ON DETERMINISTIC AND RANDOM FIXED POINTS

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ABSTRACT. Based on an extension of Aumann's measurable selection theorem due to Leese, it is shown that each fixed point theorem for $F(\omega, \cdot)$ produces a random fixed point theorem for F provided the σ-algebra Σ for Ω is a Suslin family and F has a measurable graph (in particular, when F is random continuous with closed values and X is a separable metric space). As applications and illustrations, some random fixed points in the literature are obtained or extended.

1. INTRODUCTION AND PRELIMINARIES

Random fixed point theory has received much attention in recent years (see, e.g., Bharucha-Reid [2], Bocsan [3], Chang [5], Engl [8, 9], Itoh [13, 14], Kucia and Nowak [15], Lin [17], Liu and Chen [19], Nowak [20], Papageorgiou [21], Rybinski [23], Sehgal and Singh [26], Seghal and Waters [27, 28], Tan and Yuan [29], Xu [32], etc.). In this paper, we shall deal with random non-self-maps. First, by employing a result of Chow and Teicher [6], sufficient conditions are obtained for a random non-self-map, so that the existence of a deterministic fixed point is equivalent to the existence of a random fixed point. Thus every fixed point theorem produces a random fixed point theorem. As applications and illustrations, some random fixed point theorems in the literature are obtained or extended.

A measurable space (Ω, Σ) is a pair where Ω is a set and Σ is a σ-algebra of subsets of Ω. If X is a set, $A \subset X$, and $\mathscr{D}$ is a nonempty family of subsets of X, we shall denote by $\mathscr{D} \cap A$ the family $\{D \cap A : D \in \mathscr{D}\}$ and by $\sigma_X(\mathscr{D})$ the smallest σ-algebra on X generated by $\mathscr{D}$. If X is a topological space with topology τ_X, we shall use $\mathscr{B}(X)$ to denote $\sigma_X(\tau_X)$, the Borel σ-algebra on X if there is no ambiguity on the topology τ_X. If (Ω, Σ) and (Φ, Γ) are two measurable spaces, then $\Sigma \otimes \Gamma$ denotes the smallest σ-algebra on $\Omega \times \Phi$ which contains all the sets $A \times B$, where $A \in \Sigma$, $B \in \Gamma$, i.e., $\Sigma \otimes \Gamma = \sigma_{\Omega \times \Phi}(\Sigma \times \Gamma)$. We note that the Borel σ-algebra $\mathscr{B}(X_1 \times X_2)$ contains $\mathscr{B}(X_1) \otimes \mathscr{B}(X_2)$ in general.

A function $f : \Omega \to \Phi$ is said to be (Σ, Γ) measurable if, for any $B \in \Gamma$, $f^{-1}(B) = \{x \in \Omega, f(x) \in B\} \in \Sigma$. If X is a set, 2^X denotes the family of all nonempty subsets of X. Let X be a topological space and $F : (\Omega, \Sigma) \to 2^X$ be a map. F is said to be measurable (respectively, weakly measurable) if $F^{-1}(B) = \{\omega \in \Omega, F(\omega) \cap B \neq \varnothing\} \in \Sigma$ for each closed (respectively, open) subset B of X. The map F is said to have a measurable graph if Graph $F := \{(\omega, y) \in \Omega \times X : y \in F(\omega)\} \in \Sigma \otimes \mathscr{B}(X)$. A function $f : \Omega \to X$ is a measurable selection of F if f is a measurable function such that $f(\omega) \in F(\omega)$ for all $\omega \in \Omega$.

If (Ω, Σ), (Φ, Γ) are measurable spaces and Y is a topological space, then a map $F : \Omega \times \Phi \to 2^Y$ is called jointly measurable (respectively, jointly weakly measurable) if, for every closed (respectively, open) subset B of Y, $f^{-1}(B) \in \Sigma \otimes \Gamma$. In the case $\Phi = X$, a topological space, then it is understood that Γ is the Borel σ-algebra $\mathscr{B}(X)$. Let μ be a positive measure on a measurable space (Ω, Σ). A subset N of Ω is called a μ-negligible subset of Ω if there exists a measurable subset $A \in \Sigma$ such that $N \subset A$ and $\mu(A) = 0$. The μ-completion of Σ, denoted by Σ_μ, is the σ-algebra on Ω generated by Σ and the μ-negligible subsets of Ω. The measure μ admits a unique extension $\hat{\mu}$ to Σ_μ. The σ-algebra Σ is said to be μ-complete if $\Sigma = \Sigma_\mu$. Also, (Ω, Σ) is a complete measurable space if there is a positive measure μ on (Ω, Σ) such that $\Sigma_\mu = \Sigma$.

