diagram

$$\begin{tikzcd}\mathrm{Nrd}_{\mathbb{C}_p[G]^-}(K_1(\mathbb{Z}_p[G]^-)) \arrow[r, hook] \arrow[d, "\pi''"] & \zeta(\mathbb{C}_p[G])^{-,\times} \arrow[r, two heads, "\delta^-"] \arrow[d, "\pi'"] & K_0(\mathbb{Z}_p[G]^-,\mathbb{C}_p[G]^-) \arrow[d, "\pi"] \\ \mathrm{Nrd}_{\mathbb{C}_p[G]^{\rm ss}}(K_1(\mathbb{Z}_p[G]^{\rm ss})) \arrow[r, hook] & \zeta(\mathbb{C}_p[G])^{{\rm ss},\times} \arrow[r, two heads, "\delta"] & K_0(\mathbb{Z}_p[G]^{\rm ss},\mathbb{C}_p[G]^{\rm ss}).\end{tikzcd} \tag{37}$$

Here the rows are induced by the relevant exact sequences of relative K-theory and the vertical arrows by the ring homomorphims $\mathbb{Z}_p[G]^-\to\mathbb{Z}_p[G]^{\rm ss}$ and $\mathbb{C}_p[G]^-\to\mathbb{C}_p[G]^{\rm ss}$. In addition, the map π'' is surjective as a consequence of Bass's Theorem (cf. [26, Chap. 7, (20.9)]) and the fact that $\mathbb{Z}_p[G]^-$ is semi-local.

In particular, since $\ker(\pi')=\zeta(\mathbb{C}_p[G])^{-,\times}\cap\mathbb{C}_p[G](1-e_{\rm ss})=\zeta(\mathbb{C}_p[G](e_--e_{\rm ss}))^\times$, we obtain the required description of $\ker(\pi)$ by applying the Snake lemma to the above diagram.

To deduce Corollary 3.8(ii) from Corollary 3.8(i) it suffices to note that if Conjecture 2.1 is valid for F/k, then Theorem 2.6 implies Conjecture 2.4 is valid for F/k whilst Remarks 2.3(i) and 2.5(i) combine to imply $e_{\rm ss}=e_-$ and hence that $\delta^-(\zeta(\mathbb{C}_p[G](e_--e_{\rm ss}))^\times)$ vanishes.

This completes the proof of Corollary 3.8.

7.3. We now prove Corollary 3.10.

Claim (i) asserts that for each ψ in $\mathrm{Ir}_p^-(G)$ the order of vanishing at $s=0$ of the series $L_{S,T}(\check{\psi}\omega_k,s)$ is at least $r\psi(1)$. This follows directly from Theorem 3.1(i), the explicit formula $r_{S,\psi}=\dim_{\mathbb{C}_p}(\mathrm{Hom}_{\mathbb{C}_p[G]}(V_{\check{\psi}},\mathbb{C}_p\cdot Y^-_{F,S,p})$ (implied by (1)) and the fact that $Y^-_{F,S,p}$ has a quotient $Y^-_{F,V,p}$ that a free $\mathbb{Z}_p[G]^-$-module of rank r.

Since $S\setminus V$ contains $S^{\rm ram}_{F/k}$ (and hence also S_k^∞) claim (ii) follows directly from the argument of [4, Lem. 2.8].

To prove claim (iii) we must first fix a convenient resolution of $\mathrm{Sel}_S^T(F)^-$.

To do this, for each normal subgroup N of G we set $\mathrm{Tr}_N:=\sum_{g\in N}g\in\mathbb{Z}[G]$ and identify $Y_{F^N,S}$ with $\mathrm{Tr}_N(Y_{F,S})$ by means of the map which sends each place w in S_{F^N} to $\mathrm{Tr}_N(w')$ for any choice of places w' of F above w. For each natural number m we set $[m]:=\{i\in\mathbb{Z}:1\le i\le m\}$.

We also set $n := |S|$, label (and hence order) the elements of S as $\{v_i\}_{i\in[n]}$ and for each index i choose a place w_i of F above v_i. For each Galois extension L of k in F we set $Z_L := \{i \in [n] : v_i \in V_L\}$ and, for later convenience, we assume, as we may, that the labeling of S is chosen so that $V_F = [r]$.

We then fix a surjective homomorphism $\varrho : P \to \mathrm{Sel}_S^T(F)_p^{\mathrm{tr},-}$ of $\mathbb{Z}_p[G]^-$-modules in which P is free of finite rank d and has an ordered basis $\{b_i\}_{i\in[d]}$ for which the following property is satisfied.

Write ξ for the composite of ϱ and the canonical homomorphism $\mathrm{Sel}_S^T(F)_p^{\mathrm{tr},-} \to Y_{F,S,p}^-$ induced by (6) and for each normal subgroup N of G we identify the restriction ξ^N of ξ with the composite homomorphism $P^N = Tr_N(P) \xrightarrow{\xi} \mathrm{Tr}_N(Y_{F,S,p}^-) = Y_{F^N,S,p}^-$. Then in the sequel we can (and will) assume that for each index i, and each normal subgroup N of G, the element $\mathrm{Tr}_N(b_i)$ is sent by ξ^N to $w_{v_i,F^N} - \overline{w_{v_i,F^N}}$ if $i \in [n]$ and to 0 otherwise.

