ON DETERMINING A DISTRIBUTION FUNCTION KNOWN ONLY BY ITS MOMENT AND/OR MOMENT GENERATING FUNCTION

by

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CHAPTER I

PROBLEM STATEMENT AND BACKGROUND

The problem studied in this dissertation is:

> Under what conditions can we obtain a representation for a probability density function (p.d.f.) or cumulative distribution function (c.d.f.) when the function is described in terms of:
>
> (1) a moment generating function (M.G.F.), range, and continuity, and/or
>
> (2) the first n moments, range, and continuity?
>
> Further, how do we actually achieve this representation in an efficient manner?

In Chapter II we will discuss what is meant by an efficient manner.

Motivation for the current research effort comes from the fact that system descriptions in the form of M.G.F.'s arise quite naturally when the system is described as a Semi-Markov process or as a GERT network [33], [42]. Thus, if these techniques are to be truly effective we must have a general method to obtain some sort of reasonably accurate and useful expression for the corresponding p.d.f.

or c.d.f.

Actually, this problem has been solved theoretically, at least to the mathematician's satisfaction. However, computationally no effective solution procedure has yet been proposed. Thus, we address ourselves to the computational aspects of the problem.

The problem statement implies that two distinct approaches are possible. Suppose we are given the M.G.F. Then if the function is continuous on the positive half line $(0,\infty)$ we can replace the dummy variable, s, in the M.G.F. by -s and the problem becomes one of finding the inverse Laplace transform. Alternatively the moments can be obtained from the M.G.F. and a p.d.f. representation constructed using the moments. In the latter approach there are two subproblems: First we must decide if the moments describe a p.d.f. and if there is a maximum number of moments that constitute a description. For example, assume four moments provide a complete description, then serious difficulties will be encountered if we try to approximate the p.d.f. with six moments. This portion of the problem is called the reduced problem of moments when the number of moments is finite, otherwise it is called the problem of moments. After the number of moments is decided on we can employ them to construct an expression for the p.d.f. This approach is generally referred to as the method of moments. Note that although the moments can be obtained from the M.G.F. the converse is not true in any useful sense. Both approaches are discussed below.

Historical Development

The problem of moments seems to have been first investigated by Stieltjes in 1894 [2 ,v], [18,ix]. However, a closely related problem, fundamental to probability theory, was studied by Tchebycheff in a series of papers which started in 1855 [18,ix]. Briefly stated the problem is, given

$$\int_{-\infty}^{\infty} x^n p(x)dx = \int_{-\infty}^{\infty} x^n e^{-x^2} dx \ , \quad n = 0,1,2,\ldots$$

can it be concluded that $p(x) = e^{-x^2}$? Tchebycheff's work also started the research in the general field of orthogonal polynomials. Only the classical orthogonal polynomials of Legendre, Jacobi, Laguerre and Hermite were known before Tchebycheff [18, ix]. All of the techniques that we will study in this paper either use or are strongly related to orthogonal polynomials.

As stated above we are interested in the reduced problem of moments which is the problem of moments except that we know only a finite number of the moments. Again problems related to this one were formulated by Tchebycheff and investigated by Markov [18, 77]. However, the solution to the problem in a form which is of interest to us was obtained independently by Achyeser and Krein [1] in 1938, and Verblunsky [38] in 1949.

A more detailed exposition of the history of the problem of moments can be found in Shohat and Tamarkin [18].

Many techniques have been proposed which use only the moments and range of a distribution to determine an analytical expression or table of values for the distribution. However, we will be interested only in certain of the so called generalized frequency curves [17,46] which will be discussed below. Related techniques are included in an annotated bibliography, Appendix A.

There are two distinct classes of the generalized frequency curves. The first is an infinite series in which the first term of the series is one of the common p.d.f.'s. This approach has its origin in the work of Gram which appeared in 1879 [40]. His work, based on the Normal distribution, resulted in what is now called the Gram-Charlier A series. This series also involves Hermite polynomials. In 1905 Charlier derived the Gram series from a different point of view and derived another series based on the Poisson distribution. The latter series is called the Gram-Charlier B series and involves the Poisson-Charlier polynomials [17]. In 1924 Romanovsky [35] derived series expansions based on the Pearson system of frequency curves. The one of interest to us is the expansion based on the Type III p.d.f. This expansion utilizes the Laguerre polynomials. The Type III p.d.f. is similar to the Gamma p.d.f. which also results in the Laguerre polynomials. The second class of generalized frequency curves is the above mentioned Pearson family of thirteen curves. The first seven members of the family were given in a paper by Pearson which appeared in 1895 [31]. Between 1895-1916 Pearson added the remaining six members and developed techniques for fitting these curves to data. Pearson's curves include the Normal, Exponential and Gamma p.d.f.'s as special cases.

It is much more difficult to trace the history of the problem of inverting Laplace transforms. The Laplace transform was devised by P. S. Laplace in the 1820's, but he seems to have had little interest in inverting them. The first person to take an interest in the inversion problem was Heaviside in the 1890's, and he was motivated, as we are, by the need for solving practical problems [19,vii]. His work resulted in the well known Heaviside partial fraction expansion technique. The problems of interest to us are not amenable to this solution technique.

In the 1930's mathematicians began to take an interest in inversion formulas of the type that specify a p.d.f. by evaluating its transform at selected points along the real axis, for example see [22] or [39]. However, their interest was not computational and thus the results lend us little comfort.

Two events set the stage for the efficient inversion techniques currently available. The first was the appearance of the now classical text by Gardner and Barnes [8]. This book stimulated the use of the Laplace transform to study the linear systems of interest to engineers and physicists, that is, the need was created. The second event was the advent of the digital computer. This provided the means for solving the problem. Many inversion techniques are available [3], [26], [29], [30], and [37]. The most efficient of these [26] is derived and illustrated in Chapter V.

Research Approach

The research performed for this dissertation led to a two part procedure for solving the problem. The first part is a combination, integration and extension of the theories on the reduced problem of moments and the generalized frequency curves. In the second part, we use a numerical technique for inverting the Laplace transform of the unknown p.d.f. An introductory discussion of the reasoning behind each part is given below.

It is well known that whenever a M.G.F. exists it uniquely determines the distribution function. Since we construct the M.G.F.'s of interest to us their existence is guaranteed, and in many practical situations the function of interest is defined on the positive half line $(0,\infty)$. This then is the reason for our interest in the numerical inversion of the Laplace transform.

The convenience and ease of representation of a function by a power series makes them attractive, thus let us discuss this basic approach. Suppose we wish to approximate a p.d.f., $f(x)$, by the finite series $\sum_{n=o}^{s} a_n x^n$. The coefficients, a_n, may be determined by the least-squares criterion.

That is, we choose the a_n so as to minimize

$$\int_{-\infty}^{\infty}[f(x) - \sum_{n=o}^{s} a_n x^n]^2 dx \ .$$

Differentiation with respect to a_j yields

$$-2\int_{-\infty}^{\infty}[f(x) - \sum_{n=o}^{s} a_n x^n]x^j \ dx \ .$$

Setting this equal to zero we have

$$\int_{-\infty}^{\infty} f(x)x^j \ dx = v_j = \int_{-\infty}^{\infty}\sum_{n=o}^{s} a_n x^{n+j} \ dx \quad ,$$

and the a_n are determined as a function of the moments. Note that f(x) may be zero over some part of the range $(-\infty,\infty)$.

Thus, we see that if two distributions have equal moments up to order s they must have the same least-squares approximation. The coefficients we will develop for our series expansions are equivalent to those developed by a least-squares approach.

Another approach is to use the moments of the unknown p.d.f. to fit it to one of the Pearson curves. With the Pearson system, we select a curve based primarily on the value of

$$\frac{\beta_1(\beta_2+3)^2}{4(2\beta_2-3\beta_1-6)(4\beta_2-3\beta_1)}$$

where $\beta_1 = u_3^2/\sigma^6$, and $\beta_2 = u_4/\sigma^4$.

Then the first four moments of the curve selected are then set equal to the first four moments of the unknown p.d.f.

Throughout the research it has been assumed that the amount of labor required to obtain the moments is much less than that of obtaining the M.G.F. and putting it in the proper form. This assumption adds to the attractiveness of the generalized frequency curves. For GERT networks composed entirely of Exclusive-Or nodes this assumption is appropriate, since there is a computer program which yields the moments as its output [41], [42]. At present the program gives only the first two moments but this could be modified. With this modification the entire inversion process could be automatic since the program developed in the course of this research uses the moments as input.

A basic concept used in much of the development is that of unimodality. Usually a p.d.f., $f(x)$, is said to be unimodal if there is a unique m_0, such that $f'(m_0) = 0$ and $f''(m_0)<0$. Similarly in the discrete case a probability mass function, $p(x)$, is unimodal if there is a unique m_0 such that $p(m_0) \geq p(x)$ for $x = x, x+\Delta x$, and $x<m_0<x+\Delta x$. A more general definition for the continuous case was first proposed by A. Ya. Khintchine [9,157]. The definition given below is due to Johnson and Rogers [27,433] and accounts for both the continuous and discrete cases.

Definition: The distribution function (c.d.f.) $F(x)$ is called unimodal, if there exists a number M such that for all real numbers $x_1, \ldots, x_4$ satisfying

$$x_1 < x_2 < M < x_3 < x_4 \quad ,$$

we have

$$F(\tfrac{1}{2}[x_1+x_2]) \leq \tfrac{1}{2}[F(x_1) + F(x_2)]$$

$$F(\tfrac{1}{2}[x_3+x_4]) \geq \tfrac{1}{2}[F(x_3) + F(x_4)] \quad .$$

In other words $F(x)$ is convex for $x \leq M$ and concave for $x \geq M$. If there is such an M it is the mode.

With this definition, distributions such as the uniform over a finite interval, the exponential, and the improper or constant distribution are also unimodal.

Throughout the dissertation it is assumed that the M.G.F.'s of interest are (or can be) expressed as the quotient of sums of products of the M.G.F.'s of well known distributions. That is, they take the following form

$$M_E(s) = \frac{\sum_{h=1}^{n}\left[\prod_{i=1}^{m} p_i M_i(s)\right]_h}{1 + \sum_{k=1}^{r}\left[\prod_{j=1}^{q} p_j M_j(s)\right]_k} \quad ,$$

where the components $M_i(s)$ and $M_j(s)$ are M.G.F.'s of common distributions, and p_i and p_j are probabilities. Thus, the expression for $M_E(s)$ can be very complicated.

Plan of the Dissertation

Included in this dissertation is a detailed presentation of the research of many others. The amount of theoretical research in this area is immense. The original contribution of this dissertation is the integration, extension, illustration and use of the previous research efforts to construct a useful procedure for obtaining a p.d.f. from knowledge of its M.G.F. or moments. The form of the presentation is given below.

In Chapter II we discuss the various techniques for obtaining the p.d.f. and their limitations. Then the procedure for solving the problem is presented.

Chapter III begins with a presentation of some of the known theory in the reduced problem of moments, and the reduced problem of moments for unimodal distributions. The theory on unimodal distributions is extended and then some properties of moment generating functions are discussed.

In Chapter IV three series expansions are developed. The first is the Gram-Charlier A series based on the Normal distribution. The second is a Laguerre series expansion based on the Gamma distribution. The approach and the formulas for the c.d.f. are original in the Laguerre development. The third series is the Gram-Charlier B series based on the Poisson distribution

Chapter V is a presentation of the remaining techniques, Pearson's curves and numerical inversion of the Laplace transform.

Several examples illustrating the procedure and some of its aspects are presented in Chapter VI.

Chapter VII is the summary, conclusions and suggestions for future research.

Appendix A is an annotated bibliography of published research related to the research presented in this dissertation.

Appendix B is a listing of the computer programs.

The writing style of the dissertation is conversational except in the mathematical portions which are quite formal. It is felt that this is the best approach to presenting material of this nature. To facilitate future research efforts, no steps or arguments were omitted from the mathematical presentation. The development given here is more complete than anywhere in the literature known to the author.

CHAPTER II

THE RECOMMENDED PROCEDURE

In Chapters IV and V we will develop three techniques for obtaining a p.d.f. knowing its moments and/or M.G.F. These techniques are: series expansions using orthogonal polynomials, Pearson's system of curves, and numerical inversion of the Laplace transform. There are difficulties involved with each so that no one of them will solve all the problems which arise. However, each has advantages (discussed below) which in combination with the others yields a combination capable of solving a larger portion of the problems than any one of them is capable of alone.

In this chapter we will evaluate the techniques and discuss their limitations. Then we will show how the techniques are complementary to each other and recommend a general procedure for attacking the problem we proposed to solve. The most advanced mathematics required to use the procedure is algebra.

Evaluation of the Techniques

Let us begin by considering the polynomial expansions. The various expansions have the same advantages which are: 1) they yield explicit expressions for the density and cumulative distribution functions, and 2) numerical tables of both the density and cumulative distribution functions are easily computed (a computer program called TEST does this automatically).

All of the expansions suffer from the same basic problem, that of not being positive semidefinite over the range of the variable for some set of moments. That is, a given set of moments may not yield a useful polynomial expansion. A theoretical discussion of this problem is given only for the Gram-Charlier (G-C) type A series since it is much easier and faster to simply have the computer check for negative values as it computes values for the density function, and then branch to the next stage in the approximation if a negative value occurs. When a small number of terms are used in the approximation only the Laguerre expansion is suitable for distributions that are not unimodal.

Now consider the Pearson family of curves. The advantage of the Pearson system is that we may still be able to obtain an explicit expression for the density function even when a polynomial expansion fails to give a useful answer.

However, there are three basic drawbacks to this system. First, it yields only an expression for the density function and this expression is usually not easily integrated to obtain the cumulative distribution function. We must resort to numerical integration. Second, they are currently based on only the first four moments. Thus, we are throwing away some of the information we have about the unknown distribution. This could be corrected by developing a similar system based on more moments. Third, the Pearson system does not work for multimodal distributions.

Finally, let us discuss the numerical inversion technique. The big advantage to this approach is that it can handle multimodal functions and functions whose moments are such that the other techniques fail to supply a suitable approximation. Also, this approach will usually be the most accurate.

This technique has several disadvantages. First, it requires that a considerable amount of manipulation be performed on the M.G.F. before it is in the proper form for inversion. In the case of a complicated M.G.F., this could entail many hours of labor. Second, once we have an answer it is only a table of values for the density function, i.e. we obtain no explicit expression for the density function. Third, a good approximation is obtained only after some experimentation, and cases in which the unknown p.d.f. does not have a value of zero at the beginning of its range require many more terms to obtain a suitable approximation. This will be demonstrated in Chapter V by an example using the exponential distribution. Fourth, if a table of the cumulative distribution is desired we must modify the M.G.F. by dividing it by s, and then further manipulation must be performed to obtain the form necessary for use in the program. A suitable approximation of the c.d.f. cannot be obtained by adding values in the p.d.f. table (see example in Ch.V). Finally, this approach cannot be made to handle distributions defined over the entire real line $(-\infty,\infty)$. The proof of this is given in Chapter V.

The Recommended Procedure

Using the ideas discussed in the previous section a general procedure which yields an acceptable solution to the problem while attempting to minimize the amount of labor required of the analyst has been constructed. Generally speaking, the procedure first tries to apply one of the series expansions. They yield the most information about the unknown p.d.f., but their accuracy is not always what we might desire. If the series approach fails the Pearson system is tried. Here we do not obtain as much information and the accuracy may not be too good. However, it is still relatively easy to obtain an answer. The Pearson approximation can be considered a failure if the curve selected does not agree with the range and boundary conditions of the unknown p.d.f. If the Pearson system fails we then try the numerical Laplace transform inversion which requires a great deal of work and yields only a table of values for the p.d.f. or c.d.f. However, the accuracy is very good. There may be instances where it is desirable to go directly to the numerical inversion technique.

Let us start by stating the assumptions upon which the procedure is based. It is assumed that the M.G.F. is available in explicit form, or that all of its components and the way they combine is known, and we know as many moments of the unknown distribution as desired.

Having assumed that the above information is available, let us first discuss the decision sequence comprising the inversion procedure in the large. Then we will concentrate on that portion of the procedure which is accomplished by the program TEST.

The procedure in the large. The first step should always be an attempt to recognize the M.G.F. as that of one of the well known distributions. Then the uniqueness property of the M.G.F. may be used to obtain the answer directly. At this point we might consider inversion by means of a table of Laplace transforms. If the M.G.F. is made up only of Gamma distributions this can be done, with effort. Henceforth we shall assume that this is not the case, or the labor involved is too great.

If we do not recognize the M.G.F. then we must determine the range of the unknown distribution. This is done by examining the range of each component of the M.G.F.

The unknown distribution will have finite range only if all components are of finite range, otherwise the range will be (a,∞) where $-\infty < a < \infty$ depending on the components. If the range is finite it can be determined by remembering that the range of the convolution of two functions is the sum of the range of each.

Suppose the range is finite, we must then determine if the unknown distribution is discrete or continuous. This is accomplished with the aid of Theorem 12 and Remark 1 of Chapter III. Stated briefly, the unknown distribution will be discrete only if all its components are discrete. If the unknown distribution is discrete we are forced to make some sort of term by term expansion. However, if it is continuous we can use the Laplace inversion procedure.

Next suppose the range is not finite. Here again we must determine whether the unknown distribution is continuous or discrete. This information along with the moments is then inserted into the program TEST. A summary of the decision process is given in Figure 1.

Let us now proceed to a detailed discussion of the program TEST.

The program TEST. The program must be supplied with data on the range, the moments, and continuity or discreteness of the p.d.f., see Table I. The program begins by determining if there is a maximum number of moments that specify the distribution. This is accomplished with Theorems 1, 2 and 4 of Chapter III. If a solution based on the moments does not exist for the $(-\infty,\infty)$ case, no answer is obtainable from the procedure. If a solution does not exist for the $(0,\infty)$ case, we must use a term by term expansion of the M.G.F. for the discrete case or the Laplace inversion technique for the continuous case.

If a solution exists the program then checks to see if the distribution is unimodal. This check is performed based on the results of Theorems 7 and 8, Chapter III.

If the distribution is defined over $(-\infty,\infty)$ and is not unimodal we can not obtain an approximation. If the distribution is defined over $(0,\infty)$, continuous, and not unimodal the numerical inversion technique must be used. If the distribu-

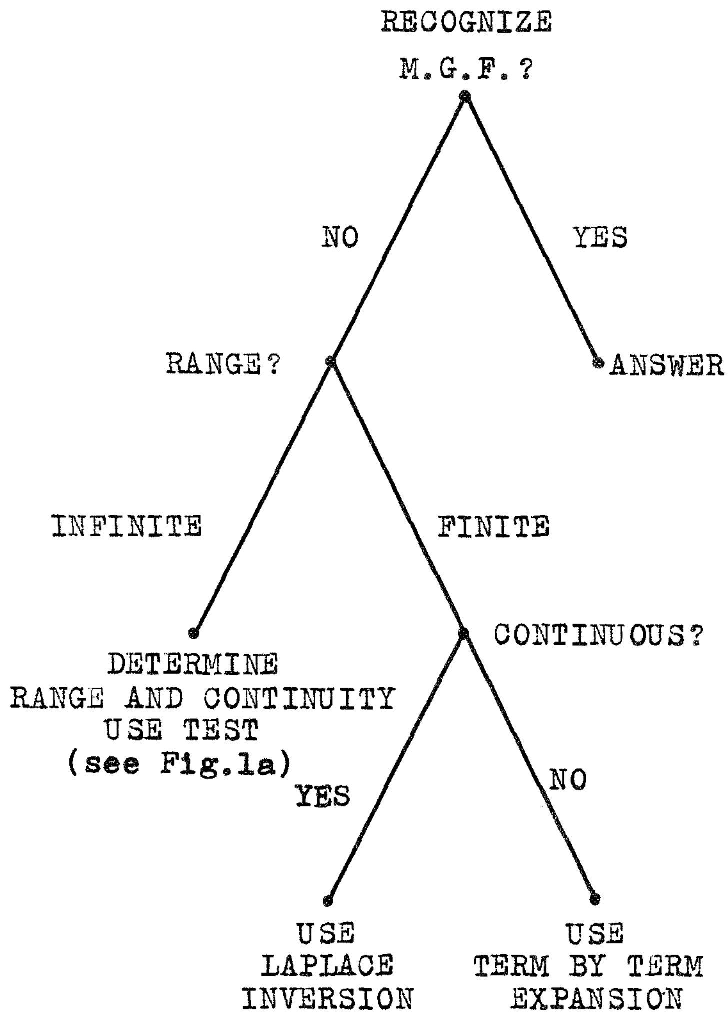


FIGURE 1 Decision tree for the procedure in the large.

TABLE I

INPUT DATA SPECIFICATIONS

Variable Name	Field Used	Variable Meaning and Comments
NOT	I4	Number of tests (set of data), one card needed
N1	I1	Number of moments plus one, one card needed for each set of data
V(J)	5E14.8	The moments to be tested
IC	I3	IC = {0 discrete distribution; 1 continuous distribution}
IR	I4	Range of distribution under test IR = {0 range $(0,\infty)$; 1 range $(-\infty,\infty)$}
R	F4.1	Value of parameter r in Laguerre expansion used only when IC = 1 and IR = 0

NOTE: Input data is listed in the order it must be assembled for use with program.

tion is defined over $(0,\infty)$, discrete, and not unimodal we must use a term by term expansion.

Suppose the unknown distribution is unimodal. Then TEST checks the range and continuity of the unknown distribution. On this basis the Gram-Charlier A series, Laguerre series, or the G-C B series is chosen.

If the B series is tried and fails we must again resort to a term by term expansion. If either of the other two series is tried and fails we then try to obtain an approximation using the Pearson system. At this point in the procedure the program TEST is exited and the analyst must take over. If the Pearson system fails to yield a reasonable answer, the analyst will have to determine the suitability, the transform inversion method must be used.

The decision process used in the program TEST is shown in Figure 1a. Note that, although the Theorems 1 and 2 (Ch.III) are not stated in the same form with regard to the number of moments that are known, the range of the indices on the quadratic forms is such that the quadratic form in Theorem 1 is the same as the first one in Theorem 2. This means that functions defined over $(0,\infty)$ must satisfy both Theorems 1 and 2.

The theory used in the construction of the procedure will be developed and illustrated in the next three chapters. Then in Chapter VI examples illustrating the procedure will be presented.

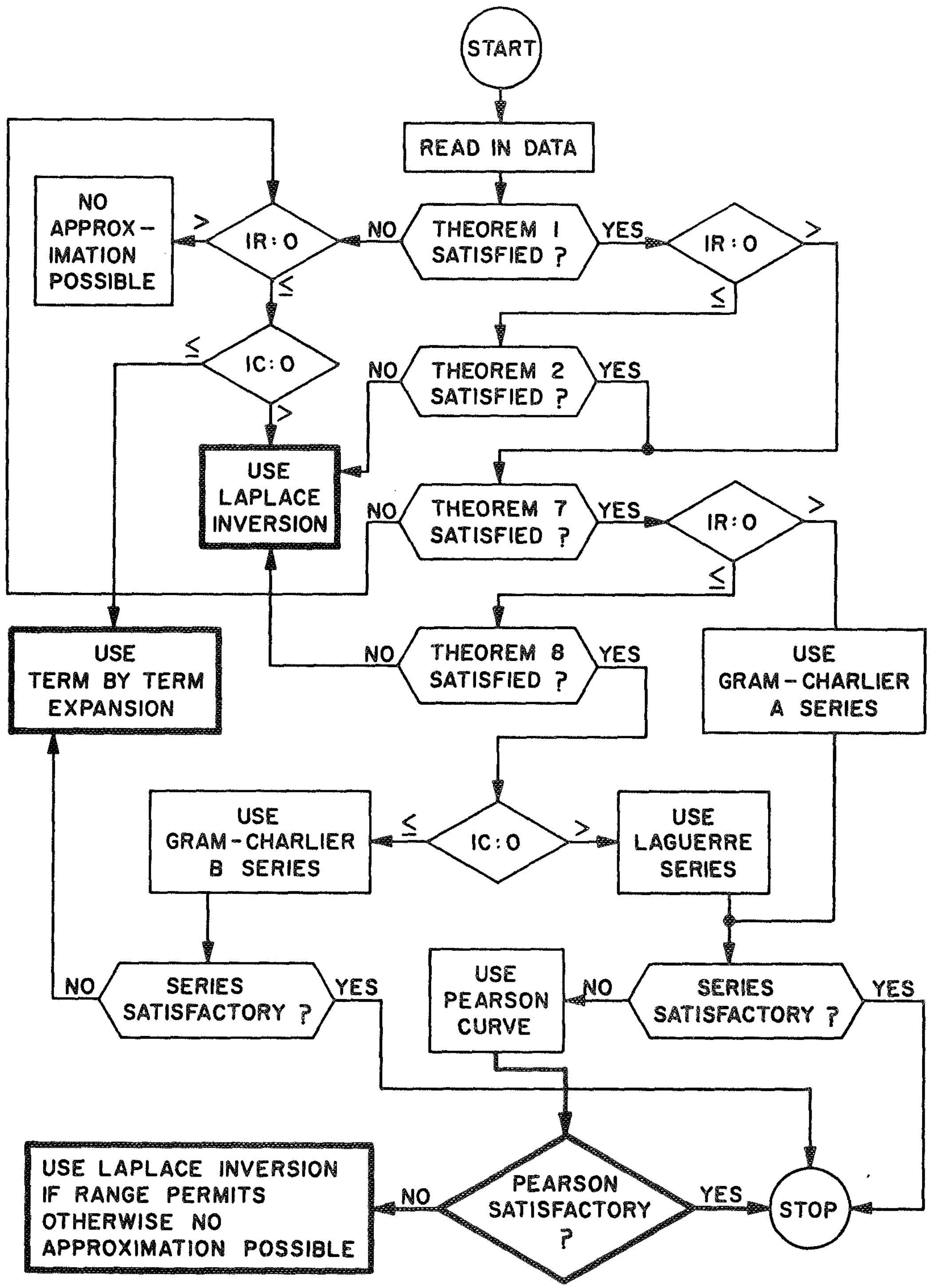


Figure 1a. Master flow diagram of the procedure. Portions of the procedure in heavy black lines are not included in the program TEST.

CHAPTER III

THE REDUCED PROBLEM OF MOMENTS, AND SOME PROPERTIES OF MOMENT GENERATING FUNCTIONS

The label 'reduced problem of moments' is by no means universally agreed upon. One is just as likely to find the problem discussed under the names 'truncated problem of moments' or 'finite problem of moments'.

Basically the reduced problem of moments is to determine under what conditions a finite sequence of moments, $v_0,\dots,v_m$, specify a cumulative distribution function F(t), where

$$v_j = \int_{-\infty}^{\infty} t^j dF(t) \quad , \qquad j = 0,1,\dots,m \qquad (1)$$

and F(t) may be zero over some portion of the range. In general a distribution function is any monotonically non-decreasing function such that the integral in Eq.1 converges with j = 0, i.e. $v_0 < \infty$. Although the theory presented in the next section is general enough to handle any value of v_0 we will use $v_0 = 1$ since we are concerned specifically with probability functions.

In the second section we address ourselves to the problem of whether or not the specified distribution F(t) is unimodal. This is a vital point in the application of the approximation techniques. The discussion is purely formal and centers around presentation of the known theorems dealing with this problem.

These theorems do not provide a means for testing an unknown distribution for unimodality. Thus, two theorems to accomplish this were constructed and they are presented in the third section.

This chapter will conclude with a discussion of some basic properties of Moment Generating Functions.

Necessary and Sufficient Conditions for the Existence of a Solution

Let us begin with some definitions which will be used in the proofs.

Definition 1. [20,102] If $P_n(t)$ is the polynomial

$$P_n(t) = \sum_{i=0}^{n} a_i t^i \;,$$

an operator $M[P_n(t)]$, called the moment of $P_n(t)$ with respect to the sequence $v_0, \ldots, v_n$, is defined as

$$M[P_n(t)] = \sum_{i=0}^{n} a_i v_i \quad .$$

If v_n has the representation Eq.1 then

$$M[P_n(t)] = \int_{-\infty}^{\infty} P_n(t)dF(t) \quad .$$

Definition 2. [2,20] $Q_{n+1}(\lambda, y)$ is a quasi-orthogonal polynomial of degree n+1, and

$$Q_{n+1}(\lambda, y) = Q_{n+1}(\lambda) - yQ_n(\lambda)$$

where y is a real number and the zeros of the polynomial are denoted by $\lambda_1 < \lambda_2 < \ldots < \lambda_{n+1}$, with $\lambda_k = \lambda_k(n, y)$.

A quasi-orthogonal polynomial is simply an orthogonal polynomial that is a linear combination of two other orthogonal polynomials. For further discussion on quasi-orthogonal polynomials see [2,10] or [18,34].

We will also need the following lemma [2,22].

Lemma 1. If $Q_{n+1}(z,y)$ and $R_{n+1}(z,y)$ are quasi-orthogonal polynomials, as specified in Definition 2, then

$$\frac{Q_{n+1}(z,y)}{R_{n+1}(x,y)} = \sum_{k=1}^{n+1} \frac{u_k}{z-\lambda_k} \quad ,$$

where

$$u_k = u_k^{(n+1)}(y) = \frac{Q_{n+1}(\lambda_k,y)}{R'_{n+1}(\lambda_k,y)} \quad .$$

Finally, suppose that u_k, λ_k and v_m are as defined above then, using a Lagrange interpolation polynomial [2,22], [18,37], it can be shown that

$$v_m = \sum_{k=1}^{n+1} u_k \lambda_k^m \quad , \ (m = 0,1,2,\ldots,2n-1) \qquad (2)$$

If $y = 0$ in Definition 2 the above relation is also valid for $m = 2n$ [2,23].

Now let us consider the case where the distribution function of interest is defined over the entire real line $(-\infty,\infty)$.

Theorem 1. A necessary and sufficient condition that there should exist at least one non-decreasing function $F(t)$ such that

$$v_j = \int_{-\infty}^{\infty} t^j dF(t) \quad , \ (j=0,1,2,\ldots,2n-1) \qquad (3)$$

is that the quadratic form

$$\sum_{i,j=0}^{n} v_{i+j} x_i x_j \tag{4}$$

be positive (definite or semidefinite), or which is the same the sequence $v_0, \ldots, v_{2n-1}$ be positive.

<u>Proof</u>: Necessity. Suppose that F(t) is a non-decreasing solution of Eq.3, and that P(t) is a non-negative polynomial,

$$P(t) = \left(\sum_{k=0}^{n} x_k t^k\right)^2 .$$

Then

$$M[P(t)] = M[(x_0 + x_1 t + \ldots + x_n t^n)^2]$$

$$= \sum_{i=0}^{n} \sum_{j=0}^{n} v_{i+j} x_i x_j .$$

But, by Definition 1

$$M[P(t)] = \int_{-\infty}^{\infty} \left(\sum_{k=0}^{n} x_k t^k\right)^2 dF(t) \geq 0 ,$$

thus the quadratic form is positive.

Sufficiency. Assume that the sequence $v_0, \ldots, v_{2n-1}$ is positive. It can be shown [18,38] that we are always able to find a quasi-orthogonal polynomial of degree (n+1), determined by the sequence $v_0, \ldots, v_{2n-1}$.

Using this fact let us select an arbitrary number y $(-\infty < y < \infty)$ and write the equation of Lemma 1

$$\frac{Q_{n+1}(z)-yQ_n(z)}{R_{n+1}(z)-yR_n(z)} = \sum_{i=1}^{n+1} \frac{u_i}{z-\lambda_i}$$

where

$$\lambda_1 < \lambda_2 < \ldots < \lambda_{n+1} ,$$

and

$$u_i = u_i^{(n+1)}(y) \quad .$$

From Eq.2 we may also write

$$v_k = \sum_{i=1}^{n+1} u_i \lambda_i^k \quad , \ (k=0,1,\ldots,2n-1). \tag{5}$$

Now consider the step function $F_n(t)$ which has n values and its only points of increase are the λ_i. At each λ_i the discontinuities are the u_i, i.e.

$$u_i = F_n(\lambda_i+0) - F_n(\lambda_i-0) \ , \ (i=1,2,\ldots,n+1);$$

then, using the Stieltjes integral, Eq.5 can be written

$$v_k = \int_{-\infty}^{\infty} t^k \, dF_n(t) \ , \quad (k=0,1,\ldots,2n-1). \tag{6}$$

Thus, we have constructed a function $F_n(t)$ which is a certain solution to Eq.3. Note that for $y < \infty$ it can be shown that Eq.5 and therefore Eq.6 are also true for k=2n [18,38].

Next consider the case where the function of interest is defined over the positive real half line $(0,\infty)$.

Theorem 2. A necessary and sufficient condition that there should exist a non-decreasing function F(t) such that

$$v_j = \int_0^\infty t^j \, dF(t) \qquad (j=0,1,\ldots,n), \tag{7}$$

is that the quadratic forms

$$\sum_{i=0}^{[n/2]} \sum_{j=0}^{[n/2]} v_{i+j} x_i x_j \tag{8}$$

and

$$\sum_{i=0}^{\left[\frac{n-1}{2}\right]} \sum_{j=0}^{\left[\frac{n-1}{2}\right]} v_{i+j+1} x_i x_j \tag{9}$$

should be positive (definite or semidefinite). The brackets on the upper limits of the indices denote the largest integer less than or equal to the value within the bracket.

The proof of this theorem follows that of Theorem 1, or alternately see [38,1].

Theorems similar to these may be found in many places in the literature [2,30], [18,83], [28,224], [38,1].

It is important to note that neither of these theorems establish the uniqueness of a solution. In fact the solution will not be unique in general [20,126]. However, this does not represent a problem because we are only interested in establishing that a solution does exist using a certain number of moments, say n, which we are given.

If we restate the problem in strictly discrete terms the uniqueness situation is considerably changed as can be seen by the following theorem for functions defined on $(0,\infty)$ [7,238].

Theorem 3. The finite problem of moments

$$v_p = \sum_{j=1}^{m} u_j s_j^p \tag{10}$$

$(p=0,1,\ldots,2m-1;\ u_1 > 0,\ldots,\ u_m > 0;\ 0 < s_1 < s_2 < \cdots < s_m)$, where v_p are given real numbers and s_j and u_j are unknown real numbers $(p=0,1,\ldots,2m-1;\ j=1,2,\ldots,m)$ has a solution if and only if the quadratic forms

$$\sum_{i,j=0}^{m-1} v_{i+j} x_i x_j$$

$$\sum_{i,j=0}^{m-1} v_{i+j+1} x_i x_j$$

are positive definite. The solution of the problem is always unique.

