

ON DEVIATIONS OF THE SAMPLE MEAN

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1. Introduction. Let X_1, X_2, \dots be a sequence of independent and identically distributed random variables. Let a be a constant, $-\infty < a < \infty$, and for each $n = 1, 2, \dots$ let

$$(1) \quad p_n = P\left(\frac{X_1 + \dots + X_n}{n} \geq a\right).$$

It is assumed throughout the paper that the distribution of X_1 and the given constant a satisfy the conditions stated in the following paragraph. These conditions imply that $p_n > 0$ for each n , and that $p_n \rightarrow 0$ as $n \rightarrow \infty$. The object of the paper is to obtain an estimate of p_n , say q_n , which is precise in the sense that

$$(2) \quad q_n/p_n = 1 + o(1) \quad \text{as } n \rightarrow \infty.$$

Let t be a real variable, and let $\varphi(t)$ denote the moment generating function (m.g.f.) of X_1 , i.e., $\varphi(t) = E(e^{tX_1})$, $0 < \varphi \leq \infty$. Define

$$(3) \quad \psi(t) = e^{-at}\varphi(t).$$

Let T denote the set of all values t for which $\varphi(t) < \infty$. We suppose that $P(X_1 = a) \neq 1$, that T is a non-degenerate interval, and that there exists a positive τ in the interior of T such that $\psi(\tau) = \inf_t \{\psi(t)\} = \rho$ (say). These conditions are satisfied if, for example, $\varphi(t) < \infty$ for all t , $E(X_1) = 0$, $a > 0$, and $P(X_1 > a) > 0$. In any case, τ and ρ are uniquely determined by

$$(4) \quad \frac{\varphi'(\tau)}{\varphi(\tau)} = a \quad \text{and} \quad \rho = \psi(\tau),$$

where $\varphi' = d\varphi/dt$, and we have $0 < \rho < 1$.

There are three separate cases to be considered.

Case 1: The distribution function (d.f.) of X_1 is absolutely continuous, or, more generally, this d.f. satisfies Cramér's condition (C) [1, p. 81].

Case 2: X_1 is a lattice variable, i.e., there exist constants x_0 and $d > 0$ such that X_1 is confined to the set $\{x_0 + rd: r = 0, \pm 1, \pm 2, \dots\}$ with probability one.

Case 3: Neither Case 1 nor Case 2 obtains.

We can now state

THEOREM 1. *There exists a sequence b_1, b_2, \dots of positive numbers b_n such that*

$$(5) \quad p_n = \frac{\rho^n}{(2\pi n)^{\frac{1}{2}}} b_n [1 + o(1)], \quad \log b_n = O(1)$$

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as $n \rightarrow \infty$. In Cases 1 and 3, b_n is independent of n . This last also holds in Case 2 if $P(X_1 = a) > 0$.

The proof of Theorem 1, and of Theorem 2 below, is given in Sections 2-5. The present determination of b_n is given by (4), (9) and (33) in Cases 1 and 3, and by (4), (8), (37), (38) and (46) in Case 2. The following refinements of Theorem 1 are available in Cases 1 and 2:

THEOREM 2. (Cases 1 and 2). *For each $j = 1, 2, \dots$ there exists a bounded (possibly constant) sequence $c_{j,1}, c_{j,2}, \dots$ such that, for any given positive integer k ,*

$$(6) \quad p_n = \frac{\rho^n}{(2\pi n)^{\frac{1}{2}}} b_n \left[1 + \frac{c_{1,n}}{n} + \frac{c_{2,n}}{n^2} + \dots + \frac{c_{k,n}}{n^k} \right] \left[1 + O\left(\frac{1}{n^{k+1}}\right) \right]$$

as $n \rightarrow \infty$.

The sequences $\{c_{j,n}\}$ are given explicitly for Cases 1 and 2 in Sections 3 and 4 respectively. It would be interesting to know whether (6) holds in Case 3 as well, perhaps with the $\{c_{j,n}\}$ determined according to the formula for Case 1.

Estimates in the form (5) or (6) were first obtained by Cramér [2, pp. 20-21] in the case when X_1 has an absolutely continuous component (so that Case 1 obtains). Cramér showed that in the latter case (6) holds for every k (with b_n and each $c_{j,n}$ independent of n), and determined b_n . Our method of proof in the general case (cf. Sections 2-5) is essentially a variant or extension of Cramér's method. Case 2 was treated recently by Blackwell and Hodges [3] by a different method. It is shown in [3] that (6) holds for $k = 1$ in Case 2, under the restriction on n and a that $P(X_1 + \dots + X_n = na) > 0$ for every admissible n , and the requisite b_n and $c_{1,n}$ (which are then independent of n) are determined explicitly. Some other references bearing on the problem under consideration are [4], [5] and [6].