Now, since our choice of T implies $\mathcal{O}_{F,S,T,p}^{\times,-}$ is $\mathbb{Z}_p$-free, the properties of the complex $C_{F,T} := R\Gamma_{\text{ét},T}(\mathcal{O}_{F,S}, \mathbb{Z}_p(1))^-$ that are described in Lemma 4.1(ii) imply the existence of an exact sequence of $\mathbb{Z}_p[G]^-$-modules

$$0 \to \mathcal{O}_{F,S,T,p}^{\times,-} \xrightarrow{\iota} P \xrightarrow{\varpi} P \xrightarrow{\varrho} \mathrm{Sel}_S^T(F)_p^{\mathrm{tr},-} \to 0 \tag{38}$$

for which there is an isomorphism $P^\bullet \to C_{F,T}$ in $D^{\mathrm{p}}(\mathbb{Z}_p[G])$ which induces the identity on cohomology in degrees one and two. Here $P^\bullet$ denotes the complex $P \xrightarrow{\varpi} P$, where the first term is placed in degree one and the cohomology groups $H^1(P^\bullet)$ and $H^2(P^\bullet)$ are identified with $\mathcal{O}_{F,S,T,p}^{\times,-}$ and $\mathrm{Sel}_S^T(F)_p^{\mathrm{tr},-}$ by using the maps ι and ϱ.

Noting that the algebra $\mathbb{Q}_p[G]$ is semisimple we choose $\mathbb{Q}_p[G]$-equivariant sections ι_1 and ι_2 to the maps $\mathbb{Q}_p \cdot P \to \mathbb{Q}_p \cdot \mathrm{im}(\varpi)$ and $\mathbb{Q}_p \cdot P \to \mathbb{Q}_p \cdot \mathrm{cok}(\varpi) = \mathbb{Q}_p \cdot \mathrm{Sel}_S^T(F)_p^{\mathrm{tr},-}$ that are induced by ϖ and ϱ respectively. We can, and will, also assume that for each Galois extension L of k in F and each i in Z_L one has $\iota_2(w_{i,L} - \overline{w_{i,L}}) = \mathrm{Tr}_H(b_i)$ with $H = G_{F/L}$.

One has $\mathbb{Q}_p \cdot P = \mathbb{Q}_p \cdot \ker(\varpi) \oplus \iota_1(\mathbb{Q}_p \cdot \mathrm{im}(\varpi))$ and for any homomorphism of $\mathbb{Q}_p[G]$-modules $\kappa : \mathbb{Q}_p \cdot \mathcal{O}_{F,S,T,p}^{\times,-} \to \mathbb{Q}_p \cdot \mathrm{Sel}_S^T(F)_p^{\mathrm{tr},-} = \mathbb{Q}_p \cdot Y_{F,S,p}^-$ we write $\langle \kappa, \varpi\rangle_{\iota_1,\iota_2}$ for the endomorphism of the $\mathbb{Q}_p[G]$-module $\mathbb{Q}_p \cdot P$ that is equal to $\iota_2 \circ \kappa \circ (\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \iota)^{-1}$ on $\mathbb{Q}_p \cdot \ker(\varpi)$ and to $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \varpi$ on $\iota_1(\mathbb{Q}_p \cdot \mathrm{im}(\varpi))$. Then [2, Lem. A.1(iii)] implies that

$$-\chi_{\mathbb{Z}_p[G]^{\mathrm{ss}}}(C_{F,T}^{\mathrm{ss}}, \lambda_{F/k,S,p}^{\mathrm{ss}}) = \delta_{\mathbb{Z}_p[G]^{\mathrm{ss}}}(\mathrm{Nrd}_{\mathbb{Q}_p[G]^{\mathrm{ss}}}(e_{\mathrm{ss}}\langle \lambda_{F,S,p}, \varpi\rangle_{\iota_1,\iota_2}))$$

and so the equality of Conjecture 3.5 combines with the exact diagram (37) to imply the existence of an element u of $\mathrm{Nrd}_{\mathbb{C}_p[G]^-}(K_1(\mathbb{Z}_p[G]^-))$ with

$$\begin{aligned} \theta_{p,F/k,S,T}^*(0) \cdot e_{\mathrm{ss}} &= u \cdot \mathrm{Nrd}_{\mathbb{Q}_p[G]}(e_{\mathrm{ss}}\langle \lambda_{F,S,p}, \varpi\rangle_{\iota_1,\iota_2}) \\ &= u\mathscr{L}_S(F/k) \cdot \mathrm{Nrd}_{\mathbb{Q}_p[G]}(e_{\mathrm{ss}}\langle \phi_{F,S,p}, \varpi\rangle_{\iota_1,\iota_2}). \end{aligned} \tag{39}$$