A topological space X is: (i) a Polish space if X is separable and metrizable by a complete metric; and (ii) a Suslin space if X is a Hausdorff topological space and the continuous image of a Polish space. A Suslin subset in a topological space is a subset which is a Suslin space. "Suslin" sets play very important roles in measurable selection theory. We remark that if X_1 and X_2 are Suslin spaces, then $\mathscr{B}(X_1 \times X_2) = \mathscr{B}(X_1) \otimes \mathscr{B}(X_2)$ (e.g., see [24, p.113]).

Denote by $\mathscr{J}$ and $\mathscr{F}$ the sets of infinite and finite sequences of positive integers respectively. Let $\mathscr{G}$ be a family of sets and $F : \mathscr{F} \to \mathscr{G}$ be a map. For each $\sigma = (\sigma_i)_{i=1}^\infty \in \mathscr{J}$ and $n \in \mathbf{N}$, we shall denote $(\sigma_1, \ldots, \sigma_n)$ by σ/n; then $\bigcup_{\sigma \in \mathscr{J}} \bigcap_{n=1}^\infty F(\sigma/n)$ is said to be obtained from $\mathscr{G}$ by the Suslin operation. Now if every set obtained from $\mathscr{G}$ in this way is also in $\mathscr{G}$, then $\mathscr{G}$ is called a Suslin family.

Note that, if μ is an outer measure on a measurable space (Ω, Σ), then Σ is a Suslin family (see [25, p. 50]). In particular, if (Ω, Σ) is a complete measurable space, then Σ is a Suslin family (for more details, see [31, p. 864]). It also implies that the σ-algebra Σ of Lebesgue measurable subsets of $[0, 1]$ is a Suslin family.

If X is a topological space, let $C(X) = \{A \in 2^X : A$ is a closed subset of $X\}$. Let X, Y be topological spaces, (Ω, Σ) be a measurable space, and $F : \Omega \times X \to 2^Y$ be a map. Then F is: (i) a random operator if for each fixed $x \in X$, the map $F(\cdot, x) : \Omega \to 2^Y$ is a measurable map; (ii) lower semicontinuous (respectively, upper semicontinuous, continuous) if for each fixed $\omega \in \Omega$, $F(\omega, \cdot) : X \to 2^Y$ is lower semicontinuous (respectively, upper semicontinuous, continuous); and (iii) random upper semicontinuous (respectively, random lower semicontinuous, random continuous) if F is both a random operator and an upper semicontinuous (respectively, lower semicontinuous, continuous) map.

Let (Ω, Σ) be a measurable space, X be a subset of a topological space Y, and $F : \Omega \times X \to 2^Y$ be a map. Note that for each $\omega \in \Omega$, unless $F(\omega, X) \subset X$ for all $x \in X$, $F(\omega, \cdot)$ is a non-self-map. The single-valued map $\varphi : \Omega \to X$ is said to be (i) a deterministic fixed point of F if $\varphi(\omega) \in F(\omega, \varphi(\omega))$ for all $\omega \in \Omega$; and (ii) a random fixed point of F if φ is a measurable map and $\varphi(\omega) \in F(\omega, \varphi(\omega))$ for all $\omega \in \Omega$. It should be noted here that some authors define a random fixed point of F to be a measurable map φ such that $\varphi(\omega) \in F(\omega, \varphi(\omega))$ for almost every $\omega \in \Omega$ (e.g., see [8, 9, 20, 21, 23]).

We remark that if the map $F : \Omega \times X \to 2^Y$ has a random fixed point, then, for each $\omega \in \Omega$, $F(\omega, \cdot)$ has a fixed point in X. The converse is not true as the following simple example illustrates.

