# ON DETERMINATION OF THE CLASS OF SATURATION IN THE THEORY OF APPROXIMATION OF FUNCTIONS II

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**1. Introduction.** Let $f(x)$ be an integrable function, with period $2\pi$ and let its Fourier series be

$$\mathfrak{S}[f] \equiv \sum_{k=0}^{\infty} A_k(x) \equiv \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx) \tag{1}$$

Let $g_k(n)$ $(k = 0, 1, 2, \ldots)$, $g_0(n) = 1$ be the "summating" function and consider a family of transforms of (1) by a method of summation $G$

$$P_n(x) = \sum_{k=0}^{\infty} g_k(n) A_k(x) \tag{2}$$

where the parameter $n$ need not be discrete.

If there are a positive non-increasing function $\varphi(n)$ and a class $K$ of functions in such a way that

(I) $\quad \|f(x) - P_n(x)\| = o(\varphi(n))$ implies $f(x) =$ constant;

(II) $\quad \|f(x) - P_n(x)\| = O(\varphi(n))$ implies $f(x) \in K$

(III) $\quad f(x) \in K$ implies $\|f(x) - P_n(x)\| = O(\varphi(n))$

then it is said that the method of summation $G$ is saturated with the order $\varphi(n)$ and with the class of saturation $K$.

Since the above definition was given by J. Favard [3], a number of authors have published their result: G. Alexits [0], P. L. Butzer, [2], J. Favard himself [4], M. Zamansky [9] and others.

The purpose of the present paper lies in giving proofs to the theorems stated in our previous paper [8].

Throughout the paper the norms should be taken *with respect to the variable* $x$, and the subscript $p$ to $L^p$-norms will generally be omitted. Another convention is that the space $(C)$ is meant the notation $L^\infty$. (or, the case $p = \infty$ of $L^p$). Thus the generalized Minkowski inequality reads

$$\left\| \int f(x, t) dt \right\| \leqq \int \| f(x, t) \| \, dt, \qquad p \geqq 1$$

and the class $\mathrm{Lip}(\alpha, p)$ with $p = \infty$ reduces to Lip $\alpha$.

**2. The inverse problem.** Let us write $\Delta_n(x) = f(x) - P_n(x)$ and suppose that for some positive function $\psi(k)$ and a positive constant $c$, we have

(3) $$\lim_{n\to\infty} \frac{1-g_k(n)}{\varphi(n)} = c\psi(k) \qquad (k=1,2\ldots\ldots,).$$[1)]

(i) If $\| \Delta_n(x) \| = o(\varphi(n))$ we have for every fixed $k \geqq 1$,

$$a_k(1-g_k(n)) = \frac{1}{\pi}\int_{-\pi}^{\pi} \Delta_n(x) \cos kx\, dx = o(\varphi(n))$$

and comparing this with (3), we see

$a_k = 0$ and similarly $b_k = 0$. $(k = 1, 2, \ldots\ldots)$.

Thus the condition (I) is verified under our assumption.

(ii) Suppose now $\| \Delta_n(x) \| = O(\varphi(n))$ and let $N < n$.

Taking the $N$-th arithmetic mean $\sigma_N[x; \Delta_n]$ of the series

$$\Delta_n(x) \sim \sum_{k=1}^{\infty} (1-g_k(n))A_k(x)$$

we have

$$\sigma_N[x; \Delta_n] = \sum_{k=1}^{N} (1-g_k(n))\left(1-\frac{k}{N+1}\right)A_k(x).$$

It is well known that $\| \Delta_n \| \geqq \| \sigma_N[x; \Delta_n] \|$ and our hypothesis on $\Delta_n(x)$ yields

$$\left\| \sum_{k=1}^{N} (1-g_k(n))\left(1-\frac{k}{N+1}\right)A_k(x) \right\| = O(\varphi(n))$$

or, equivalently,

$$\left\| \sum_{k=1}^{n} \frac{1-g_k(n)}{\varphi(n)}\left(1-\frac{k}{N+1}\right)A_k(x) \right\| = O(1),$$

which implies (using Fatou's lemma if necessary)

$$\left\| \lim \sum_{k=1}^{N} \frac{1-g_k(n)}{\varphi(n)}\left(1-\frac{k}{N+1}\right)A_k(x) \right\| = O(1)$$

i. e.