Here the second equality follows directly by comparing the explicit definition of $\mathscr{L}_S(F/k)$ with the fact that $e_{\mathrm{ss}}\langle \phi_{F,S,p}, \varpi\rangle_{\iota_1,\iota_2}$ and $e_{\mathrm{ss}}\langle \lambda_{F,S,p}, \varpi\rangle_{\iota_1,\iota_2}$ coincide on $\iota_1(\mathbb{Q}_p \cdot \mathrm{im}(\varpi))$ whilst on $\mathbb{Q}_p \cdot \ker(\varpi)$ there is a commutative diagram

$$\begin{array}{ccc} \mathbb{Q}_p\cdot\ker(\varpi) & \xrightarrow{\langle\lambda_{F,S,p},\varpi\rangle_{\iota_1,\iota_2}} & \iota_2(\mathbb{Q}_p\cdot Y^-_{F,S,p}) \\ {\scriptstyle\mu}\big\downarrow & & \big\| \\ \mathbb{Q}_p\cdot\ker(\varpi) & \xrightarrow{\langle\phi_{F,S,p},\varpi\rangle_{\iota_1,\iota_2}} & \iota_2(\mathbb{Q}_p\cdot Y^-_{F,S,p}) \end{array}$$

with $\mu := (\mathbb{Q}_p\otimes_{\mathbb{Z}_p}\iota)\circ(\mathbb{Q}\otimes_{\mathbb{Z}_p}\lambda_{F,S,p})\circ(\mathbb{Q}\otimes_{\mathbb{Z}_p}\phi_{F,S,p})^{-1}\circ(\mathbb{Q}_p\otimes_{\mathbb{Z}_p}\iota)^{-1}$.

For each normal subgroup N of G with $\tau\notin N$ we regard $\mathrm{Ir}_p^-(G/N)$ as a subset of $\mathrm{Ir}_p^-(G)$ in the obvious way and then set $\Upsilon_{S,N} := \{\psi\in\mathrm{Ir}_p^-(G/N) : r_{S,\psi,p} = r_{F^N}\cdot\psi(1)\}$.

Since $\Upsilon_{S,N}$ is closed under the action of $\mathrm{Aut}(\mathbb{C}_p)$ on $\mathrm{Ir}_p^-(G)$ the idempotent $e_{S,N} := \sum_{\chi\in\Upsilon_{S,N}} e_\chi$ belongs to $\zeta(\mathbb{Q}_p[G])$. In addition, Theorem 5.2(i) and (ii) combine to imply $e_{S,N} = e_{\mathrm{ss}}\cdot e_{S,N}$ and so one can multiply the last displayed equality by $e_{S,N}$ to deduce that

$$\begin{aligned} \theta^{r_{F^N}}_{p,F^N/k,S,T}(0) &= \theta^*_{p,F/k,S,T}(0)\cdot e_{S,N} \\ &= \theta^*_{p,F/k,S,T}(0)\cdot e_{\mathrm{ss}}e_{S,N} \\ &= u\mathscr{L}_S(F/k)\cdot e_{S,N}\mathrm{Nrd}_{\mathbb{Q}_p[G]}(e_{\mathrm{ss}}\langle\phi_{F,S,p},\varpi\rangle_{\iota_1,\iota_2}) \\ &= u\cdot\mathscr{L}_S(F/k)\mathrm{Nrd}_{\mathbb{Q}_p[G]}(e_{S,N}\langle\phi_{F,S,p},\varpi\rangle_{\iota_1,\iota_2}). \end{aligned}$$

Taken into account both this equality and the result of Theorem 3.6, the first claim in Corollary 3.10(iii) follows directly from the observation in Proposition 7.2 below (with N taken to be the identity subgroup of G).

Turning to the second claim of Corollary 3.10(iii) we note that $S\setminus V$ contains $S^{\mathrm{ram}}_{F/k}$ (and hence, in particular, S^k_∞). To deduce the second claim from the validity of the first claim of Corollary 3.10(iii) it is thus clearly enough to show that for any finite set of places Σ of k that contains $S^{\mathrm{ram}}_{F/k}$ one has

$$\mathrm{Fit}_{\mathbb{Z}_p[G]}(\mathrm{Sel}^T_\Sigma(F)_p) = \mathrm{Fit}_{\mathbb{Z}_p[G]}(\mathrm{Sel}^T_\Sigma(F)^{\mathrm{tr}}_p)^\#. \tag{40}$$

To prove this we recall from [4, Prop. 2.4] that $\mathbb{Z}_p\otimes_{\mathbb{Z}} R\Gamma_{c,T}((\mathcal{O}_{F,\Sigma})_{\mathcal{W}},\mathbb{Z})$ can be represented by a complex $P'\xrightarrow{\theta}P'$, where P' is a finitely generated free $\mathbb{Z}_p[G]$-module and the first term is placed in degree one. This representative combines with [4, Prop. 2.4(iii)] and [4, Def. 2.6] to imply that the $\mathbb{Z}_p[G]$-modules $\mathrm{Sel}^T_\Sigma(F)_p$ and $\mathrm{Sel}^T_\Sigma(F)^{\mathrm{tr}}_p$ are respectively isomorphic to the cokernels of θ and of $\theta^* := \mathrm{Hom}_{\mathbb{Z}_p}(\theta,\mathbb{Z}_p)$.