It is interesting to note the similarity between Eq.10 and Eq.2.

In order to realize a useful procedure for testing a given set of moments we use the following well known theorem [11,260], [20,133].

Theorem 4. A set of necessary and sufficient conditions for the form

$$\sum_{i=0}^{n} \sum_{j=0}^{n} v_{i+j} x_i x_j = \bar{x}' V \bar{x}$$

to be positive is that all the principal minors of V should be positive, i.e.

$$v_0 \geq 0, \quad \begin{vmatrix} v_0 v_1 \\ v_1 v_2 \end{vmatrix} \geq 0, \ldots, \quad \begin{vmatrix} v_0 v_1 \cdots v_n \\ v_1 v_2 \cdots v_{n+1} \\ \vdots \\ v_n v_{n+1} \cdots v_{2n} \end{vmatrix} \geq 0. \qquad (11)$$

Theorems 1 and 2 have been implemented in the program TEST with the aid of the Theorem 4. These two theorems allow us to determine if there is a maximum number of moments that specify the unknown p.d.f. This is possible because the theorems tell us that n moments specify a distribution if and only if the corresponding quadratic forms are positive which is true if and only if the resulting determinants are non-negative. Thus, the maximum value of n, if there is one, can be determined by allowing n to sequentially assume the values 2, 3,... up to the number of moments available. If for some value of n one of the determinants is negative then the previous value of n is the maximum number of moments that constitute a description of the unknown p.d.f.

The conditions under which a distribution function is unimodal will be examined in the next section.

Necessary and Sufficient Conditions for a Unimodal Solution

Very little has been published on this facet of the problem. The principal works being Gnedenko and Kolmogorov [9 ,157], Johnson and Rogers [27,433], and Mallows [28,224]. Johnson and Rogers' work is a rediscovery of the earlier work by A. Ya. Khintchine as given in Gnedenko and Kolmogorov [9]. Mallows' work is an extension to distributions other than those defined over $(-\infty,\infty)$.

Theorems 5 and 6, and Corollary 1 will be presented next, they are composites made up from [27] and [28]. Also, it is important to note that even though v_0 is not always specifically mentioned it is assumed to be equal to one.

<u>Theorem 5</u>. Let n be one of the numbers 3,5,7,... . Also let a real number v_r be given for each integer r with $1 \leq r < n$. Then there is a unimodal distribution function F(t) with mode M and with

$$v_r = \int_{-\infty}^{\infty} t^r dF(t) , \qquad r = 1,2,\ldots,(n-1), \tag{12}$$

if and only if there is a distribution function G(t) such that

$$(r+1)v_r - rMv_{r-1} = \int_{-\infty}^{\infty} t^r dG(t) , \; r = 1,2,\ldots,(n-1). \tag{13}$$

The proof of a very similar theorem is given by Johnson and Rogers [27,434] except that they use the equation

$$G(t) = F(t) - (t)F'(t)$$

to relate F(t) and G(t) instead of using

$$G(t) = F(t) - (t-M)F'(t)$$

which would be required to prove Theorem 5.

Corollary 1. A necessary and sufficient condition that there be a unimodal distribution function F(t) with Mode M and

$$v_r = \int_{-\infty}^{\infty} t^r dF(t) , \; r = 1,2,\ldots,(n-1) \tag{14}$$

is that

$$\begin{vmatrix} 1 & 2v_1-M & \cdots & (s+1)v_s-sMv_{s-1} \\ 2v_1-M & 3v_2-2Mv_1 & \cdots & (s+2)v_{s+1}-(s+1)Mv_s \\ \vdots & \vdots & & \vdots \\ (s+1)v_s-sMv_{s-1} & (s+2)v_{s+1}-(s+1)Mv_s & \cdots & (2s+1)v_{2s}-2sMv_{2s-1} \end{vmatrix} \geq 0 \qquad (15)$$

for all integers s with $2 \leq 2s < n$ or, what is the same, that

$$\begin{vmatrix} 1 & M & M^2 & \cdots & M^{s+1} \\ 0 & v_0 & 2v_1 & \cdots & (s+1)v_s \\ v_0 & 2v_1 & 3v_2 & \cdots & (s+2)v_{s+1} \\ \vdots & \vdots & & & \vdots \\ sv_{s-1} & (s+1)v_s & (s+2)v_{s+1} & \cdots & (2s+1)v_{2s} \end{vmatrix} \geq 0 , \qquad (16)$$

which will be denoted by

$$\Delta_s(v) = \left| M^j ; (i+j-1)v_{i+j-2} \right|_{i,j=0}^{s+1} ,$$

where i is the row number and j is the column number.

For functions defined over $(0,\infty)$ we have the following theorem.

Theorem 6. A necessary and sufficient condition that there be a unimodal distribution function F(t) with mode M and

$$v_r = \int_0^\infty t^r dF(t) \quad , \quad r = 0,1,\ldots, n \qquad (17)$$

is that the determinants

$$\Delta_r(v) \; , \qquad r = 0,1,\ldots,[n/2] \qquad (18)$$

and

$$\left| M^j ; (i+j) v_{i+j-1} \right|_{i,j=0}^{r+1} , \quad r = 0,1,\ldots,[(n-1)/2] \tag{19}$$

be positive.

Let us now consider two theorems which will allow us to test for unimodality without knowing the value of the mode.

Two Theorems Useful in Testing for Unimodality

The theorems of the previous section establish conditions under which a distribution F(t) is unimodal but they are not suitable for testing a set of moments from a distribution whose mode is unknown. Theorem 2 of Johnson and Rogers [27,434] suggested an approach which will allow us to circumvent this difficulty. The following two theorems are the result.

Theorem 7. A necessary and sufficient condition that the distribution function F(t) with mode M and

$$v_j = \int_{-\infty}^{\infty} t^j \, dF(t) , \quad j = 0,1,\ldots,n \tag{20}$$

be unimodal is that there exist a real number M such that

$$v_1 - \sqrt{3(v_2 - v_1^2)} \leq M \leq v_1 + \sqrt{3(v_2 - v_1^2)} \tag{21}$$

and that the roots r_1, r_2, and r_3 of the cubic y,

$$y = aM^3 + bM^2 + cM + d \geq 0 \tag{22}$$

where

$$a = 4v_3 - 12v_1v_2 + 8v_1^3$$

$$b = 8v_1v_3 + 9v_2^2 - 12v_1^2v_2 - 5v_4$$

$$c = 10v_1v_4 - 12v_2v_3 - 16v_1^2v_3 + 18v_1v_2^2$$

$$d = 15v_2v_4 + 48v_1v_2v_3 - 27v_2^3 - 16v_3^2 - 20v_1^2v_4$$

satisfy the conditions:

$$r_1 \leq M \tag{23}$$

if y has only one real root and $a > 0$, or

$$M \leq r_1 \tag{23a}$$

if y has only one real root and $a < 0$; or if all roots are real ($r_1 \leq r_2 \leq r_3$)

$$r_3 \leq M \text{ or } r_1 \leq M \leq r_2 \tag{24}$$

when $a > 0$, or

$$M \leq r_1 \text{ or } r_2 \leq M \leq r_3 \tag{25}$$

when $a < 0$.

In the case where $a = 0$, M must satisfy Relation 21, and the real roots r_1 and r_2 ($r_1 \leq r_2$) of the remaining quadratic must satisfy the following conditions:

$$r_1 \leq M \leq r_2 \tag{26}$$

when $b < 0$, or

$$M \leq r_1 \text{ or } r_2 \leq M \tag{27}$$

when $b > 0$, or if both roots are complex

$$b > 0 \tag{28}$$

must hold.

In the case where both a and b in Eq.22 are zero, M must satisfy Relation 21 and

$$M \geq \frac{-d}{c} \tag{29}$$

for $c > 0$, or

$$M \leq \frac{-d}{c} \tag{29a}$$

for $c < 0$.

In the case where a, b and c are zero, M must satisfy Relation 21 and

$$d \geq 0 \ . \tag{30}$$

<u>Proof</u>. Necessity. Suppose F(t) is unimodal with mode M and moments v_1, v_2, v_3, v_4. Then by Corollary 1 with n=5 we have

$$\begin{vmatrix} 1 & M & M^2 \\ 0 & 1 & 2v_1 \\ 1 & 2v_1 & 3v_2 \end{vmatrix} \geq 0 \tag{31}$$

and

$$\begin{vmatrix} 1 & M & M^2 & M^3 \\ 0 & 1 & 2v_1 & 3v_2 \\ 1 & 2v_1 & 3v_2 & 4v_3 \\ 2v_1 & 3v_2 & 4v_3 & 5v_4 \end{vmatrix} \geq 0 \ . \tag{32}$$

To show that Relation 31 leads to the Relation 21 we proceed as follows.

Assume that the mode M is positive or zero (if mode is negative the curve is easily shifted). Then we write

$$z = \begin{vmatrix} 1 & M & M^2 \\ 0 & 1 & 2v_1 \\ 1 & 2v_1 & 3v_2 \end{vmatrix} \geq 0 \ ,$$

which yields

$$z = -M^2 + 2v_1M + 3v_2 - 4v_1^2 \geq 0$$

when the determinant is expanded.
Since the coefficient of M^2 is negative, $d^2z/dM^2 < 0$ and z is a concave function, see Fig.2. Thus, the M of interest must lie between the roots of z, and Relation 21 follows by use of the quadratic formula.

In order to show that Relation 32 leads to the Relations 23, 24, and 25 we argue as follows.

Again assume that the mode M is positive or zero. Then write

$$y = \begin{vmatrix} 1 & M & M^2 & M^3 \\ 0 & 1 & 2v_1 & 3v_2 \\ 1 & 2v_1 & 3v_2 & 4v_3 \\ 2v_1 & 3v_2 & 4v_3 & 5v_4 \end{vmatrix} \geq 0$$

which yields Eq.22 when the determinant is expanded.
We know that a cubic has either one real root and two imaginary roots, or three real roots. If all of the roots are real the two possible cases are shown in Fig.3a and Fig.3b. With any given set of moments we can determine the applicable case by taking the derivative of y, y'. Now y' is a quadratic and it will be concave or convex according to the sign of its second derivative $[y']''$. Also y' is the slope of y at any point. Thus if y' is convex it will be negative between its roots and as a result y will be similar to Fig.3a. The correspondence between y and y' is shown in Fig.4a. On the other hand if y' is concave y will be similar to Fig.3b and

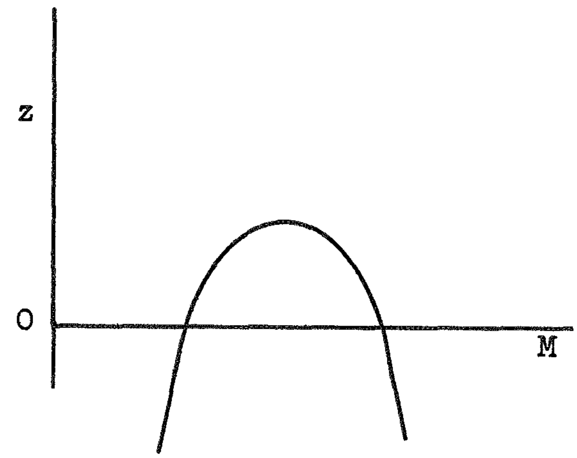


Figure 2. A concave quadratic function

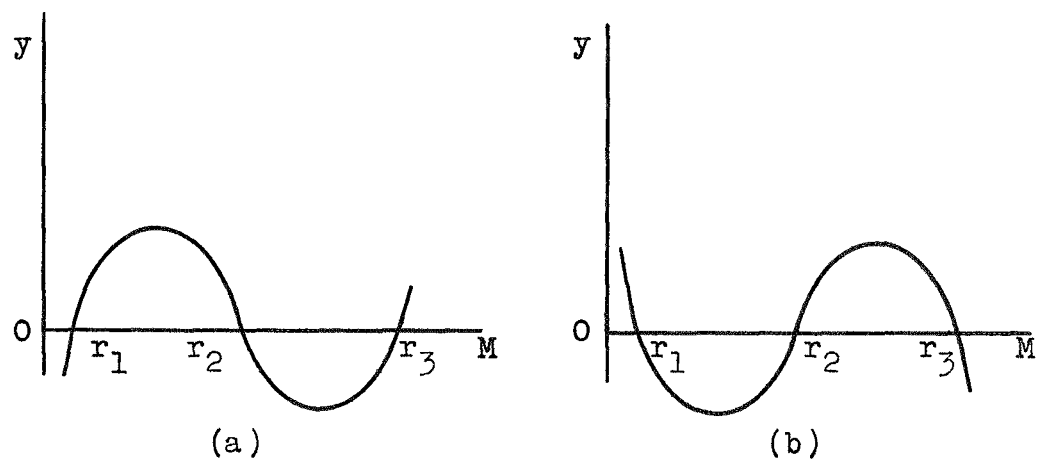


Figure 3. The two possible configurations for a cubic with three real roots.

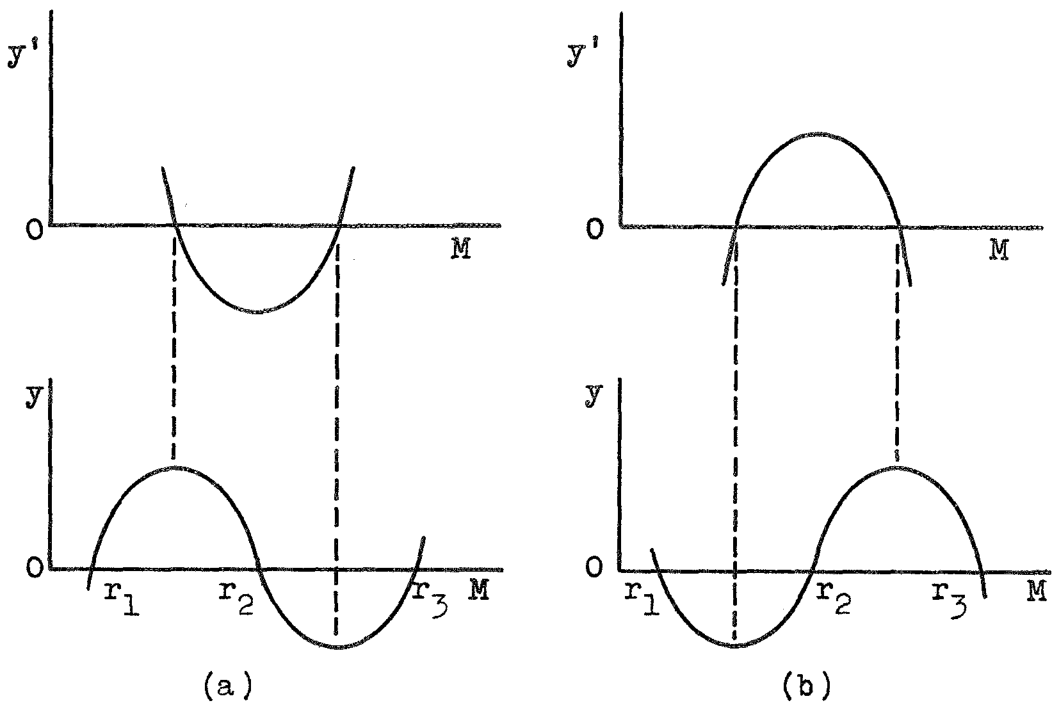


Figure 4. The correspondence between the shape of the cubic y and the convexity or concavity of y'.

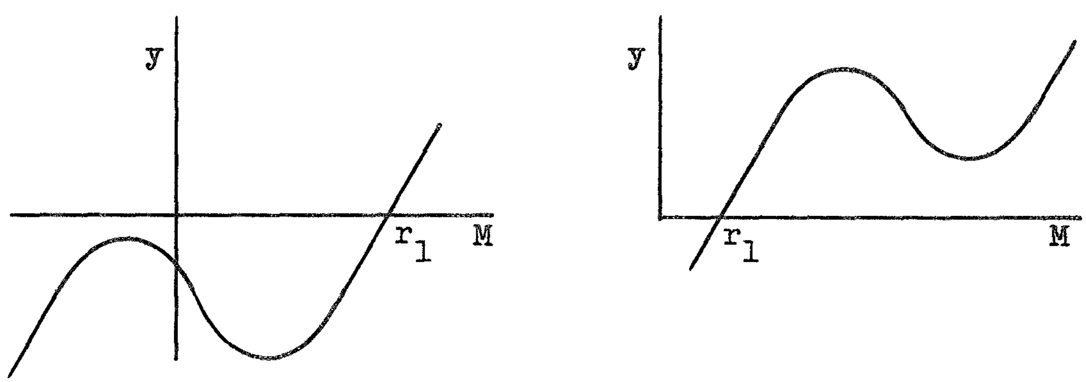


Figure 5. The two possible cases when y has one real root and $a > 0$.

the correspondence is shown in Fig.4b. Differentiation of Eq.22 shows that

$$y''' = 6(4v_3-12v_1v_2+8v_1^3) = 6a \quad .$$

Therefore this relation along with the preceding argument results in Relation 24 when y''' is positive and Relation 25 when y''' is negative. The case of only one real root can be explained by examining the four possible situations as shown in Fig.5 and 6. Arguing exactly as above we see that $a > 0$ corresponds to Fig.5 and also that $y \geq 0$ occurs only when $M \geq r_1$ which is Relation 23. Pursuing a similar argument using Fig.6 leads to Relation 23a.

Let us now consider the case $a = 0$, whence y becomes the quadratic

$$y = bM^2 + cM + d \geq 0 \quad .$$

Now it can be shown, also it is implied by above argument, that if $b > 0$, y is convex and if $b < 0$, y is concave, see Figs.7a and 7b. Thus, if the roots are real and $b < 0$, M must lie between the roots, Relation 26, or if $b > 0$, then $M \leq r_1$ or $M \geq r_2$, Relation 27, must hold for $y \geq 0$. If the roots are complex we want the entire curve y to lie above the M axis which occurs when y is convex, i.e. $b > 0$, Relation 28.

Next, suppose both a and b are zero, then y becomes

$$y = cM + d \geq 0 \ .$$

If $c > 0$ the slope is positive (Fig.8a) and M must be greater than $-d/c$, Relation 29. If $c < 0$ the slope is negative (Fig.8b) and Relation 29 holds.

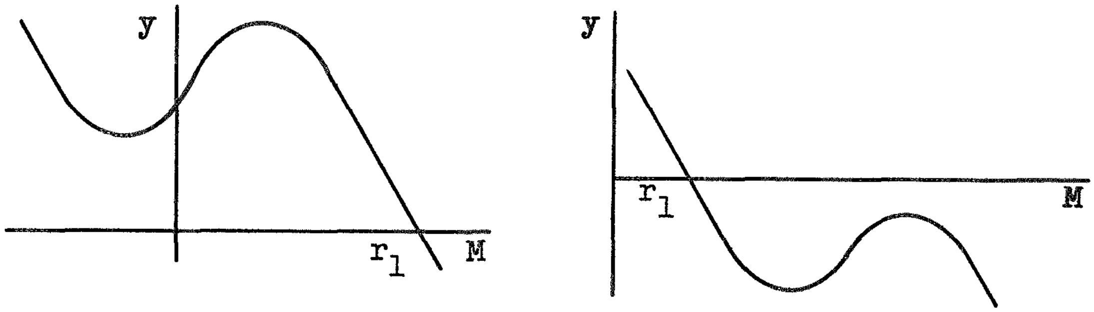
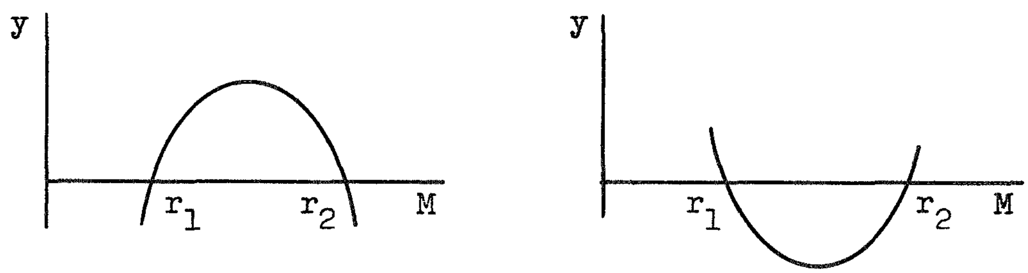
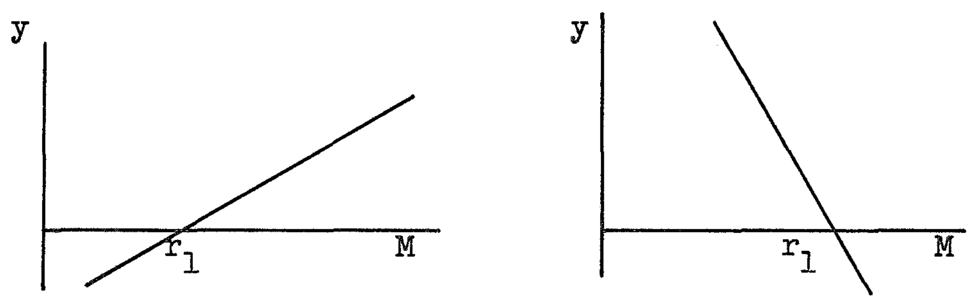


Figure 6. The two possible cases when y has one real root and $a < 0$.



(a) $b < 0$ (b) $b > 0$

Figure 7. The two possible cases when $a = 0$.



(a) $c > 0$ (b) $c < 0$

Figure 8. The two possible cases when $a = b = 0$.

Finally suppose $a = b = c = 0$, then for $y \geq 0$ all we need is $d \geq 0$, Relation 30.

Sufficiency. Assume that M, v_1, v_2, v_3, and v_4 are real numbers satisfying Relations 31 and 32.

Proof of the existence of a unimodal distribution function $F(t)$ with mode M and moments v_1, v_2, v_3, v_4 can be accomplished by proving the existence of a distribution function $G(t)$ having $2v_1-M$, $3v_2-2Mv_1$, $4v_3-3Mv_2$, and $5v_4-4Mv_3$ for its first four moments respectively.

From Theorems 1 and 4, $G(t)$ is a distribution function if and only if

$$\begin{vmatrix} 1 & 2v_1-M \\ 2v_1-M & 3v_2-2Mv_1 \end{vmatrix} \geq 0 \tag{33}$$

and

$$\begin{vmatrix} 1 & 2v_1-M & 3v_2-2Mv_1 \\ 2v_1-M & 3v_2-2Mv_1 & 4v_3-3Mv_2 \\ 3v_2-2Mv_1 & 4v_3-3Mv_2 & 5v_4-4Mv_3 \end{vmatrix} \geq 0 \quad . \tag{34}$$

But relation 33 is equivalent to Relation 31 and Relation 34 is equivalent to Relation 32 by virtue of the equivalence of Relations 15 and 16 in Corollary 1. Thus, the conditions of Theorem 1 are satisfied and $G(t)$ is a distribution function.

Since $G(t)$ is a distribution function Theorem 5 assures the existence of the desired distribution function $F(t)$ and the proof is complete.

Note that in the proof of Theorem 7 we assumed that the number of moments n was 5, $n = 5$. We must account for this and the result is that Theorem 8 involves the use of the first five moments of a distribution.

<u>Theorem 8</u>. A necessary and sufficient condition that the distribution function $F(t)$ with mode M and

$$v_j = \int_0^\infty t^j \, dF(t) \ , \quad j = 0,1,\ldots,n \tag{35}$$

be unimodal is that there exist a real number M which satisfies Theorem 7 and also satisfies all of the following relations:

1. $$M \leq 2v_1 \quad . \tag{36}$$

2. The roots p_1 and p_2 ($p_1 \leq p_2$) of the quadratic z

$$z = a_2 M^2 + a_1 M + a_0 \geq 0 \tag{37}$$

where

$$\begin{aligned} a_2 &= 3v_2 - 4v_1^2 \\ a_1 &= 6v_1 v_2 - 4v_3 \\ a_0 &= 8v_1 v_3 - 9v_2^2 \ , \end{aligned} \tag{38}$$

satisfy one of the conditions:

I. Both roots real ($p_1 \leq p_2$)

$$p_1 \leq M \leq p_2 \tag{39}$$

if $a_2 < 0$, or

$$M \leq p_1 \quad \text{or} \quad p_2 \leq M \tag{40}$$

if $a_2 > 0$.

II. Both roots complex

$$2(3v_2 - 4v_1^2) > 0 \ , \tag{41}$$

that is the entire curve z must lie above the M axis.

III. In the case where $a_2 = 0$

$$M \geq \frac{-a_0}{a_1} \tag{42}$$

if $a_1 > 0$, or

$$M \leq \frac{-a_0}{a_1} \tag{42a}$$

if $a_1 < 0$.

IV. In the case where $a_2 = a_1 = 0$

$$a_0 \geq 0 . \tag{43}$$

3. The roots r_1, r_2 and r_3 of the cubic y,

$$y = aM^3 + bM^2 + cM + d \geq 0 \tag{44}$$

where

$$\begin{aligned} a &= 27v_2^3 + 16v_3^2 + 20v_1^2v_4 - 15v_2v_4 - 48v_1v_2v_3 \\ b &= 18v_2v_5 + 30v_1v_2v_4 + 30v_1v_3^2 - 36v_2^2v_3 \\ &\qquad - 20v_3v_4 - 24v_1^2v_5 \\ c &= 48v_2v_3^2 + 36v_1v_2v_5 + 25v_4^2 - 24v_3v_5 \\ &\qquad - 45v_2^2v_4 - 40v_1v_3v_4 \\ d &= 48v_1v_3v_5 + 120v_2v_3v_4 - 64v_3^3 - 50v_1v_4^2 - 54v_2^2v_5 , \end{aligned} \tag{45}$$

satisfy the Relations 23 through 30 of Theorem 7.

<u>Proof</u>. The proof is the same as for Theorem 7 except we use Theorem 6 instead of Corollary 1.

Theorem 7 and 8 have also been incorporated in the FORTRAN program TEST.

It is important to note that Theorems 7 and 8 do not say anything about uniqueness. Thus, it is entirely possible that a set of moments from a non-unimodal distribution could satisfy them. This would indicate that there also exists a unimodal distribution with the same moments.

Some Useful Properties of Moment Generating Functions

The properties, theorems, etc. discussed here are taken from characteristic function theory, but they apply equally well to M.G.F.'s, if we assume the M.G.F.'s of interest always exist.

First consider a theorem, for characteristic functions [9 ,160], which is analogous to Theorem 5. Although the theorem is not used here it does relate to the previous discussion.

Theorem 9. The function F(s) is the M.G.F. of a unimodal distribution function if and only if it can be represented in the form

$$F(s) = \frac{1}{s}\int_0^s V(u)du,$$

where V(u) is some M.G.F.

Now let us examine a group of theorems and remarks [15] that are either useful when we are deciding on an approximation technique, or contribute to our understanding of the type of M.G.F.'s we are interested in.

Theorem 10. Suppose that the real numbers $a_1, a_2, \ldots, a_n$ satisfy the conditions

$$a_j \geq 0, \quad \sum_{j=1}^{n} a_j = 1$$

and that $F_1(s)$, $F_2(s),\ldots,F_n(s)$ are M.G.F. Then

$$G(s) = \sum_{j=1}^{n} a_j F_j(s)$$

is also a M.G.F.

Theorem 11. The product of two moment generating functions is also a moment generating function.

Theorem 12. Let $F = F_1 * F_2$ be the convolution of two distribution functions F_1 and F_2. If either F_1 or F_2 is a continuous distribution, then the convolution is also a continuous distribution.

Remark 1. Let $F = F_1 * F_2$ be the convolution of two distributions F_1 and F_2 and suppose that F is a discrete distribution. Then both F_1 and F_2 are also discrete distributions.

Thus, the only way we can obtain a discrete distribution is by starting with all distributions discrete.

Finally, consider a much argued theorem concerning the convolution of unimodal distributions. Proofs of earlier versions of this theorem contained subtle flaws which were not repairable. However, a correct statement and proof of the theorem was finally given by Chung [24].

Theorem 13. Suppose that the distributions F_1 and F_2 are unimodal with peak at 0 and absolutely continuous.

Then the function $F_1 * F_2$ is unimodal with peak at p if and only if

$$V_1 * F_2 + V_2 * F_1 - F_1 * F_2 + p(F_1' * F_2)$$

is a distribution function (continuous to the left), where *

$$V_i = F_i(x) - D_-F_i(x) \qquad (i=1,2)$$

is a distribution function.

---

*$D_-F_i(x)$ is the left derivative of $F_i(x)$.

CHAPTER IV

SERIES EXPANSIONS

In this chapter we will study series expansions of the form

$$f(x) = \sum_{n=0}^{\infty} a_n P_n(x)\ h(x) \quad ,$$

where $h(x)$ is a probability density function,

$P_n(x)$ is an orthogonal polynomial derived from $h(x)$, and

a_n is an appropriate coefficient.

Three types of series are developed, each suitable for representing a function over a specific range. In each case the series yields an exact representation of the unknown function. However, we will use only a finite number of terms and thus the series representation will be an approximation.

Gram-Charlier A Type Series

When it is desired to represent a p.d.f. defined over the entire real line $(-\infty,\infty)$, and only its moments are known, the Gram-Charlier (G-C) series offers a convenient way to accomplish the representation. There are restrictions, however, and these will be discussed along with the development of the technique.

The G-C series is based upon the Normal distribution and its derivatives, and although there are many ways to derive this series a straightforward approach will be used here.

We begin by differentiating the Normal p.d.f.

$$\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \tag{1}$$

as follows

$$\phi^{(1)}(x) = - x\phi(x)$$

$$\phi^{(2)}(x) = (x^2-1)\phi(x)$$

$$\phi^{(3)}(x) = - (x^3-3x)\phi(x) \quad ,$$

$$\vdots$$

and in general

$$\phi^{(n)}(x) = (-1)^n H_n(x)\phi(x) \quad . \tag{2}$$

The expressions $H_n(x)$ are known as Hermite polynomials and may also be defined by the relation [12,181]

$$\exp[xt-t^2/2] = \sum_{n=0}^{\infty} \frac{H_n(x)\ t^n}{n!} \quad . \tag{3}$$

This definition allows us to develop a general formula for $H_n(x)$. We proceed by expanding $\exp(xt-t^2/2)$ in two infinite series as follows

$$\exp[xt-t^2/2] = \sum_{n=0}^{\infty} \frac{(xt)^n}{n!} \sum_{n=0}^{\infty} (-\frac{1}{2})^n \frac{t^{2n}}{n!}$$

$$= \sum_{n=0}^{\infty} t^n \sum_{k=0}^{[n/2]} (-\frac{1}{2})^k \frac{x^{n-2k}}{k!(n-2k)!}$$

by use of the following lemma [16,125].

Lemma. Within the common interval of convergence of the series being multiplied,

$$\sum_{n=0}^{\infty} a_n y^{2n} \sum_{n=0}^{\infty} b_n y^n = \sum_{n=0}^{\infty} \sum_{k=0}^{[n/2]} a_k b_{n-2k} y^n \quad .$$

The above expression along with Eq.3 implies that

$$H_n(x) = \sum_{k=0}^{[n/2]} (-\frac{1}{2})^k \frac{x^{n-2k} n!}{k!(n-2k)!} \quad , \tag{4}$$

where $[n/2]$ denotes the largest integer $\leq n/2$.

Note that $H_0 = 1$ as a direct consequence of Eq.2.

The first six Hermite polynomials, computed from Eq.4, are

$$H_0(x) = 1$$

$$H_1(x) = x$$

$$H_2(x) = x^2 - 1$$

$$H_3(x) = x^3 - 3x$$

$$H_4(x) = x^4 - 6x^2 + 3$$

$$H_5(x) = x^5 - 10x^3 + 15x$$

$$H_6(x) = x^6 - 15x^4 + 45x^2 - 15 \ .$$

Many other interesting properties of Hermite polynomials may be derived from Eq.3. Two which are pertinent to our development are shown below.

First, differentiate both sides of Eq.3 with respect to t. From the LHS, we obtain

$$\frac{d}{dt}\exp[xt-t^2/2] = (x-t)\exp[xt-t^2/2] \quad ,$$

and from the RHS, we have

$$\frac{d}{dt}\sum_{n=0}^{\infty}\frac{H_n(x)t^n}{n!} = \sum_{n=0}^{\infty}\frac{H_{n+1}(x)t^n}{n!} \quad .$$

Equating the derivatives, expressed in series form, we have

$$x\sum_{n=0}^{\infty}\frac{H_n(x)t^n}{n!} - t\sum_{n=0}^{\infty}\frac{H_n(x)t^n}{n!} = \sum_{n=0}^{\infty}\frac{H_{n+1}(x)t^n}{n!}$$

or

$$x\sum_{n=0}^{\infty}\frac{H_n(x)t^n}{n!} - \sum_{n=1}^{\infty}\frac{H_{n-1}(x)t^n}{(n-1)!} = \sum_{n=0}^{\infty}\frac{H_{n+1}(x)t^n}{n!} \quad .$$

Arranging all terms on the left side of the equal sign and comparing coefficients of t^n gives

$$\frac{H_{n+1}(x)}{n!} - \frac{xH_n(x)}{n!} + \frac{H_{n-1}(x)}{(n-1)!} = 0$$

or

$$H_{n+1}(x) - xH_n(x) + nH_{n-1}(x) = 0 \quad , \qquad m=1,2,\ldots \qquad (5)$$

This recurrence relation is satisfied by all Hermite polynomials and is useful in computing the higher orders of $H_n(x)$.