(4) $$\left\| \sum_{k=1}^{N} \psi(k)\left(1-\frac{k}{N+1}\right)A_k(x) \right\| = O(1).$$

If $K$ denotes the class of functions satisfying (4) we have (II) for this class $K$.

For most of methods of summation, the function $\psi(k)$ has the form $k^\rho$, where $\rho$ is a positive integer, and the degree of approximation has been studied for those classes, resulting the relation (III). If we denote by $f^{[\rho]}(x)$

1) This assumption can be slightly relaxed.

the trigonometric series $\Sigma k^p A_k(x)$, the class $K$ will be the set of functions with $\| \sigma_N[x; f^{[p]}] \| = O(1)$ and this is equivalent to the assertions

$f^{[p]}(x)$ is the Fourier series of a bounded function ($p = \infty$)

$f^{[p]}(x)$ is the Fourier series of a function in $L^p$ ($1 < p < \infty$)

$f^{[p]}(x)$ is the Fourier-Stieltjes series of a function of bounded variation ($p = 1$)

respectively. See for example [10, §§ 4. 31 – 4. 33].

These classes are also characterized by the property that the indefinite integral of $f^{[p]}(x)$ belongs to the class Lip$(1, p)$. (See for example [10, § 4. 7; examples 7 and 8])

### 3. Determination of the class of Saturation.

**3. 1.** We have, Cesàro-Fejér method of summability,

$$P_n(x) = \sum_{k=0}^{n} \left(1 - \frac{k}{n+1}\right) A_k(x) = \frac{1}{2(n+1)\pi} \int_{-\pi}^{\pi} f(x+t) \left\{ \frac{\sin (n+1)t/2}{\sin t/2} \right\}^2 dt$$

$$g_k(n) = \left(1 - \frac{k}{n+1}\right), \ \lim_{n \to \infty} n(1 - g_k(n)) = k.$$

The considerations of the preceding section give

(i) if $\| \Delta_n(x) \| = o(1/n)$ then $f(x) = $ constant;

(ii) if $\| \Delta_n(x) \| = O(1/n)$ then $\| \sigma_n(x; f^{[1]}) \| = O(1)$

or, equivalently, $f(x) \in$ Lip$(1, p)$.

Since the condition (III) was already proved by A. Zygmund [12], we have

THEOREM 1. *The method of Cesàro-Fejér summation is saturated; its order of saturation is* $1/n$, *its class of saturation is the class of functions* $f(x)$ *for which* $\tilde{f}(x) \in$ Lip $(1, p)$, *where* $p$ *is the suffix to the norm considered.*

More generally,

THEOREM 1′. *The method of approximation by* $(C, \alpha)$ $(\alpha > 0)$ *means is saturated with the same order and the same class to the case of Cesàro-Fejér summation.*

Indeed, we can easily verify the first two conditions, and the third was also proved in [12].

**3. 2.** The Abel-Poisson mean of $\mathfrak{S}[f]$ is

$$P_r(x) = \sum_{k=0}^{\infty} A_k(x) r^k = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x+t) \frac{(1-r^2)\, dt}{1 - 2r \cos t + r^2} \quad (0 \leqq r < 1)$$

$$g_k(r) = r^k \qquad \text{and} \qquad \lim_{r\to 1-0} \frac{1 - g_k(r)}{1 - r} = k,$$

we have

THEOREM 2. *The method of Abel-Poisson summability is saturated with the order* $(1 - r)$ *and the same class to the Cesàro-Fejér summation.*