The explicit definition of Fitting invariants therefore implies that $\mathrm{Fit}_{\mathbb{Z}_p[G]}(\mathrm{Sel}^T_\Sigma(F)_p)$ and $\mathrm{Fit}_{\mathbb{Z}_p[G]}(\mathrm{Sel}^T_\Sigma(F)^{\mathrm{tr}}_p)$ are respectively generated over $\xi(\mathbb{Z}_p[G])$ by $\mathrm{Nrd}_{\mathbb{Q}_p[G]}(\theta)$ and $\mathrm{Nrd}_{\mathbb{Q}_p[G]}(\theta^*)$ and so (40) is true if $\mathrm{Nrd}_{\mathbb{Q}_p[G]}(\theta^*) = \mathrm{Nrd}_{\mathbb{Q}_p[G]}(\theta)^\#$.

The latter equality is then easily deduced from the fact that for any element x of $\mathbb{C}_p[G]$ and any character ψ in $\mathrm{Ir}_p(G)$ the matrix, with respect to any fixed $\mathbb{C}_p$-basis $\mathcal{X}$ of V_ψ, of the action of x on V_ψ is the transpose of the matrix, with respect to the basis that is dual to $\mathcal{X}$, of the action of $x^\#$ on $\mathrm{Hom}_{\mathbb{C}_p}(V_\psi,\mathbb{C}_p)$.

This completes the proof of Corollary 3.10.

Proposition 7.2. *For each normal subgroup N of G as above one has*

$$\xi(\mathbb{Z}_p[G/N])\cdot\mathscr{L}_S(F/k)\mathrm{Nrd}_{\mathbb{Q}_p[G]}(e_{S,N}\langle\phi_{F,S,p},\varpi\rangle_{\iota_1,\iota_2}) = \mathscr{L}_S(F/k)\cdot\mathrm{Fit}_{\mathbb{Z}_p[G/N]}(\mathrm{Sel}^T_{S\setminus V}(F^N)^{\mathrm{tr}}_p).$$

Proof. The exact sequence (38) implies that $\mathrm{cok}(\iota)$ is $\mathbb{Z}_p$-free and hence that the corresponding restriction map $\mathrm{Hom}_{\mathbb{Z}_p[G]}(P, Y^-_{F,S,p}) \to \mathrm{Hom}_{\mathbb{Z}_p[G]}(\mathcal{O}^{\times,-}_{F,S,T,p}, Y^-_{F,S,p})$ is surjective. We choose a lift $\hat{\phi}$ of $\phi_{F,S,p}$ through this restriction map and consider the composite homomorphism

$$\hat{\phi}_{(N)} : P^N = \mathrm{Tr}_N(P) \to \mathrm{Tr}_N(Y^-_{F,S,p}) = Y^-_{F^N,S,p} \to Y^-_{F^N,V_{F^N},p} \tag{41}$$

where the first arrow is the restriction of $\hat{\phi}$ and the second the natural projection map to the free $\mathbb{Z}_p[G/N]^-$-module $Y^-_{F^N,V_{F^N},p}$.

For any integer pair (s,t) in $[d]\times Z_{F^N}$ we define a unique element $M_{N,st}$ of $\mathbb{Z}_p[G/N]^-$ by means of the equality

$$\hat{\phi}_{(N)}(\mathrm{Tr}_N(b_s)) = \sum_{t\in Z_N} M_{N,st}\cdot(w_{i,F^N} - \overline{w_{i,F^N}}).$$

In addition, since the exact sequence (38) (and our explicit choice of ϱ) implies that

$$\mathrm{im}(\varpi^N) \subseteq \sum_{j\in[d]\setminus Z_N} \mathbb{Z}_p[G]^-\cdot\mathrm{Tr}_N(b_j),$$

for each (s,t) in $[d]\times([d]\setminus Z_{F^N})$ there exists a unique element $M_{N,st}$ of $\mathbb{Z}_p[G/N]^-$ which satisfies

$$\varpi(\mathrm{Tr}_N(b_s)) = \sum_{t\in[d]\setminus Z_{F^N}} M_{N,st}\cdot\mathrm{Tr}_N(b_t).$$

We then define M_N to be the matrix in $\mathrm{M}_d(\mathbb{Q}[G/N]^-)$ with each (i,j)-entry equal to the given element $M_{N,ij}$ and claim that

$$\mathscr{L}_S(F/k)\cdot\mathrm{Nrd}_{\mathbb{Q}_p[G/N]^-}(e_{S,N}\langle\phi_{F,S,p},\varpi\rangle_{\iota_1,\iota_2}) = \mathscr{L}_S(F/k)\cdot\mathrm{Nrd}_{\mathbb{Q}_p[G/N]^-}(M_N). \tag{42}$$

It is enough to prove this equality after multiplication by e_ψ for each ψ in $\mathrm{Ir}^-_p(G/N)$ and we assume first that $e_\psi e_{S,N} = 0$.

In this case the left hand side of (42) obviously vanishes. In addition, if $e_\psi e_{\mathrm{ss}} = 0$, then $e_\psi\mathscr{L}_S(F/k) = 0$ and so the right hand side vanishes. It is thus enough in this case to show that if both $e_\psi e_{\mathrm{ss}} = e_\psi$ and $e_\psi e_{S,N} = 0$, then $e_\psi\mathrm{Nrd}_{\mathbb{Q}_p[G/N]^-}(M_N)$ vanishes. But for any such character ψ one has $r_{S,\psi} = r_{S,\psi,p} > r_{F^N}\cdot\psi(1)$ and so the exactness of (38) implies $\dim_{\mathbb{C}_p}(\mathrm{Hom}_{\mathbb{C}_p[G]}(V_{\check{\psi}}, \mathbb{C}_p\cdot\mathrm{im}(\varpi))) = \dim_{\mathbb{C}_p}(\mathrm{Hom}_{\mathbb{C}_p[G]}(V_{\check{\psi}}, \mathbb{C}_p\cdot P)) - r_{S,\psi} < d - r_{F^N}$. This in turn implies that the matrix obtained by splitting $e_\psi M_N$ is singular and hence has zero determinant, as required.