Example 1.1. Let $\Omega = X = [0, 1]$, Σ be the σ-algebra of Lebesgue measurable subsets of $[0, 1]$, and A be a non-Lebesgue measurable subset of $[0, 1]$ (see [22, p. 63]). Define $F : \Omega \times X \to 2^X$ for any $(\omega, x) \in \Omega \times X$ by

$$F(\omega, x) := \begin{cases} \{0\} & \text{for } \omega \in A, \\ \{1\} & \text{for } \omega \notin A. \end{cases}$$

Define $\varphi : \Omega \to X$ by

$$\varphi(\omega) := \begin{cases} 1 & \text{if } \omega \notin A, \\ 0 & \text{if } \omega \in A. \end{cases}$$

Then φ is not a measurable map and is the only function such that

$$\varphi(\omega) \in F(\omega, \varphi(\omega))$$

for all $\omega \in \Omega$. Hence F has no random fixed point.

The following measurable selection theorem due to Lesse in [16, Corollary to Theorem 7] which generalizes the earlier result of Aumann's Selection Theorem in [2] plays a very important role:

Theorem A. *Let (Ω, Σ) be a measurable space, Σ be a Suslin family, X be a Suslin space, and $F : (\Omega, \Sigma) \to 2^X$ be a map such that $\operatorname{Graph} F \in \Sigma \otimes \mathscr{B}(X)$. Then there exists a sequence $\{\gamma_n\}_{n=1}^{\infty}$ of measurable selections of F such that for each $\omega \in \Omega$ the set $\{\gamma_n(\omega), n = 1, 2, \dots\}$ is dense in $F(\omega)$.*

For convenience, we shall list below Theorem 1.3.3 of Chow and Teicher [6, p. 8]:

Lemma 1.2. *For every nonempty family $\mathscr{D}$ of subsets of Ω and every nonempty set $A \subset \Omega$, $\sigma_{\Omega}(\mathscr{D}) \cap A = \sigma_A(\mathscr{D} \cap A)$.*

2. MAIN RESULTS

In order to consider the existence of random fixed points for random non-self-maps, we first have the following:

Lemma 2.1. *If (Ω, Σ) is a measurable space and X_0 is a nonempty subset of a topological space X, then $\Sigma \otimes \mathscr{B}(X) \cap (\Omega \times X_0) = \Sigma \otimes \mathscr{B}(X_0)$.*

Proof. Let τ_X denote the topology on X and $\tau_{X_0} = \tau_X \cap X_0$, the relative topology of τ_X to X_0. First we note that by Lemma 1.2

$$\mathscr{B}(X) \cap X_0 = \sigma_X(\tau_X) \cap X_0 = \sigma_{X_0}(\tau_X \cap X_0) = \sigma_{X_0}(\tau_{X_0}) = \mathscr{B}(X_0),$$

and it follows that

$$(\Sigma \times \mathscr{B}(X)) \cap (\Omega \times X_0) = \Sigma \times (\mathscr{B}(X) \cap X_0) = \Sigma \times \mathscr{B}(X_0).$$

By Lemma 1.2 again,

$$\Sigma \otimes \mathscr{B}(X) \cap (\Omega \times X_0) = \sigma_{\Omega \times X}(\Sigma \times \mathscr{B}(X)) \cap (\Omega \times X_0)$$
$$= \sigma_{\Omega \times X_0}((\Sigma \times \mathscr{B}(X)) \cap (\Omega \times X_0)) = \sigma_{\Omega \times X_0}(\Sigma \times \mathscr{B}(X_0)) = \Sigma \otimes \mathscr{B}(X_0). \quad \square$$

Now we shall prove the following useful random fixed point theorem for random non-self-maps:

Theorem 2.2. *Let* (Ω, Σ) *be a measurable space,* Σ *be a Suslin family,* X *be a topological space, and* X_0 *be a nonempty Suslin subset of* X. *Suppose* $K : \Omega \to 2^X$ *has a measurable graph and* $F : \Omega \times X_0 \to 2^X$ *is such that:*
  (a) *for each* $w \in \Omega$, *the set* $\{x \in X_0 : x \in F(\omega, x)\} \cap K(\omega) \neq \varnothing$, *and*
  (b) Graph $F \in \Sigma \otimes \mathscr{B}(X_0 \times X)$.
Then F *has a family* $\{\gamma_n\}_{n=1}^{\infty}$ *of random fixed points such that for each* $w \in \Omega$
  (i) $\gamma_n(\omega) \in F(\omega, \gamma_n(\omega)) \cap K(\omega)$, $n = 1, 2, \ldots$, *and*
  (ii) $\{\gamma_n(\omega)\}_{n=1}^{\infty}$ *is a dense subset of* $\{x \in X_0 : x \in F(\omega, x)\} \cap K(\omega)$.