PROOF. We have only to prove the assertion (III), and for this purpose it is sufficient to prove $\|\tilde{f}(x) - \tilde{f}(r, x)\| = O(1 - r)$ under the assumption $\|\psi_x(t)\| \equiv \|f(x + t) - f(x - t)\| = O(t)$. But, an elementary computation shows

$$f(r, x) - f(x) = \frac{4(1-r)^2}{\pi}\int_0^\pi \frac{\psi_x(t)\,dt}{(1-2r\cos t + r^2)2\tan t/2}$$
$$= \frac{2(1-r)^2}{\pi}\int_0^\pi \frac{\psi_x(t)dt}{\{(1-r)^2 + 4r\sin^2 t/2\}\tan t/2}.$$

Thus we have

$$\|\tilde{f}(r, x) - \tilde{f}(x)\| \leqq A(1-r)^2\int_0^\pi \frac{\|\psi_x(t)\|dt}{\{(1-r)^2 + 4r\sin^2 t/2\}\tan t/2}$$
$$= A(1-r)^2\left(\int_0^{1-r} + \int_{1-r}^\pi\right) \equiv A(1-r)^2(J_1 + J_2),\text{[2)]}$$

where

$$J_1 \equiv \int_0^{1-r} \frac{\|\psi_x(t)\|dt}{\{(1-r)^2 + 4r\sin^2 t/2\}\tan t/2}$$
$$\leqq A\int_0^{1-r} \frac{t}{(1-r)^2 t}\,dt = \frac{A}{1-r}$$

and

$$J_2 \equiv \int_{1-r}^\pi \frac{\|\psi_x(t)\|dt}{\{(1-r)^2 + 4r\sin^2 t/2\}\tan t/2}$$
$$\leqq A\int_{1-r}^\pi \frac{t}{t^2 t}\,dt \leqq A\int_{1-r}^\infty \frac{dt}{t^2} = \frac{A}{1-r}$$

which was to be proved.[3)]

**3. 3.** The Riesz mean $(R, n^\rho, \lambda)$ of $\mathfrak{S}[f]$ is

$$R_n(x) = \sum_{k=0}^{n-1}\left(1 - \left(\frac{k}{n}\right)^\rho\right)^\lambda A_k(x) \text{ and } g_k(n) = \left(1 - \left(\frac{k}{n}\right)^\rho\right)^\lambda$$

THEOREM 3. *For spaces* $L^p$, $1 < p \leqq \infty$, *the method of Riesz summabity* $(R, n^\rho, \lambda)$ *is saturated; its order of saturation is* $1/n^\rho$, *its class of*

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2) $A$ denotes a constant which need not the same at different contexts.
3) This is also deduced from the equation of Cauchy-Riemann.

*saturation is the class of functions $f(x)$ for which*

$$f^{[\rho]}(x) \in B \qquad (p = \infty)$$
$$f^{[\rho]}(x) \in L^p \qquad (1 < p < \infty)$$

*where $f^{[\rho]}(x)$ denotes the trigonometric series $\sum k^\rho A_k(x)$.*

PROOF. Since the assertions (I) and (II) are obviously verified, we may confine ourselves to the proof of (III). The case $p = \infty$ i.e., the fact $f^{[\rho]}(x) \in B$ implies $\| \Delta_n(x) \| \leqq \frac{M}{n^\rho} \| f^{[\rho]} \|$ is due to S. Nagy [6]. Now, if $f \in L^2$, we have by Parseval's identity.

$$\begin{aligned} \|\Delta_n(x)\|_2^2 &= \sum_{k=0}^{n} \left\{1 - \left(1 - \left(\frac{k}{n}\right)^\rho\right)^\lambda\right\}(a_k^2 + b_k^2) + \sum_{k=n+1}^{\infty} (a_k^2 + b_k^2) \\ &\leqq A(\lambda, \rho) \sum_{k=0}^{n} \frac{k^{2\rho}}{n^{2\rho}}(a_k^2 + b_k^2) + \sum_{k=n+1}^{\infty} (a_k^2 + b_k^2) \\ &\leqq A(\lambda, \rho) \sum_{k=0}^{\infty} \frac{k^{2\rho}}{n^{2\rho}}(a_k^2 + b_k^2) \\ &= \frac{A(\lambda, \rho)}{n^{2\rho}} \|f^{[\rho]}\|_2^2. \end{aligned}$$