We now assume $e_\psi e_{S,N} = e_\psi$, and hence $\psi\in\Upsilon_{N,S}$, and in this case we prove the validity of (42) after multiplication by e_ψ by showing that

$$\mathrm{Nrd}_{\mathbb{C}_p[G/N]e_\psi}(e_\psi(\mathbb{C}_p\otimes_{\mathbb{Q}_p}\langle\phi_{F,S,p},\varpi\rangle_{\iota_1,\iota_2})) = \mathrm{Nrd}_{\mathbb{C}_p[G/N]e_\psi}(e_\psi M_N).$$

To do this we note that for any ψ in $\Upsilon_{N,S}$ the second arrow in (41) is bijective and so

$$e_\psi(\mathbb{C}_p\otimes_{\mathbb{Q}_p}\langle\phi_{F,S,p},\varpi\rangle_{\iota_1,\iota_2}) = e_\psi(\iota_2\circ(\mathbb{C}_p\otimes_{\mathbb{Z}_p}\hat{\phi}_{(N)})\circ\hat{\iota}^N_1) + e_\psi(\mathbb{C}_p\otimes_{\mathbb{Z}_p}\varpi)$$

where $\hat{\iota}_1$ is the projection $\mathbb{C}_p\cdot P \to \mathbb{C}_p\cdot\ker(\varpi)$ induced by the chosen splitting ι_1, whilst $e_\psi(M_N)$ is the matrix, with respect to the basis $\{e_\psi(\mathrm{Tr}_N(b_i))\}_{i\in[d]}$ of $e_\psi(\mathbb{C}_p\cdot P)$, of the endomorphism

$$e_\psi(\iota_2\circ(\mathbb{C}_p\otimes_{\mathbb{Z}_p}\hat{\phi}_{(N)}))+e_\psi(\mathbb{C}_p\otimes_{\mathbb{Z}_p}\varpi).$$

To complete the proof of (42) it is enough to show that the last two displayed endomorphisms have the same reduced norm. This is true because they restrict to $e_\psi(\mathbb{C}_p\cdot\ker(\varpi))$ to give the same isomorphism $e_\psi(\mathbb{C}_p\cdot\ker(\varpi))\cong e_\psi(\iota_2(\mathbb{C}_p\cdot Y^-_{F,V_{F^N},p}))$ and also restrict to $e_\psi(\mathbb{C}_p\cdot\mathrm{im}(\iota_1))$ to induce the same composite homomorphism

$$e_\psi(\mathbb{C}_p\cdot\mathrm{im}(\iota_1)^N)\to e_\chi(\mathbb{C}_p\cdot P^N)\to e_\psi((\mathbb{C}_p\cdot P^N)/\iota_2(\mathbb{C}_p\cdot Y^-_{F,V_{F^N},p})).$$

Having proved (42) the claimed equality will now follow if we can show $\mathrm{Nrd}_{\mathbb{Q}_p[G/N]^-}(M_N)$ is a generator of the $\xi(\mathbb{Z}_p[G/N])$-module $\mathrm{Fit}_{\mathbb{Z}_p[G/N]^-}(\mathrm{Sel}^T_{S\setminus V}(F^N)^{\mathrm{tr},-}_p)$.

But, if we identify M_N as an endomorphism of $\mathbb{Z}_p[G/N]^{-,d}$ (by means of the fixed $\mathbb{Z}_p[G/N]^-$-basis $\{\mathrm{Tr}_N(b_i)\}_{i\in[d]}$ of P^N), then the tautological exact sequence

$$\mathbb{Z}_p[G/N]^{-,d}\xrightarrow{M_N}\mathbb{Z}_p[G/N]^{-,d}\to\mathrm{cok}(M_N)\to 0$$

combines with the definition of non-commutative Fitting invariants to imply directly that $\mathrm{Nrd}_{\mathbb{Q}_p[G/N]^-}(M_N)$ is a generator of the $\xi(\mathbb{Z}_p[G/N])$-module $\mathrm{Fit}_{\mathbb{Z}_p[G/N]^-}(\mathrm{cok}(M_N))$ and hence the claimed equality is a consequence of the isomorphism described in Lemma 7.3 below. $\square$

Lemma 7.3. *The $\mathbb{Z}_p[G/N]$-module $\mathrm{cok}(M_N)$ is isomorphic to $\mathrm{Sel}^T_{S\setminus V}(F^N)^{\mathrm{tr},-}_p$.*

Proof. The isomorphism $P^\bullet\to C_{F,T}$ in $D^{\mathrm{p}}(\mathbb{Z}_p[G])$ discussed just after (38) combines with the natural descent isomorphism $\mathbb{Z}_p[G/N]\otimes_{\mathbb{Z}_p[G]}C_{F,T}\cong C_{F^N,T}$ to induce an isomorphism in $D^{\mathrm{p}}(\mathbb{Z}_p[G/N])$ between $C_{F^N,T}$ and the complex $P^N\to P^N$, where the first module is placed in degree one and the differential is ϖ^N. Hence, after replacing F/k by F^N/k it is enough to consider the case that N is the trivial group.