Proof. For each $\omega \in \Omega$, define

$$\Phi(\omega) = \{(x, x) \in X_0 \times X_0 : x \in F(\omega, x) \cap K(\omega)\};$$

then $\Phi(\omega)$ is a nonempty subset of $X_0 \times X_0$ by (a). Thus $\Phi : \Omega \to 2^{X_0 \times X_0}$. Let $\Delta = \{(x, x) : x \in X_0\}$; then $\Delta \in \mathscr{B}(X_0 \times X_0)$ since Δ is closed in X_0 as X_0 is Hausdorff so that $\Omega \times \Delta \in \Sigma \otimes \mathscr{B}(X_0 \times X_0)$. Since Graph $F \in \Sigma \otimes \mathscr{B}(X_0 \times X)$ by (b), (Graph $F) \cap (\Omega \times X_0 \times X_0) \in \Sigma \otimes \mathscr{B}(X_0 \times X) \cap (\Omega \times X_0 \times X_0) = \Sigma \otimes \mathscr{B}(X_0 \times X_0)$ by Lemma 2.1. By assumption, Graph $K \in \Sigma \otimes \mathscr{B}(X)$ so that (Graph $K) \times X_0 \in (\Sigma \otimes \mathscr{B}(X)) \otimes \mathscr{B}(X_0) \subset \Sigma \otimes \mathscr{B}(X \times X_0)$ and hence by Lemma 2.1 again, (Graph $K) \times X_0 \cap (\Omega \times X_0 \times X_0) \in \Sigma \otimes \mathscr{B}(X \times X_0) \cap (\Omega \times X_0 \times X_0) = \Sigma \otimes \mathscr{B}(X_0 \times X_0)$. It follows that

$$\text{Graph } \Phi = (\Omega \times \Delta) \cap (\text{Graph } F) \cap ((\text{Graph } K) \times X_0)$$
$$= (\Omega \times \Delta) \cap ((\text{Graph } F) \cap (\Omega \times X_0 \times X_0))$$
$$\cap (((\text{Graph } K) \times X_0) \cap (\Omega \times X_0 \times X_0))$$
$$\in \Sigma \otimes \mathscr{B}(X_0 \times X_0).$$

By Theorem A, there exists a sequence $\{\gamma_n'\}_{n=1}^{\infty}$ of measurable selections of Φ, where $\gamma_n' : \Omega \to X_0 \times X_0$, such that, for each $\omega \in \Omega$, $\{\gamma_n'(\omega)\}_{n=1}^{\infty}$ is dense in $\Phi(\omega)$. But then for each $n = 1, 2, \ldots$, there exists $\gamma_n : \Omega \to X_0$ such that $\gamma_n'(\omega) = (\gamma_n(\omega), \gamma_n(\omega))$ for all $\omega \in \Omega$. Now, if A is a closed subset of X_0, then $A \times A$ is a closed subset of $X_0 \times X_0$; thus for each $n = 1, 2, \ldots$,

$$\gamma_n^{-1}(A) = \{\omega \in \Omega : \gamma_n(\omega) \in A\} = \{\omega \in \Omega : \gamma_n'(\omega) \in (A \times A)\} \in \Sigma$$

and hence each γ_n is measurable. Moreover, for each $\omega \in \Omega$, since $\{\gamma_n'(\omega)\}_{n=1}^{\infty}$ is dense in $\Phi(\omega)$, $\{\gamma_n(\omega)\}_{n=1}^{\infty}$ is also dense in $\{x \in X_0 : x \in F(\omega, x) \cap K(\omega)\}$. $\quad \square$

We remark that in the proof of Theorem 2.2, we applied the measurable selection theorem to the mapping Φ defined by $\Phi(\omega) = \{(x, x) \in X_0 \times X_0 : x \in F(\omega, x) \cap K(\omega)\}$ which is different than the usual technique of applying the

measurable selection theorem to the set-valued mapping Φ defined by $\Phi(\omega) = \{x \in X : x \in F(\omega, x)\}$ (e.g., see [8, 14, 21, 23, 32]).