Next, consider the differentiation of Eq.3 with respect to x

$$t \exp[xt - t^2/2] = \sum_{n=0}^{\infty} \frac{d}{dx} H_n(x) \frac{t^n}{n!} \quad .$$

Expanding the left side as before we have

$$t \exp[xt - t^2/2] = \sum_{n=0}^{\infty} \sum_{k=0}^{[n/2]} \left(-\frac{1}{2}\right)^k \frac{x^{n-2k} n!}{k!(n-2k)!} \frac{t^{n+1}}{n!} \quad .$$

Comparing the coefficients of t^n with those in the previous expression yields

$$\frac{d}{dx} H_n(x) = nH_{n-1}(x) \quad . \tag{6}$$

Hermite polynomials also possess the important property of orthogonality, that is

$$\int_{-\infty}^{\infty} H_m(x) H_n(x) \phi(x) dx = \begin{cases} 0, & m \neq n \\ n!, & m = n \end{cases} \quad .$$

This can be demonstrated using integration by parts. Suppose $m \leq n$, then*

$$I = \int_{-\infty}^{\infty} H_m(x) H_n(x) \phi(x) dx = (-1)^n \int_{-\infty}^{\infty} H_m \frac{d^n \phi}{dx^n} dx$$

$$= (-1)^n \left[H_m \frac{d^{n-1}\phi}{dx^{n-1}} \right]_{-\infty}^{\infty} + (-1)^{n-1} \int_{-\infty}^{\infty} \frac{dH_m}{dx} \frac{d^{n-1}\phi}{dx^{n-1}} dx \quad .$$

Now, using Eq. 6 and the fact that the term in brackets is zero because

$$\phi(-\infty) = \phi(\infty) = 0$$

we have

---

*Note that for the sake of brevity the arguments have been dropped from $\phi(x)$ and $H_n(x)$.

$$I = m(-1)^{n-1} \int_{-\infty}^{\infty} H_{m-1} \frac{d^{n-1}}{dx^{n-1}} \phi dx \quad .$$

By continuing this process we find zero for m < n and for m = n

$$I = \int_{-\infty}^{\infty} H_n^2 \, \phi dx \quad .$$

This can be evaluated as follows.

We replace n by n-1 in Eq.5, then multiply by $\phi(x)H_n(x)dx$, integrate from $-\infty$ to ∞ and

$$\int_{-\infty}^{\infty} \phi(x)H_n^2(x)dx - \int_{-\infty}^{\infty} \phi(x)xH_n(x)H_{n-1}(x)dx = 0$$

by orthogonality when $m \neq n$.

Next, multiply Eq.5 by $\phi(x)H_{n-1}(x)dx$, integrate from $-\infty$ to ∞ and

$$-\int_{-\infty}^{\infty} \phi(x)xH_n(x)H_{n-1}(x)dx + n \int_{-\infty}^{\infty} \phi(x)H_{n-1}^2(x)dx = 0$$

again using orthogonality when $m \neq n$.

Combining the above two expressions, we have the recurrence relation

$$\int_{-\infty}^{\infty} \phi(x)H_n^2(x)dx = n \int_{-\infty}^{\infty} \phi(x)H_{n-1}^2(x)dx \quad .$$

Letting n = 1,2,..., n recursively we find that

$$\int_{-\infty}^{\infty} \phi(x)H_n^2(x)dx = n(n-1)\ldots \int_{-\infty}^{\infty} \phi(x)H_0^2(x)dx$$

$$= n! \quad ,$$

since $H_0(x) = 1$

Now, our objective is to be able to expand a p.d.f. f(x) in a series as follows

$$f(x) = \sum_{j=0}^{\infty} c_j H_j(x) \phi(x) \quad . \tag{7}$$

However, we still must determine the c_j, and this can be accomplished in the same manner as the coefficients of a Fourier series are obtained. Multiply both sides of Eq.7 by $H_n(x)dx$ and integrate from $-\infty$ to ∞

$$\int_{-\infty}^{\infty} H_n(x) f(x) dx = \int_{-\infty}^{\infty} \sum_{j=0}^{\infty} c_j H_n(x) H_j(x) \phi(x) dx \quad .$$

Using the orthogonal property we have

$$\int_{-\infty}^{\infty} H_n(x) f(x) dx = c_n \int_{-\infty}^{\infty} H_n^2(x) \phi(x) dx$$

or

$$c_n = \frac{1}{n!} \int_{-\infty}^{\infty} H_n(x) f(x) dx. \tag{8}$$

Denoting moments about the origin as v_j and substituting Eq.4 into Eq.8 a general expression for c_n is

$$c_n = \frac{1}{n!} \sum_{k=0}^{[n/2]} (-\tfrac{1}{2})^k \frac{n! v_{n-2k}}{k!(n-2k)!} \quad . \tag{9}$$

The convergence of Eq.8 allows us to interchange the summation and integration operations. In obtaining Eq.9 the summation and integration operations were interchanged. This was possible because f(x) is a p.d.f. with finite moments and this assures the convergence of the integral in Eq.8.

These results yield the desired expansion

$$f(x) = \emptyset(x)\left[1 + v_1H_1 + \frac{v_2-1}{2}H_2 + \frac{v_3-3v_1}{6}H_3 + \frac{v_4-6v_2+3}{24}H_4 + \dots\right] \quad (10)$$

In order to obtain the Gram-Charlier A series we simply let $\emptyset(x) = \exp(-\frac{1}{2}[(x-u)/\sigma]^2)/\sqrt{2\pi}\,\sigma$ in Eq.1 which means we replace x by $(x-u)/\sigma$ in the subsequent development. Now the c_n are

$$c_n = \frac{1}{n!}\sum_{k=0}^{[n/2]}(-\tfrac{1}{2})^k \frac{n!}{k!(n-2k)!}\left(\frac{u_{n-2k}}{\sigma^{n-2k}}\right), \quad (11)$$

and the series becomes

$$f(x) = \emptyset(x)\left[1 + \frac{u_3}{6\sigma^3}H_3(x) + \frac{u_4-3\sigma^4}{24\sigma^4}H_4(x) + \dots\right], \quad (12)$$

where the u_j are the moments about the mean. Eq.12 is the Gram-Charlier type A series.

Let us now consider under what conditions this expansion may represent a p.d.f.

Many necessary conditions have been discussed in the literature when the series involved is infinite [14,161]. However, in practice we are interested in obtaining a good approximation using as few terms as possible [14,162].

Some attention has been given to the case where only the first five c_j are considered [21,425], [36,59]. Although different reasons are advanced for examining this particular case, the fact remains that this is the largest easily solvable case.

Thus, let us examine

$$f(x) = \phi(x) \sum_{i=0}^{4} c_i H_i(x) \quad . \tag{13}$$

Since all p.d.f.'s are positive we desire f(x) to be positive, and we know that $\phi(x)$ is always positive thus, we only need to examine the series expression. The desired conditions for the series to be positive definite may be derived by determining for what values of the c_j the series will equal zero. Since $u_0=1$, $u_1=0$ and $u_2=\sigma^2$, $c_1=c_2=0$ from Eq.11, then setting the series equal to zero, we have that

$$1 + c_3H_3(x) + c_4H_4(x) = 0 \ .$$

Multiplying through by 24 and expressing the c_j and $H_j(x)$ explicitly yields

$$(\beta_2-3)x^4 + 4\sqrt{\beta_1}x^3 + 6(-1)(\beta_2-3)x^2 + 4(-3\sqrt{\beta_1})x + 3(\beta_2-3) + 24 = 0 \quad , \tag{14}$$

where $\beta_2 = u_4/\sigma^4$ and $\sqrt{\beta_1} = u_3/\sigma^3$.

This is the form of the general quartic equation [4,56].

Since we are interested in the nature of the roots of this equation let us transform Eq.14 into a new equation lacking the second term [4,56]. Using the notation from [4] we have

$$y^4 + 6Hy^2 + 4Gy + (\beta_2-3)^2 I - 3H^2 = 0 \ , \tag{15}$$

where

$$y = (\beta_2-3)x + \sqrt{\beta_1}$$

$$H = -(\beta_2-3)^2 - \beta_1$$

$$G = 2(\sqrt{\beta_1})^3$$

$$I = 6(\beta_2-3)^2 + 24(\beta_2-3) + 12\beta_1 .$$

Finally, define

$$J = [(\beta_2-3)^2 HI - G^2 - 4H^3]/(\beta_2-3)^3 .$$

Now, because there is no cube term in Eq.15 it will be positive for all y if all its roots are imaginary. All the roots will be imaginary if [4,58]

$$I^3 - 27J^2 > 0 \tag{16}$$

and $$3(\beta_2-3)J - 2HI < 0 .$$

These conditions allow us to determine whether or not we wish to obtain a G-C expansion based on the values of the first four moments of the unknown distribution.

Substituting the values of H, G, I and J into Eq.16 and rearranging yields the conditions in explicit form

$$[8(\beta_2-3) + 2(\beta_2-3)^2 + 4\beta_1]^3$$

$$> [24(\beta_2-3)^2 + 2(\beta_2-3) + 24\beta_1 + 6\beta_1(\beta_2-3)]^2$$

and $$4(\beta_2-3)^3 + 4\beta_1(\beta_2-3) > 4\beta_1^2 + 3\beta_1(\beta_2-3)^2 + (\beta_2-3)^4 . \tag{17}$$

When these inequalities are satisfied the G-C expansion Eq.13 will be positive definite for all x in the range $-\infty < x < \infty$.

We may also obtain an expression for the distribution function, if desired. This is accomplished as follows

$$F(x) = \int_{-\infty}^{x} f(t)dt$$

$$= \int_{-\infty}^{x} \sum_{j=0}^{\infty} c_j H_j(t)\phi(t)dt$$

using Eq.7. Assuming convergence

$$F(x) = \sum_{j=0}^{\infty} c_j \int_{-\infty}^{x} H_j(t)\phi(t)dt ,$$

which may be integrated using Eq.2, whence

$$F(x) = \sum_{j=0}^{\infty} c_j \int_{-\infty}^{x} (-1)^{-j}\phi^{(j)}(t)dt$$

$$= \sum_{j=0}^{\infty} c_j(-1)^{-j}\phi^{(j-1)}(t) \Big]_{-\infty}^{x}$$

$$= \sum_{j=0}^{\infty} c_j(-1)^{-j}(-1)^{j-1}H_{j-1}(x)\phi(x)$$

$$= - \sum_{j=0}^{\infty} c_j H_{j-1}(x)\phi(x) \quad . \qquad (18)$$

The calculations necessary for a G-C expansion may be accomplished by a FORTRAN subprogram called ASER which operates in conjunction with TEST.

As an example of the G-C expansion consider the following problem. Suppose we convolve an exponential distribution with a normal distribution, then the resulting M.G.F. is

$$M(s) = \frac{ae^{ms+\sigma^2 s^2/2}}{a - s} .$$

Let $a = 4$, $m = 2$, and $\sigma = 1$, then the first six moments are

$$v_1 = 2.250000 \qquad v_4 = 61.59375$$

$$v_2 = 6.125000 \qquad v_5 = 218.9922$$

$$v_3 = 18.59375 \qquad v_6 = 797.4883 .$$

We know the range is $(-\infty,\infty)$ and that the resulting distribution is continuous.

The moments along with the range and continuity information are now inserted into the program TEST.

The output of the program tells us that only four moments can be used in the approximation because

$$\begin{vmatrix} v_0 & v_1 & v_2 & v_3 \\ v_1 & v_2 & v_3 & v_4 \\ v_2 & v_3 & v_4 & v_5 \\ v_3 & v_4 & v_5 & v_6 \end{vmatrix} = - 54.6675 ,$$

which violates Theorem 1.

The program also tells us the resulting distribution is unimodal.

The coefficients for the expansion are

$$c_1 = 0.0 \qquad c_3 = .004756$$

$$c_2 = 0.0 \qquad c_4 = .000865 ,$$

and values of the resulting curve are shown in Table II along with the actual curve.

TABLE II

VALUES OF THE ACTUAL, f(t), AND APPROXIMATE, $\hat{f}(t)$, DISTRIBUTION RESULTING FROM THE CONVOLUTION OF AN EXPONENTIAL DISTRIBUTION WITH A NORMAL DISTRIBUTION

t	f(t)	$\hat{f}(t)$	t	f(t)	$\hat{f}(t)$
-3.10	.000000	.000000	.90	.168205	.164825
-3.00	.000001	.000001	1.00	.190135	.186540
-2.90	.000001	.000001	1.10	.212859	.209080
-2.80	.000002	.000002	1.20	.236010	.232084
-2.70	.000003	.000003	1.30	.259169	.255137
-2.60	.000005	.000005	1.40	.281875	.277780
-2.50	.000008	.000008	1.50	.303638	.299525
-2.40	.000012	.000013	1.60	.323957	.319870
-2.30	.000019	.000020	1.70	.342341	.338321
-2.20	.000030	.000031	1.80	.358324	.354409
-2.10	.000046	.000047	1.90	.371489	.367712
-2.00	.000069	.000070	2.00	.381485	.377874
-1.90	.000104	.000104	2.10	.388042	.384618
-1.80	.000154	.000154	2.20	.390982	.387764
-1.70	.000227	.000226	2.30	.390230	.387229
-1.60	.000331	.000327	2.40	.385817	.383040
-1.50	.000478	.000470	2.50	.377874	.375325
-1.40	.000683	.000669	2.60	.366631	.364309
-1.30	.000966	.000943	2.70	.352402	.350305
-1.20	.001354	.001318	2.80	.335572	.333696
-1.10	.001879	.001825	2.90	.316581	.314920
-1.00	.002581	.002504	3.00	.295901	.294448
-.90	.003512	.003402	3.10	.274022	.272769
-.80	.004731	.004579	3.20	.251430	.250366
-.70	.006311	.006105	3.30	.228589	.227705
-.60	.008337	.008062	3.40	.205929	.205212
-.50	.010906	.010546	3.50	.183831	.183268
-.40	.014128	.013665	3.60	.162622	.162199
-.30	.018123	.017536	3.70	.142567	.142268
-.20	.023022	.022289	3.80	.123867	.123676
-.10	.028961	.028059	3.90	.106662	.106563
.00	.036078	.034985	4.00	.091035	.091011
.10	.044507	.043201	4.10	.077015	.077050
.20	.054373	.052833	4.20	.064586	.064664
.30	.065782	.063990	4.30	.053693	.053800
.40	.078814	.076758	4.40	.044253	.044378
.50	.093515	.091185	4.50	.036163	.036294
.60	.109885	.107279	4.60	.029301	.029431
.70	.127873	.124996	4.70	.023543	.023664
.80	.147371	.144233	4.80	.018760	.018868

TABLE II (continued)

t	f(t)	$\hat{f}(t)$
4.90	.014827	.014918
5.00	.011623	.011697
5.10	.009039	.009096
5.20	.006974	.007015
5.30	.005339	.005366
5.40	.004057	.004070
5.50	.003059	.003063
5.60	.002290	.002285
5.70	.001702	.001692
5.80	.001256	.001242
5.90	.000921	.000904
6.00	.000671	.000653
6.10	.000486	.000468
6.20	.000349	.000332
6.30	.000250	.000234
6.40	.000178	.000164
6.50	.000126	.000113
6.60	.000088	.000078
6.70	.000062	.000053
6.80	.000043	.000036
6.90	.000030	.000024
7.00	.000021	.000016
7.10	.000014	.000010
7.20	.000010	.000007
7.30	.000007	.000004
7.40	.000005	.000003
7.50	.000003	.000002
7.60	.000002	.000001
7.70	.000001	.000001
7.80	.000001	.000000
7.90	.000001	.000000
8.00	.000000	.000000

A tabulation of the cumulative distribution function is also provided using Eq.18, see Table III. Note that the approximation is such that cumulative distribution does not attain the value of one. This might indicate that the approximation is not too good in the tails of the distribution. However, examination of Table II reveals that the approximation is good in the tails, and the greatest deviation, about .004, occurs to the left of the mean and the mode. Since the area under the curve is most sensitive to deviations in the curve at the mode the resulting error in the cumulative distribution function is not surprising. The exact nature of the deviation is displayed in Fig.9, and clearly illustrates the difficulty. The approximate c.d.f. could be made to attain the value of one simply by normalization. This should work quite well in this case since the approximation underestimates the actual p.d.f. for most values in the range of interest.

In actually evaluating the goodness of the approximation it would be more meaningful to plot the percent deviation, which would correspond to performing a chi-square test. This is not done here since in general the actual p.d.f. is not available for comparison, and this particular example cannot be claimed to be completely representative of the error generated by the G-C A series approximation.

Laguerre Series

When the p.d.f. of interest is defined over the positive half line $(0,\infty)$ we may use an expansion analogous to

TABLE III

VALUES OF THE APPROXIMATE CUMULATIVE

DISTRIBUTION FUNCTION

t	$\hat{F}(t)$	t	$\hat{F}(t)$	t	$\hat{F}(t)$
-3.00	.000000	.60	.052218	4.20	.940896
-2.90	.000000	.70	.063472	4.30	.946629
-2.80	.000000	.80	.076520	4.40	.951380
-2.70	.000001	.90	.091502	4.50	.955283
-2.60	.000001	1.00	.108541	4.60	.958462
-2.50	.000002	1.10	.127731	4.70	.961029
-2.40	.000003	1.20	.149135	4.80	.963086
-2.30	.000004	1.30	.172778	4.90	.964720
-2.20	.000007	1.40	.198643	5.00	.966007
-2.10	.000011	1.50	.226666	5.10	.967012
-2.00	.000016	1.60	.256734	5.20	.967791
-1.90	.000025	1.70	.288687	5.30	.968389
-1.80	.000037	1.80	.322315	5.40	.968845
-1.70	.000056	1.90	.357369	5.50	.969190
-1.60	.000082	2.00	.393562	5.60	.969448
-1.50	.000121	2.10	.430577	5.70	.969641
-1.40	.000176	2.20	.468082	5.80	.969782
-1.30	.000254	2.30	.505728	5.90	.969886
-1.20	.000363	2.40	.543130	6.00	.969961
-1.10	.000515	2.50	.579944	6.10	.970016
-1.00	.000724	2.60	.615847	6.20	.970054
-.90	.001010	2.70	.650536	6.30	.970082
-.80	.001395	2.80	.683739	6.40	.970101
-.70	.001911	2.90	.715225	6.50	.970114
-.60	.002596	3.00	.744806	6.60	.970123
-.50	.003495	3.10	.772338	6.70	.970130
-.40	.004665	3.20	.797727	6.80	.970134
-.30	.006173	3.30	.820925	6.90	.970137
-.20	.008098	3.40	.841928	7.00	.970139
-.10	.010532	3.50	.860771	7.10	.970140
.00	.013581	3.60	.877523	7.20	.970141
.10	.017362	3.70	.892283	7.30	.970142
.20	.022008	3.80	.905172	7.40	.970142
.30	.027661	3.90	.916327	7.50	.970142
.40	.034474	4.00	.925897		
.50	.042606	4.10	.934035		

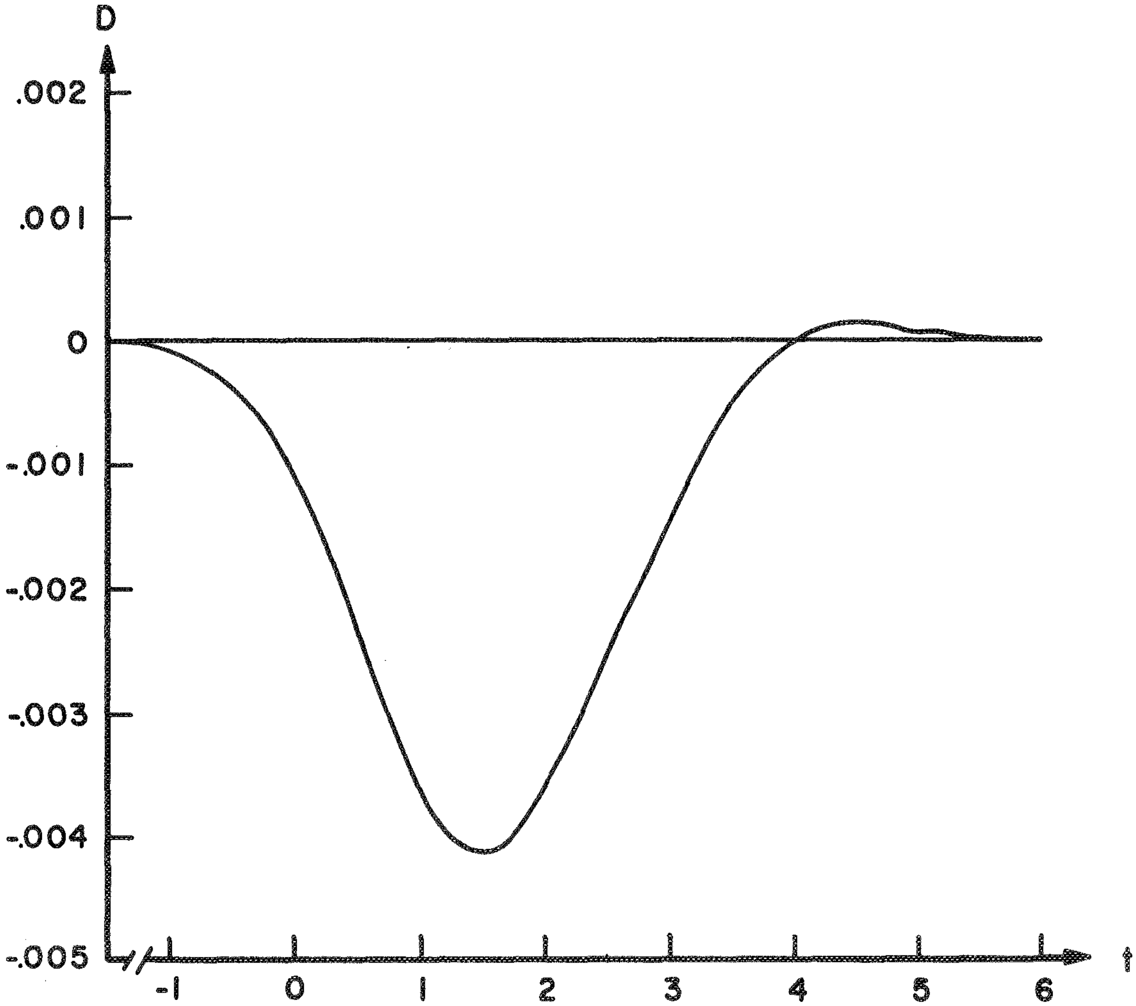


Figure 9. Graph of the deviation D, $D=\hat{f}(t)-f(t)$, vs t for the A - series of the convolution of an exponential with a normal.

the Gram-Charlier A series. This expansion is based on a form of the Gamma distribution. Once again a direct approach will be used in the derivation.

First define

$$f(x) = \begin{cases} \dfrac{\lambda^{r+1} x^r e^{-\lambda x}}{\Gamma(r+1)} & , \ 0 < x < \infty, \quad r > -1, \ \lambda > 0 \\ 0 & \text{otherwise,} \end{cases}$$

the Gamma density function.

If we multiply through by x^n and set $\lambda = 1$ we have the function on which the expansion is based, thus

$$\gamma(x) = \frac{x^{r+n} e^{-x}}{\Gamma(r+1)} \quad . \tag{19}$$

Actually λ could be given any finite value or determined as a function of the moments of the unknown p.d.f. However, in the limited testing of the Laguerre series approximation during the research reported herein it was found that $\lambda = 1$ gave the best approximation, in the least-squares sense, with fewer terms. Also define

$$g(x) = \frac{x^r e^{-x}}{\Gamma(r+1)} \quad . \tag{19a}$$

The Laguerre polynomial of order n is obtained as the coefficient of g(x) by differentiating Eq.19 n times. This is done for $n = 1,2,\ldots,n$. Thus,

$$\gamma^{(1)}(x) = g(x)[-x + r + 1]$$

$$\gamma^{(2)}(x) = g(x)[x^2 - 2(r+2)x + (r+1)(r+2)]$$

$$\gamma^{(3)}(x) = g(x)[-x^3+3(r+3)x^2-3(r+2(r+3)x+(r+1)(r+2)(r+3)]$$

.

.

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and in general

$$\gamma^{(n)}(x) = (-1)^n \, g(x) L_n^{(r)}(x) \quad . \tag{20}$$

The expressions $L_n^{(r)}(x)$ are known as Laguerre Polynomials of type r, and are usually defined by the formula [12,184].

$$L_n^{(r)}(x) = (-1)^n \frac{e^x}{x^r} \frac{d^n}{dx^n} [x^{r+n} e^{-x}] \; , \quad n = 0,1,\ldots \tag{21}$$

which follows from (20).

Application of Leibniz's rule to Eq.21 with $n = 1,2,\ldots$ leads to the general relation

$$L_n^{(r)}(x) = (-1)^n \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{(1+r)_n}{(1+r)_k} x^k \quad , \tag{22}$$

where $(1+r)_m = (r+1)(r+2)(r+3) \ldots (r+m)$.

The first four Laguerre polynomials, computed from Eq.22, are

$$L_0^{(r)}(x) = 1$$

$$L_1^{(r)}(x) = x - r - 1$$

$$L_2^{(r)}(x) = x^2 - 2(r+2)x + (r+1)(r+2)$$

$$L_3^{(r)}(x) = x^3 - 3(r+3)x^2 + 3(r+2)(r+3)x - (r+1)(r+2)(r+3)$$

$$L_4^{(r)}(x) = x^4 - 4(r+4)x^3 + 6(r+3)(r+4)x^2$$
$$- 4(r+2)(r+3)(r+4)x$$
$$+ (r+1)(r+2)(r+3)(r+4)$$

Laguerre polynomials also possess the property of orthogonality, which may be demonstrated as follows.

Suppose m and n are positive integers, $m \leq n$, and

$$I = \int_0^\infty g(x)L_m^{(r)}(x)L_n^{(r)}(x)dx = (-1)^n \int_0^\infty L_m^{(r)}(x)\gamma^{(n)}(x)dx .$$

Let us now perform m successive integrations by parts with

$$u = \frac{d^{k-1}}{dx^{k-1}} L_m^{(r)}(x) \text{ and } dv = \gamma^{(n-k+1)}(x)dx$$

as the factors for the k^{th} step. It can be seen from Eq.19 that the expression uv with $v = \gamma^{(n-k)}(x)$ vanishes at both 0 and ∞, which leaves

$$I = (-1)^{m+n} \int_0^\infty [L_m^{(r)}(x)]^{(m)}\gamma^{(n-m)}(x)dx \quad , \qquad (23)$$

at the m^{th} step.

If $m < n$ another integration by parts implies that $I = 0$ because

$$\frac{d^{m+1}}{dx^{m+1}} [L_m^{(r)}(x)] = 0.$$

That is, the $(m+1)^{st}$ derivative of an m^{th} order polynomial is zero.

When $n = m$, Eq.23 becomes

$$I = \int_0^\infty \frac{d^n}{dx^n} [L_n^{(r)}(x)] \gamma(x)dx.$$

Since the coefficient of x^n in $L_n^{(r)}(x)$ is 1,

$$\frac{d^n}{dx^n}\,[L_n^{(r)}(x)] = n!$$

Let V(I) denote the value of I in this case, and

$$V(I) = n!\int_0^\infty \gamma(x)dx$$

$$= \frac{n!}{\Gamma(r+1)}\int_0^\infty x^{r+n}e^{-x}dx$$

$$= \frac{n!\,\Gamma(r+n+1)}{\Gamma(r+1)} \quad .$$

Thus, Laguerre polynomials are orthogonal since

$$I = \int_0^\infty g(x)L_m^{(r)}(x)L_n^{(r)}(x)dx = \begin{cases} 0 & , n \neq m \\ \dfrac{n!\,\Gamma(r+n+1)}{\Gamma(r+1)} & , n = m \quad . \end{cases}$$

Now consider the development of a recurrence relation useful in computing one Laguerre polynomial in terms of two others. We begin with the generating function definition of Laguerre polynomials [12,187],

$$G(t) = \frac{\exp\left[-\frac{xt}{1-t}\right]}{(1-t)^{r+1}} = \sum_{n=0}^\infty (-1)^n \frac{L_n^{(r)}(x)\ t^n}{n!} \quad .$$

Note that this definition differs from the standard generating function form in that the series contains the factor $(-1)^n$. Taking the partial of each expression with respect to t we have

$$\frac{\partial G}{\partial t} = \frac{e^{-xt/(1-t)}}{(1-t)^{r+1}}\left[\frac{r+1}{1-t} - \frac{x}{(1-t)^2}\right]$$

$$= -\sum_{n=0}^{\infty} (1-n)^n \frac{L_{n+1}^{(r)}(x)t^n}{n!} \quad .$$

The first relation above may be rewritten as

$$(1-t)^2 \frac{\partial G}{\partial t} = [(r+1)(1-t) - x]\, G \quad ,$$

or

$$(1-2t+t^2) \frac{\partial G}{\partial t} = [r+1-x]G - (r+1)tG \quad .$$

Substituting the series expressions for G and $\partial G/\partial t$ in the previous equation yields

$$-\sum_{n=0}^{\infty} (-1)^n \frac{L_{n+1}^{(r)}(x)t^n}{n!} + 2\sum_{n=0}^{\infty} (-1)^n \frac{L_{n+1}^{(r)}(x)t^{n+1}}{n!}$$

$$-\sum_{n=0}^{\infty} (-1)^n \frac{L_{n+1}^{(r)}(x)t^{n+2}}{n!} = [r+1-x]\sum_{n=0}^{\infty} (-1)^n \frac{L_n^{(r)}(x)t^n}{n!}$$

$$- (r+1)\sum_{n=0}^{\infty} (-1)^n \frac{L_n^{(r)}(x)t^{n+1}}{n!} \, .$$

Expressing each series as a series in t^n we have

$$-\sum_{n=0}^{\infty} (-1)^n \frac{L_{n+1}^{(r)}(x)t^n}{n!} - 2\sum_{n=1}^{\infty} (-1)^n \frac{L_n^{(r)}(x)t^n}{(n-1)!}$$

$$-\sum_{n=2}^{\infty} (-1)^n \frac{L_{n-1}^{(r)}(x)t^n}{(n-2)!} = [r+1-x]\sum_{n=0}^{\infty} (-1)^n \frac{L_n^{(r)}(x)t^n}{n!}$$

$$+ (r+1)\sum_{n=1}^{\infty} (-1)^n \frac{L_{n-1}^{(r)}(x)t^n}{(n-1)!} \, .$$

Finally, equating the coefficient of t^n in each series it is found that

$$-\frac{L_{n+1}^{(r)}(x)}{n!} - 2\,\frac{L_n^{(r)}(x)}{(n-1)!} - \frac{L_{n-1}^{(r)}(x)}{(n-2)!}$$

$$= \left[r+1-x\right]\frac{L_n^{(r)}(x)}{n!} + (r+1)\,\frac{L_{n-1}^{(r)}(x)}{(n-1)!}\;,$$

which upon rearrangement gives the recurrence relation

$$L_{n+1}^{(r)}(x) - \left[x-r-1-2n\right] L_n^{(r)}(x) + n\left[n+r\right] L_{n-1}^{(r)}(x) = 0. \qquad (24)$$

The Laguerre series expansion of a function takes the form

$$f(x) = \sum_{k=0}^{\infty} d_k\, L_k^{(r)}(x)\; g(x)\;, \qquad (25)$$

where the d_k are coefficients that must be determined. The d_k are determined by multiplying both sides of Eq.25 by $L_j^{(r)}(x)dx$, and integrating over the entire range of x. This yields

$$d_j = \frac{\Gamma(r+1)}{j!\,\Gamma(r+j+1)}\int_0^{\infty} L_j^{(r)}(x)\; f(x)dx$$

by the orthogonal property.

A more useful expression for d_j may be obtained by substituting for $L_j^{(r)}(x)$ the expression given in Eq.22 and integrating term by term, whence

$$d_j = \frac{(-1)^j}{j!\,(1+r)_j}\sum_{k=0}^{j}\binom{j}{k}(-1)^k\,\frac{(1+r)_j}{(1+r)_k}\,v_k\;, \qquad (26)$$

where the v_k are the moments about the origin.

Before Eq.25 can be used to correctly represent a function, known by its M.G.F. and moments, we must be able to select the appropriate value for the parameter r.

At first this might seem to present an insurmountable problem since we do not even know what criterion we should use to select r. However, a little thought reveals that there are really only two types of p.d.f.'s that we can encounter. That is, either the p.d.f. has a value of zero at its left hand boundary or it does not. In any case the series representation should possess the same attribute as the unknown p.d.f. This requirement is critical because we are interested in using only the first few terms of the series. Thus, our approach is as follows: (1) use the initial value theorem from Laplace transform theory to determine the nature of the unknown p.d.f.; (2) if the initial value is not zero we set $r = 0$ which makes $g(0) = 1$ (see Eq.19a), and the $\Sigma d_k L_k(0)$ will take care of the initial value; (3) if the initial value is zero, we set $r = 1$ which makes $g(0) = 0$ and again the correct initial value results. Note that we need not consider $r > 1$ since we are interested only in unimodal distributions.

An expression for the distribution function may also be derived as follows

$$F(x) = \int_0^x f(t)dt$$

$$= \int_0^x \sum_{k=0}^{\infty} d_k \; L_k^{(r)}(t)g(t)dt,$$

Using Eq.25 Since the integral converges

$$F(x) = \sum_{k=0}^{\infty} d_k \int_0^x L_k^{(r)}(t)g(t)dt$$

$$= \sum_{k=0}^{\infty} d_k \int_0^x \frac{(-1)^k}{\Gamma(r+1)} \frac{d^k}{dt^k}\left[t^{r+k}e^{-t}\right] dt$$

from Eqs. 19a and 21. Now

$$F(x) = \frac{1}{\Gamma(r+1)}\left[d_0 \int_0^x t^r e^{-t}dt + \sum_{k=1}^{\infty} d_k(-1)^k \int_0^x \frac{d^k}{dt^k}\left[t^{r+k}e^{-t}\right]dt\right]$$

$$= \frac{1}{\Gamma(r+1)}\left[d_0 \int_0^x t^r e^{-t}dt + \sum_{k=1}^{\infty} d_k(-1)^k \frac{d^{k-1}}{dt^{k-1}}\left[t^{r+k}e^{-t}\right]\Bigg|_0^x\right],$$

but

$$\frac{d^{k-1}}{dt^{k-1}}\left[t^{r+k}e^{-t}\right] = \frac{d^{k-1}}{dt^{k-1}}\left[t^{r+1+(k-1)}e^{-t}\right]$$

$$= (-1)^{-(k-1)} t^{r+1} e^{-t} L_{k-1}^{(r+1)}(t)$$

from Eq.21, thus

$$F(x) = \frac{1}{\Gamma(r+1)}\left[d_0 \int_0^x t^r e^{-t}dt - \sum_{k=1}^{\infty} d_k x^{r+1} e^{-x} \; L_{k-1}^{(r+1)}(x)\right] .$$

The two cases of interest to us are $r = 0$ and $r = 1$.