To do this we set $r:=r_F$, write P_1 for the (free rank r) $\mathbb{Z}_p[G]^-$-submodule of P with basis $\{b_i\}_{i\in Z_F}$ and identify this with $Y^-_{F,V,p}$ by sending each b_i to $w_i-\overline{w_i}$. We also write $P_1^\bullet$ for the complex $P_1\xrightarrow{0}P_1$ with the first term in degree one.

We then choose a morphism of complexes of $\mathbb{Z}_p[G]$-modules $\alpha: P^\bullet\to P_1^\bullet$ that represents the composite morphism in $D^{\mathrm{p}}(\mathbb{Z}_p[G])$

$$P^\bullet\cong C_{F,T}\to\mathbb{Z}_p\otimes_{\mathbb{Z}}R\mathrm{Hom}_{\mathbb{Z}}(C,\mathbb{Z})[-3]^-\to R\mathrm{Hom}_{\mathbb{Z}_p}(\bigoplus_{i\in Z_F}R\Gamma_{\text{ét}}(\kappa_{w_i},\mathbb{Z}_p))^-,\mathbb{Z}_p[-2])\cong P_1^\bullet.$$

Here C denotes $R\Gamma_{c,T}((\mathcal{O}_{F,S})_{\mathcal{W}},\mathbb{Z})$, the first arrow is the isomorphism induced by Lemma 4.1(iii), the second is induced by applying the exact functor $\mathbb{Z}_p\otimes_{\mathbb{Z}}R\mathrm{Hom}_{\mathbb{Z}}(-,\mathbb{Z})[-3]^-$ to the triangle in [4, Prop. 2.4(ii)] and the third is the canonical isomorphism induced by our identification of P_1 with $Y^-_{F,V,p}$ and the fact that for each i in Z_F the place v_i splits completely in F/k.

With this definition $H^1(\alpha)$ coincides with the composite $\mathcal{O}^{\times,-}_{F,S,T,p}\xrightarrow{\phi_{F,S,p}}Y^-_{F,S,p}\to Y^-_{F,V,p}=P_1$ and so α^1 is a suitable choice for the morphism $\hat{\phi}$ that is fixed at the beginning of the proof of Proposition 7.2.

We note next that there is a short exact sequence of complexes (with horizontal differentials and the first term in the upper complex placed in degree two)

$$\begin{tikzcd} & P_1 \arrow[r, "\mathrm{id}"] \arrow[d, hook, "(-\mathrm{id},\iota_1)"'] & P_1 \arrow[d, "\mathrm{id}"] \\ P \arrow[r, "(\alpha^1,\varpi)"] \arrow[d, "\mathrm{id}"'] & P_1 \oplus P \arrow[r, "(0,\pi_1)"] \arrow[d, "(\iota_1,\mathrm{id})"] & P_1 \\ P \arrow[r, "M"] & P. & \end{tikzcd}$$

Here ι_1 and π_1 denote the natural inclusion and projection and M denotes the matrix M_N (with N the trivial group) constructed after fixing $\hat{\phi}$ to be α^0. Now the first complex in this sequence is obviously acyclic and the second is $\mathrm{Cone}(\alpha)[-1]$ and so the sequence induces an isomorphism of $H^1(\mathrm{Cone}(\alpha))$ with $\mathrm{cok}(M_N)$.

In addition, the exact triangle in [4, Prop. 2.4(ii)] induces an isomorphism in $D^{\mathrm{p}}(\mathbb{Z}_p[G])$

$$\mathrm{Cone}(\alpha) \cong \mathbb{Z}_p \otimes_{\mathbb{Z}} R\,\mathrm{Hom}_{\mathbb{Z}}(R\Gamma_{c,T}((\mathcal{O}_{F,S\setminus V})_{\mathcal{W}}, \mathbb{Z}), \mathbb{Z})[-2]^-$$

and hence an isomorphism of $\mathbb{Z}_p[G]$-modules

$$H^1(\mathrm{Cone}(\alpha)) \cong \mathbb{Z}_p \otimes_{\mathbb{Z}} H^{-1}(R\,\mathrm{Hom}_{\mathbb{Z}}(R\Gamma_{c,T}((\mathcal{O}_{F,S\setminus V})_{\mathcal{W}}, \mathbb{Z}), \mathbb{Z}))^-.$$

It therefore suffices to recall that $\mathrm{Sel}^T_{S\setminus V}(F)^{\mathrm{tr}}$ is defined (in [4, Def. 2.6]) to be equal to the cohomology group $H^{-1}(R\,\mathrm{Hom}_{\mathbb{Z}}(R\Gamma_{c,T}((\mathcal{O}_{F,S\setminus V})_{\mathcal{W}}, \mathbb{Z}), \mathbb{Z}))$. □

7.4. We now prove Corollary 3.11.

This proof relies on two facts concerning annihilators of finitely generated $\mathbb{Z}_p[G]$-modules M (that we shall use in the sequel without explicit comment). Firstly, the natural contragredient action of G on $M^\vee := \mathrm{Hom}_{\mathbb{Z}_p}(M, \mathbb{Z}_p)$ implies that $\mathrm{Ann}_{\mathbb{Z}_p[G]}(M^\vee) = \mathrm{Ann}_{\mathbb{Z}_p[G]}(M)^\#$. Secondly, if M has a quadratic presentation, then for any element a of $\mathcal{A}(G)$ there is an inclusion $a \cdot \mathrm{Fit}_{\mathbb{Z}_p[G]}(M) \subseteq \mathrm{Ann}_{\mathbb{Z}_p[G]}(M)$ (see [23, Th. 4.2]).