When X is a Polish space and $X_0 = X$, Theorem 2.2 has been given in [20, Theorem 1]. We remark that in Theorem 2.2, if we define $F_1(\omega, x) = F(\omega, x) \cap X_0$ for each $(\omega, x) \in \Omega \times X_0$, since Σ is a Suslin family and X_0 is a Suslin space, F_1 is weakly measurable (see, e.g., [12, Corollary 5.1]); if in addition, F_1 is closed-valued, then Graph $F_1 \in \Sigma \otimes \mathscr{B}(X_0 \times X_0)$ (see, e.g., [31, Theorem 4.2]). Moreover, when F_1 is single-valued, then F_1 is measurable if and only if Graph $F_1 \in \Sigma \otimes \mathscr{B}(X_0 \times X_0)$. In this case, Theorem 2.2 was also given by [15]. Thus Theorem 2.2 unifies and generalizes the corresponding results of [15] and [20] to set-valued and non-self-maps in Suslin spaces.

By taking $K(\omega) = X$ for all $\omega \in \Omega$ in Theorem 2.2 we have:

Theorem 2.3. *Let (Ω, Σ) be a measurable space, Σ be a Suslin family, X be a topological space, and X_0 be a Suslin subset of X. Suppose $F : \Omega \times X_0 \to 2^X$ is such that Graph $F \in \Sigma \otimes \mathscr{B}(X_0 \times X)$. Then F has a random fixed point if and only if F has a deterministic fixed point, i.e., for each $\omega \in \Omega$, $F(\omega, \cdot)$ has a fixed point in X_0.*

We remark that if $X_0 = X$, Theorem 2.2 (respectively, Theorem 2.3) reduces to Theorem 2.1 (respectively, Theorem 2.2) in [29].

Example 1.1 shows that the condition "Graph $F \in \Sigma \otimes \mathscr{B}(X_0 \times X)$" cannot be omitted in Theorem 2.3.

The following result is Theorem 2 of [20]; for completeness we include the proof:

Lemma 2.4. *Let (Ω, Σ) be a measurable space and X and Y be separable metric spaces. Suppose $F : \Omega \times X \to C(Y)$ is such that for each $(\omega, x) \in \Omega \times X$*
  (i) *$F(\cdot, x) : \Omega \to 2^Y$ is weakly measurable, and*
  (ii) *$F(\omega, \cdot) : X \to 2^Y$ is continuous.*
Then F is jointly weakly measurable and Graph $F \in (\Sigma \otimes \mathscr{B}(X)) \otimes \mathscr{B}(Y)$.

Proof. For each $z \in Y$, define $d_z(\omega, x) = d(z, F(\omega, x))$ for all $(\omega, x) \in \Omega \times X$, where d is the given metric on the space Y. Note that for each $\omega \in \Omega$, the continuity of $x \mapsto F(\omega, x)$ implies the continuity of $x \mapsto d_z(\omega, x)$.

Next for an arbitrarily fixed $x \in X$, since $F(\cdot, x) : \Omega \to 2^Y$ is weakly measurable, $d_z(\cdot, x)$ is measurable by [11, Theorem 3.3]. Thus by [11, Theorem 6.1] (also [4, Lemma III-14, p. 70]), the map $(\omega, x) \mapsto d_z(\omega, x)$ is jointly measurable, so that by [11, Theorem 3.3] again, the map $F : \Omega \times X \to 2^Y$ is jointly weakly measurable. Finally, as F is closed-valued, by [11, Theorem 3.3], Graph $F \in (\Sigma \otimes \mathscr{B}(X)) \otimes \mathscr{B}(Y)$. $\square$

Theorem 2.5. *Let (Ω, Σ) be a measurable space, Σ be a Suslin family, and X_0 be a Suslin subset of a separable metric space X. Suppose $F : \Omega \times X_0 \to C(X)$ is a random continuous map. Then F has a random fixed point if and only if, for each $\omega \in \Omega$, $F(\omega, \cdot)$ has a fixed point in X_0.*