Therefore we use:

$$F(x) = 1 - e^{-x} - xe^{-x} \sum_{k=1}^{\infty} d_k \, L_{k-1}^{(1)}(x) \quad , \tag{27}$$

if $r = 0$, or

$$F(x) = 1 - (x+1)e^{-x} - x^2 e^{-x} \sum_{k=1}^{\infty} d_k \, L_{k-1}^{(2)}(x) \quad , \tag{28}$$

if $r = 1$.

The calculations necessary for the Laguerre expansion are performed by a FORTRAN subprogram called LAG.

The Laguerre expansion will be illustrated by approximating the distribution of the convolution of two exponential distributions. The resulting M.G.F. is

$$M(s) = \frac{ab}{(a-s)(b-s)} \quad .$$

Let $a = 1.5$, and $b = 2.5$, then the first eight moments are

$$\begin{aligned} v_1 &= 1.0666667 & v_5 &= 37.6629730 \\ v_2 &= 1.7422222 & v_6 &= 153.60101 \\ v_3 &= 3.8684444 & v_7 &= 725.06225 \\ v_4 &= 10.930252 & v_8 &= 3893.4228 \quad . \end{aligned}$$

The resulting distribution is continuous and has range $(0,\infty)$. Also, since the initial value is zero, that is, $\lim_{s\to\infty} sM(s) = 0$, we set $r = 1$. The above information is used as input data to TEST.

The tests performed by the program tell us that a unimodal distribution is specified by the eight moments given above. The coefficients for the expansion are

$$d_1 = -.46666665 \qquad d_5 = -.00023585$$
$$d_2 = .11185183 \qquad d_6 = .00002049$$
$$d_3 = -.01832098 \qquad d_7 = -.00000155$$
$$d_4 = .00230140 \qquad d_8 = .00000010 \quad .$$

Note that the last few coefficients are quite small in magnitude which indicates that the approximation using eight moments should be fairly good. The resulting curve and the actual are shown in Table IV, and the approximate cumulative distribution function in Table V.

The approximation remains positive up to $t = 10.0$ where the actual value is .000001, and the approximate c.d.f. sums to one, thus we can say that a reasonable approximation was attained.

<u>The Gram-Charlier Type B Series</u>

Let us now consider the case where the p.d.f. of interest is discrete and defined over the positive half line $(0,\infty)$.

The Type B, or simply B, series is based upon the Poisson distribution and its differences. Again we use a direct approach in the derivation.

First, define

$$p(x,m) = \begin{cases} e^{-m} \dfrac{m^x}{x!} , & x = 0,1,2,3,\ldots \\ 0 & \text{otherwise} \end{cases} \qquad (29)$$

the Poisson distribution. Next we difference the function

TABLE IV

VALUES OF DENSITY FUNCTION f(t) AND LAGUERRE

APPROXIMATION $\hat{f}(t)$ WITH r = 1

t	f(t)	$\hat{f}(t)$	t	f(t)	$\hat{f}(t)$
0.00	0.000000	0.000000	4.00	.009125	.009356
.10	.307152	.304311	4.10	.007868	.007988
.20	.503578	.501301	4.20	.006783	.006802
.30	.619731	.619201	4.30	.005847	.005774
.40	.678496	.679729	4.40	.005039	.004887
.50	.696982	.699470	4.50	.004342	.004123
.60	.687898	.690989	4.60	.003741	.003467
.70	.660614	.663707	4.70	.003223	.002905
.80	.621971	.624608	4.80	.002777	.002427
.90	.576904	.578796	4.90	.002392	.002021
1.00	.528919	.529934	5.00	.002060	.001677
1.10	.480458	.480589	5.10	.001774	.001389
1.20	.433169	.432505	5.20	.001528	.001148
1.30	.388124	.386812	5.30	.001316	.000948
1.40	.345971	.344187	5.40	.001133	.000782
1.50	.307056	.304983	5.50	.000976	.000647
1.60	.271509	.269323	5.60	.000840	.000537
1.70	.239315	.237169	5.70	.000723	.000449
1.80	.210362	.208380	5.80	.000623	.000379
1.90	.184472	.182749	5.90	.000536	.000323
2.00	.161434	.160033	6.00	.000462	.000280
2.10	.141017	.139974	6.10	.000397	.000247
2.20	.122986	.122311	6.20	.000342	.000222
2.30	.107111	.106795	6.30	.000295	.000204
2.40	.093169	.093186	6.40	.000254	.000190
2.50	.080952	.081265	6.50	.000218	.000180
2.60	.070269	.070832	6.60	.000188	.000173
2.70	.060943	.061706	6.70	.000162	.000168
2.80	.052814	.053727	6.80	.000139	.000165
2.90	.045737	.046750	6.90	.000120	.000162
3.00	.039585	.040652	7.00	.000103	.000159
3.10	.034241	.035320	7.10	.000089	.000157
3.20	.029604	.030658	7.20	.000076	.000154
3.30	.025583	.026582	7.30	.000066	.000151
3.40	.022100	.023019	7.40	.000057	.000148
3.50	.019084	.019905	7.50	.000049	.000144
3.60	.016474	.017184	7.60	.000042	.000139
3.70	.014218	.014809	7.70	.000036	.000134
3.80	.012267	.012736	7.80	.000031	.000129
3.90	.010581	.010929	7.90	.000027	.000122

TABLE IV (continued)

t	f(t)	$\hat{f}(t)$
8.00	.000023	.000116
8.10	.000020	.000109
8.20	.000017	.000102
8.30	.000015	.000095
8.40	.000013	.000087
8.50	.000011	.000080
8.60	.000009	.000072
8.70	.000008	.000065
8.80	.000007	.000058
8.90	.000006	.000051
9.00	.000005	.000044
9.10	.000004	.000038
9.20	.000004	.000032
9.30	.000003	.000026
9.40	.000003	.000021
9.50	.000002	.000016
9.60	.000002	.000011
9.70	.000002	.000007
9.80	.000002	.000003
9.90	.000001	-.000000
10.00	.000001	-.000003

TABLE V

VALUES OF THE c.d.f. FOR EIGHT TERM

LAGUERRE APPROXIMATION WITH r = 1

t	$\hat{F}(t)$	t	$\hat{F}(t)$	t	$\hat{F}(t)$
0.00	0.000000	3.40	0.985104	6.80	0.999761
0.10	0.016248	3.50	0.987247	6.90	0.999778
0.20	0.057294	3.60	0.989098	7.00	0.999794
0.30	0.113879	3.70	0.990695	7.10	0.999810
0.40	0.179227	3.80	0.992070	7.20	0.999825
0.50	0.248470	3.90	0.993251	7.30	0.999841
0.60	0.318184	4.00	0.994264	7.40	0.999855
0.70	0.386043	4.10	0.995129	7.50	0.999870
0.80	0.450534	4.20	0.995867	7.60	0.999884
0.90	0.510743	4.30	0.996495	7.70	0.999898
1.00	0.566193	4.40	0.997027	7.80	0.999911
1.10	0.616714	4.50	0.997476	7.90	0.999924
1.20	0.662353	4.60	0.997855	8.00	0.999935
1.30	0.703296	4.70	0.998173	8.10	0.999947
1.40	0.739818	4.80	0.998439	8.20	0.999957
1.50	0.772247	4.90	0.998661	8.30	0.999967
1.60	0.800933	5.00	0.998845	8.40	0.999976
1.70	0.826229	5.10	0.998998	8.50	0.999985
1.80	0.848479	5.20	0.999125	8.60	0.999992
1.90	0.868010	5.30	0.999229	8.70	0.999999
2.00	0.885126	5.40	0.999315	8.80	1.000005
2.10	0.900105	5.50	0.999387	8.90	1.000011
2.20	0.913201	5.60	0.999446	9.00	1.000015
2.30	0.924639	5.70	0.999495	9.10	1.000020
2.40	0.934623	5.80	0.999536	9.20	1.000023
2.50	0.943332	5.90	0.999571	9.30	1.000026
2.60	0.950926	6.00	0.999601	9.40	1.000028
2.70	0.957542	6.10	0.999627	9.50	1.000030
2.80	0.963305	6.20	0.999651	9.60	1.000032
2.90	0.968321	6.30	0.999672	9.70	1.000032
3.00	0.972684	6.40	0.999692	9.80	1.000033
3.10	0.976477	6.50	0.999710	9.90	1.000033
3.20	0.979771	6.60	0.999728	10.00	1.000033
3.30	0.982628	6.70	0.999745		

with respect to x as follows

$$\Delta p(x,m) = p(x,m) - p(x-1,m) = -\frac{1}{m}[x-m]p(x,m)$$

$$\Delta^2 p(x,m) = \frac{2}{m^2}\left[\frac{x(x-1)}{2} - xm + \frac{m^2}{2}\right]p(x,m)$$

$$\vdots$$

and in general

$$\Delta^n p(x,m) = (-1)^n G_n(x)p(x,m) \quad . \tag{30}$$

The expressions $G_n(x)$ are known as either Poisson-Charlier or G polynomials. They have also been defined [13,473] by

$$G_n(x) = (-1)^n \frac{\frac{d^n}{dm^n}p(x,m)}{p(x,m)} \quad .$$

A general formula for $G_n(x)$ may be developed from the generating function definition of the polynomials [14,154],

$$e^{-t}\left(1+\frac{t}{m}\right)^x = \sum_{r=0}^{\infty} G_r(x)\,\frac{t^r}{r!} \quad . \tag{31}$$

We begin by expanding the left hand side of Eq.31 in two infinite series

$$e^{-t}\left(1 + \frac{t}{m}\right)^x = \sum_{r=0}^{\infty} (-1)^r \frac{t^r}{r!} \sum_{j=0}^{\infty} \binom{x}{j}\left(\frac{t}{m}\right)^j$$

$$= \sum_{r=0}^{\infty} \sum_{j=0}^{r} \frac{(-1)^j}{j!}\binom{x}{r-j}\frac{1}{m^{r-j}}\, t^r$$

$$= \sum_{r=0}^{\infty} \left[\frac{r!}{m^r} \sum_{j=0}^{r} (-1)^j \binom{x}{r-j} \frac{m^j}{j!} \right] \frac{t^r}{r!} \quad .$$

The last expression along with Eq.31 implies that

$$G_r(x) = \frac{r!}{m^r} \sum_{j=0}^{r} (-1)^j \binom{x}{r-j} \frac{m^j}{j!} \quad . \tag{32}$$

The first four G polynomials computed from Eq.32 are

$$G_0(x) = 1$$

$$G_1(x) = \frac{1}{m} [x-m]$$

$$G_2(x) = \frac{2}{m^2} [\binom{x}{2} - \binom{x}{1}m + \frac{m^2}{2}]$$

$$G_3(x) = \frac{3!}{m^3} [\binom{x}{3} - \binom{x}{2}m + \binom{x}{1}\frac{m^2}{2} - \frac{m^3}{3!}]$$

$$G_4(x) = \frac{4!}{m^4} [\binom{x}{4} - \binom{x}{3}m + \binom{x}{2}\frac{m^2}{2} - \binom{x}{1}\frac{m^3}{3!} + \frac{m^4}{4!}] \quad .$$

Although the G polynomials possess many interesting properties only the orthogonal property is useful to us.

The orthogonal property may be stated as follows

$$\sum_{x=0}^{\infty} G_r(x)G_s(x)p(x,m) = \begin{cases} 0 \, , & r \neq s \\ \frac{r!}{m^r} , & r = s \, , \end{cases} \tag{33}$$

and is proven in Jordan [13,476].

Our desire is to be able to expand a p.d.f., known only by its moments, in an infinite series of the form

$$f(x) = \sum_{j=0}^{\infty} b_j G_j(x) p(x,m) \quad . \tag{34}$$

To accomplish this we must determine the b_j in terms of the moments.

The b_j may be determined, as before, by multiplying both sides of Eq.34 by G_j, then summing from $x = 0$ to $x = \infty$. Then we have

$$\sum_{x=0}^{\infty} G_j(x) f(x) = \sum_{x=0}^{\infty} \sum_{j=0}^{\infty} b_j [G_j(x)]^2 p(x,m)$$

$$= b_j \frac{j!}{m^j}$$

by the orthogonal property. Thus,

$$b_j = \frac{m^j}{j!} \sum_{x=0}^{\infty} G_j(x) f(x) \quad . \tag{35}$$

By substitution of Eq.32 into Eq.35 we can develop a general expression for b_j, so

$$b_j = \frac{m^j}{j!} \sum_{x=0}^{\infty} \frac{j}{m^j} \sum_{i=0}^{j} (-1)^i \binom{x}{j-i} \frac{m^i}{i!} f(x)$$

$$= \sum_{i=0}^{j} (-1)^i \frac{m^i}{i!} \sum_{x=0}^{\infty} \binom{x}{j-i} f(x) \quad , \tag{36}$$

assuming convergence so that we may interchange the sums. Now, the expression

$$\sum_{x=0}^{\infty} \binom{x}{j-i} f(x)$$

is known as the $(j-i)^{\underline{th}}$ binomial moment of f(x), and can be expressed in terms of moments about the origin via factorial moments. We write

$$\sum_{x=0}^{\infty}\binom{x}{j-i}f(x) = \frac{1}{(j-i)!}\sum_{x=0}^{\infty} (x)_{j-i}f(x)$$

where $(x)_{j-i} = x(x-1)\ldots(x-j+i-1)$ and the sum on the right hand side is the $(j-i)^{\underline{th}}$ factorial moment which may be written in terms of the moments about the origin by expanding the factorial expression. Thus, from Eq.36, we have the desired expression for b_j

$$b_j = \sum_{i=0}^{j}(-1)^i\frac{m^i}{i!(j-i)!}\sum_{x=0}^{\infty}(x)_{j-i}f(x) \quad . \qquad (37)$$

The first four b_j, computed from Eq.37 are

$$b_0 = 1$$

$$b_1 = v_1 - m$$

$$b_2 = \frac{1}{2}\left[v_2-v_1(1+2m) + m^2\right]$$

$$b_3 = \frac{1}{3!}\left[v_3-3v_2+2v_1-3m(v_2-v_1) + 3m^2v_1 - m^3\right]$$

$$b_4 = \frac{1}{4!}\left[v_4-6v_3+11v_2-6v_1-4m(v_3-3v_2+2v_1) + 6m^2(v_2-v_1) - 4m^3v_1 + m^4\right] .$$

To facilitate approximation it is common to set the parameter $m = v_1$.

The distribution function is obtained by summation of f(x) over the desired range.

The use of the B series is demonstrated with the following example. Suppose we wish to approximate the Binomial distribution

$$p(x) = \binom{n}{x} p^x (1-p)^{n-x} \quad , \; x = 0,1,\ldots,20 \quad ,$$

with $n = 20$ and $p = .5$. The first five moments are

$$v_1 = 10 \qquad v_3 = 1150 \qquad v_5 = 153625$$

$$v_2 = 105 \qquad v_4 = 13072.5$$

The distribution is discrete and defined over $(0,\infty)$.

Using the above as input data for TEST we find that a unimodal distribution exists having the specified moments, as expected. The coefficients in the expansion are

$$b_1 = 0 \qquad b_3 = .83333$$

$$b_2 = -2.5 \qquad b_4 = 2.8125 \quad ,$$

and the resulting curve along with the actual curve is shown in Table VI. The actual computation for the approximation was accomplished by a FORTRAN subprogram called BSER.

As can be seen the approximation is not too good using only four moments. This is due to the fact the Binomial has a zero ordinate at zero and the first term of the series does not. So that in trying to compensate for this the expansion is not positive semidefinite for values of t in the tails of the distribution. The problem is similar to that encountered with the Laguerre expansion.

TABLE VI

VALUES OF APPROXIMATE AND ACTUAL DENSITY FUNCTION, $\hat{f}(t)$ AND $f(t)$ RESPECTIVELY, FOR BINOMIAL DISTRIBUTION WITH $n = 20$, $p = .5$

t	f(t)	$\hat{f}(t)$
0.00	.0000	.000013
1.00	.0000	.000130
2.00	.0002	-.000202
3.00	.0011	.000083
4.00	.0046	.003044
5.00	.0148	.014676
6.00	.0370	.040539
7.00	.0739	.080208
8.00	.1201	.124047
9.00	.1602	.156971
10.00	.1762	.166918
11.00	.1602	.151459
12.00	.1201	.118041
13.00	.0739	.078771
14.00	.0370	.044183
15.00	.0148	.019717
16.00	.0046	.005732
17.00	.0011	-.000394
18.00	.0002	-.001974
19.00	.0000	-.001615
20.00	.0000	-.000825

CHAPTER V

PEARSON'S CURVES AND NUMERICAL INVERSION OF THE LAPLACE TRANSFORM

In this chapter we will consider two techniques, Pearson's curves and numerical inversion of the Laplace transform.

Pearson's system was developed in the late 1800's for fitting curves to empirical data. However, since it uses the moments of the distribution to be approximated it should fit our needs just as well.

On the other hand the numerical inversion technique is quite new, being published in January 1968. It is, in general, the most accurate of the techniques and requires the most effort to obtain an answer.

Pearson Distributions

One of the most widely accepted techniques for curve fitting is the Pearson system which consists of a family of curves and rules for selecting that curve which best fits the given data. The development given in this section follows that given by Elderton [5]. An alternate development has been given by Craig [25].

Pearson's family of curves are generated by the solutions to the differential equation

$$\frac{df(x)}{dx} = \frac{dy}{dx} = \frac{(x+a)y}{b_0+b_1x+b_2x^2} \quad . \tag{1}$$

It is interesting to note, although it will not be pursued, that all orthogonal polynomials also satisfy this differential equation [12,162].

Instead of solving for all the distribution types let us consider the general results that are common to all types, and the criteria for selecting each type. Then we will simply list the types and discuss each. Rearranging Eq.1 we have

$$(b_0+b_1x+b_2x^2)dy = (x+a)y\ dx$$

or

$$x^n(b_0+b_1x+b_2x^2)\ \frac{dy}{dx}dx = x^n(x+a)y\ dx \quad .$$

Integrating both sides over the entire real line, using parts on the left hand side and assuming that the integrals exist, yields

$$\left[x^n(b_0+b_1x+b_2x^2)y\right]\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty}\left\{nb_0x^{n-1}+(n+1)b_1x^n +(n+2)b_2x^{n+1}\right\}y\ dx$$

$$= \int_{-\infty}^{\infty} x^{n+1}y\ dx+a\int_{-\infty}^{\infty} x^ny\ dx \ .$$

Assume that the expression in square brackets vanishes at the extremes of the distribution, that is $\lim_{x\to\pm\infty} x^{n+2}y \to 0$ if the range is infinite. Then, substituting moments gives

$$-nb_0v_{n-1} - (n+1)b_1v_n - (n+2)b_2v_{n+1} = v_{n+1} + av_n \quad ,$$

or $av_n + nb_0v_{n-1} + (n+1)b_1v_n + (n+2)b_2v_{n+1} = -v_{n+1}$. (2)

Putting n = 0,1,2,3 in Eq.2 we obtain four equations in the four unknowns a, b_0, b_1, b_2, and the first four moments of the distribution we wish to approximate

$$\begin{aligned} a\,v_0 &+ b_1v_0 + 2b_2v_1 = -v_1 \\ a\,v_1 + b_0v_0 &+ 2b_1v_1 + 3b_2v_2 = -v_2 \\ a\,v_2 + 2b_0v_1 &+ 3b_1v_2 + 4b_2v_3 = -v_3 \\ a\,v_3 + 3b_0v_2 &+ 4b_1v_3 + 5b_2v_4 = -v_4 \end{aligned} \qquad (3)$$

Let us now make a transformation that carries the origin into the mean of the distribution, with the result that $v_1 = 0$, and denote the moments about the mean as u_i. Then Eqs.3 become

$$\begin{aligned} a &+ b_1 = 0 \\ + b_0 &+ 3b_2u_2 = -u_2 \\ au_2 &+ 3b_1u_2 + 4b_2u_3 = -u_3 \\ au_3 + 3b_0u_2 &+ 4b_1u_3 + 5b_2u_4 = -u_4 \end{aligned} \qquad (4)$$

Solving these simultaneously gives

$$a = -\frac{u_3(u_4+3u_2^2)}{D}$$

$$b_0 = -\frac{u_2(4u_2u_4-3u_3^2)}{D}$$

$$b_1 = -a$$

$$b_2 = -\frac{2u_2u_4-3u_3^2-6u_2^3}{D}$$

where

$$D = 10u_2u_4 - 18u_2^3 - 12u_3^2 \quad .$$

If we use the first and second moment ratios

$$\beta_1 = \frac{u_3^2}{u_2^3} \quad \text{and} \quad \beta_2 = \frac{u_4}{u_2^2}$$

the expressions for a, b_0, b_1, and b_2 become

$$\begin{aligned} a &= \frac{\sqrt{u_2}\,\sqrt{\beta_1}(\beta_2+3)}{D'} \\ b_0 &= -\frac{u_2(4\beta_2-3\beta_1)}{D'} \\ b_1 &= -\frac{\sqrt{u_2}\,\sqrt{\beta_1}(\beta_2+3)}{D'} \\ b_2 &= -\frac{(2\beta_2-3\beta_1-6)}{D'} \end{aligned} \tag{5}$$

where

$$D' = 2(5\beta_2-6\beta_1-9) \quad .$$

Substituting these expressions into Eq.1 yields a general equation for integration. The form of the integral depends on the particular values of the coefficients of x in the denominator, i.e. the values of b_0, b_1, b_2. An equivalent way of determining the form is to examine the nature of the roots of $b_0+b_1x+b_2x^2=0$. This can be done by computing the ratio $b_1^2/(4b_0b_2)$, which in specific terms is

$$\frac{\beta_1(\beta_2+3)^2}{4(2\beta_2-3\beta_1-6)(4\beta_2-3\beta_1)} \quad . \tag{6}$$

Let us denote this ratio as k. Thus, according to the value of k we may obtain one of the three main types.

The ten transitional types are determined by the k values which do not specify one of the main types and other criteria which will be given with each specific type.

First let us consider the three main types.

If k is negative the roots are real and opposite in sign, and we get the Type 1 frequency function

$$y = y_0\left(1 + \frac{x}{a_1}\right)^{m_1}\left(1 - \frac{x}{a_2}\right)^{m_2} \tag{7}$$

where $m_1/a_1 = m_2/a_2$ and the mode is at the origin. To completely determine the distribution we must calculate the following quantities in order:

$$r = 6(\beta_2-\beta_1-1)/(6+3\beta_1-2\beta_2) \quad ,$$

$$a_1+a_2 = \frac{1}{2}\sqrt{u_2}\sqrt{[\beta_1(r+2)^2+16(r+1)]} \quad ,$$

$$m_{1,2} = \frac{1}{2}\left[r-2 \pm r(r+2)\sqrt{\frac{\beta_1}{\beta_1(r+2)^2+16(r+1)}}\right]$$

where m_2 is the positive root when u_3 is positive, and

$$y_0 = \frac{a_1^{m_1} a_2^{m_2}}{(a_1+a_2)^{m_1+m_2+1}} \cdot \frac{\Gamma(m_1+m_2+2)}{\Gamma(m_1+1)\Gamma(m_2+1)} \quad .$$

The mode is computed by

$$\text{Mode} = \text{mean} - \frac{u_3}{2u_2}\,\frac{r+2}{r-2} \quad .$$

The exact shape of this curve varies according to the values of m_1 and m_2. A detailed account can be found in

Elderton [5,59]. The range is from $-a_1$ to a_2.

If k is positive but less than one the roots are complex and this gives us the Type IV frequency function

$$y = y_0\left(1 + \frac{x^2}{a^2}\right)^{-m} e^{-v \tan^{-1} x/a} \qquad (8)$$

where the parameters are

$$r = \frac{6(\beta_2-\beta_1-1)}{2\beta_2-3\beta_1-6} \quad ,$$

$$m = \frac{1}{2}(r+2) \quad ,$$

$$v = \frac{r(r-2)\sqrt{\beta_1}}{\sqrt{16(r-1)-\beta_1(r-2)^2}} \quad ,$$

$$a = \sqrt{\frac{u_2}{16}\sqrt{16(r-1) - \beta_1(r-2)^2}} \quad ,$$

and

$$\frac{1}{y_0} = a\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos\theta)^{2m-2} e^{-v\theta} d\theta \quad .$$

The mean is at $-va/r$ and the mode is given by

$$\text{Mode} = \text{mean} - \frac{u_3(r-2)}{2u_2(r+2)} \quad .$$

The curve is skewed and ranges over the entire real line, $(-\infty,\infty)$.

If k is positive and greater than one the roots are real and of the same sign, and this yields the Type VI frequency function

$$y = y_0(x-a)^{q_2}\,(x)^{-q_1} \tag{9}$$

where the parameters are given by

$$r = \frac{6(\beta_2-\beta_1-1)}{6+3\beta_1-2\beta_2} \quad ,$$

$$a = \frac{1}{2}\sqrt{u_2}\,\sqrt{\beta_1(r+2)^2+16(r+1)} \quad ,$$

$$q_2, -q_1 = \frac{r-2}{2} \pm \frac{r(r+2)}{2}\sqrt{\frac{\beta_1}{\beta_1(r+2)^2+16(r+1)}} \quad ,$$

and

$$y_0 = \frac{a^{q_1-q_2-1}\,\Gamma(q_1)}{\Gamma(q_1-q_2-1)\,\Gamma(q_2+1)} \quad .$$

The origin is located by

$$\text{Origin} = \text{mean} - \frac{a(q_1-1)}{q_1-q_2-2} \quad ,$$

and the mode by

$$\text{Mode} = \text{mean} - \frac{u_3}{2u_2}\;\frac{r+2}{r-2} \quad .$$

The range of this curve is from a to ∞ if u_3 is positive or from $-\infty$ to $-a$ if u_3 is negative. r is always negative and $q_1 > q_2$. If $q_2 < 0$ the curve is J shaped otherwise it is unimodal with zero ordinates at the terminals.

Next, let us consider the so called transition types.

If $k = 0$, $\beta_1 = 0$ and $\beta_2 = 3$ we have the normal curve of error

$$y = y_0 \exp(-x^2/2u_2) \tag{10}$$

where

$$y_0 = 1/\sqrt{2\pi u_2} \quad .$$

If $k = 0$, $\beta_1 = 0$ and $\beta_2 < 3$ we have the Type II frequency function

$$y = y_0\left(1 - \frac{x^2}{a^2}\right)^m \tag{11}$$

where

$$m = \frac{5\beta_2 - 9}{2(3-\beta_2)} \quad ,$$

$$a^2 = \frac{2u_2\beta_2}{3-\beta_2} \quad ,$$

and

$$y_0 = \frac{\Gamma(m+1\tfrac{1}{2})}{a\sqrt{\pi}\,\Gamma(m+1)} \quad .$$

The origin is at the mode which coincides with the mean. If m is positive the curve is unimodal with zero ordinates at both ends of the range. However, if m is negative the curve is U shaped.

If $k = 0$, $\beta_1 = 0$ and $\beta_2 > 3$ the resulting curve is the Type VII

$$y = y_0\left(1 + \frac{x^2}{a^2}\right)^{-m} \tag{12}$$

where

$$m = \frac{5\beta_2 - 9}{2(\beta_2 - 3)} \quad ,$$

$$a^2 = \frac{2u_2\beta_2}{\beta_2-3} \quad ,$$

and

$$y_0 = \frac{\Gamma(m)}{a\sqrt{\pi}\,\Gamma(m-\frac{1}{2})} \quad .$$

The origin is at the mode which equals the mean. The curve is symmetrical and of unlimited range in both directions.

If k is very large, which occurs when $2\beta_2 = 3\beta_1+6$, the result is the Type III frequency curve

$$y = y_0(1+\frac{x}{a})^{\delta a} \; e^{-\delta x} \tag{13}$$

where

$$\delta = \frac{2u_2}{u_3} \quad ,$$

$$p = \delta a = \frac{4}{\beta_1} - 1 \quad ,$$

$$a = \frac{2u_2^{\,2}}{u_3} - \frac{u_3}{2u_2} \quad ,$$

and

$$y_0 = \frac{p^{p+1}}{ae^p\,\Gamma(p+1)} \quad .$$

The origin is at the mode. The range is unlimited in one direction (positive or negative as δ is positive or negative). If $p > 0$ the curve is unimodal, but if $p < 0$ the curve is J shaped. Some experience indicates that the Type III should be used whenever $k > 5$.

There is one special case of the Type III which is of particular interest, it arises when $\beta_1 = 4$. This means that $p = 0$ and we then have the Type X frequency curve

$$y = \begin{cases} \delta e^{-\delta x} & , \ x > 0 \quad \delta > 0 \\ 0 & \text{otherwise} \end{cases} \tag{13a}$$

This is the familiar exponential distribution.

If k is equal to one we have the Type V frequency function

$$y = y_0 \ x^{-p} \ e^{-\delta/x} \tag{14}$$

where

$$p = 4 + \frac{8+4\sqrt{4+\beta_1}}{\beta_1} \quad ,$$

$$\delta = (p-2) \ \sqrt{u_2(p-3)} \quad ,$$

and

$$y_0 = \frac{\delta^{p-1}}{\Gamma(p-1)} \quad .$$

The curve starts at the origin which may be located by

$$\text{Origin} = \text{mean} - \frac{\delta}{p-2} \quad .$$

The mode may be found from

$$\text{Mode} = \text{mean} - \frac{2\delta}{p(p-2)} \quad .$$

The curve is unimodal and ranges from 0 to ∞.

The remaining types VIII through XII are special cases of the previous curves and will not be discussed here. An account of them may be found in Elderton [5] or Kendall [14,142].

In fitting one of the Pearson curves, under the assumptions of our problem, we would perform the following steps.

1. Compute the first four moments of the unknown distribution, and from these β_1 and β_2.
2. Compute k and other criteria needed to determine type of curve.
3. Determine the parameters of the selected curve.

The computations necessary for fitting one of Pearson's curves to a set of moments are performed by FORTRAN subprogram called PSON. Examples of the Pearson approximation are given in Chapter VI.

Inversion of Laplace Transforms

Recently many techniques for numerical inversion of Laplace transforms have been proposed [3], [26], [29], [30], and [37]. Each has some attribute to recommend it over the others. Among all the available techniques the one which appears easiest to apply is that of Dubner and Abate [26,115]. Their technique also has the advantage that the order of magnitude of error is easily preassigned by selection of one parameter in the computation. The computer program needed to realize this technique requires less than twenty cards, an obvious advantage. Let us proceed with the development of the technique, and then examine one method of implementing it for use on a computer.

Development of the Technique. The Laplace transform pair is defined as follows

$$F(s) = \int_0^\infty e^{-st} f(t)\, dt \quad , \tag{15}$$

$$f(t) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} e^{st} F(s)\, ds \ . \tag{16}$$

Where $a > 0$ is arbitrary, but must be chosen so that it is greater than the real parts of all singularities of $F(s)$. We assume that the integrals exist for $\text{Re } s \geq a > 0$.

We are free to choose the contour over which the inverse transform Eq.16 is evaluated, in fact any vertical line with abscissa $a > 0$ provides a suitable contour. This ability to assign the value of a is the basis of the computational technique, since the error in the approximation can be made as small as desired by the appropriate choice of a.

We begin by letting $s = a + iw$ in Eq.15, then

$$F(s) = \int_0^\infty e^{-(a+iw)t} f(t)\, dt \ ,$$

$$= \int_0^\infty e^{-at} f(t)\, [\cos wt - i \sin wt]\, dt$$

by Euler's identity.

Now, $f(t)$ is always a real function, thus Eq.15 becomes

$$\text{Re}\left[F(s)\right] = \int_0^\infty e^{-at} f(t) \cos wt\, dt \ , \tag{17}$$

the Fourier cosine transform of a function having an attenuation factor.

In general we are interested in approximating f(t) over some finite range, say (0,T). This suggests the use of the Fourier half range cosine series (the finite Fourier transform pair)

$$c_n = \frac{2}{T}\int_0^T f(x)\cos\left[\frac{n\pi x}{T}\right]dx, \tag{18}$$

and

$$f(x) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n \cos\left(\frac{n\pi x}{T}\right) \quad . \tag{19}$$

Thus, our approach will be to relate Eq.17 to Eq.18 and then use Eq.19 as the inversion formula. Since, intuitively speaking, the inversion of Eq.17 requires a knowledge of the continuous frequency spectrum of Re F(s) while Eq.19 requires only a discrete spectrum the relationship between Eq.17 and Eq.18 will contain an error which is calculable. Therefore, we will be able to approximate f(t) within some calculable error.