To prove claim (i) of Corollary 3.11 we note that the long exact cohomology sequence of the exact triangle constructed in [4, Prop. 2.4(ii)] induces a canonical surjective homomorphism $\mathrm{Sel}^T_{S\setminus V}(F)_p \to \mathrm{Sel}^T_{S_k^\infty}(F)_p$ and hence that the exact sequence (5) (with $S = S_k^\infty$) identifies $\mathrm{Cl}^T(F)_p^\vee$ as a subquotient of $\mathrm{Sel}^T_{S\setminus V}(F)_p$. This shows that

$$\mathrm{Ann}_{\mathbb{Z}_p[G]}(\mathrm{Sel}^T_{S\setminus V}(F)_p)^\# \subseteq \mathrm{Ann}_{\mathbb{Z}_p[G]}(\mathrm{Cl}^T(F)_p^\vee)^\# = \mathrm{Ann}_{\mathbb{Z}_p[G]}(\mathrm{Cl}^T(F)_p) \tag{43}$$

and hence that the second displayed equality in Corollary 3.10(iii) implies

$$\begin{aligned} a \cdot \theta^{(r)}_{p,F/k,S,T}(0) &= (a^\# \cdot \theta^{(r)}_{p,F/k,S,T}(0)^\#)^\# \\ &\in \mathscr{L}_S(F/k)(a^\# \cdot \mathrm{Fit}_{\mathbb{Z}_p[G]}(\mathrm{Sel}^T_{S\setminus V}(F)_p))^\# \\ &\in \mathscr{L}_S(F/k) \cdot \mathrm{Ann}_{\mathbb{Z}_p[G]}(\mathrm{Sel}^T_{S\setminus V}(F)_p)^\# \\ &\subseteq \mathscr{L}_S(F/k) \cdot \mathrm{Ann}_{\mathbb{Z}_p[G]}(\mathrm{Cl}^T(F)_p), \end{aligned} \tag{44}$$

where the second containment uses the fact that $\mathcal{A}(G)$ is stable under the involution $x \mapsto x^{\#}$. This proves claim (i).

To prove claim (ii) we note $\theta_{F/k,S\setminus V,T}(0) = \theta^{(r)}_{F/k,S,T}(0)\cdot R_S(F/k)^{-1}$ since each place in V splits completely in F/k.

The assumed validity of Conjecture 2.4 for F/k therefore implies that for each field isomorphism $j : \mathbb{C}_p \cong \mathbb{C}$ and each a in $\mathcal{A}(G)\cap\mathbb{Q}_p[G]^{\rm ss}$ one has

$$\begin{aligned} a\cdot j_*^{-1}(\theta_{F/k,S\setminus V,T}(0)) &= a\cdot j_*^{-1}(e_{\rm ss}\theta^{(r)}_{F/k,S,T}(0)\cdot R_S(F/k)^{-1})\\ &= a\cdot\theta^{(r)}_{p,F/k,S,T}(0)(\mathscr{L}_S(F/k)e_{\rm ss})^{-1}\\ &\in a\cdot e_{\rm ss}\mathrm{Fit}_{\mathbb{Z}_p[G]}(\mathrm{Sel}^T_{S\setminus V}(F)_p)^{\#}\\ &\subseteq \mathrm{Ann}_{\mathbb{Z}_p[G]}(\mathrm{Sel}^T_{S\setminus V}(F)_p)^{\#}\\ &\subseteq \mathrm{Ann}_{\mathbb{Z}_p[G]}(\mathrm{Cl}^T(F)_p)\end{aligned}$$

where the first equality and inclusion use $a = a\cdot e_{\rm ss}$, the second equality follows from (36), the containment from the first containment in (44) and the final inclusion from (43).

This proves the assertions in claim (ii) concerning $a\cdot\theta_{F/k,S\setminus V,T}(0)$ and the assertions concerning $a\cdot\theta_{F/k,S\setminus V,T}(0)^{\#}$ are then easily deduced by using the equality (40) with $\Sigma = S\setminus V$.

To prove claim (iii) we use the complex $R\Gamma_{c,T}((\mathcal{O}_{F,\Sigma})_{\mathcal{W}},\mathbb{Z})$. In particular, we recall (from [4, Prop. 2.4]) that this complex belongs to $D^{\rm p}(\mathbb{Z}[G])$, is acyclic outside degrees one and two and that its cohomology in these degrees identifies with $Y_{F,\Sigma}/\Delta_\Sigma(\mathbb{Z})$ and $\mathrm{Sel}^T_\Sigma(F)$ respectively, where Δ_Σ denotes the natural diagonal map.