The operation $Tf^{[\rho]}(x) = \Delta_n(x)$ being linear, the well-known convexity theorem of M. Riesz gives

$$\| \Delta_n(x) \|_p \leqq \frac{A(\lambda, \rho)}{n^\rho} \|f^{[\rho]}\|_p \qquad (2 \leqq p \leqq \infty).$$

The case $1 < p \leqq 2$ can be treated by the familiar "conjugacy" argument. Let $1 < p \leqq 2$ and $1/p + 1/q = 1$, (so that $2 \leqq q < \infty$) and let $g^{[\rho]}(x) = \sum_{k=1}^{N} k^\rho B_k(x)$ be a trigonometric polynomial with $\|g^{[\rho]}\|_q \leqq 1$. Then we have, by Hölder's inequality and the theorem already proved for the exponent $q$,

$$\begin{aligned} (5) \qquad \left|\int_{-\pi}^{\pi} \Delta_n(x; f) g^{[\rho]}(x)\, dx\right| &= \left|\int_{-\pi}^{\pi} T f^{[\rho]}(x) g^{[\rho]}(x)\, dx\right| \\ &= \left|\int_{-\pi}^{\pi} f^{[\rho]}(x) T g^{[\rho]}(x)\, dx\right| \leqq \| f^{[\rho]}\|_p \, \| T g^{[\rho]}\|_q \\ &\leqq \frac{A(\lambda, \rho)}{n^\rho} \| f^{[\rho]}\|_p \, \| g^{[\rho]}\|_q \leqq \frac{A(\lambda, \rho)}{n^\rho} \| f^{[\rho]}\|_p. \end{aligned}$$

Since (5) holds for every trigonometric polynomial $g^{[\rho]}(x)$, $\|g^{[\rho]}(x)\|_q \leqq 1$, this implies $\|\Delta_n(x; f)\|_p \leqq \frac{A(\lambda, \rho)}{n^\rho} \| f^{[\rho]}\|_p$ and our assertion is proved.

COROLLARY. *If $\rho$ is a positive integer, the class of saturation of the* *method of Riesz summability* $(R, n^{\rho}, 1)$ *is the class of those functions* $f(x)$ *for which* $f^{(\rho-1)}(x) \in \mathrm{Lip}\,(1, p)$ *if* $\rho$ *is even* $f^{(\rho-1)}(x) \in \mathrm{Lip}\,(1, p)$ *if* $\rho$ *is odd.*

The part (III) of this was proved by A. Zygmund [11], but we give another proof for the case $p = 1$[4].

Let $P_n(x)$, $\sigma_n(x)$, and $K_n(x)$ be the $(R, n^{\rho}, 1)$ means of the series (1), $\sum k^{\rho} A_k(x)$ and $\frac{1}{2} + \sum \cos nx$ respectively. Assuming that $f^{[\rho]}(x)$ is of bounded variation, we have

$$\sigma_n(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} K_n(x-t)\,df^{[\rho]}(t)$$

and

$$\| \sigma_n(x) \| \leqq \frac{1}{\pi}\int_{-\pi}^{\pi} \| K_n(x-t)\| \,|\, df^{[\rho]}(t) \,| = O(1). \tag{6}$$

Write

$$\Lambda_n \equiv n^{\rho},\ \lambda_n \equiv \Lambda_n - \Lambda_{n-1} > 0, \qquad k^{\rho}A_k(x) \equiv B_k(x).$$

$$s_n \equiv \sum_{k=0}^{n} B_k(x) \ \text{ and } \ s_n^* = \sum_{k=0}^{-1} \lambda_{k+1}s_k \quad (\lambda_{-1} \equiv 0).$$