Define a function h(t) as follows

$$h(t) = \begin{cases} 0 & t < 0 \\ e^{-at}f(t) & t \geq 0 \end{cases} \quad . \tag{20}$$

Let us now separately consider h(t) in the intervals (nT,[n+1]T), n = 0,1,... . Using h(t) we construct an infinite set of even periodic functions $g_n(t)$ by reflecting each section of h(t) indefinitely through its boundaries, see Figure 10. Each $g_n(t)$ has a period of 2T. That is, for n = 0,1,...

$$g_n(t) = \begin{cases} h(t) & nT \leq t \leq (n+1)T \\ h(2nT-t) & (n-1)T \leq t \leq nT \end{cases} \quad (21)$$

In order to give a Fourier representation of $g_n(t)$ we rewrite Eq.21 so that the functions are defined on the interval (-T,T). Then for n = 0,2,4,...

$$g_n(t) = \begin{cases} h(nT+t) & 0 \leq t \leq T \\ h(nT-t) & -T \leq t \leq 0 \end{cases} \quad (22a)$$

and for n = 1,3,5,...

$$g_n(t) = \begin{cases} h([n+1]T-t) & 0 \leq t \leq T \\ h([n+1]T+t) & -T \leq t \leq 0 \end{cases} \quad (22b)$$

Thus, we have the Fourier representation

$$g_n(t) = \frac{A_{n,o}}{2} + \sum_{k=1}^{\infty} A_{n,k} \cos\left(\frac{k\pi t}{T}\right) , \quad (23)$$

where

$$A_{n,k} = \begin{cases} \frac{2}{T} \int_0^T h(nT+x) \cos\left(\frac{k\pi x}{T}\right) dx & n = 0,2,4,\ldots \\ \frac{2}{T} \int_0^T h([n+1]T-x) \cos\left(\frac{k\pi x}{T}\right) dx & n = 1,3,5,\ldots . \end{cases} \quad (24)$$

Next let t = x+nT in the first expression and t = -[nT-x+T] in the second, then Eq.24 becomes

$$A_{n,k} = \frac{2}{T} \int_{nT}^{(n+1)T} h(t) \cos\left(\frac{k\pi t}{T}\right) dt . \quad (25)$$

In order to convert Eq.25, a finite transform, to a true integral transform we must sum over all n, so

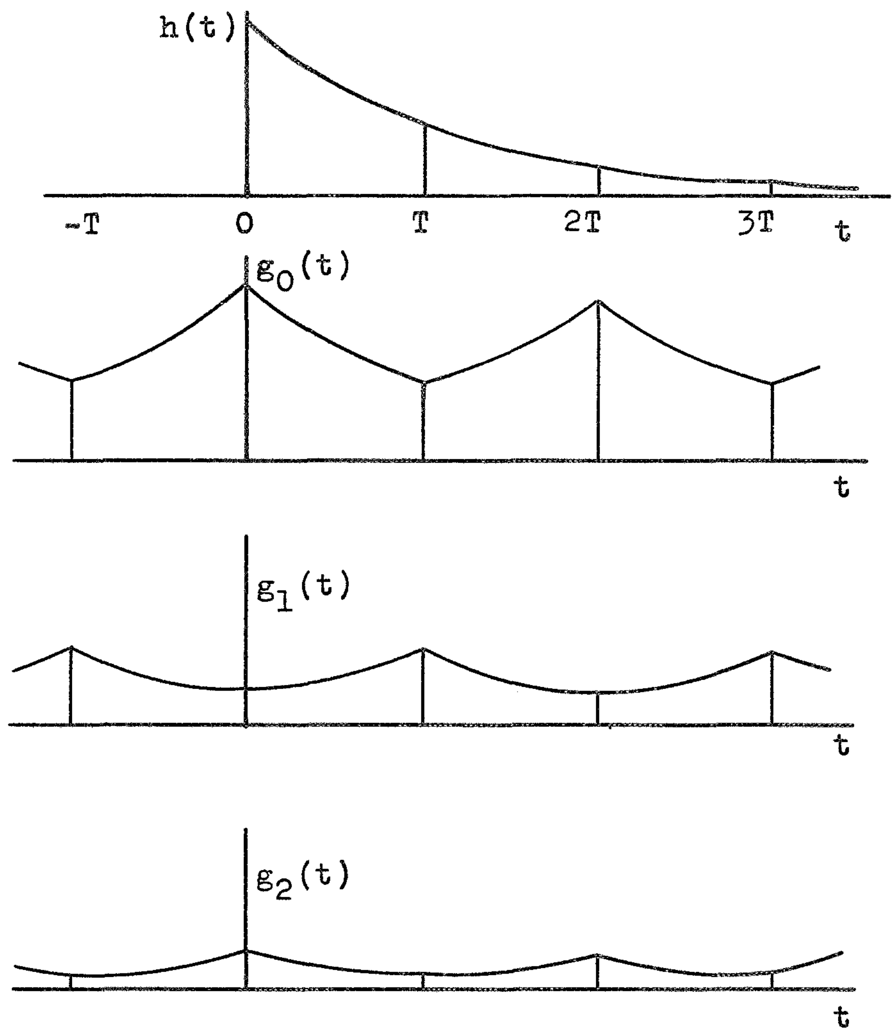

Figure 10. The function $h(t) = e^{-at}f(t)$ and its reflections $g_n(t)$.

$$\sum_{n=0}^{\infty} A_{n,k} = \frac{2}{T} \sum_{n=0}^{\infty} \int_{nT}^{(n+1)T} h(t) \cos\left(\frac{k\pi t}{T}\right) dt$$

$$= \frac{2}{T} \int_{0}^{\infty} h(t) \cos\left(\frac{k\pi t}{T}\right) dt \qquad (26)$$

$$= \frac{2}{T} A(w_k) \ ,$$

and $A(w_k)$ is a Fourier cosine transform. Now sum Eqs.22 over all n and

$$\sum_{n=0}^{\infty} g_n(t) = \frac{2}{T}\left[\frac{A(w_o)}{2} + \sum_{k=1}^{\infty} A(w_k) \cos\left(\frac{k\pi t}{T}\right)\right] . \qquad (27)$$

Substituting Eq.20 into Eq.26 we see that $A(w_k) = \mathrm{Re}\left[F(s)\right]$, where $s = a + i(k\pi/T)$. Thus, Eq.27 becomes

$$\sum_{n=0}^{\infty} e^{at} g_n(t) = \frac{2e^{at}}{T}\left[\frac{1}{2}\left[F(a)\right] + \sum_{k=1}^{\infty} \mathrm{Re}\left[F(a+\frac{k\pi i}{T})\right] \cos\left(\frac{k\pi t}{T}\right)\right], \qquad (28)$$

after multiplying both sides by e^{at}.

The l.h.s. of Eq.28 is almost the inverse Laplace transform of F(s) in the interval (0,T), but it contains an error. Using Eqs.22 we have

$$\sum_{n=0}^{\infty} e^{at} g_n(t) = \sum_{n=0}^{\infty} e^{at} h(2nT+t) + \sum_{n=1}^{\infty} e^{at} h(2nT-t)$$

$$= f(t) + \sum_{n=1}^{\infty} e^{-2anT}\left[f(2nT+t) + e^{2at} f(2nT-t)\right] , \qquad (29)$$

by virtue of Eq.20.

Hence, the desired approximation is

$$f(t) \simeq \frac{2e^{at}}{T}\left[\frac{1}{2}\,\mathrm{Re}\left[F(a)\right] + \sum_{k=1}^{\infty} \mathrm{Re}\left[F\left(a+\frac{k\pi i}{T}\right)\right] \cos\left(\frac{k\pi t}{T}\right)\right]. \qquad (30)$$

It can be shown [26,119] that the error term becomes sufficiently small only over the interval (0,T/2). Thus, we must keep this in mind when selecting T.

Dubner and Abate [26,119] show that if in general $f(t) \leq C\, t^m$, where C is some constant and m is any nonnegative integer, then the error in representing f(t) over (0,T/2) by Eq.30 is given by

$$\text{ERROR} \sim C(1.5T)^m e^{-aT} \quad . \qquad (31)$$

Our procedure will be as follows: (1) Determine the largest value of t for which we desire to know f(t), then set T equal to twice this value. (2) Having T and knowing how f(t) behaves at infinity we then may use Relation 17 to choose the value of a which yields the desired numerical accuracy. For our problems $f(t) \rightarrow 0$ as $t \rightarrow \infty$ in all cases. This means we may let C = 1 and m = 0.

There is also an error which arises out of truncating the series in Eq.30. This error is, roughly speaking, about the value of the term at which the truncation is made. This error is illustrated below.

Let us now consider some examples demonstrating the application of the technique.

First, suppose we wish to approximate the Gamma distribution

$$f(t) = \frac{t^2 e^{-t}}{\Gamma(3)} \quad .$$

The M.G.F. is

$$M(s) = (1 - s)^{-3} \quad ,$$

and the Laplace transform may be obtained by replacing s with -s, thus

$$F(s) = \frac{1}{(1+s)^3} \quad .$$

Let

$$s = a + i\,\frac{k\pi}{T} = A + i\,Q \quad , \tag{32}$$

then

$$F(s) = \frac{1}{\left[(a+1) + i\,Q\right]^3} \quad , \tag{33}$$

or in the most general form

$$F(s) = \frac{Z + i\,Y}{U + i\,V} \quad . \tag{34}$$

Now, we are interested in the Re F(s) , so we must put Eq.33 in the form of Eq.34 and then rationalize it. Expanding the denominator of Eq.33 we find

$$U = (A+1)^3 - 3(A+1)Q^2 + 3(A+1)^2 - Q^2 + 3(A+1) + 1$$

and

$$V = Q\left[3(A+1)^2 - Q^2 + 2(A+1) + 3\right] \, .$$

Rationalizing yields,

$$\mathrm{Re}\left[F(s)\right] = \frac{UZ + VY}{U^2+V^2}$$

$$= \frac{U}{U^2+V^2} \quad ,$$

for this example.

Assume we desire the approximation to have an error of the order of magnitude of 10^{-6}, and we want values of f(t) over (0,15). Then using Relation 31 we have

$$10^{-6} \sim e^{-30a}$$
$$\Rightarrow a \approx .5 \quad .$$

A FORTRAN program which will perform the calculations is shown in Fig.11, and the approximation is listed in Table VII along with the true values.

The total computer time required was 65.28 seconds including input and the printing of 150 values of the approximation. This time is for a CDC 6500.

As a second example consider the approximation of

$$f(t) = .25e^{-.25t} \quad .$$

The Laplace transform of this function is

$$F(s) = \frac{.25}{(s+.25)} \quad .$$

Substituting for s as before we find

$$\text{Re}\left[F(s)\right] = \frac{.25(.25+A)}{(.25+A)^2 + Q^2} \quad .$$

A FORTRAN program very similar to the previous one was written to perform the calculations and the results are listed in Table VIII along with the actual values.

Although the value of T and a remain unchanged from the previous example the accuracy is not as good. The reason of course is the second type of error, truncation error. We see that even with 2000 terms we have not reached the desired accuracy. Functions such as this one with low contact at one or both ends of the interval of approxi-

```
      PROGRAM BUNKIE(OUTPUT,TAPE 6 = OUTPUT)
      A=.5
      XU=15.0
      DX=.1
      T=30.
      FL=1.0
      FA=(FL/(A+FL))**3
      B=A+FL
      F=0.0
      X=0.0
      WRITE(6,950)
      NI=1000
   20 DO 40 J=1,NI
      FJ=J
      Q=FJ*3.14159/T
      U=B**3-3.*B*Q**2
      V=Q*(3.*B**2-(Q**2))
      Z=FL
      Y=0.0
      G=(Z*U+Y*V)/(U**2+V**2)
   40 F=G*COSF(FJ*3.14159*X/T)+F
      DF=(2.*EXPF(A*X)/T)*(.5*FA+F)
      WRITE(6,951)X,DF
      X=X+DX
      F=0.0
      IF(X-XU)20,20,50
   50 STOP
  950 FORMAT(11X,4HTIME,6X,4HF(T),/)
  951 FORMAT(10X,F6.2,F12.6)
      END
```

Figure 11. Fortran program for GAMMA distribution example.

TABLE VII

VALUES OBTAINED BY NUMERICAL INVERSION, $\hat{f}(t)$, AND THE TRUE VALUES OF $f(t) = t^2e^{-t}/2$ 1000 TERMS USED IN THE APPROXIMATION

t	f(t)	$\hat{f}(t)$
0.0	.000000	.000001
1.0	.183940	.183940
2.0	.270671	.270671
3.0	.224042	.224042
4.0	.146525	.146525
5.0	.084224	.084224
6.0	.044618	.044618
7.0	.022341	.022341
8.0	.010735	.010735
9.0	.004998	.004998
10.0	.002270	.002270
15.0	.000034	.000032

mation will require more terms to achieve the desired accuracy.

Finally, let us approximate the c.d.f. of the previous example, that is approximate

$$F(t) = 1 - e^{-.25t} \quad .$$

The Laplace transform of this function is obtained from the previous transform by dividing by s, thus

$$F(s) = \frac{.25}{s(s+.25)} \quad .$$

Again substituting for s and manipulating yields

$$Re\left[F(s)\right] = \frac{.25[A(A+.25) - Q^2]}{[A(A+.25)-Q^2]^2 + [Q(2A+.25)]^2} \quad .$$

Values obtained by using this quantity are given in Table IX along with the true values. Note that the accuracy is better than that for the p.d.f., at the beginning of the range, but worse for large values of t.

This technique is not without limitations. It can require a great deal of effort in terms of algebraic manipulation and can also require a lot of computer time to obtain a suitable solution to complicated problems. For example, consider the added algebraic complications when terms such as e^s are included in the denominator of the M.G.F. This requires the use of Euler's identity. Also, the results are strictly numerical, that is no explicit analytical expressions for the p.d.f. and c.d.f. are obtained directly. Finally, the technique is only able to handle functions defined on $[0,\infty)$. To illustrate the difficulty suppose that f(t) is defined on $[-a,\infty)$ and we wish to determine its Laplace transform. First we shift f(t) by adding a to each value of the argument which yields f(t+a), a function defined on $[0,\infty)$. Now we may take the Laplace transform of f(t+a) which is

$$L[f(t+a)] = \int_0^\infty f(t+a)e^{-st}\,dt \quad .$$

Next, let u = t + a, then

$$L[f(t+a)] = \int_a^\infty f(u)e^{-s(u-a)}\,du$$

$$= e^{as}\int_a^\infty f(u)e^{-su}\,du$$

$$= e^{as}\left[\int_0^\infty f(u)e^{-su}\,du - \int_0^a f(u)e^{-su}\,du\right].$$

The second term in the brackets is the finite Laplace transform and herein lies the difficulty. For the numerical inversion procedure to work the finite transform must have been included in the original M.G.F. from which we are working.

TABLE VIII

TRUE AND APPROXIMATE VALUES OF $f(t) = .25e^{-.25t}$

		Number of terms		
		2000	1500	1000
t	f(t)	$\hat{f}(t)$	$\hat{f}(t)$	$\hat{f}(t)$
0.0	.250000	.249430	.249241	.248861
1.0	.194700	.194704	.194701	.194685
2.0	.151633	.151629	.151663	.151645
3.0	.118092	.118092	.118093	.118094
4.0	.091970	.091974	.091972	.091951
5.0	.071626	.071620	.071629	.071645
6.0	.055783	.055785	.055788	.055794
7.0	.043444	.043452	.043452	.043392
8.0	.033834	.033815	.033848	.033879
9.0	.026335	.026363	.026373	.026401
10.0	.020521	.020542	.020559	.020352
15.0	.005880	.006136	.006336	.006906

TABLE IX

ACTUAL AND APPROXIMATE VALUES
OF $F(t) = 1 - e^{-.25t}$
USING 1500 TERMS

t	F(t)	$\hat{F}(t)$
0.0	.000000	.001013
1.0	.221199	.221199
2.0	.393469	.393469
3.0	.527633	.527632
4.0	.632121	.632119
5.0	.713495	.713492
6.0	.776870	.776864
7.0	.826226	.826216
8.0	.864665	.864647
9.0	.894661	.894571
10.0	.917915	.917866
15.0	.976482	.975875

CHAPTER VI

EXAMPLES

In this chapter we will examine five examples in detail. Each example demonstrates one or more of the properties of the recommended procedure. The sequence of the discussion in the examples follows the flow diagram given in Fig.1a of Chapter II and repeated here as Fig.12 for ease of reference. Also the theorems of Chapter III will be referred to frequently. These examples combined with those in the previous two chapters provide a complete demonstration of the recommended procedure.

Example 1.

Suppose we use a convex combination of the moments of two unimodal distributions, that is we mix the two density functions. Specifically let us mix an exponential having a mean of one with a gamma of order 2 having a mean of three, i.e. let

$$f(t) = .1e^{-t} + .9\left[\frac{t^2e^{-t}}{2}\right].$$

Since the function has a value of .1 at t = 0 we use the Laguerre approximation with r = 0. Each of the components of f(t) is reproduced exactly by the Laguerre approximation. Thus, let us demonstrate that the convex combination is also reproduced exactly. The first five moments are

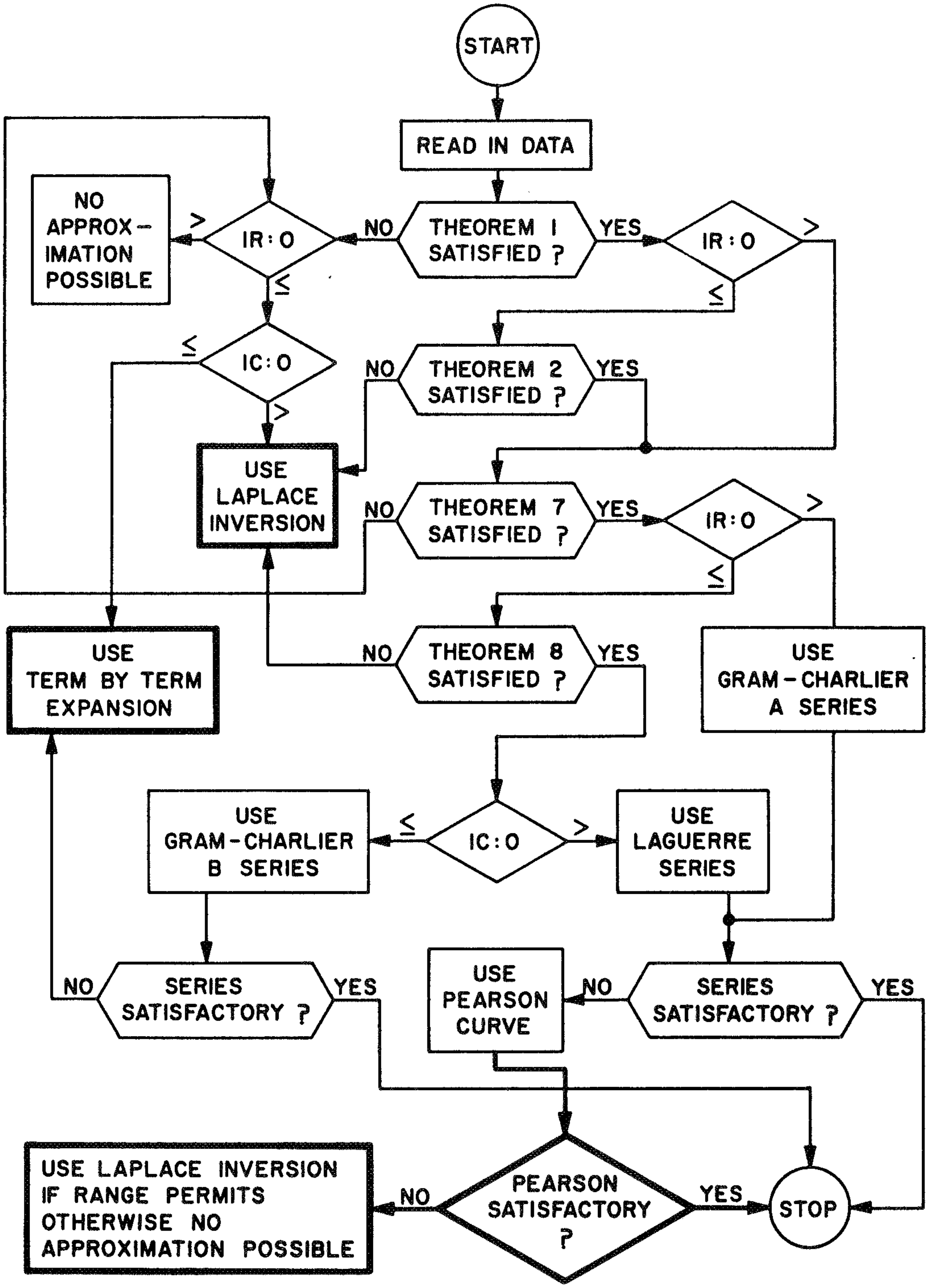

Figure 12. Master flow diagram of the procedure. Portions of the procedure in heavy black lines are not included in the program TEST.

$$v_1 = 2.8 \qquad v_3 = 54.6 \qquad v_5 = 2280.0$$

$$v_2 = 11.0 \qquad v_4 = 326.4 \quad .$$

Using the above five moments the program TEST constructs and evaluates the following determinants

$$\begin{vmatrix} 1 & 2.8 \\ 2.8 & 11 \end{vmatrix} , \qquad \begin{vmatrix} 1 & 2.8 & 11 \\ 2.8 & 11 & 54.6 \\ 11 & 54.6 & 326.4 \end{vmatrix} .$$

Both are greater than zero, thus Theorem 1 is satisfied. Since the distribution is defined on $(0,\infty)$, IR = 0, the moments must also satisfy Theorem 2. Therefore, the program checks the determinant

$$\begin{vmatrix} 2.8 & 11 \\ 11 & 54.6 \end{vmatrix} ,$$

finds that it is positive definite, and so Theorem 2 is satisfied. This means that there exists at least one distribution which has the given moments for its first five moments.

Following Fig.12, we see that TEST will check to see if the conditions of Theorem 7 are satisfied. This is done as follows. The quantities a, b, c and d of Theorem 7 (called D(1),...,D(4) in TEST) are computed as

$$a = 24.416$$

$$b = -354.84$$

$$c = 1181.376$$

$$d = -238.44 \quad .$$

The corresponding cubic equation is solved and the roots are

$$r_1 = .215585266$$

$$r_2 = 4.71968031$$

$$r_3 = 9.59782748$$

Since a > 0 and all roots are real the following inequalities must be satisfied

$$v_1 - \sqrt{3(v_2-v_1^2)} \le M \le v_1 + \sqrt{3(v_2-v_1^2)}$$

or

$$-.278960864 \le M \le 5.87896086 \quad ,$$

and

$$r_3 \le M \quad \text{or} \quad r_1 \le M \le r_2 \quad .$$

We see that any M such that $r_1 \le M \le v_1 + \sqrt{3(v_2-v_1^2)}$ satisfies Theorem 7, thus since IR = 0 the program checks to see if an M in this interval can also satisfy the conditions of Theorem 8.

The program now computes the coefficients a_2, a_1, and a_0 (called C(1), C(2) and C(3) in TEST) of the quadratic equations in Theorem 8. We find that

$$a_2 = 1.64$$

$$a_1 = -33.6$$

$$a_0 = 134.04 \quad ,$$

and

$$p_1 = 5.42665947$$

and

$$p_2 = 15.0611454$$

Since both roots are real and $a_2 > 0$, M must also satisfy the inequalities

$$M \leq 2v_1 = 5.6$$

and

$$M \leq p_1 \quad \text{or} \quad M \geq p_2 \quad .$$

Now since p_1 for the quadratic is less than $2v_1$ and easily falls in the interval imposed by Theorem 7 the program next computes the coefficients a, b, c, and d (called D(1),...., D(4) in TEST) of the cubic of Theorem 8. The values are

$$a = 238.440$$

$$b = -3125.664$$

$$c = 4579.199$$

$$d = 25428.095 \quad ,$$

and

$$r_1 = -2.09132645$$

$$r_2 = 4.99857821$$

$$r_3 = 10.2015554 \quad .$$

Since $a > 0$

$$r_1 \leq M \leq r_2 \quad \text{or} \quad r_3 \leq M$$

We can easily find an M in the interval $[r_1, r_2]$ that satisfies all of the previous inequalities and thus a unimodal distribution can be specified by the moments.

The distribution of interest satisfies both theorems and is continuous (IC=1) so TEST calls the subroutine LAG which calculates the appropriate coefficients, and these are

$d_1 = 1.8$ $\qquad d_3 = 0.0 \qquad d_5 = 0.0$

$d_2 = 0.45 \qquad d_4 = 0.0$.

Substituting these into the Laguerre expansion yields

$$\hat{f}(t) = e^{-t} + 1.8(t-1)e^{-t} + .45(t^2-4t+2)e^{-t}$$

$$= .1\ e^{-t} + .9\left[\frac{t^2e^{-t}}{2}\right] \quad ,$$

which is exactly the distribution we wished to approximate. The distribution is shown in Fig.13 and a graph of the cumulative distribution function is shown in Fig.14.

Now, examining Figs.13 and 14 it is not clear that the function is unimodal according to the definition given in Chapter I. Fig.13 would suggest it is not while Fig.14 suggests that it is. Thus, let us analyze the situation in an attempt to resolve the problem. We must check to see how many points of inflection the cumulative distribution function has, since the definition of unimodality allows only one.

A theorem in calculus tells us that a sufficient condition for $x = a$ to be a point of inflection of $F(x)$ is that $F''(a) = 0$ and $F'''(a) \neq 0$. Using this result we differentiate $f(t)$ twice to obtain $F''(t)$ and $F'''(t)$ respectively. Now

$$F''(t) = -.1\ e^{-t} + .45(2t-t^2)e^{-t} \quad ,$$

and

$$F'''(t) = .1\ e^{-t} + .45(t^2-4t+2)e^{-t} \quad .$$

Setting $F''(t) = 0$ and solving for t yields

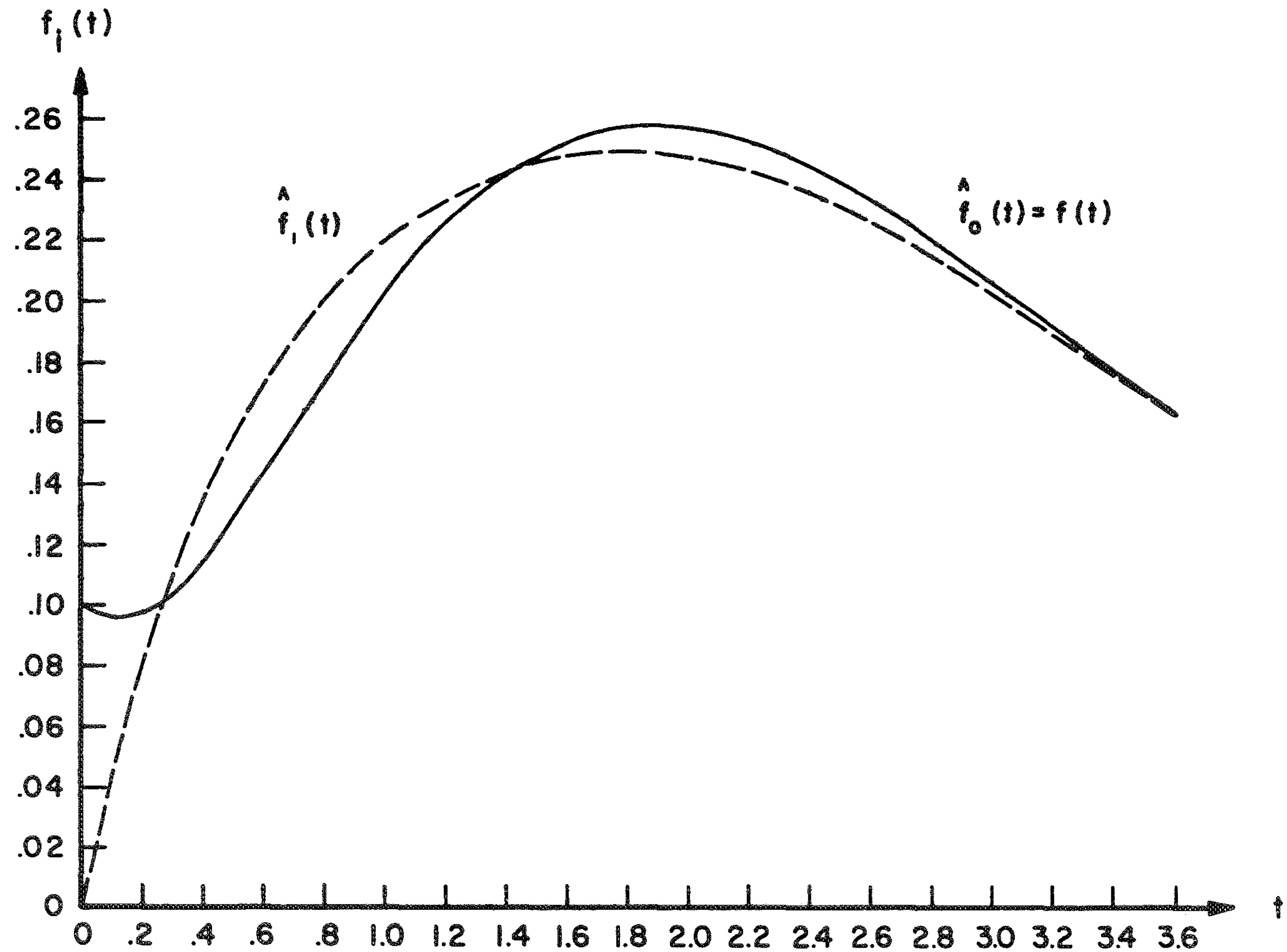

Figure 13. Graph of $f(t) = .1\,e^{-t} + .9[\frac{t^2e^{-t}}{2}]$ and its Laguerre approximation using five moments for $r = 0$ ($\hat{f}_0(t)$) and $r = 1$ ($\hat{f}_1(t)$).

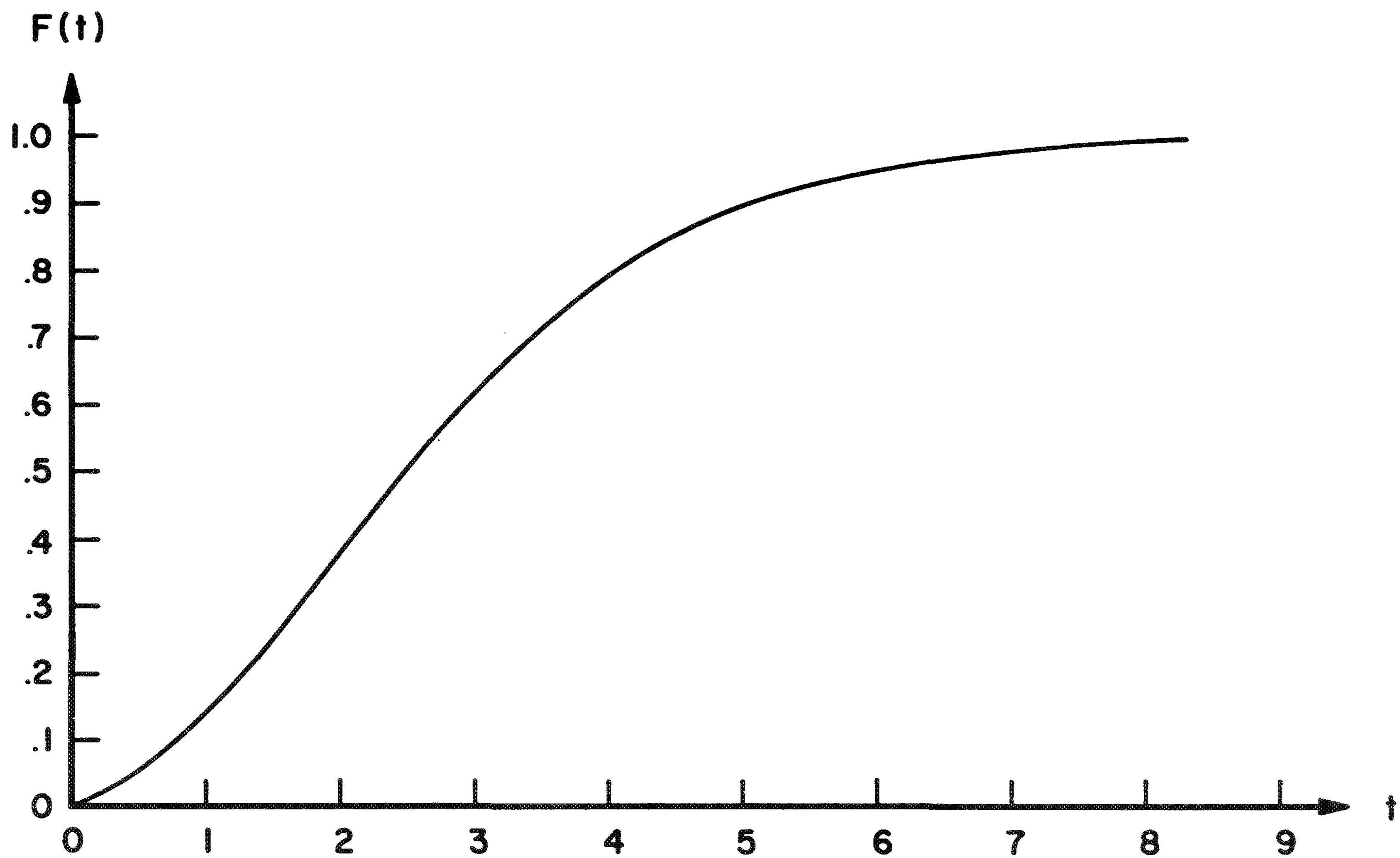

Figure 14. Graph of c.d.f., F(t), for p.d.f., f(t), shown in Fig. 13.

$$t = 1 \pm \sqrt{7}/3 \quad .$$

Substituting these values into $F'''(t)$ we find that neither makes the expression zero. Thus, we have two points of inflection and the distribution is not unimodal. Let us pursue this discussion a little further. From the previous analysis and by examining Fig.13 we see that the distribution has a mode at $t = 0$ and at $t = 1 + \sqrt{7}/3$, i.e. the modes are separated by $1 + \sqrt{7}/3$ units. The relationship of the distance between modes and the ability to distinguish unimodal functions from bimodal functions via their moments will be investigated in Example 4.

Finally, we note that the shape of the p.d.f. is that of a function which is very close to being unimodal. Thus, it is not hard to imagine that there exists a unimodal distribution having exactly the same moments. In fact the approximation with $r = 1$ is a possible example, see Fig.13. This observation serves to illustrate the uniqueness problem mentioned in Chapter III.

<u>Example 2</u>.

Next, let us examine the approximation of the following distribution

$$f(t) = .5(.375)\,[e^{-1.5t} - e^{-2.5t}] + .5e^{-t} \quad ,$$

which is the convolution of two exponentials mixed with another exponential. The first eight moments are

$v_1 = 1.0335$	$v_5 = 78.8300$
$v_2 = 1.8710$	$v_6 = 436.8000$
$v_3 = 4.9340$	$v_7 = 2882.5000$
$v_4 = 17.4650$	$v_8 = 22106.0000$

Based on these moments and the other data the program TEST tells us that a unimodal distribution exists and that the coefficients for the Laguerre expansion with r = 0 are

$d_1 = .03350000$	$d_5 = .00045174$
$d_2 = -.06575000$	$d_6 = -.00004774$
$d_3 = .01938889$	$d_7 = .00000424$
$d_4 = -.00344271$	$d_8 = -.00000032$.

The operations performed by TEST were exactly the same as in the previous example. Here again the magnitude of the last few coefficients indicates that the approximation should be reasonably good. The actual curve and the approximation are shown in TABLE X. The deviation of the approximation from the true curve is plotted in Fig.15. The actual and approximate c.d.f. are shown in TABLE XI, and the deviation is plotted in Fig.16.