We also note that a straightforward (non-abelian) extension of the argument used to prove [4, Prop. 3.4] shows that the equivariant Tamagawa number conjecture is valid for the pair $(h^0(\mathrm{Spec}(F)),\mathbb{Z}_p[G]^-)$ if and only for every isomorphism j as above one has

$$\delta_{\mathbb{Z}_p[G]^-,\mathbb{C}_p[G]^-}(j_*^{-1}(e_-\theta^*_{F/k,\Sigma,T}(0)^{\#})) = -\chi_{\mathbb{Z}_p[G]^-,\mathbb{C}_p[G]^-}(R\Gamma_{c,T}((\mathcal{O}_{F,\Sigma})_{\mathcal{W}},\mathbb{Z})^-_p, R^*_{\Sigma,j}) \tag{45}$$

where $R^*_{\Sigma,j}$ denotes the $\mathbb{C}_p$-linear dual of the exact sequence (28) (with S replaced by Σ).

In particular, under the hypotheses of claim (iii), Corollary 3.8(ii) implies that the equality (45) is valid.

To analyse this equality we represent $R\Gamma_{c,T}((\mathcal{O}_{F,\Sigma})_{\mathcal{W}},\mathbb{Z})_p$ by the complex $P'\xrightarrow{\theta}P'$ used in the proof of (40) and write e_0 for the idempotent of $\zeta(\mathbb{Q}_p[G])$ obtained by summing e_ψ over all ψ in $\mathrm{Ir}^-_p(G)$ for which the space $\mathrm{Hom}_{\mathbb{C}_p[G]}(V_{\check\psi},\mathbb{C}_p\cdot\ker(\theta))$ vanishes.

Then it is clear that $\mathrm{Nrd}_{\mathbb{Q}_p[G]}(\theta^-) = e_0\mathrm{Nrd}_{\mathbb{Q}_p[G]}(\theta)$ whilst [2, Lem. A.1(iii)] implies

$$-\chi_{\mathbb{Z}_p[G]^-,\mathbb{C}_p[G]^-}(R\Gamma_{c,T}((\mathcal{O}_{F,\Sigma})_{\mathcal{W}},\mathbb{Z})^-_p, R^*_{\Sigma,j}) = \delta_{\mathbb{Z}_p[G]^-,\mathbb{C}_p[G]^-}(\mathrm{Nrd}_{\mathbb{C}_p[G]^-}(\langle\theta\rangle))$$

for an automorphism $\langle\theta\rangle$ of the $\mathbb{C}_p[G]$-module $e_-(\mathbb{C}_p\cdot P')$ which agrees with $\mathbb{C}_p\otimes_{\mathbb{Z}_p}\theta$ on $e_0(\mathbb{C}_p\cdot P')$ and hence satisfies $e_0\mathrm{Nrd}_{\mathbb{C}_p[G]}(\langle\theta\rangle) = e_0\mathrm{Nrd}_{\mathbb{Q}_p[G]}(\theta) = \mathrm{Nrd}_{\mathbb{Q}_p[G]}(\theta^-)$.

The last displayed equality combines with (45) and the exactness of the upper row in (37) to imply the existence of an element u of $\mathrm{Nrd}_{\mathbb{Q}_p[G]^-}(K_1(\mathbb{Z}_p[G]^-))$ with $j_*^{-1}(e_-\theta^*_{F/k,\Sigma,T}(0)^{\#}) =$

$u \cdot \mathrm{Nrd}_{\mathbb{C}_p[G]^-}(\langle\theta\rangle)$ and hence also

$$j_*^{-1}(e_-\theta_{F/k,\Sigma,T}(0)^{\#}) = e_0(j_*^{-1}(e_-\theta^*_{F/k,\Sigma,T}(0)^{\#}) = u \cdot e_0\mathrm{Nrd}_{\mathbb{C}_p[G]^-}(\langle\theta\rangle) = u \cdot \mathrm{Nrd}_{\mathbb{Q}_p[G]^-}(\theta^-)$$

where the first equality follows from (1) and the isomorphism $\mathrm{Hom}_{\mathbb{C}_p[G]}(V_{\check{\psi}}, \mathbb{C}_p \cdot \ker(\theta)) \cong \mathrm{Hom}_{\mathbb{C}_p[G]}(V_{\check{\psi}}, \mathbb{C}_p \cdot Y^-_{F,\Sigma})$.

In particular, since $\mathrm{Sel}^T_\Sigma(F)_p$ is naturally isomorphic to $\mathrm{cok}(\theta)$, the above equality implies that $j_*^{-1}(e_-\theta_{F/k,\Sigma,T}(0)^{\#})$ is a generator over $\xi(\mathbb{Z}_p[G])$ of $\mathrm{Fit}_{\mathbb{Z}_p[G]^-}(\mathrm{Sel}^T_\Sigma(F)^-_p)$, as required to prove the displayed equality in Corollary 3.11(iii).

The remaining assertions of claim (iii) are then derived from these facts by the same arguments used to prove claim (ii).

This completes the proof of Corollary 3.11.

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King's College London, Department of Mathematics, London WC2R 2LS, U.K.
E-mail address: `david.burns@kcl.ac.uk`