In the following Section 2 it is shown that p_n can be expressed as $\rho^n I_n$, where I_n is a certain integral; $0 < I_n < 1$, and $I_n = O(n^{-1})$ as $n \rightarrow \infty$. I_n can be estimated by application of certain refinements [1], [7] of the central limit theorem. This estimation of I_n is carried out in Sections 3, 4 and 5 for Cases 1, 2 and 3 respectively. It may be added here that, as was pointed out in [2], direct application of the central limit theorem (or refinements thereof) to p_n defined by (1) does not, in general, yield approximations q_n which satisfy (2).

In Section 6 we describe certain numerical approximations to p_n which are suggested by Theorems 1 and 2 and their proofs.

2. Lemmas. Let $Y_1 = X_1 - a$, and let F be the (left-continuous) distribution function (d.f.) of Y_1 , $F(y) = P(Y_1 < y)$. Let G be defined by $G(z) = \int_{-\infty < y < z} \rho^{-1} e^{\tau y} dF(y)$. Since $E(e^{\tau Y_1}) = \psi(\tau) = \rho$, it is clear that G is a probability d.f.. Let Z_1 be a random variable distributed according to G .

LEMMA 1. *The m.g.f. of Z_1 exists in a neighborhood of the origin. We have*

$$(7) \quad E(Z_1) = 0, \quad 0 < \text{Var}(Z_1) < \infty.$$

PROOF. Let $\xi(t)$ denote the m.g.f. of Z_1 . Then $\xi(t) = \psi(\tau + t)/\rho$ for all t ,

by (3) and the definition of Z_1 . Since $\psi(t) < \infty$ in a neighborhood of $t = \tau$, it follows that $\xi(t) < \infty$ in a neighborhood of $t = 0$. Consequently, $E|Z_1|^r < \infty$ for $r = 1, 2, 3, \dots$ and $E(Z_1^r) = \{d^r \xi/dt^r\}_{t=0}$. In particular, $E(Z_1) = \{d\xi/dt\}_{t=0} = \psi'(\tau)/\rho = 0$, since $\psi(t)$ is minimum at $t = \tau$, and τ is in the interior of T . It remains to show that $\text{Var}(Z_1) > 0$. Suppose to the contrary that $\text{Var}(Z_1) = 0$; then $P(Z_1 = 0) = 1$; hence $P(Y_1 = 0) = 1$, i.e., $P(X_1 = a) = 1$, which is contrary to our assumptions. This completes the proof.

Let $\text{Var}(Z_1)$ be denoted by σ^2 . It follows from the preceding paragraph and (4) that

$$(8) \quad \sigma^2 = \frac{\varphi''(\tau)}{\varphi(\tau)} - a^2.$$

Define

$$(9) \quad \alpha = \sigma\tau, \quad (0 < \alpha < \infty).$$

Let Z_1, Z_2, \dots be a sequence of independent and identically distributed random variables. For each n , let

$$(10) \quad U_n = \frac{Z_1 + \dots + Z_n}{n^{1/2}\sigma}$$

and

$$(11) \quad H_n(x) = P(U_n < x), \quad (-\infty < x < \infty).$$

LEMMA 2. $p_n = \rho^n I_n$, where

$$(12) \quad I_n = n^{1/2} \alpha \int_0^\infty e^{-n^{1/2} \alpha x} [H_n(x) - H_n(0)] dx.$$

PROOF. Let $Y_j = X_j - a$ for $j = 1, 2, \dots, n$. Then

$$\begin{aligned} p_n &= P(Y_1 + \dots + Y_n \geq 0) && \text{by (1)} \\ &= \int \dots \int_{y_1 + \dots + y_n \geq 0} dF(y_1) \dots dF(y_n) \\ (13) \quad &= \rho^n \int \dots \int_{z_1 + \dots + z_n \geq 0} e^{-\tau(z_1 + \dots + z_n)} dG(z_1) \dots dG(z_n) \\ &= \rho^n \int_{0 \leq x < \infty} e^{-n^{1/2} \alpha x} dH_n(x) && \text{by (9), (10), (11)} \\ &= \rho^n I_n^* \text{ say.} \end{aligned}$$

It follows by integration by parts that I_n^* defined in (13) is equal to I_n , and this completes the proof.