We have

$$\sigma_n(x) = \sum_{k=0}^{n}\Big(1 - \frac{\Lambda_k}{\Lambda_n}\Big)B_k(x) = \frac{s_n^*}{\Lambda_n} \tag{7}$$

$$P_n(x) = \sum_{k=0}^{n}\Big(1 - \frac{\Lambda_k}{\Lambda_n}\Big)A_k(x) = \sum_{k=0}^{n}\Big(1 - \frac{\Lambda_k}{\Lambda_n}\Big)\frac{B_k(x)}{\Lambda_k}$$

thus

$$P_n - P_{n-1} = \sum_{k=0}^{n}\Big(1 - \frac{\Lambda_k}{\Lambda_n}\Big)\frac{B_k}{\Lambda_k} - \sum_{k=0}^{n-1}\Big(1 - \frac{\Lambda_k}{\Lambda_{n-1}}\Big)\frac{B_k}{\Lambda_k}$$

$$= s_{n-1}\Big(\frac{1}{\Lambda_{n-1}} - \frac{1}{\Lambda_n}\Big) = \frac{\lambda_n s_{n-1}}{\Lambda_n\Lambda_{n-1}}$$

Summing up this equality for $N < n \leqq M$, we see

$$P_M - P_N = \sum_{n=N+1}^{M} \frac{\lambda_n s_{n-1}}{\Lambda_n\Lambda_{n-1}}$$

$$= \sum_{n=N+1}^{M-1} s_n^*\Big(\frac{1}{\Lambda_n\Lambda_{n-1}} - \frac{1}{\Lambda_{n+1}\Lambda_n}\Big) + \frac{s_M^*}{\Lambda_M\Lambda_{M-1}} - \frac{s_N^*}{\Lambda_N\Lambda_{N-1}}.$$

Consequently, using (6) and (7),

$$\|P_M - P_N\| = \sum_{n=N+1}^{M-1} \frac{O(\Lambda_n)(\Lambda_{n+1} - \Lambda_{n-1})}{\Lambda_{n+1}\Lambda_n\Lambda_{n-1}} + \frac{O(\Lambda_M)}{\Lambda_M\Lambda_{M-1}} + \frac{O(\Lambda_N)}{\Lambda_N\Lambda_{N-1}}$$

4) $\rho$ need not be an integer in the following proof.

$$= O(1)\left(\sum_{n=N+1}^{M-1}\left(\frac{1}{\Lambda_{n-1}} - \frac{1}{\Lambda_{n+1}}\right) + \frac{1}{\Lambda_{M-1}} + \frac{1}{\Lambda_{N-1}}\right).$$

Making $M \to \infty$ we have

$$\|f(x) - P_N(x)\| = O(1/\Lambda_{N-1}) = O(1/(N-1)^\rho) = O(1/N^\rho)$$

which was to be proved.

**3. 4.** The Gauss-Weierstrass integral of $f(x)$ is

$$W(x;\xi) = \sum_{k=0}^{\infty} \exp(-k^2\xi/4)\,A_k(x) = \sqrt{\frac{\pi}{\xi}}\int_{-\pi}^{\pi} f(x+t)\exp(-t^2/\xi)\,dt$$

$g_k(\xi) = \exp(-k^2\xi/4)$, the parameter $\xi$ tending to 0. We have

THEOREM 4. *The method of approximation by the Gauss-Weierstrass integral is saturated; its order of saturation is $\xi$; its class of saturation is the class of functions $f(x)$ for which*

$$f'(x) \in \mathrm{Lip}(1, p).$$

PROOF. Only the assertion (III) requires the proof. But,

$W(x;\xi) - f(x)$

$$= \sqrt{\frac{\pi}{\xi}}\int_{-\pi}^{\pi}(f(x+t) + f(x-t) - 2f(x))\exp(-t^2/\xi)\,dt - R(x,\xi),$$

say, where $R(x,\xi) = \left(\int_{-\infty}^{\pi} + \int_{\pi}^{\infty}\right)\frac{2\pi}{\sqrt{\xi}} f(x)\exp(-t^2/\xi)\,dt$

$$\|R(x,\xi)\| = 2\|f(x)\|\sqrt{\frac{\pi}{\xi}}\int_{\pi}^{\infty}\exp(-t^2/\xi)\,dt$$

$$\leqq \|f(x)\|\int_{\pi^2/\xi}^{\infty} e^{-u}\,du = \|f(x)\|\exp(-\pi^2/\xi) = o(\xi).$$