Proof. By assumption, for each $x \in X_0$, $F(\cdot, x) : \Omega \to 2^X$ is measurable; as X is metrizable, [11, Proposition 2.1] shows that $F(\cdot, x)$ is weakly measurable. Hence by Lemma 2.4, Graph $F \in (\Sigma \otimes \mathscr{B}(X_0)) \otimes \mathscr{B}(X) \subset \Sigma \otimes \mathscr{B}(X_0 \times X)$. The conclusion now follows from Theorem 2.3. $\square$

We remark that if $X_0 = X$, Theorem 2.5 reduces to [29, Theorem 2.6]. Also, Theorem 2.5 generalizes [32, Theorem 1(i)] of Xu in the following ways: (i) F is a set-valued random continuous map (instead of a single-valued nonexpansive random map); and (ii) X_0 is a Suslin subset of a separable metric space X (instead of $X = X_0$ and is a closed bounded convex separable subset of a reflexive Banach space).

In view of Theorems 2.3 and 2.5, every fixed point theorem for single- or set-valued maps gives rise to some random fixed point theorems; this idea was implicitly utilized, for example, by Bocsan [3], Chang [5], Engl [8, 9], Itoh [13, 14], Kucia and Nowak [15], Lin [17], Nowak [20], Rybinski [23], Sehgal and Singh [26], Sehgal and Waters [28], Xu [32], etc., and explicitly utilized by Tan and Yuan [29]. As an illustration, we have:

Theorem 2.6. *Let (Ω, Σ) be a measurable space and Σ be a Suslin family. Let S be a nonempty closed convex subset of a separable Hilbert space X and $F : \Omega \times S \to X$ be a random map such that, for each $\omega \in \Omega$, $F(\omega, \cdot)$ has bounded range and $F(\omega, \cdot)$ is either a continuous densifying map [10] or a nonexpansive map satisfying any of the following conditions:*

(1) For each $x \in S$, there is a number λ (real or complex, depending on whether the vector space X is real or complex) such that $|\lambda| < 1$ and $\lambda x + (1 - \lambda)F(\omega, x) \in S$.

(2) For each $x \in S$ with $x \neq F(\omega, x)$, there exists $y \in I_S(x) := \{x + c(z - x) : z \in S, c > 0\}$ such that $\|y - F(\omega, x)\| < \|x - F(\omega, x)\|$.

(3) $F(\omega, \cdot)$ is weakly inward (i.e., for each $x \in S$, $F(\omega, x) \in \overline{I_S(x)}$, the closure of $I_S(x)$ in X).

(4) For each u in the boundary of S, if $\|F(\omega, u) - u\| = \inf\{\|F(\omega, u) - s\| : s \in S\}$, then $F(\omega, u) = u$.

(5) For each x in the boundary of S, there exists $y \in S$, such that $\|F(\omega, x) - y\| \leq \|x - y\|$.
Then F has a random fixed point.

Proof. By [18, Corollary 4], F has a deterministic fixed point. By Theorem 2.5, F has a random fixed point. □

Theorem 2.6 extends [17, Theorem 6′]. As another illustration, we have the following

Theorem 2.7. *Let (Ω, Σ) be a measurable space and Σ be a Suslin family. Let S be a nonempty bounded closed convex subset of a separable uniformly convex Banach space X and $F : \Omega \times S \to 2^X$ be a random map such that for each $\omega \in \Omega$*

(i) $F(\omega, x)$ is compact for each $x \in S$;

(ii) $F(\omega, \cdot)$ is nonexpansive, i.e., $D(F(\omega, x), F(\omega, y)) \leq \|x - y\|$ for all $x, y \in S$ where D is the Hausdorff metric induced by the norm $\|\cdot\|$ on X;

(iii) $F(\omega, x) \subset I_S(x)$ for each $x \in S$.
Then F has a random fixed point.

Proof. For each $\omega \in \Omega$, $F(\omega, \cdot)$ has a fixed point in S by [7, Theorem 4]; thus F has a deterministic fixed point. Also, for each $\omega \in \Omega$, since $F(\omega, \cdot)$ is nonexpansive, $F(\omega, \cdot)$ is continuous; thus F is also random continuous. By Theorem 2.5, F has a random fixed point. □

We remark that throughout this paper, the σ-algebra Σ is always assumed to be a Suslin family. For related results where Σ is not assumed to be a Suslin family, we refer to [29, Theorem 2.8 and 3.6; 30].

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