Let us compare the deviations. Although the curves are of different period, the most striking feature is the difference in the magnitudes of the deviations. The maximum deviation in the case of the p.d.f.'s is approximately four times that for the c.d.f.'s. This is not surprising since it is reasonable to assume that it should be easier to approxi-

TABLE X

VALUES OF $f(t) = .5(.375)[e^{-1.5t} - e^{-2.5t}] + .5e^{-t}$

AND ITS LAGUERRE APPROXIMATION $\hat{f}(t)$

t	f(t)	$\hat{f}(t)$	t	f(t)	$\hat{f}(t)$
0.00	.500000	.513192	4.20	.010889	.011248
.10	.605995	.606324	4.30	.009708	.010058
.20	.661155	.656259	4.40	.008658	.008993
.30	.680275	.674420	4.50	.007726	.008039
.40	.674408	.669717	4.60	.006897	.007185
.50	.651756	.649044	4.70	.006159	.006420
.60	.618355	.617686	4.80	.005503	.005734
.70	.578600	.579653	4.90	.004919	.005119
.80	.535650	.537948	5.00	.004399	.004568
.90	.491737	.494780	5.10	.003936	.004075
1.00	.448399	.451740	5.20	.003522	.003632
1.10	.406664	.409943	5.30	.003154	.003236
1.20	.367182	.370136	5.40	.002825	.002882
1.30	.330328	.332787	5.50	.002531	.002565
1.40	.296284	.298158	5.60	.002269	.002282
1.50	.265093	.266355	5.70	.002035	.002029
1.60	.236703	.237377	5.80	.001825	.001803
1.70	.210999	.211142	5.90	.001638	.001602
1.80	.187830	.187520	6.00	.001470	.001423
1.90	.167020	.166345	6.10	.001320	.001263
2.00	.148385	.147436	6.20	.001186	.001122
2.10	.131737	.130602	6.30	.001065	.000996
2.20	.116895	.115654	6.40	.000958	.000884
2.30	.103685	.102408	6.50	.000861	.000786
2.40	.091943	.090689	6.60	.000774	.000698
2.50	.081519	.080333	6.70	.000696	.000621
2.60	.072271	.071191	6.80	.000627	.000553
2.70	.064074	.063123	6.90	.000564	.000492
2.80	.056812	.056005	7.00	.000508	.000439
2.90	.050380	.049724	7.10	.000457	.000392
3.00	.044686	.044181	7.20	.000412	.000351
3.10	.039645	.039287	7.30	.000371	.000315
3.20	.035183	.034961	7.40	.000334	.000283
3.30	.031233	.031136	7.50	.000301	.000255
3.40	.027737	.027749	7.60	.000271	.000230
3.50	.024641	.024747	7.70	.000244	.000208
3.60	.021899	.022084	7.80	.000220	.000189
3.70	.019471	.019718	7.90	.000199	.000172
3.80	.017319	.017614	8.00	.000179	.000156
3.90	.015411	.015740	8.10	.000162	.000143
4.00	.013720	.014070	8.20	.000146	.000131
4.10	.012220	.012579	8.30	.000132	.000120

TABLE X (Continued)

t	f(t)	$\hat{f}(t)$	t	f(t)	$\hat{f}(t)$
8.40	.000119	.000111	11.80	.000004	.000006
8.50	.000107	.000102	11.90	.000003	.000005
8.60	.000097	.000095	12.00	.000003	.000005
8.70	.000087	.000088	12.10	.000003	.000004
8.80	.000079	.000081	12.20	.000003	.000004
8.90	.000071	.000076	12.30	.000002	.000003
9.00	.000064	.000070	12.40	.000002	.000003
9.10	.000058	.000065	12.50	.000002	.000002
9.20	.000052	.000061	12.60	.000002	.000002
9.30	.000047	.000057	12.70	.000002	.000001
9.40	.000043	.000053	12.80	.000001	.000001
9.50	.000039	.000049	12.90	.000001	.000001
9.60	.000035	.000046	13.00	.000001	.000001
9.70	.000032	.000043	13.10	.000001	.000000
9.80	.000028	.000040	13.20	.000001	.000000
9.90	.000026	.000037	13.30	.000001	-.000000
10.00	.000023	.000034	13.40	.000001	-.000000
10.10	.000021	.000032	13.50	.000001	-.000000
10.20	.000019	.000030	13.60	.000001	-.000000
10.30	.000017	.000027	13.70	.000001	-.000000
10.40	.000016	.000025	13.80	.000001	-.000001
10.50	.000014	.000023	13.90	.000000	-.000001
10.60	.000013	.000021	14.00	.000000	-.000001
10.70	.000011	.000020	14.10	.000000	-.000001
10.80	.000010	.000018	14.20	.000000	-.000001
10.90	.000009	.000016	14.30	.000000	-.000001
11.00	.000008	.000015	14.40	.000000	-.000001
11.10	.000008	.000014	14.50	.000000	-.000001
11.20	.000007	.000012	14.60	.000000	-.000001
11.30	.000006	.000011	14.70	.000000	-.000001
11.40	.000006	.000010	14.80	.000000	-.000001
11.50	.000005	.000009	14.90	.000000	-.000001
11.60	.000005	.000008	15.00	.000000	-.000001
11.70	.000004	.000007			

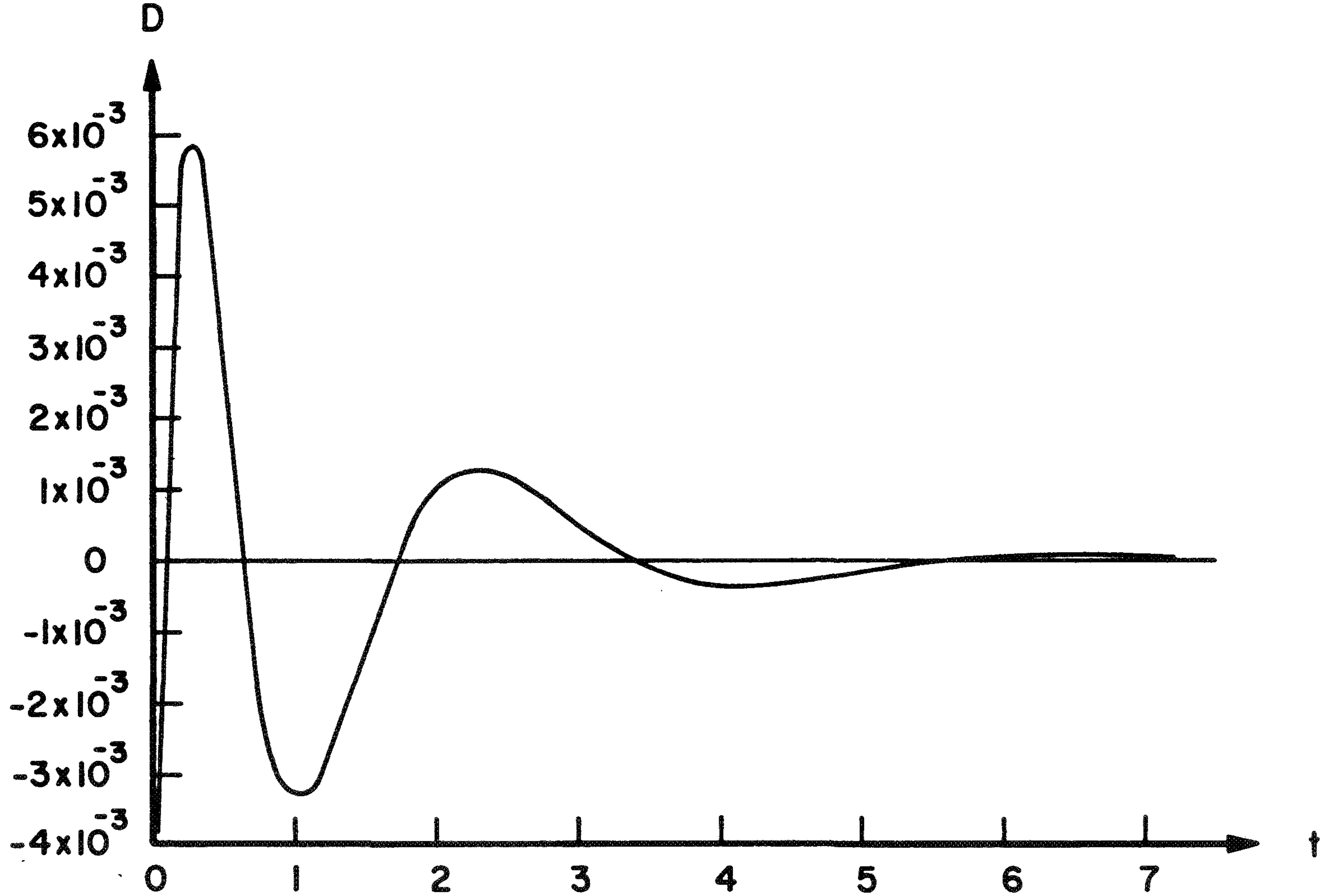

Figure 15. Graph of deviation D, $D = f(t) - \hat{f}(t)$, for Laguerre approximation of p.d.f. in Example 2.

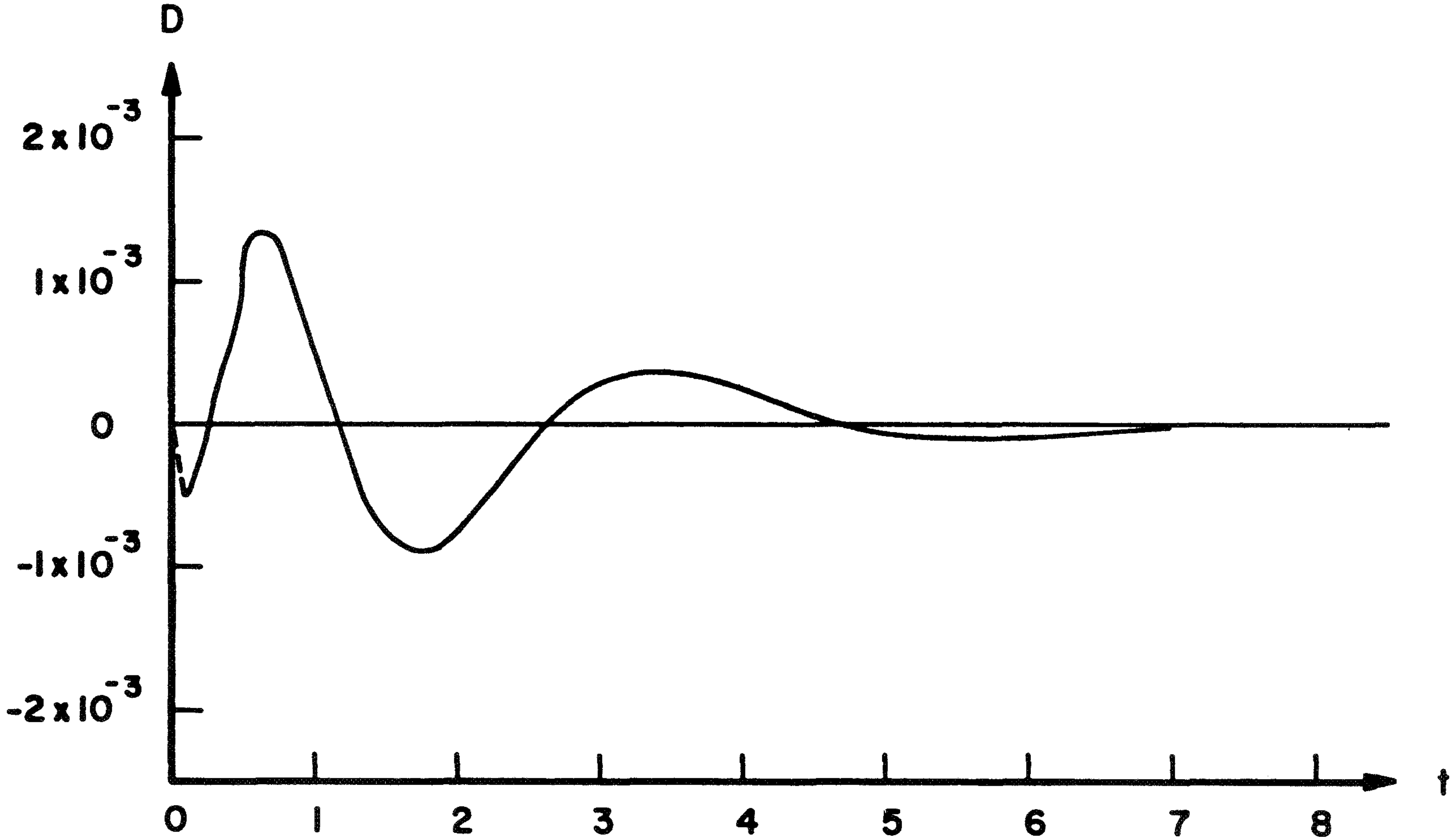

Figure 16. Graph deviation D, $D = F(t) - \hat{F}(t)$, for Laguerre approximation to the c.d.f. in Example 2.

TABLE XI

VALUES OF THE ACTUAL c.d.f. F(t) AND

THE LAGUERRE APPROXIMATION $\hat{F}(t)$

t	F(t)	$\hat{F}(t)$	t	F(t)	$\hat{F}(t)$
.10	.055797	.056391	4.30	.991256	.991125
.20	.119510	.119828	4.40	.992173	.992076
.30	.186831	.186586	4.50	.992992	.992927
.40	.254735	.253952	4.60	.993722	.993688
.50	.321155	.319999	4.70	.994374	.994367
.60	.384730	.383406	4.80	.994957	.994974
.70	.444616	.443315	4.90	.995477	.995516
.80	.500344	.499216	5.00	.995942	.996000
.90	.551714	.550857	5.10	.996359	.996432
1.00	.598711	.598176	5.20	.996731	.996817
1.10	.641448	.641247	5.30	.997065	.997160
1.20	.680120	.680232	5.40	.997363	.997465
1.30	.714972	.715356	5.50	.997631	.997738
1.40	.746279	.746880	5.60	.997871	.997980
1.50	.774324	.775082	5.70	.998086	.998195
1.60	.799391	.800245	5.80	.998278	.998386
1.70	.821754	.822649	5.90	.998451	.998556
1.80	.841675	.842561	6.00	.998607	.998707
1.90	.859399	.860234	6.10	.998746	.998842
2.00	.875152	.875905	6.20	.998871	.998961
2.10	.889142	.889791	6.30	.998984	.999066
2.20	.901560	.902089	6.40	.999085	.999160
2.30	.912576	.912978	6.50	.999175	.999244
2.40	.922345	.922621	6.60	.999257	.999318
2.50	.931008	.931161	6.70	.999331	.999384
2.60	.938688	.938728	6.80	.999397	.999442
2.70	.945497	.945435	6.90	.999456	.999494
2.80	.951534	.951384	7.00	.999510	.999541
2.90	.956888	.956664	7.10	.999558	.999583
3.00	.961635	.961354	7.20	.999601	.999620
3.10	.965846	.965522	7.30	.999640	.999653
3.20	.969583	.969230	7.40	.999675	.999683
3.30	.972900	.972531	7.50	.999707	.999710
3.40	.975845	.975472	7.60	.999736	.999734
3.50	.978461	.978093	7.70	.999762	.999756
3.60	.980785	.980432	7.80	.999785	.999776
3.70	.982851	.982520	7.90	.999806	.999793
3.80	.984688	.984385	8.00	.999825	.999810
3.90	.986323	.986051	8.10	.999842	.999825
4.00	.987778	.987540	8.20	.999857	.999839
4.10	.989073	.988871	8.30	.999871	.999851
4.20	.990227	.990061	8.40	.999883	.999863

TABLE XI (Continued)

t	F(t)	$\hat{F}(t)$	t	F(t)	$\hat{F}(t)$
8.50	.999895	.999873	11.40	.999994	.999995
8.60	.999905	.999883	11.50	.999995	.999996
8.70	.999914	.999892	11.60	.999995	.999997
8.80	.999922	.999901	11.70	.999996	.999997
8.90	.999930	.999909	11.80	.999996	.999998
9.00	.999937	.999916	11.90	.999997	.999999
9.10	.999943	.999923	12.00	.999997	.999999
9.20	.999948	.999929	12.10	.999997	1.000000
9.30	.999953	.999935	12.20	.999997	1.000000
9.40	.999958	.999940	12.30	.999998	1.000000
9.50	.999962	.999945	12.40	.999998	1.000001
9.60	.999965	.999950	12.50	.999998	1.000001
9.70	.999969	.999955	12.60	.999998	1.000001
9.80	.999972	.999959	12.70	.999998	1.000001
9.90	.999974	.999963	12.80	.999999	1.000001
10.00	.999977	.999966	12.90	.999999	1.000001
10.10	.999979	.999970	13.00	.999999	1.000001
10.20	.999981	.999973	13.10	.999999	1.000001
10.30	.999983	.999975	13.20	.999999	1.000001
10.40	.999985	.999978	13.30	.999999	1.000002
10.50	.999986	.999980	13.40	.999999	1.000001
10.60	.999987	.999983	13.50	.999999	1.000001
10.70	.999989	.999985	13.60	.999999	1.000001
10.80	.999990	.999987	13.70	.999999	1.000001
10.90	.999991	.999988	13.80	.999999	1.000001
11.00	.999992	.999990	13.90	1.000000	1.000001
11.10	.999992	.999991	14.00	1.000000	1.000001
11.20	.999993	.999993	14.10	1.000000	1.000001
11.30	.999994	.999994			

mate a function that is monotonically non-decreasing and asymptotic to one than a function which does not have these properties.

Note that the approximate p.d.f. attained a negative value at t = 13.3 and remained negative, but never went more negative than -.000001. Thus, for practical purposes we would use the approximation up to t = 13.2.

Example 3.

Consider now the convolution of an exponential distribution with a Gamma distribution of order two. The resulting M.G.F. is

$$M(s) = \frac{ab^2}{(a-s)(b-s)^2} \quad .$$

If we let a = 1.5 and b = 2.5 the first eight moments are

$$v_1 = 1.4666667 \qquad v_5 = 83.098236$$

$$v_2 = 2.9155554 \qquad v_6 = 353.03679$$

$$v_3 = 7.3671111 \qquad v_7 = 1713.5653$$

$$v_4 = 22.717630 \qquad v_8 = 9376.8324 \quad .$$

We know the distribution is continuous, defined over $(0,\infty)$, and has a value of zero when the independent variable is zero (so we set r = 1.0 in the input data).

The program TEST, following the same sequence as in the previous examples, tells us that a unimodal distribution exists having the given moments and that the appropriate coefficients for the Laguerre expansion are

$d_1 = -.26666670$ $\qquad$ $d_5 = .00038236$

$d_2 = .00962965$ $\qquad$ $d_6 = -.00004598$

$d_3 = .00819753$ $\qquad$ $d_7 = .00000446$

$d_4 = -.00234650$ $\qquad$ $d_8 = -.00000037$.

The magnitude of the last few coefficients might lead us to believe that a good approximation will be obtained. However, examination of TABLE XII indicates that the approximation is not good in the tail of the distribution and the approximation was terminated after $t = 6.9$ due to negative values. The lesson here is that if we have reason to expect the unknown p.d.f. to be asymptotic to the t axis as $t \to \infty$ and the approximation terminates, due to negativity, before some reasonable value of t, depending on the mean, we can assume that the approximation is not too good. This is in contrast to the previous example where the p.d.f. went only slightly negative and the largest c.d.f. values (TABLE XI) were very close to one. In the current example the largest c.d.f. values, TABLE XIII, are much greater than one.

Since the approximation was terminated the program TEST called the subroutine PSON to obtain the appropriate Pearson curve. Using the first four moments PSON computes

$$\beta_1 = 1.61198510$$

$$\beta_2 = 5.5538114 \quad ,$$

which yields

$$k = 6.24526069 \quad .$$

Thus we use the Type III curve

$$f(x) = \frac{p^{p+1}}{ae^{p}\Gamma(p+1)}(1+\frac{x}{a})^{\delta a} e^{-\delta x}$$

$$= Y_0(1+\frac{x}{a})^{\delta a} e^{-\delta x}$$

where

$\delta = 1.80167523$

$p = 1.48141251$

$a = .822241704$

$Y_0 = .558665557$.

The range of the curve is $(-a,\infty)$ and this really doesn't suit our purpose. We can translate the curve to one beginning at the origin simply by replacing x with x - a, then

$$f(x) = \frac{p^{p+1}}{a^{p+1}\Gamma(p+1)} x^p e^{-\delta x} \quad .$$

Values of this curve are given in TABLE XII, and the mean is $a + 1/\delta$, i.e. $v_1 = 1.377$.

In order to compare the two approximations a graph, Fig.17, was constructed. From this graph we see that the Laguerre approximation is superior to the Pearson curve except in the tail of the distribution where the Laguerre approximation overestimates the true p.d.f. until going negative at t = 6.9. This overestimation is reflected in the c.d.f. approximation given in TABLE XIII.

Since there is disagreement in the approximations the next logical step is to use the numerical inversion procedure. Replacing s by -s in the M.G.F. yields the

TABLE XII

VALUES OF TRUE p.d.f. f(t), LAGUERRE APPROXIMATION USING EIGHT TERMS WITH r = 1, THE SHIFTED PEARSON TYPE III, AND THE NUMERICAL INVERSION TECHNIQUE

t	f(t)	$\hat{f}(t)$		
		Laguerre	Type III Pearson	Laplace Inversion
0.00	.000000	.000000	.000000	.000014
.10	.037754	.052050	.090528	.037757
.20	.121701	.132569	.211100	.121699
.30	.220797	.222510	.321444	.220797
.40	.316691	.309596	.411098	.316693
.50	.399463	.386448	.477819	.399462
.60	.464638	.449163	.522779	.464639
.70	.511144	.496233	.548590	.511145
.80	.539913	.527752	.558357	.539913
.90	.552954	.544818	.555194	.552954
1.00	.552752	.549110	.541981	.552753
1.10	.541888	.542585	.521265	.541888
1.20	.522819	.527267	.495214	.522819
1.30	.497751	.505111	.465633	.497752
1.40	.468588	.477921	.433988	.468588
1.50	.436921	.447294	.401439	.436921
1.60	.404037	.414605	.368889	.404038
1.70	.370952	.381003	.337019	.370952
1.80	.338441	.347421	.306325	.338441
1.90	.307073	.314591	.277154	.307073
2.00	.277249	.283066	.249733	.277249
2.10	.249233	.253248	.224192	.249233
2.20	.223177	.225402	.200588	.223177
2.30	.199148	.199689	.178919	.199148
2.40	.177150	.176177	.159145	.177150
2.50	.157136	.154867	.141192	.157136
2.60	.139027	.135706	.124968	.139027
2.70	.122720	.118598	.110366	.122720
2.80	.108098	.103424	.097272	.108098
2.90	.095036	.090045	.085569	.095035
3.00	.083406	.078310	.075142	.083406
3.10	.073083	.068067	.065877	.073084
3.20	.063945	.059165	.057665	.063945
3.30	.055875	.051456	.050404	.055875
3.40	.048764	.044802	.043997	.048765
3.50	.042510	.039071	.038356	.042510

TABLE XII (Continued)

t	f(t)	f̂(t)		
		Laguerre	Type III Pearson	Laplace Inversion
3.60	.037021	.034146	.033397	.037021
3.70	.032210	.029916	.029046	.032211
3.80	.028000	.026285	.025235	.028000
3.90	.024321	.023166	.021901	.024321
4.00	.021110	.020482	.018989	.021111
4.10	.018311	.018166	.016449	.018310
4.20	.015873	.016162	.014237	.015873
4.30	.013752	.014420	.012311	.013753
4.40	.011908	.012897	.010638	.011907
4.50	.010306	.011558	.009185	.010306
4.60	.008916	.010373	.007924	.008917
4.70	.007710	.009319	.006832	.007710
4.80	.006665	.008374	.005886	.006665
4.90	.005760	.007522	.005068	.005761
5.00	.004976	.006750	.004361	.004975
5.10	.004297	.006046	.003751	.004297
5.20	.003710	.005402	.003224	.003711
5.30	.003202	.004812	.002769	.003201
5.40	.002763	.004269	.002378	.002763
5.50	.002384	.003769	.002040	.002385
5.60	.002057	.003309	.001750	.002056
5.70	.001774	.002885	.001500	.001773
5.80	.001530	.002496	.001286	.001531
5.90	.001319	.002140	.001101	.001318
6.00	.001137	.001814	.000943	.001136
6.10	.000980	.001518	.000807	.000981
6.20	.000845	.001249	.000690	.000844
6.30	.000728	.001007	.000590	.000727
6.40	.000627	.000790	.000505	.000629
6.50	.000540	.000597	.000431	.000539
6.60	.000466	.000426	.000368	.000465
6.70	.000401	.000577	.000315	.000403
6.80	.000345	.000147	.000269	.000344
6.90	.000298	.000036	.000229	.000297
7.00	.000256		.000196	.000258
7.10	.000221		.000167	.000219
7.20	.000190		.000142	.000189
7.30	.000164		.000121	.000166
7.40	.000141		.000103	.000140
7.50	.000121		.000088	.000121
7.60	.000105		.000075	.000107
7.70	.000090		.000064	.000088
7.80	.000077		.000054	.000076

TABLE XII (Continued)

t	f(t)	$\hat{f}(t)$ Laguerre	$\hat{f}(t)$ Type III Pearson	$\hat{f}(t)$ Laplace Inversion
7.90	.000067		.000046	.000069
8.00	.000057		.000039	.000056
8.10	.000049		.000033	.000048
8.20	.000043		.000028	.000046
8.30	.000037		.000024	.000035
8.40	.000032		.000021	.000030
8.50	.000027		.000017	.000031
8.60	.000023		.000015	.000021
8.70	.000020		.000013	.000019
8.80	.000017		.000011	.000021
8.90	.000015		.000009	.000013
9.00	.000013		.000008	.000011
9.10	.000011		.000007	.000015

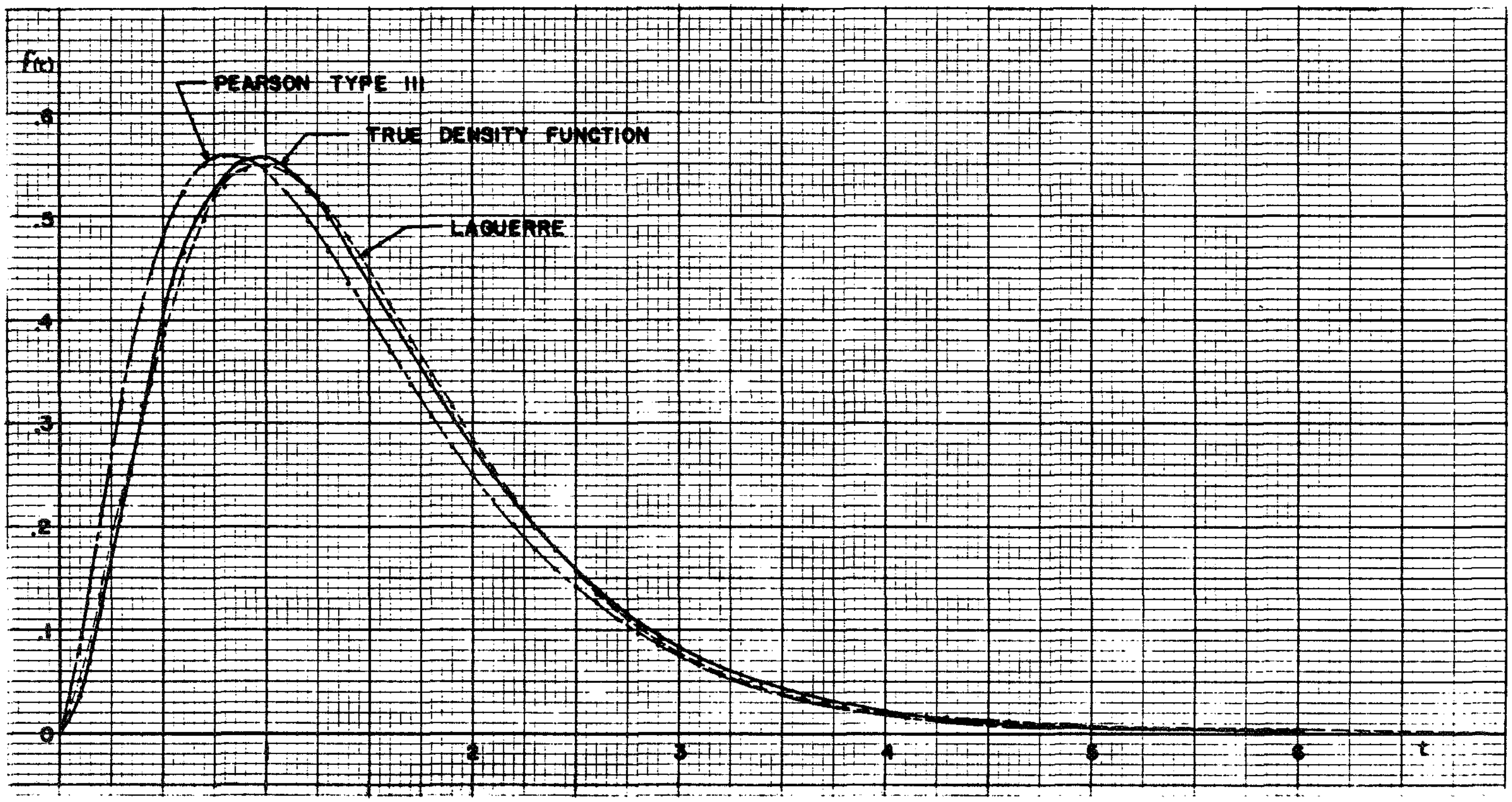

Figure 17. Comparison of the Laguerre and Pearson Type III approximations with the true p.d.f. of Example 3.

TABLE XIII

VALUES OF THE c.d.f. USING AN EIGHT TERM

LAGUERRE APPROXIMATION WITH r = 1

t	$\hat{F}(t)$	t	$\hat{F}(t)$
.00	.000000	3.50	.969254
.10	.002264	3.60	.972908
.20	.011349	3.70	.976106
.30	.029084	3.80	.978911
.40	.055750	3.90	.981380
.50	.090658	4.00	.983559
.60	.132566	4.10	.985489
.70	.179968	4.20	.987203
.80	.231293	4.30	.988730
.90	.285036	4.40	.990094
1.00	.339831	4.50	.991315
1.10	.394497	4.60	.992411
1.20	.448055	4.70	.993394
1.30	.499723	4.80	.994278
1.40	.548909	4.90	.995072
1.50	.595193	5.00	.995785
1.60	.638300	5.10	.996424
1.70	.678083	5.20	.996996
1.80	.714501	5.30	.997507
1.90	.747593	5.40	.997960
2.00	.777463	5.50	.998362
2.10	.804263	5.60	.998715
2.20	.828178	5.70	.999025
2.30	.849415	5.80	.999294
2.40	.868190	5.90	.999525
2.50	.884724	6.00	.999722
2.60	.899235	6.10	.999889
2.70	.911933	6.20	1.000027
2.80	.923019	6.30	1.000139
2.90	.932678	6.40	1.000229
3.00	.941083	6.50	1.000298
3.10	.948390	6.60	1.000349
3.20	.954741	6.70	1.000384
3.30	.960263	6.80	1.000405
3.40	.965067	6.90	1.000414

Laplace transform

$$F(s) = \frac{ab^2}{(a+s)(b+s)^2} \quad .$$

Next, we let $s = A + iQ$, expand, rationalize, and take the real part of the result. Thus

$$\text{Re}\left[F(s)\right] = \frac{ab^2 U}{U^2 + V^2}$$

where

$$U = (A+a)\left[(A+b)^2 - Q^2\right] - 2Q^2(A+b) \quad ,$$

$$V = Q\left[2(A+b)(A+a) + (A+b)^2 - Q^2\right] \quad .$$

Using a program similar to the one given in Chapter V with $A = .5$ and taking 1000 terms in the sum yields the desired approximation which is also listed in TABLE XII.

Example 4.

Let us now investigate the problem of multimodality. To expedite the discussion, and give some insight as to how the problem might arise, consider the GERT network shown in Fig.18.

The resulting M.G.F. is

$$M(s) = \frac{.6e^{t_1 s} + .4e^{t_2 s}}{(1+s)^2} \quad ,$$

and the corresponding p.d.f. is

$$f(t) = .6u(t-t_1)(t-t_1)e^{-(t-t_1)} + .4u(t-t_2)(t-t_2)e^{-(t-t_2)} \quad ,$$

where $u(t)$ is the unit step function. By varying the parameters t_1 and t_2 we can change the distance between the modes

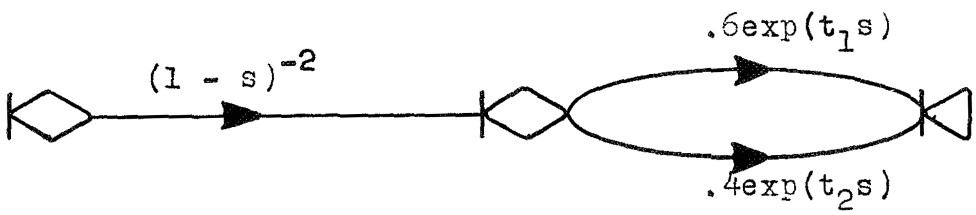

Figure 18. GERT network with bimodal output.

and the amount the distribution is shifted from the origin.