Consequently

$$\left\|\sqrt{\frac{\pi}{\xi}}\int_0^{\pi}(f(x+t) + f(x-t) - 2f(x))\exp(-t^2/\xi)\,dt\right\|$$

$$\leqq \sqrt{\frac{\pi}{\xi}}\int_0^{\pi}\|f(x+t) + f(x-t) - 2f(x)\|\exp(-t^2/\xi)\,dt$$

$$\leqq \frac{A}{\sqrt{\xi}}\int_0^{\pi} t^2\exp(-t^2/\xi)\,dt = O(\xi)$$

which was to be proved.

The Bernstein-Rogosinski mean of $\mathfrak{S}(f)$ is defined by

$$B_n(x) = \frac{1}{2}\left\{S_n\left(x + \frac{\pi}{2n+1}\right) + S_n\left(x - \frac{\pi}{2n+1}\right)\right\} = \sum_{k=0}^{n}\cos\frac{k\pi}{2n+1}A_k(x).$$

THEOREM 5. *The method of Bernstein-Rogosinski summation is saturated; its order of saturation is $1/n$, and its class of saturation is the class of functions $f(x)$ for which*

$$f'(x) \in \mathrm{Lip}(1, p) \qquad 1 \leqq p \leqq \infty.$$

We omit the proof since the reader will find no difficulty in modifying the proof of (III) of the case $p = \infty$ in Natanson [7, p. 192] and a generalization of our theorem was published recently (see F. Harsiladze [5]).

The integral of de la Vallée Poussin is defined by

$$V_n(x) = \frac{h_n}{2\pi}\int_{-\pi}^{\pi} f(x+t)\cos^{2n}\frac{t}{2}\,dt$$

$$= \sum_{k=0}^{n} \frac{(n!)^2}{(n-k)!(n+k)!} A_k(x), \qquad h_n = \frac{2n(2n-2)\ldots 4.\,2}{(2n-1)(2n-3)\ldots 3.\,1},$$

$$g_k(n) = \frac{(n!)^2}{(n-k)!(n+k)!} = 1 - \frac{k^2}{n} + O\left(\frac{1}{n^2}\right).$$

THEOREM 6. *The method of approximation by the integral of de la Vallée Poussin is saturated; its order of saturation is* $1/n$, *its class of saturation is the class of functions* $f(x)$ *for which*

$$f'(x) \in \mathrm{Lip}(1, p) \qquad 1 \leqq p \leqq \infty.$$

PROOF. The assertion (III) is due to P. L. Butzer [1], and the other two are evidently verified by the consideration of §2.

The integral of Jackson-de la Vallée Poussin is defined by

$$I_n(x) = \frac{1}{2\pi\tau_4}\int_{-\infty}^{\infty} f\left(x + \frac{2t}{n}\right)\left(\frac{\sin t}{t}\right)^4 dt \qquad \left(\tau_4 = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{\sin t}{t}\right)^4 dt\right)$$

$$= \sum_{k=0}^{2n-1} h\left(\frac{k}{n}\right) A_k(x)$$

where

$$h(x) = \begin{cases} 1 - \frac{3}{2}x^2 + \frac{3}{4}|x|^3 & |x| \leqq 1 \\ \frac{1}{4}(2 - |x|)^3 & 1 \leqq |x| \leqq 2 \\ 0 & |x| \geqq 2. \end{cases}$$

THEOREM 7. *The method of approximation by the Jackson-de la Vallée-Poussin integral is saturated; its order of saturation is* $1/n^2$, *its class of saturation is the class of functions* $f(x)$ *for which*

$$f'(x) \in \mathrm{Lip}(1, p) \qquad 1 \leqq p \leqq \infty.$$

To verify the condition (III), we have only to calculate directly, starting from the assumption that

$$\|f(x+u)+f(x-u)-2f(x)\| = O(u^2).$$

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