First, suppose we let $t_1 = 5$ and $t_2 = 0$, then

$$v_1 = 5 \qquad v_4 = 1921$$

$$v_2 = 33 \qquad v_5 = 16245 \quad .$$

$$v_3 = 243$$

The program TEST tells us that a solution exists for the given moments and then begins the test for unimodality. Performing the necessary calculations yields the coefficients of the cubic equation y in Theorem 7 as

$$a = -8$$

$$b = 16$$

$$c = 632$$

$$d = -128 \quad .$$

The roots of y are

$$r_1 = -8.055$$

$$r_2 = .202$$

$$r_3 = 9.853 \quad .$$

Since $a < 0$ and all roots are real there must be an M such that

$$M \leq r_1 \quad \text{or} \quad r_2 \leq M \leq r_3$$

and M must also satisfy Relation 21, that is

$$.101 \leq M \leq 9.899 \quad .$$

This is easily done and the strictest restriction is

$$r_2 \leq M \leq r_3 \quad .$$

We now proceed to Theorem 8, and compute the coefficients of the quadratic Z as

$$a_2 = -1$$

$$a_1 = 18$$

$$a_0 = -81 \quad .$$

Since the roots are equal ($p_1 = p_2 = 9$) and $a_2 < 0$ the curve lies below the M axis but touches the axis at $M = 9$. This value of M lies in the above interval and is less than $2v_1 = 10$, thus we compute the coefficients of the cubic y. This yields

$$a = 128$$

$$b = 3312$$

$$c = 44896$$

$$d = -245488 \quad ,$$

and the corresponding roots are

$$r_1 = 9.984$$

$$r_2 = 7.95 + i\ 11.36$$

$$r_3 = 7.95 - i\ 11.36 \quad .$$

There is one real root and $a > 0$ so $r_1 \leq M$ must hold, that is

$$9.984 \leq M \quad .$$

This requirement is not compatable with the interval imposed by Theorem 7, thus the p.d.f. is not unimodal.

Let us determine the distance between the modes. This is done by setting the first derivative of the p.d.f. equal to zero. We find that there is a mode at $t = 1$ and at $t = 5.97$. Thus, the modes are 4.97 units apart.

Now suppose we let $t_1 = 0$ and $t_2 = 3$, then the first eight moments are

$v_1 = 3.2$	$v_5 = 3373.2$
$v_2 = 14.4$	$v_6 = 26262$
$v_3 = 78$	$v_7 = 226879.4$
$v_4 = 483.6$	$v_8 = 2161926.8$.

This time we find that there exists a unimodal distribution having these moments. The appropriate coefficients for the Laguerre expansion with r = 1 are

$d_1 = .6$	$d_5 = .00029167$
$d_2 = .1$	$d_6 = -.00006845$
$d_3 = -.025$	$d_7 = .00000734$
$d_4 = .00125$	$d_8 = -.00000045$

and the resulting approximation along with the actual p.d.f. is given in TABLE XIV. The approximate and true c.d.f. are given in TABLE XV.

Examining TABLE XIV we find that although the approximation is bimodal it is not very good. Of course more terms would make it better. However, the c.d.f. approximation, TABLE XV, is quite good as can be seen from the graph Fig.19.

For this case the modes are separated by 2.78 units. This combined with the previous case implies that if the modes of a bimodal distribution are close together then the moments could also have come from a unimodal distribution.

TABLE XIV

ACTUAL BIMODAL p.d.f. AND ITS EIGHT TERM LAGUERRE APPROXIMATION

t	f(t)	$\hat{f}(t)$	t	f(t)	$\hat{f}(t)$
.10	.054290	.028753	2.60	.115867	.140671
.20	.098248	.074706	2.70	.108873	.145562
.30	.133347	.124030	2.80	.102161	.150607
.40	.160877	.168607	2.90	.095740	.155587
.50	.181959	.204257	3.00	.089617	.160315
.60	.197572	.229435	3.10	.119985	.164632
.70	.208566	.244272	3.20	.143762	.168413
.80	.215678	.249887	3.30	.161927	.171560
.90	.219548	.247911	3.40	.175333	.174004
1.00	.220728	.240155	3.50	.184721	.175701
1.10	.219695	.228399	3.60	.190734	.176628
1.20	.216860	.214264	3.70	.193930	.176784
1.30	.212575	.199135	3.80	.194791	.176181
1.40	.207141	.184137	3.90	.193731	.174849
1.50	.200817	.170132	4.00	.191109	.172826
1.60	.193821	.157735	4.10	.187232	.170161
1.70	.186337	.147340	4.20	.182362	.166909
1.80	.178523	.139153	4.30	.176723	.163128
1.90	.170508	.133224	4.40	.170506	.158881
2.00	.162402	.129485	4.50	.163872	.154229
2.10	.154295	.127772	4.60	.156957	.149237
2.20	.146260	.127861	4.70	.149873	.143964
2.30	.138357	.129484	4.80	.142717	.138471
2.40	.130634	.132354	4.90	.135565	.132813
2.50	.123127	.136178			

TABLE XV

ACTUAL c.d.f. OF BIMODAL DISTRIBUTION AND ITS LAGUERRE APPROXIMATION USING EIGHT TERMS

t	F(t)	$\hat{F}(t)$	t	F(t)	$\hat{F}(t)$
.10	.002807	.001218	2.60	.439569	.423978
.20	.010514	.006315	2.70	.450804	.438288
.30	.022162	.016265	2.80	.461353	.453096
.40	.036931	.030959	2.90	.471246	.468407
.50	.054122	.049686	3.00	.480511	.484205
.60	.073141	.071459	3.10	.491050	.500456
.70	.093483	.095227	3.20	.504288	.517113
.80	.114725	.120005	3.30	.519616	.534117
.90	.136511	.144951	3.40	.536515	.551402
1.00	.158545	.169395	3.50	.554549	.568893
1.10	.180582	.192849	3.60	.573347	.586516
1.20	.202424	.214996	3.70	.592602	.604193
1.30	.223906	.235669	3.80	.612055	.621848
1.40	.244900	.254827	3.90	.631496	.639405
1.50	.265305	.272530	4.00	.650750	.656794
1.60	.285041	.288908	4.10	.669676	.673949
1.70	.304053	.304144	4.20	.688163	.690807
1.80	.322298	.318450	4.30	.706123	.707313
1.90	.339751	.332050	4.40	.723488	.723417
2.00	.356396	.345168	4.50	.740210	.739076
2.10	.372231	.358014	4.60	.756253	.754252
2.20	.387258	.370782	4.70	.771596	.768914
2.30	.401487	.383638	4.80	.786226	.783037
2.40	.414935	.396720	4.90	.800139	.796602
2.50	.427622	.410140			

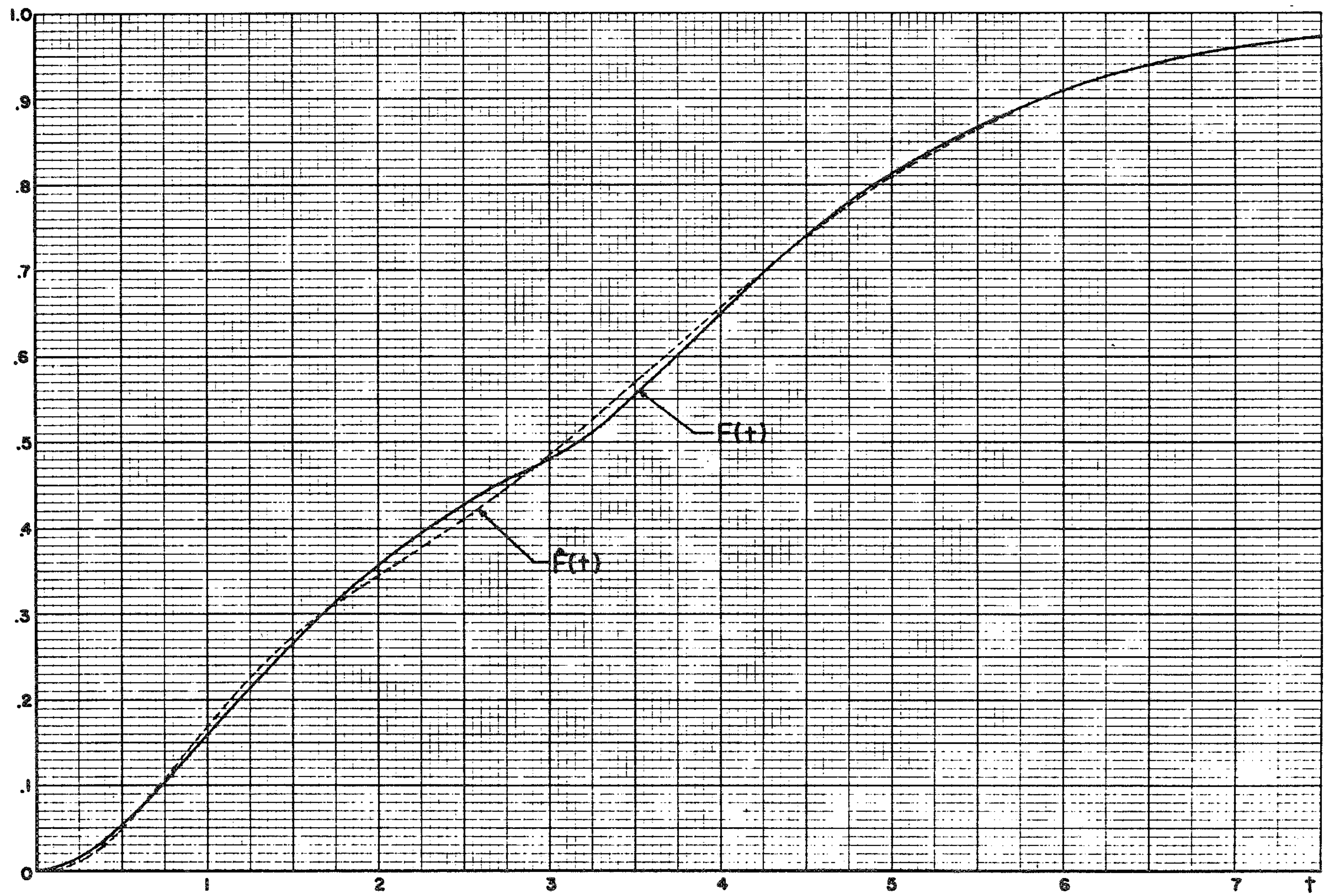

Figure 19. Comparison of the actual, F(t), and eight term Laguerre approximation, $\hat{F}(t)$, of the bimodal p.d.f. in Example 4.

That is, we encounter the uniqueness problem.

Finally, suppose we shift the above p.d.f. five units to the right, that is let $t_1 = 5$ and $t_2 = 8$. Now the moments are

$$v_1 = 8.2 \qquad v_4 = 6428.6$$

$$v_2 = 71.4 \qquad v_5 = 66088.2 \quad .$$

$$v_3 = 659$$

Again the program tells us that a unimodal distribution exists having these moments. However, the magnitude of the moments causes the Laguerre approximation to become negative at $t = .4$. This results in the calling of subroutine PSON.

The attempted approximation continues with PSON computing β_1, β_2 and k. The values are

$$\beta_1 = .389597311$$

$$\beta_2 = 3.19914933$$

$$k < 0 \quad .$$

Thus, we use the Type I curve

$$f(x) = \frac{a_1^{m_1} a_2^{m_2} \Gamma(m_1+m_2+2)}{(a_1+a_2)^{m_1+m_2+1} \Gamma(m_1+1)\Gamma(m_2+1)} (1+\frac{x}{a_1})^{m_1} (1-\frac{x}{a_2})^{m_2} \quad ,$$

where

$$m_1 = -.652928462$$

$$m_2 = 12.7443068$$

$$a_1 = -1.01891088$$

$$a_2 = 19.8878033 \quad .$$

Note that we must compute $a_1^{m_1}$, an imaginary number. This indicates that our approximation has encountered difficulty.

This difficulty can be circumvented by a linear transformation which shifts the range of the distribution from the interval $(-a_1,a_2)$ to $(0,a_1+a_2)$. We replace x with $x-a_1$, then

$$f(x) = \frac{a_2^{m_2}\Gamma(m_1+m_2+2)}{(a_1+a_2)^{m_1+m_2+1}\Gamma(m_1+1)\Gamma(m_2+1)}\, x^{m_1}\left(1-\frac{x-a_1}{a_2}\right)^{m_2}$$

This is the form actually used in PSON. Note that since $m_1 < 0$, $f(x) \to \infty$ as $x \to 0$. Also $f(x) = 0$ when $x = a_1+a_2$. So we see that this curve behaves somewhat like a negative exponential (see Fig.20), hardly a bimodal distribution. Thus, in this case we have actually found a unimodal distribution which has the same first four moments as the given bimodal distribution.

Example 5.

For the final example let us investigate the problem of multimodality for functions defined over $(-\infty,\infty)$.

Suppose we take the mixture of two distributions each made up of the convolution of a normal with an exponential. Specifically, we have

$$M(s) = .5\,\frac{a_1\exp(m_1s+\sigma_1^2s^2/2)}{a_1-s} + .5\,\frac{a_2\exp(m_2s+\sigma_2^2s^2/2)}{a_2-s}\quad .$$

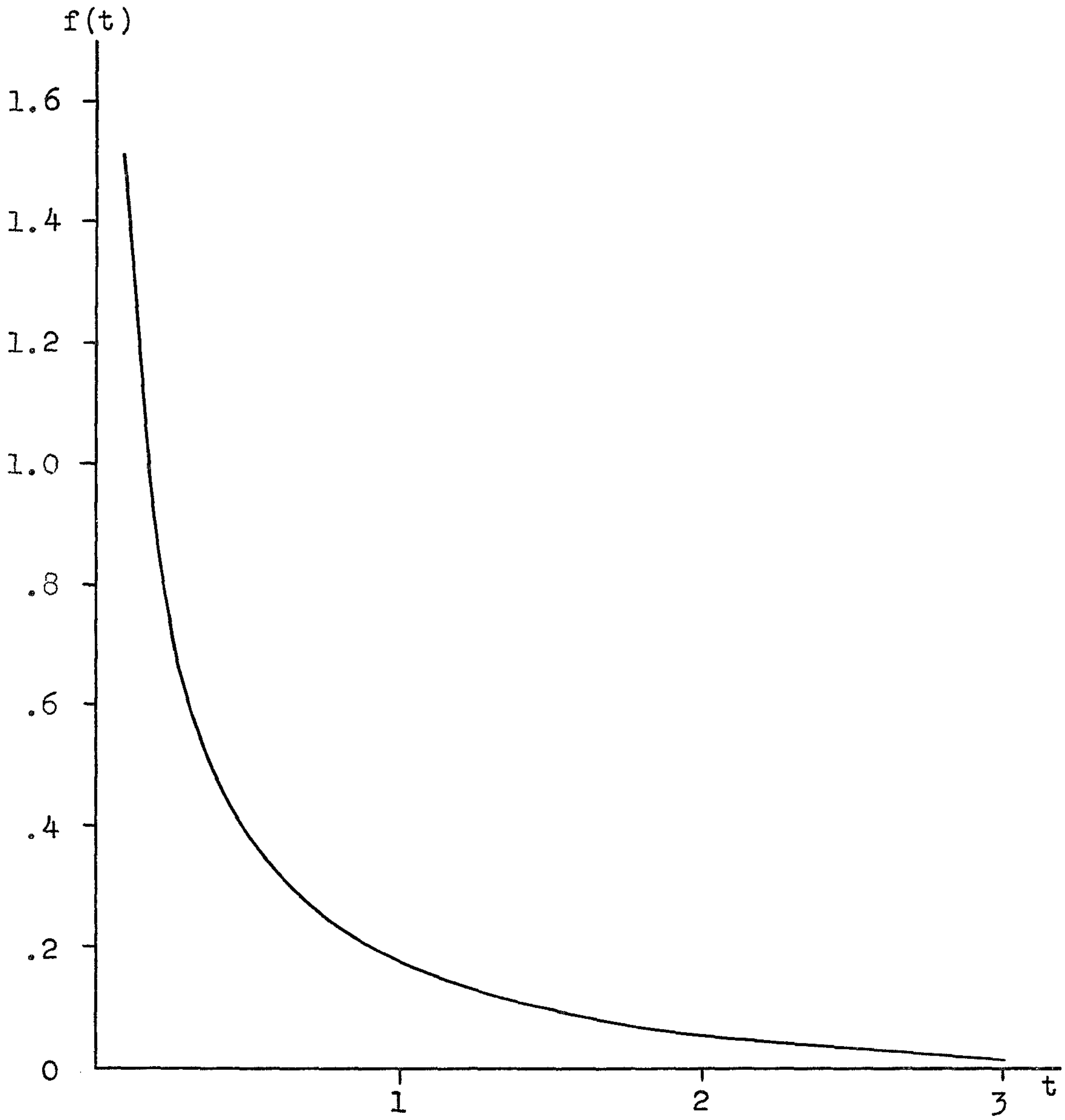

Figure 20. Pearson Type I approximation resulting from the moments of a bimodal distribution.

Now let

$$a_1 = 4 \qquad a_2 = 2$$

$$m_1 = 2 \qquad m_2 = 10$$

$$\sigma_1 = 1 \qquad \sigma_2 = 2 \quad ,$$

then

$$v_1 = .5(2.25) + .5(10.5) = 6.375$$

$$v_2 = 60.3125$$

$$v_3 = 655.171875$$

$$v_4 = 7996.4635 \quad .$$

As before, these moments along with the continuity and range information constitute the input data for TEST. Based on the data TEST tells that a distribution does exist having the above moments and begins to check for unimodality using Theorem 7. First the coefficients of the cubic, Eq.22, Chapter III, are computed as

$$a = 79.453145$$

$$b = -3243.82506$$

$$c = 26982.7161$$

$$d = 34678.1011 \quad ,$$

and the resulting roots are

$$r_1 = -1.12800495$$

$$r_2 = 13.6889205$$

$$r_3 = 28.2659775 \quad .$$

Since $a > 0$ M must lie between r_1 and r_2, or be greater than r_3. *At the same time Relation 21 Chapter III tells us that*

M must lie between -1.30716278 and 14.0571628. There are an infinite number of values of M that will satisfy both restrictions so the theorem tells us that there is a unimodal distribution with the given moments. Thus, we have again encountered the uniqueness problem.

This situation prompts the following question. Can we make any stronger statement concerning the moments of a multimodal distribution defined on $(-\infty,\infty)$? Let us begin the investigation of this question by reexamining Theorem 7. First we express the coefficients of the cubic y in terms of moments about the mean, whence

$$a = 4u_3$$

$$b = -3ua - [5u_4-9u_2^2]$$

$$c = -3u_2a - ub + u[5u_4-9u_2^2]$$

$$d = -a^2 - u^3a - u^2b - uc + 3u_2[5u_4-9u_2^2] \quad ,$$

where $u = v_1$ (the mean).

Rather than examine all possible cases let us consider only one which will demonstrate the approach. Suppose that the distribution of interest is symmetric which implies that $u_3 = 0$ and thus $a = 0$. Now, we are to examine $y = bM^2 + cM + d \geq 0$ and according to Theorem 7 two situations are of interest to us $b < 0$ and $b > 0$. Note that if a and b are both zero all of the coefficients are zero. Next we rewrite the coefficients as

$$b = -[5u_4 - 9u_2^2]$$

$$c = -ub + u[5u_4 - 9u_2^2] = -2ub$$

$$d = -(u^2 + 3u_2)b \quad .$$

Using the quadratic formula we find the roots of y are

$$M = u \pm \sqrt{3u_2} \quad .$$

Thus, if $b < 0$ we see that Relation 21 and Relation 26 of Theorem 7 are exactly the same, and if $b > 0$ the extreme points of the interval described by Relation 21 coincide with the beginning points of the two intervals described by Relation 27. Also note that complex roots are impossible since this would imply a negative variance. This means we have been able to show that the first four moments of all symmetric distributions will satisfy Theorem 7. Thus, the first four moments from a symmetric multimodal distribution are also the first four moments of a unimodal distribution. Since $u_3 = 0$ the unimodal distribution is also symmetric. According to Theorem 7 only the first four moments are needed to determine unimodality, hence, we may arbitrarily set the higher order moments of a unimodal distribution equal to the higher order moments of a symmetric multimodal distribution, and the following theorem results.

<u>Theorem</u> <u>14</u>. For any symmetric multimodal distribution defined on $(-\infty,\infty)$ there always exists a symmetric unimodal distribution, also defined on $(-\infty,\infty)$, having the same moments.

CHAPTER VII

CONCLUSIONS AND RECOMMENDATIONS

The conclusions fall into two categories. The first category is composed of those definite statements that can be made as a result of the research. The second category is made up of those statements that must be linked with the recommendations for future research.

First let us consider the concrete implications that may be drawn from the research.

The collection of the techniques is important in that it illustrates their interconnection, and the resulting procedure is a useful research tool for understanding the inversion problem.

In using the moments of a p.d.f. to construct an approximation we must keep the uniqueness problem in mind. The importance of this was clearly demonstrated in the examples concerned with identifying bimodal distributions. That is to say, although we developed necessary and sufficient conditions that a set of moments from a unimodal distribution must satisfy, the moments from a bimodal distribution may also satisfy them, indicating that there was also a unimodal distribution possessing the same moments. In Theorem 14 (Chapter VI) we see one example of just how far reaching the uniqueness problem can be.

Examples 3 and 4 illustrate the fact that more than eight moments are needed to provide a good Laguerre series approximation. At least when the value of the p.d.f. was not zero for t = 0.

Example 4 shows that trouble can be encountered in using Pearson curves when a general idea of the shape of the p.d.f. is not available. That is the Pearson approximation of a bimodal p.d.f. does not make sense. This difficulty is of course involved with the uniqueness problem.

If a p.d.f. has large moments (i.e. the mean is large) the series will require more terms to obtain a useful approximation, and thus more moments will be needed. Whenever a p.d.f. is shifted to the right so that it does not start at the origin, as was done in Example 4, it should be shifted back so that it does. This translation will reduce the magnitude of the moments about the origin and make the approximation more accurate for a given number of terms.

The numerical inversion technique is accurate, time consuming, and requires a great deal of effort on the part of the analyst. At the present time it should be used when great accuracy is needed, or the other methods fail.

The solution procedure as presented is not the ultimate answer. However, it is strongly felt that the basic approach is sound, and that the inclusion of the recommendations given below would yield a very useful tool for solving the problem.

Instead of terminating a series approximation, when the p.d.f. values go negative, it should be completed and then a Pearson approximation made. These two approximations should then be compared to see if they agree in kind. If they do not agree one or both is inaccurate and we will be forced to use numerical inversion. This was illustrated in Example 3.

Tests should be made using the Laguerre series with $r > 1$, particularly with respect to multimodal distributions and obtaining approximations with fewer terms.

Three areas for future research are recommended.

Pearson developed his system using only four moments because of the large error involved in higher moments obtained from a sample. However, we have assumed that we know as many moments as desired with complete accuracy. Thus, it might be desirable to develop a system of curves similar to Pearson's but based on more moments.

In actually fitting the curves it was found that strict adherence to Pearson's rules for selection did not always yield the most appropriate curve. This situation should be investigated.

The relationships that exist between the moments of a unimodal distribution should be more thoroughly investigated. In particular the limits on the rate of growth of the moments of a unimodal distribution should be determined, since the growth rate may be critical only between the third and fourth moments, for example.

The GERT program [42], [43] should be expanded to yield the first ten moments as output. Then an investigation of network types and output distributions made, along with approximating these p.d.f.'s by the procedure suggested in this dissertation.

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APPENDIX A

ANNOTATED BIBLIOGRAPHY

Bohman, H. "To Compute the Distribution Function When the Characteristic Function Is Known," Skand. Akluartidskr., Vol.46, pp.41-46, 1964.

A function which is nonzero only in a finite interval is constructed. This function is then multiplied by the characteristic function of interest and the result integrated. The integration yields two functions which bound the actual distribution function from above and below.

_______. "What Is the Reason that Esscher's Method of Approximation Is as Good as It Is," Skand. Akluartidskr., Vol.46, pp.87-94, 1964.

Esscher's method (see reference 4 below) is derived from a different point of view and the reasons for its accuracy for large arguments discussed.

Charlier, C. V. L. "A New Form of the Frequency Function," Meddelande Fran Lunds Astronomiska Observatorium, Ser.II No.51, 1928.

The Type A and B series sometimes suffer from two difficulties, (1) negative values for the p.d.f. and (2) the coefficients in the series do not decrease in regular succession. In order to circumvent these difficulties the Type C series is proposed

$$f(x) = \exp\left(\sum_{r=0}^{\infty} y_r H_r(x)\right) ,$$

where $H_r(x)$ is the Hermite polynomial of order r. This approach eliminates the problems mentioned, however, if the A or B series approximation has difficulties the C series will not yield a good approximation.

Esscher, F. "On Approximate Calculation of Distribution Functions When the Corresponding Characteristic Functions Are Known," Skand. Akluartidskr., Vol.46, pp.78-86, 1964.

The approach advocated in this paper is to approximate the distribution function, F(x), as follows:

$$F(x) = e^{-hx} \phi(h) \int_{-\infty}^{x} e^{-h(z-x)} d\bar{F}(z)$$

where $\phi(h) = \int_{-\infty}^{\infty} e^{hx}\, dF(x)$, and h a real number,

and $\bar{F}(z)$ is the first two terms in the Edgeworth series expansion. The limited results presented seem quite good.

Hartley, H. O. and S. H. Khamis. "A Numerical Solution of the Problem of Moments," Biometrica, Vol.34, pp.340-351, 1947.

Using the first R moments, Sheppard's corrections, and finite-difference calculus a table of R + 1 equidistant values for the distribution function is constructed. The method is not easy to apply when the function is defined on an infinite interval and the technique is good only for unimodal functions.

Khamis, S. H. "On the Reduced Moment Problem," Ann. Math. Stat., Vol.25, No.1, pp.113-122, 1954.

The main result in this paper is a proof that if one continuous c.d.f. is a solution to a reduced moment problem, over a finite range, then there exists an infinite number of continuous c.d.f.'s which are also solutions to the problem.

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The numerical details of the Hartley-Khamis procedure are extended.

Mallows, C. L. "Generalizations of Tchebycheff's Inequalities," J.Royal Stat.Soc., Ser.B, Vol.18, No.2, pp.139-168, 1956.

This paper contains the best known inequalities on distribution functions. However, the bounds are good only for unimodal distributions that are fairly smooth and continuous.

Medgyessy, P. "On the Interconnection Between the Representation Theorems of Characteristic Functions of Unimodal Distribution Functions and of Convex Characteristic Functions," Publ.Math.Inst.Hung.Acad.Sci., Vol.8, pp.425-430, 1963.

Using the definition of unimodality, given in Chap.I of this dissertation, it is shown that Theorem 9 Chap.III and the following theorem are intimately connected.

Theorem. The function $\phi(s)$ is a convex characteristic function if and only if for $s > 0$ it can be represented in the form

$$\phi(s) = \int_s^\infty (1 - \frac{s}{x})\, dG(x)$$

where $G(x)$ is some distribution function for which $G(x) = 0$ if $x \leq 0$ and $G(+0) = 0$, and for $s < 0$, $\phi(s) = \phi(-s)$.

Mulholland, H. P. "On Distributions for Which the Hartley-Khamis Solution of the Moment-Problem Is Exact," Biometrica, Vol.38, pp.74-89, 1951.
In this paper it is shown that despite its apparent dissimilarity, the Hartley-Khamis procedure is akin to the series expansion methods, especially the Type A.

Zelen, M. "Bounds on a Distribution Function That Are Functions of Moments to Order Four," J. of Research of the National Bureau of Standards, Vol.53, No.6, pp.377-381, December, 1954.
Explicit expressions are given for the bounds on any distribution function, defined on [a,b], when the moments to order four are known. The cases when $a = -\infty$, and/or $b = \infty$ are also discussed.

APPENDIX B

```
      PROGRAM TEST(INPUT,OUTPUT,TAPE5=INPUT,TAPE6=OUTPUT)
      DIMENSION A(15,15),V(11),B(11),C(11),D(11),XR(3),AQ(3),XS(4),
     1XI(4),AC(4),AR(3),BR(3),RT(3),R(4),RI(4),XL(11),F(11),
     2S(11),CF(11),H(11),VP(11),U(11),G(11)
      V(1)=1.0
      NT=0
      READ(5,985)NOT
    9 NT=NT+1
      IZ2=0
      IZZ=0
      IF(NT-NOT)12,12,900
   12 IF(NT-1)8,8,14
   14 WRITE(6,989)
    8 READ(5,990)N1
  585 READ(5,975)(V(J),J=2,N1)
  590 READ(5,950)IC
  595 READ(5,985)IR
  600 IT=0
      IF(IR)55,55,56
   55 WRITE(6,981)
      GO TO 57
   56 WRITE(6,982)
   57 WRITE(6,987)
      DO 11 IJ=1,N1
      M=IJ-1
   11 WRITE(6,988)M,V(IJ)
C
C         CHECK FOR EXISTENCE OF SOLUTION
C
    2 NU=(N1+1)/2
    4 ND=2
    5 DO 20 I=1,ND
      DO 20 J=1,ND
      IF(IR)15,15,17
   15 IF(IT)17,17,16
   16 K=I+J
      GO TO 6
   17 K=I+J-1
    6 A(I,J)=V(K)
   20 CONTINUE
      N=ND
      CALL CRAMER(A,N,DETERM)
      IF(DETERM)99,99,30
   30 ND=ND+1
      IF(ND-NU)5,5,45
   45 IF(IT)46,46,48
   46 IF(IR)47,47,48
   47 NU=N1/2
      IT=1
      GO TO 4
   48 WRITE(6,994)
      NM=N1-1
      GO TO 1
   51 IF(IT)49,49,104
   49 IF(IR)50,50,106
   50 NU=(ND-1)/2
      IT=1
      GO TO 4
   99 IF(IT)101,101,102
  101 NM=K-3
      NC=NM
```

```
      GO TO 103
  102 NM=K-2
      NE=NM+1
  103 WRITE(6,996)NM
      WRITE(6,992)N,DETERM
      IF(NM-2)105,105,51
  105 IF(IR)300,300,400
  104 IF((NC+1)-NE)106,106,107
  106 NM=NC
      GO TO 100
  107 NM=NE
C
C         CHECK FOR UNIMODALITY
C
  100 WRITE(6,994)
    1 D(1)=4.*V(4)-12.*V(2)*V(3)+8.*(V(2)**3)
      WRITE(6,972)D(1)
      D(2)=8.*V(2)*V(4)+9.*(V(3)**2)-12.*(V(2)**2)*V(3)-5.*V(5)
      D(3)=10.*V(2)*V(5)-12.*V(4)*V(3)-16.*V(4)*(V(2)**2)
     1+18.*V(2)*(V(3)**2)
      D(4)=15.*V(3)*V(5)+48.*V(4)*V(2)*V(3)-27.*(V(3)**3)-16.*(V(4)**2)
     1-20.*V(5)*(V(2)**2)
      CM=V(2)-SQRT(3.*(V(3)-(V(2)**2)))
      CP=V(2)+SQRT(3.*(V(3)-(V(2)**2)))
      WR+TE(6,972)(D(I),I=2,4)
      IF(D(1))130,110,130
  110 IF(D(2))115,111,115
  111 IF(D(3))112,114,113
  112 IF((-D(4)/D(3))-CM)197,330,331
  113 IF((-D(4)/D(3))-CP)334,337,197
  330 CL=CM
      CU=CM
      GO TO 198
  331 CL=CM
      IF((-D(4)/D(3))-CP)332,332,333
  332 CU=-D(4)/D(3)
      GO TO 198
  333 CU=CP
      GO TO 198
  334 IF((-D(4)/D(3))-CM)335,335,336
  335 CL=CM
      GO TO 333
  336 CL =-D(4)/D(3)
      GO TO 333
  337 CL=CP
      GO TO 333
  114 IF(D(4))197,108,108
  108 CL=CM
      CU=CP
      GO TO 198
  115 C(1)=D(2)
      C(2)=D(3)
      C(3)=D(4)
      CALL QUAD(C,XR(1),XR(2),XI)
      XI(2)=-XI
      WRITE(6,972)XR(1),XR(2),XI(2)
      WRITE(6,972)CM,CP
      IF(XI(2))116,117,116
  116 IF(C(1))197,198,198
  117 IF(XR(1)-XR(2))118,118,119
  118 R1=XR(1)
```

```
      R2=XR(2)
      GO TO 120
  119 R1=XR(2)
      R2=XR(1)
  120 IF(C(1))121,121,128
  121 IF(CM-R1)122,122,123
  122 CL=CM
      GO TO 124
  123 CL=R1
  124 IF(CP-R2)125,125,126
  125 CU=CP
      GO TO 127
  126 CU=R2
  127 IF(CL-CU)198,198,197
  128 IF(CM-R1)198,198,129
  129 IF(CP-R2)197,198,198
  130 CALL CUBIC(D,XR,XI)
      XI(1)=0.0
      XI(3)=-XI(2)
      WRITE(6,972)(XR(I),I=1,3)
      WRITE(6,972)XI(3)
      WRITE(6,972)CM,CP
      IF(XI(3))150,160,150
  150 IF(D(1))151,110,187
  151 IF(XR(1)-CM)197,152,152
  152 CL=XR(1)
      CU=CP
      GO TO 198
C
C     RANK ROOTS
C
  160 IF(XR(1)-XR(2))163,163,164
  163 IF(XR(2)-XR(3))165,165,166
  164 IF(XR(1)-XR(3))169,169,170
  165 R1=XR(1)
      R2=XR(2)
      R3=XR(3)
      GO TO 173
  166 R3=XR(2)
      IF(XR(1)-XR(3))167,167,168
  167 R1=XR(1)
      R2=XR(3)
      GO TO 173
  168 R1=XR(3)
      R2=XR(1)
      GO TO 173
  169 R1=XR(2)
      R2=XR(1)
      R3=XR(3)
      GO TO 173
  170 R3=XR(1)
      IF(XR(2)-XR(3))171,171,172
  171 R1=XR(2)
      R2=XR(3)
      GO TO 173
  172 R1=XR(3)
      R2=XR(2)
  173 IZ=0
      IF(IZ2)174,174,224
  174 IF(D(1))161,161,162
  161 IF(R1-CM)180,500,500
```

```
500 CL=CM
    CU=R1
    IF(R2-CP)501,501,198
501 CL1=R2
    IF(R3-CP)502,502,504
502 CU1=R3
503 IZ=1
    GO TO 198
504 CU1=CP
    GO TO 503
180 IF(R2-CM)182,505,183
505 CL=CM
    IF(R3-CP)506,506,507
506 CU=R3
    GO TO 198
507 CU=CP
    GO TO 198
182 IF(R3-CM)197,505,505
183 IF(R2-CP)509,509,197
509 CL=R2
    IF(R3-CP)506,506,507
162 IF(R3-CP)510,510,184
510 IF(R3-CM)511,511,512
511 CU=CP
    CL=CM
    GO TO 198
512 IF(R2-CM)511,511,513
513 CL1=CP
    CU1=R3
    IZ=1
    IF(R1-CM)514,514,515
514 CL=CM
    CU=R2
    GO TO 198
515 CL=R1
    CU=R2
    GO TO 198
184 IF(R1-CM)185,516,186
516 IF(R2-CP)515,515,517
517 CL=R1
    CU=CP
    GO TO 198
185 IF(R2-CM)197,518,518
186 IF(R1-CP)516,516,197
518 IF(R2-CP)514,514,511
187 IF(XR(1)-CP)188,188,197
188 CL=CM
    CP=XR(1)
    GO TO 198
197 IF(IZZ)297,297,206
297 IF(IR)206,206,400
198 IF(IZZ)298,298,307
298 IF(IR)200,200,204
200 CS=2.*V(2)
    WRITE(6,972)R1,R2,R3
    C(1)=3.*V(3)-4.*(V(2)**2)
    C(2)=6.*V(3)*V(2)-4.*V(4)
    C(3)=8.*V(2)*V(4)-9.*(V(3)**2)
    WRITE(6,972)(3(I),I=1,3)
    IF(C(1))305,301,305
301 IF(C(2))320,310,315
```

```
310 IF(C(3))206,207,207
315 IF(-C(3)/C(2)-CS)207,207,206
320 IF(-C(3)/C(2))206,207,207
305 CALL QUAD(C,XR(1),XR(2),XI)
    XI(2)=-XI
    WRITE(6,972)XR(1),XR(2),XI(2)
    IF(XI(2))205,199,205
205 IF(C(1))206,207,207
199 IF(XR(1)-XR(2))201,201,202
201 R1=XR(1)
    R2=XR(2)
    GO TO 203
202 R1=XR(2)
    R2=XR(1)
203 IF(C(1))216,216,520
216 IF(CS-R1)206,524,524
520 IF(R1-CL)521,521,207
521 IF(R2-CU)207,207,522
522 IF(IZ-1)523,206,206
523 CL=CL1
    CU=CU1
    IZ=1
    GO TO 520
524 IF(CS-R2)525,525,529
525 IF(R1-CU)526,526,527
526 IF(CS-CL)527,207,207
527 IF(IZ)206,206,528
528 CL=CL1
    CU=CU1
    IZ=0
    GO TO 525
529 CS=R2
    GO TO 525
204 WRITE(6,993)
    WRITE(6,972)R1,R2,R3
    CALL ASER(V,IK,NM)
    IF(IK)212,212,999
206 IF(IC)210,210,300
207 D(1)=-15.*V(3)*V(5)-48.*V(2)*V(3)*V(4)+27.*(V(3)**3)
   1+16.*(V(4)**2)+20.*(V(2)**2)*V(5)
    WRITE(6,972)D(1)
    D(2)=18.*V(3)*V(6)+30.*V(2)*V(3)*V(5)+32.*V(2)*(V(4)**2)
   1-36.*V(4)*(V(3)**2)-20.*V(4)*V(5)-24.*V(6)*(V(2)**2)
    D(3)=-24.*V(4)*V(6)-45.*V(5)*(V(3)**2)-40.*V(2)*V(4)*V(5)
   1+48.*V(3)*(V(4)**2)+36.*V(2)*V(3)*V(6)+25.*(V(5)**2)
    D(4)=48.*V(2)*V(4)*V(6)+120.*V(3)*V(4)*V(5)-64.*(V(4)**3)
   1-50.*V(2)*(V(5)**2)-54.*V(6)*(V(3)**2)
    WRITE(6,972)(D(I),I=2,4)
    CALL CUBIC(D,XR,XI)
    XI(1)=0.0
    XI(3)=-XI(2)
    WRITE(6,972)(XR(I),I=1,3)
    WRITE(6,972)XI(3)
    IF(XI(3))220,223,220
220 IF(D(1))221,221,222
221 IF(XR(1)-CL)206,307,307
222 IF(XR(1)-CU)307,307,206
223 IZ2=1
    GO TO 160
224 CM=CL
    CP=CU
```

```
      IZZ=5
      GO TO 174
  307 WRITE(6,993)
      IF(IC)211,211,208
  208 CALL LAG(V,IK,NM)
      WRITE(6,972)R1,R2,R3
      IF(IK)900,212,9
  210 WRITE(6,971)
      WRITE(6,972)R1,R2,R3
      GO TO 999
  211 CALL BSER(V,IK,NM)
      IF(IK)210,210,999
  212 CALL PSON(V)
      GO TO 999
  300 WRITE(6,970)
      READ(5,997)R(1)
      GO TO 9
  400 WRITE(6,998)
      GO TO 9
  900 STOP
  999 GO TO 9
C
  950 FORMAT(I3)
  970 FORMAT(//14X,32HTRANSFORM INVERSION MUST BE USED)
  971 FORMAT(//14X,31HMUST USE TERM BY TERM EXPANSION)
  972 FORMAT(/ E16.8/)
  975 FORMAT(5E14.8)
  980 FORMAT(/6F10.2/)
  981 FORMAT(/17X,25HTHE RANGE IS (0,INFINITY))
  982 FORMAT(/12X,33HTHE RANGE IS (-INFINITY,INFINITY))
  985 FORMAT(I4)
  987 FORMAT(//21X,16HTHE MOMENTS USED/21X,16HIN THE TESTS ARE,/)
  988 FORMAT(/18X,2HV(,I2,1H),2H =,F16.8)
  989 FORMAT(1H1)
  990 FORMAT(I1)
  992 FORMAT(/14X,28HTHE VALUE OF THE DETERMINANT/,
     114X,8HOF ORDER,I3,3H IS,1X,F16.8)
  993 FORMAT(/14X,37HA UNIMODAL DISTRIBUTION EXISTS HAVING/
     122X,21HTHE SPECIFIED MOMENTS//)
  994 FORMAT(/21X,17HA SOLUTION EXISTS/,19X,21HFOR SPECIFIED MOMENTS/)
  996 FORMAT(//15X,28HA SOLUTION IS POSSIBLE USING/,
     115X,22HONLY THE MOMENTS UP TO,I3,9H, BECAUSE,/)
  997 FORMAT(F4.1)
  998 FORMAT(//14X,20HNO SOLUTION POSSIBLE)
C
      END
      SUBROUTINE PSON(V)
      DIMENSION A(15,15),V(11),B(11),C(11),D(11),XR(3),AQ(3),XS(4),
     1XI(4),AC(4),AR(3),BR(3),RT(3),R(4),RI(4),XL(11),F(11),
     2S(11),CF(11),H(11),VP(11),U(11),G(11)
  212 B1=((V(4)-3.*V(2)*V(3)+2.*(V(2)**3))**2)/((V(3)-(V(2)**2))**3)
  213 B2=(V(5)-4.*V(2)*V(4)+6.*V(3)*(V(2)**2)-3.*(V(2)**4))/((V(3)
     1-(V(2)**2))**2)
      WRITE(6,972)B1,B2
      FK=(B1*((B2+3.)**2))/(4.*(2.*B2-3.*B1-6.)*(4.*B2-3.*B1))
      AOK=4.*(2.*B2-3.*B1-6.)*(4.*B2-3.*B1)
      IF(AOK-.000001)216,215,214
  216 IF(AOK)217,215,215
  217 FK=-10.0
      GO TO 214
  215 FK=2000.
```

```
214 WRITE(6,975)FK
    IF(FK)1,10,2
1   R=6.*(B2-B1-1.)/(6.+3.*B1-B2*2.)
    A1PA2=.5*SQRT (V(3)-V(2)**2)*SQRT (B1*(R+2.)**2+16.*(R+1.))
    Z=R*(R*2.)*SQRT (B1/(B1*(R+2.)**2+16.*(R+1.)))
    FM1=.5*(R-2.-Z)
    FM2=.5*(R-2.+Z)
    A1=(FM1*(A1PA2)/(FM1+FM2))
    A2=A1PA2-A1
    Y0=(A2**FM2)/(A1PA2**(FM1+FM2+1.))
    Y0=Y0*GAMMA(FM1+FM2+2.)/(GAMMA(FM1+1.)*GAMMA(FM2+1.))
    ID=1
    WRITE(6,955)ID
    WRITE(6,967)R(1)
    WRITE(6,960)FM1,FM2,A1,A2,Y0
    U3=V(4)-3.*V(2)*V(3)+2.*(V(2)**3)
    WRITE(6,970)U3
    RETURN
2   IF(FK-1.)3,14,4
3   R=6.*(B2-B1-1.)/(2.*B2-3.*B1-6.)
    FM=.5*(R+2.)
    Z=SQRT (16.*(R-1.)-B1*(R-2.)**2)
    V1=R*(R-2.)*SQRT (B1)/Z
    W=SQRT (Z*(V(3)-V(2)**2)/16.)
    ID=4
    WRITE(6,955)ID
    WRITE(6,967)R(1)
    WRITE(6,961)FM,W,V1
    RETURN
4   IF(FK-5.)5,5,6
5   R=6.*(B2-B1-1.)/(6.+3.*B1-2.*B2)
    Z=SQRT (B1*(R+2.)**2+16.*(R+1.))
    W=.5*Z*SQRT (V(3)-V(2)**2)
    Q2=  (R-2.)/2.+(.5*R*(R+2.))*SQRT (B1/(Z**2))
    Q1=-((R-2.)/2.-(.5*R*(R+2.))*SQRT (B1/(Z**2)))
    Y0=(W**(Q1-Q2-1.))*GAMMA(Q1)/(GAMMA(Q1-Q2-1.)*GAMMA(Q2+1.))
    ID=6
    WRITE(6,955)ID
    WR+TE(6,967)R(1)
    WRITE(6,962)W,Q2,Q1,Y0
    RETURN
6   IF(B1-4.)8,7,8
7   ID=10
    WRITE(6,955)ID
    WRITE(6,968)V(2)
    GO TO 9
8   X=2.*(V(3)-V(2)**2)/(V(4)-3.*V(3)*V(2)+2.*V(2)**3)
    P=4./B1-1.
    W=(2.*(V(3)-V(2)**2)**2)/(V(4)-3.*V(3)*V(2)+2.*V(2)**3)-1./X
    Y0=P**(P+1.)/(W*EXP (P)*GAMMA(P+1.))
    ID=3
    WRITE(6,955)ID
    WRITE(6,963)X,P,W,Y0
9   RETURN
10  IF(B2-3.)11,12,13
11  FM=(5.*B2-9.)/(2.*(3.-B2))
    A2=2.*B2*(V(3)-V(2)**2)/(3.-B2)
    Y0=GAMMA(FM+1.5)/(SQRT (A2*3.14159)*GAMMA(FM+1.))
    ID=2
    WRITE(6,955)ID
    WRITE(6,964)FM,A2,Y0
```

```
      RETURN
   12 U2=V(3)-V(2)**2
      WRITE(6,965)U2
      RETURN
   13 FM=(5.*B2-9.)/(2.*(B2-3.))
      A2=2.*B2*(V(3)-V(2)**2)/(B2-2.)
      YO=GAMMA(FM)/(GAMMA(FM-.5)*SQRT (A2*3.14159))
      ID=7
      WRITE(6,955)ID
      WRITE(6,966)FM,A2,YO
      RETURN
   14 P=4.+(8.+4.*SQRTF(4.+B1))/B1
      DEL=(P-2.)*SQRTF((P-3.)*(V(3)-(V(2)**2)))
      YO=(DEL**(P-1.))/GAMMA(P-1.)
      ID=5
      WRITE(6,955)ID
      WRITE(6,969)P,DEL,YO
      RETURN
C
  955 FORMAT(//14X,16HUSE PEARSON TYPE,I3,/14X,15HWITH PARAMETERS)
  960 FORMAT(//14X,6HM(1) =,E16.8,/14X,6HM(2) =,E16.8,/14X,6HA(1) =,
     1E16.8,/14X,6HA(2) =,E16.8,/14X,6HY(0) =,E16.8)
  961 FORMAT(//14X,3HM =,E16.8,/14X,3HA =,E16.8,/14X,6HV(1) =,E16.8)
  962 FORMAT(//14X,3HA =,E16.8,/14X,6HQ(2) =,E16.8,/14X,6HQ(1) =,
     1E16.8,/14X,6HY(0) =,E16.8)
  963 FORMAT(//14X,3HD =,E16.8,/14X,3HP =,E16.8,/14X,3HA =,E16.8,
     1/14X,6HY(0) =,E16.8)
  964 FORMAT(//14X,3HM =,E16.8,/14X,6HA(2) =,E16.8,/14X,6HY(0) =,E16.8)
  965 FORMAT(//14X,15HUSE STD. NORMAL,/14X,11HWITH U(2) =,E16.8)
  966 FORMAT(//14X,3HM =,E16.8,/14X,6HA(2) =,E16.8,/14X,6HY(0) =,E16.8)
  967 FORMAT(//14X,3HR =,E16.8)
  968 FORMAT(//14X,28HUSE EXPONENTIAL DISTRIBUTION/14X,
     112HWITH MEAN OF,E16.8)
  969 FORMAT(//14X,3HP =,E16.8,/14X,3HD =,E16.8,/14X,6HY(0) =,E16.8)
  970 FORMAT(//14X,6HU(3) =,E16.8)
  972 FORMAT(//20X,9HBETA(1) =,E16.8/20X,9HBETA(2) =,E16.8)
  975 FORMAT(//20X,3HK =,E16.8)
C
      END
      SUBROUTINE BSER(V,IK,NM)
      DIMENSION A(15,15),V(11),B(11),C(11),D(11),XR(3),AQ(3),XS(4),
     1XI(4),AC(4),AR(3),BR(3),RT(3),R(4),RI(4),XL(11),F(11),
     2S(11),CF(11),H(11),VP(11),U(11),G(11)
      TD=1.0
      TU=20.0
      KZ=0
      M=0
      NU=NM
      NU=4
      DO 10 I=1,NU
   10 CF(I)=0.0
      WRITE(6,900)
      B(1)=0.0
      B(2)=.5*(V(3)-V(2)*(2.*V(2)+1.)+V(2)**2)
      B(3)=(V(4)-3.*V(3)+2.*V(2)-3.*V(2)*(V(3)-V(2))+2.*V(2)**3)/6.
      B(4)=(V(5)-6.*V(4)+11.*V(3)-6.*V(2)-4.*V(2)*(V(4)-3.*V(3)
     1+2.*V(2))+6.*(V(2)**2)*(V(3)-V(2))-3.*V(2)**4)/24.
      WRITE(6,902)NU
      WRITE(6,903)(B(JK),JK=2,NU)
      WRITE(6,9)
      WRITE(6,12)
```

```
   25 T=0.0
   30 G(1)=(T-V(2))/V(2)
      G(2)=2.*(FACT(T,2.)-V(2)*FACT(T,1.)+.5*V(2)**2)/(V(2)**2)
      G(3)=6.*(FACT(T,3.)-V(2)*FACT(T,2.)+.5*(V(2)**2)*FACT(T,1.)
     1-(V(2)**3)/6.)/(V(2)**3)
      G(4)=24.*(FACT(T,4.)-V(2)*FACT(T,3.)+.5*(V(2)**2)*FACT(T,2.)
     1-(V(2)**3)*FACT(T,1.)/6.+(V(2)**4)/24.)/(V(2)**4)
      IF(T)34,34,35
   34 P=EXP(-V(2))
      GO TO 36
   35 P=(V(2)**T)*EXP(-V(2))/(GAMMA(T+1.))
   36 S(2)=0.0
      IF(M)40,40,70
   40 DO 45 J=3,NU
      K=J-1
      S(J)=S(K)+B(J)*G(J)
   45 F(J)=P*(1.+B(2)*G(2)+S(J))
   55 WRITE(6,901)T,(F(JK),JK=3,NU)
      T=T+TD
      IF(T-TU)30,30,60
   60 M=1
      WRITE(6,904)
      WRITE(6,905)
      WRITE(6,9)
      WRITE(6,12)
      GO TO 25
   70 DO 75 L=3,NU
      N=L-1
      S(L)=S(N)+B(L)*G(L)
      F(L)=(1.+B(2)*G(2)+S(L))*P
   75 CF(L)=CF(L)+F(L)
      WRITE(6,901)T,(CF(KJ),KJ=3,NU)
      T=T+TD
      IF(T-TU)30,30,100
  100 IF(KZ)101,101,102
  101 IK=5
      RETURN
  102 IK=0
      RETURN
C
    9 FORMAT(35X,15HNUMBER OF TERMS/)
   12 FORMAT(6X,5H TIME,6X,1H3,9X,1H4,9X,1H5,9X,1H6,9X,1H7,9X,1H8//)
  900 FORMAT(//14X,22HGRAM-CHARLIER B SERIES//)
  901 FORMAT(5X,F6.2,10F10.6)
  902 FORMAT(//4X,22HVALUE OF B(J) FOR J=2,,I3/)
  903 FORMAT(/4X,8F12.8)
  904 FORMAT(1H1)
  905 FORMAT(/14X,35HVALUES OF THE DISTRIBUTION FUNCTION//)
C
      END
      SUBROUTINE LAG(V,IK,NM)
      DIMENSION A(15,15),V(11),B(11),C(11),D(11),XR(3),AQ(3),XS(4),
     1XI(4),AC(4),AR(3),BR(3),RT(3),R(4),RI(4),XL(11),F(11),
     2S(11),CF(11),H(11),VP(11),U(11),G(11)
    2 FORMAT(5X,F6.2,10F10.6)
    4 FORMAT(5X,33HLAGUERRE POLYNOMIAL APPROXIMATION/)
    5 FORMAT(1H1)
    6 FORMAT(//14X,35HVALUES OF THE DISTRIBUTION FUNCTION//)
    7 FORMAT(//5X,25HCOEFFICIENTS D(N), N = 1,,I2)
    8 FORMAT(/8F12.8//)
    9 FORMAT(35X,15HNUMBER OF TERMS/)
```

```
   12 FORMAT(6X,5H TIME,6X,1H3,9X,1H4,9X,1H5,9X,1H6,9X,1H7,9X,1H8//)
  900 FORMAT(F4.1)
      WRITE(6,4)
      MC=0
      TU=15.
      KZ=0
      NU=NM
   98 FL=1.
      T=0.0
      READ(5,900)R(1)
   97 DO 99 I=1,NU
      J=I+1
   99 V(I)=V(J)
      D(1)=(FL*V(1)-R-1.)*FL/(R+1.)
      D(2)=(V(2)*FL**2-FL*V(1)*2.*(R+2.)+R**2+3.*R+2.)
      D(2)=D(2)*(FL**2/(2.*(R+1.)*(R+2.)))
  100 D(3)=(V(3)*(FL**3)-V(2)*(FL**2)*3.*(R+3.)+V(1)*3.*FL*(R+2.)*(R+3.)
     1-(R+1.)*(R+2.)*(R+3.))*FL**3/(6.*(R+1.)*(R+2.)*(R+3.))
  101 D(4)=(V(4)*(FL**4)-4.*(FL**3)*V(3)*(R+4.)
     1+6.*(FL**3)*V(2)*(R+3.)*(R+4.)-4.*FL*V(1)*(R+2.)*(R+3.)*(R+4.)
     2+(R+1.)*(R+2.)*(R+3.)*(R+4.))*(FL**4)
      D(4)=D(4)/(24.*(R+1.)*(R+2.)*(R+3.)*(R+4.))
      D(5)=FL**5*(V(5)*FL**5-5.*V(4)*(R+5.)*FL**4
     1+10.*V(3)*(R+4.)*(R+5.)*FL**3-10.*V(2)*(R+3.)*(R+4.)*(R+5.)*FL**2
     2+5.*V(1)*(R+2.)*(R+3.)*(R+4.)*(R+5.)*FL
     3-(R+1.)*(R+2.)*(R+3.)*(R+4.)*(R+5.))
      D(5)=D(5)/(120.*(R+1.)*(R+2.)*(R+3.)*(R+4.)*(R+5.))
      D(6)=FL**6*(V(6)*FL**6-6.*(R+6.)*V(5)*FL**5
     1+15.*(R+5.)*(R+6.)*V(4)*FL**4
     2-20.*(R+4.)*(R+5.)*(R+6.)*V(3)*FL**3
     3+15.*(R+3.)*(R+4.)*(R+5.)*(R+6.)*V(2)*FL**2
     4-6.*(R+2.)*(R+3.)*(R+4.)*(R+5.)*(R+6.)*V(1)*FL
     5+(R+1.)*(R+2.)*(R+3.)*(R+4.)*(R+5.)*(R+6.))
      D(6)=D(6)/(720.*(R+1.)*(R+2.)*(R+3.)*(R+4.)*(R+5.)*(R+6.))
      D(7)=FL**7*(V(7)*FL**7-7.*(R+7.)*V(6)*FL**6
     1+21.*(R+6.)*(R+7.)*V(5)*FL**5-35.*(R+5.)*(R+6.)*(R+7.)*V(4)*FL**4
     2+35.*(R+4.)*(R+5.)*(R+6.)*(R+7.)*V(3)*FL**3
     3-21.*(R+3.)*(R+4.)*(R+5.)*(R+6.)*(R+7.)*V(2)*FL**2
     4+7.*(R+2.)*(R+3.)*(R+4.)*(R+5.)*(R+6.)*(R+7.)*V(1)*FL
     5-(R+1.)*(R+2.)*(R+3.)*(R+4.)*(R+5.)*(R+6.)*(R+7.))
      D(7)=D(7)/(5040.*(R+1.)*(R+2.)*(R+3.)*(R+4.)*(R+5.)*(R+6.)*(R+7.))
      D(8)=FL**8*(-56.*(R+4.)*(R+5.)*(R+6.)*(R+7.)*(R+8.)*V(3)*FL**3
     1-8.*(R+2.)*(R+3.)*(R+4.)*(R+5.)*(R+6.)*(R+7.)*(R+8.)*V(1)*FL
     2+28.*(R+3.)*(R+4.)*(R+5.)*(R+6.)*(R+7.)*(R+8.)*V(2)*FL**2
     3+(R+1.)*(R+2.)*(R+3.)*(R+4.)*(R+5.)*(R+6.)*(R+7.)*(R+8.)
     4+70.*(R+5.)*(R+6.)*(R+7.)*(R+8.)*V(4)*FL**4
     5-56.*(R+6.)*(R+7.)*(R+8.)*V(5)*FL**5+28.*(R+7.)*(R+8.)*V(6)*FL**6
     6-8.*(R+8.)*V(7)*FL**7+V(8)*FL**8)/40320.
      DENOM=(R+1.)*(R+2.)*(R+3.)*(R+4.)*(R+5.)*(R+6.)*(R+7.)*(R+8.)
      D(8)=D(8)/DENOM
      DO 105 I=1,NU
      J=NU+2-I
      K=NU+1-I
  105 V(J)=V(K)
      V(1)=1.0
      WRITE(6,7)NU
      WRITE(6,8)(D(IJ),IJ=1,NU)
      WRITE(6,9)
      WRITE(6,12)
  150 XL(1)=FL*T-R-1.
      XL(2)=(FL**2)*(T**2)-2.*FL*T*(R+2.)+R**2+3.*R+2.
```

```
      NL=NU+1
      DO 130 JK=2,NL
      JK1=JK-1
      JK2=JK+1
      FJ=JK
  130 XL(JK2)=(FL*T-R-2.*FJ-1.)*XL(JK)-FJ*(R+FJ)*XL(JK1)
  230 IF(R)131,131,132
  131 GT=EXPF(-T)
      GO TO 133
  132 GT=T*EXPF(-T)
  133 S(2)=0.0
      IF(MC)30,30,40
   30 DO 140 K1=3,NU
      K2=K1-1
      S(K1)=S(K2)+D(K1)*XL(K1)
      F(K1)=1.+D(1)*XL(1)+D(2)*XL(2)+S(K1)
  140 F(K1)=F(K1)*GT
      IF(T-.1)146,146,144
  144 IF(F(NU)+.00001)145,146,146
  145 TU=T-.1
      KZ=1
      GO TO 160
  146 WRITE(6,2)T,(F(JK),JK=3,NU)
      T=T+.1
      IF(T-TU)150,150,160
  160 WRITE(6,5)
      WRITE(6,6)
      WRITE(6,9)
      WRITE(6,12)
      T=0.0
      MC=1
      R=R+1.
      GO TO 150
C     CUMULATIVE DISTRIBUTION FUNCTION
   40 DO 45 K=3,NU
      J=K-1
      S(K)=S(J)+D(K)*XL(J)
      IF(R-1.)41,41,42
   41 CF(K)=1.-EXPF(-T)-GT*(D(1)+D(2)*XL(1)+S(K))
      GO TO 45
   42 CF(K)=1.-EXPF(-T)*(T+1.)-T*GT*(D(1)+D(2)*XL(1)+S(K))
   45 CONTINUE
   47 WRITE(6,2)T,(CF(JK),JK=3,NU)
      T=T+.1
      IF(T-TU)150,150,199
  199 IF(KZ)200,200,201
  200 IK=5
      RETURN
  201 IK=0
      RETURN
      END
      SUBROUTINE ASER(V,IK,NM)
      DIMENSION A(15,15),V(11),B(11),C(11),D(11),XR(3),AQ(3),XS(4),
     1XI(4),AC(4),AR(3),BR(3),RT(3),R(4),RI(4),XL(11),F(11),
     2S(11),CF(11),H(11),VP(11),U(11),G(11)
    2 FORMAT(5X,F6.2,10F10.6)
    3 FORMAT(/5X,34HGRAM-CHARLIER A TYPE APPROXIMATION//)
    4 FORMAT(/5X,21HVALUE OF C(J), J = 2,,I2//)
    5 FORMAT(/5X,9F12.6/)
    7 FORMAT(21X,8F12.6/)
    8 FORMAT(2X,4F12.4//)
```

```
    9 FORMAT(35X,15HNUMBER OF TERMS/)
   10 FORMAT(1H1)
   11 FORMAT(/5X,35HVALUES OF THE DISTRIBUTION FUNCTION//)
   12 FORMAT(6X,5H TIME,6X,1H3,9X,1H4,9X,1H5,9X,1H6,9X,1H7,9X,1H8//)
      KZ=0
      NU=NM
      M=0
  600 TU=20.0
      TD=.1
      VP(1)=V(2)
      DO 32 I=2,NU
      J=I-1
   32 VP(I)=VP(J)*V(2)
      U(1)=0.0
      U(2)=V(3)-VP(2)
      U(3)=V(4)-3.*V(2)*V(3)+2.*VP(3)
      U(4)=V(5)-4.*V(2)*V(4)+6.*VP(2)*V(3)-3.*VP(4)
      U(5)=V(6)-5.*V(2)*V(5)+10.*VP(2)*V(4)-10.*VP(3)*V(3)+4.*VP(5)
   33 U(6)=V(7)-6.*V(2)*V(6)+15.*VP(2)*V(5)-20.*VP(3)*V(4)
     1-5.*VP(6)+15.*VP(4)*V(3)
      S=SQRT (V(3)-(V(2)**2))
  495 DO 500 JA=1,NU
  500 U(JA)=U(JA)/(S**JA)
      WRITE(6,3)
      C(1)=U(1)
      C(2)=(1./2.)*(U(2)-1.)
      C(3)=(U(3)-3.*U(1))/6.
      C(4)=(U(4)-6.*U(2)+3.)/24.
      C(5)=(U(5)-10.*U(3)+15.*U(1))/120.
      C(6)=(U(6)-15.*U(4)+45.*U(2)-15.)/720.
      WRITE(6,4)NU
      WRITE(6,5)(C(JK),JK=2,NU)
      WRITE(6,9)
      WRITE(6,12)
   29 T=-10.0
   30 TS=(T-V(2))/S
      H(2)=TS**2-1.
      H(3)=TS**3-3.*TS
      DO 36 L=3,NU
      J=L-1
      J1=L+1
      FL=L
   36 H(J1)=TS*H(L)-FL*H(J)
      AX=(EXP (-(TS**2)/2.))/(SQRT (2.*22./7.))
      S(2)=0.0
      IF(M)37,37,38
   37 DO 40 K=3,NU
      K1=K-1
      S(K)=S(K1)+C(K)*H(K)
      F(K)=1.+S(K)
   40 F(K)=AX*F(K)/S
      IF(F(NU)+.00001)41,42,42
   41 TU=T-.1
      KZ=1
      GO TO 580
   42 WRITE(6,2)T,(F(J),J=3,NU)
      T=T+TD
      IF(T-TU)30,30,580
  580 M=1
      WRITE(6,10)
      WRITE(6,11)
```

```
      WRITE(6,9)
      WRITE(6,12)
      GO TO 29
   38 DO 39 K=3,NU
      K1=K-1
      S(K)=S(K1)+C(K)*H(K1)
      CF(K)=-(S(K))
      X=TS
      IF(X-0.0)50,51,51
   50 X=-X
      Z=1./(1.+.33267*X)
      P=(EXPF(-(X**2)/2.)/SQRTF(6.283186))*(.4361836*Z-.1201676*(Z**2)
     1+.9372980*(Z**3))
      GO TO 39
   51 Z=1./(1.+.33267*X)
      P=1.-(EXPF(-(X**2)/2.)/SQRTF(6.283186))*(.4361836*Z
     1-.1201676*(Z**2)+.9372980*(Z**3))
   39 CF(K)=(P+AX*CF(K))/S
      WRITE(6,2)T,(CF(J),J=3,NU)
      T=T+TD
      IF(T-TU)30,30,999
  999 IF(KZ)700,700,701
  700 IK=5
      RETURN
  701 IK=0
      RETURN
      END
      SUBROUTINE CUBIC(D,XR,XI)
C
      DIMENSION A(15,15),V(11),B(11),C(11),D(11),XR(3),AQ(3),XS(4),
     1XI(4),AC(4),AR(3),BR(3),RT(3),R(4),RI(4),XL(11),F(11),
     2S(11),CF(11),H(11),VP(11),U(11),G(11)
      IPATH=2
      EX=1./3.
      IF(D(4))6,4,6
    4 XR(1)=0.
      GO TO 34
    6 A2=D(1)*D(1)
      Q=(27.*A2*D(4)-9.*D(1)*D(2)*D(3)+2.*D(2)**3)/(54.*A2*D(1))
      IF(Q)10,8,14
    8 Z=0.
      GO TO 32
   10 Q=-Q
      IPATH=1
   14 P=(3.*D(1)*D(3)-D(2)*D(2))/(9.*A2)
      ARG=P*P*P+Q*Q
      IF(ARG)16,18,20
   16 Z=-2.*SQRT (-P)*COS (ATAN (SQRT (-ARG)/Q)/3.)
      GO TO 28
   18 Z=-2.*(Q**EX)
      GO TO 28
   20 SARG=SQRT (ARG)
      IF(P)22,24,26
   22 Z=-((Q+SARG)**EX+(Q-SARG)**EX)
      GO TO 28
   24 Z=-((2.*Q)**EX)
      GO TO 28
   26 Z=(SARG-Q)**EX-((SARG+Q)**EX)
   28 GO TO(30,32),IPATH
   30 Z=-Z
   32 XR(1)=(3.*D(1)*Z-D(2))/(3.*D(1))
```

```
   34 AQ(1)=D(1)
      AQ(2)=D(2)+XR(1)*D(1)
      AQ(3)=D(3)+XR(1)*AQ(2)
      CALL QUAD(AQ,XR(2),XR(3),XI)
      XI(2)=XI
      RETURN
  900 FORMAT(4X,E10.5)
      END
      SUBROUTINE QUAD(C,XR1,XR2,XI)
C
      DIMENSION A(15,15),V(11),B(11),C(11),D(11),XR(3),AQ(3),XS(4),
     1XI(4),AC(4),AR(3),BR(3),RT(3),R(4),RI(4),XL(11),F(11),
     2S(11),CF(11),H(11),VP(11),U(11),G(I1)
      X1=-C(2)/(2.*C(1))
      DISC=X1*X1-C(3)/C(1)
      IF(DISC)10,20,20
   10 X2=SQRT (-DISC)
      XR1=X1
      XR2=X1
      XI=X2
      GO TO 30
   20 X2=SQRT (DISC)
      XR1=X1+X2
      XR2=X1-X2
      XI=0.
   30 RETURN
      END
      SUBROUTINE CRAMER(A,N,DETERM)
C
      DIMENSION A(15,15)
      K=2
      L=1
    5 DO 10 I=K,N
      RATIO=A(I,L)/A(L,L)
      DO 10 J=K,N
   10 A(I,J)=A(I,J)-A(L,J)*RATIO
      IF(K-N)15,20,20
   15 L=K
      K=K+1
      GO TO 5
   20 DETERM=1.0
      DO 25 L=1,N
   25 DETERM=DETERM*A(L,L)
      RETURN
      END
      FUNCTION GAMMA(A)
      KI=0
      IF(A-1.)20,2,1
    1 N=A-0.0001
      C=1.
      DO 10 K=1,N
   10 C=C*(A-FLOATF(K))
    3 X=A-FLOATF(N)
      G=C*(1.-.577191652*X+.988205891*X**2-.897056937*X**3
     1+.918206857*X**4-.756704078*X**5+.482199394*X**6
     2-.193527818*X**7+.035868343*X**8)
      GAMMA=G/(A-FLOATF(N))
      IF(KI)4,4,22
   22 GAMMA=-3.14159/(A*SINF(3.14159*A)*GAMMA)
    4 RETURN
    2 N=0
```

```
      C=1.
      GO TO 3
   20 IF(A)21,2,2
   21 KI=1
      A=-A
      GO TO 1
      END
      FUNCTION FACT(X,Y)
      IF(X-Y)2,2,1
    1 FACT=GAMMA(X+1.)/(GAMMA(Y+1.)*GAMMA(X-Y+1.))
      RETURN
    2 FACT=1.
      RETURN
      END
```

BIOGRAPHICAL SKETCH

Thomas William Hill, Jr. was born in Phoenix, Arizona, on June 2, 1939. He received his elementary and secondary education in Phoenix Public Schools. He received a Bachelor of Science degree from Arizona State University in 1962, a Master of Science in Industrial Engineering from Arizona State University in 1966. He has been employed in engineering at Douglas Aircraft, 1962-1963. He is currently employed as assistant professor of Industrial Engineering at Purdue University, since 1967. He is a member of Alpha Pi Mu and Sigma Xi, national honorary societies. He is also a member of Operations Research Society of America, The Society for Industrial and Applied Mathematics, and The Institute of Management Sciences.