A theorem of Chernoff [4] states that $p_n \leq \rho^n$ for every n , and that for any

given positive $\rho_0 < \rho$, we have $p_n \geq \rho_0^n$ for all sufficiently large n . A simple proof of Chernoff's theorem can be given as follows. Since $0 \leq H_n(x) - H_n(0) \leq 1$ for every n and $x \geq 0$, we have $I_n \leq 1$ and hence $p_n \leq \rho^n$ for every n , by Lemma 2. To establish the second part of the theorem, we note first that $\lim_{n \rightarrow \infty} H_n(x) = \Phi(x)$ for every x , where

$$(14) \quad \Phi(x) = \int_{-\infty}^x (2\pi)^{-\frac{1}{2}} e^{-\frac{1}{2}t^2} dt \quad (-\infty < x < \infty),$$

by (7), (10), (11) and the central limit theorem. Let ϵ be a positive constant. Then

$$\begin{aligned} I_n &\geq n^{\frac{1}{2}} \alpha \int_{\epsilon}^{\infty} e^{-n^{\frac{1}{2}}\alpha x} [H_n(x) - H_n(0)] dx \\ &\geq [H_n(\epsilon) - H_n(0)] n^{\frac{1}{2}} \alpha \int_{\epsilon}^{\infty} e^{-n^{\frac{1}{2}}\alpha x} dx \\ &= [H_n(\epsilon) - H_n(0)] e^{-n^{\frac{1}{2}}\alpha \epsilon}. \end{aligned}$$

Hence $\liminf_{n \rightarrow \infty} \{n^{-\frac{1}{2}} \log I_n\} \geq -\alpha\epsilon$. Since $I_n \leq 1$ for every n , and since ϵ is arbitrary, it follows that $n^{-\frac{1}{2}} \log I_n = o(1)$. Hence $n^{-1} \log p_n = \log \rho + o(1)$, by Lemma 2, and this is equivalent to the conclusion desired.

The preceding argument depends only on the central limit theorem. In the following sections we estimate I_n more accurately by substituting the expansions of $H_n(x)$ due to Cramér [1] and Esseen [7] in the right side of (12). The remainder of this section is concerned with preparations for this application of the Cramér-Esseen expansions. Almost all the considerations of the following paragraphs are well known, and we include them here only for the sake of completeness.

Let $\eta(w)$ denote the m.g.f. of Z_1/σ . According to Lemma 1, $\eta < \infty$ in a neighborhood of $w = 0$. For $j = 2, 3, \dots$ let λ_j be defined by

$$(15) \quad \lambda_2 = \frac{1}{2}; \quad \lambda_j = (j!\sigma^j)^{-1} (d^j/dt^j)\{\log \varphi(t)\}_{t=\tau} \quad (j = 3, 4, \dots).$$

It should be noted that $j!\lambda_j$ is the j th cumulant of the distribution of Z_1/σ . The m.g.f. of U_n , with U_n defined by (10), is

$$[\eta(w/n^{\frac{1}{2}})]^n = \exp \left[n \sum_{j=2}^{\infty} \lambda_j (w/n^{\frac{1}{2}})^j \right].$$

Clearly, $[\eta(w/n^{\frac{1}{2}})]^n \exp(-w^2/2)$ is analytic in a domain independent of n , and can be expanded there as a power series in w . By regrouping the terms of this series according to powers of n we shall have

$$(16) \quad [\eta(w/n^{\frac{1}{2}})]^n e^{-w^2/2} = \sum_{j=0}^{\infty} n^{-\frac{1}{2}j} P_j(w)$$

where the P_j are polynomials. P_j is of degree $3j$, and P_j is even or odd according as j is even or odd. The first few polynomials are

$$\begin{aligned}
 P_0(w) &= w^0 \equiv 1, \\
 P_1(w) &= \lambda_3 w^3, \\
 (17) \quad P_2(w) &= \lambda_4 w^4 + \frac{1}{2} \lambda_3^2 w^6, \\
 P_3(w) &= \lambda_5 w^5 + \lambda_3 \lambda_4 w^7 + \frac{1}{6} \lambda_3^3 w^9, \\
 P_4(w) &= \lambda_6 w^6 + \left(\frac{1}{2} \lambda_4^2 + \lambda_3 \lambda_5\right) w^8 + \lambda_3^2 \lambda_4 w^{10} + \frac{1}{24} \lambda_3^4 w^{12}.
 \end{aligned}$$

Write $\Phi^{(0)}(x) = \Phi(x)$ and $\Phi^{(r)}(x) = (d^r/dx^r)\Phi(x)$ for $r = 1, 2, \dots$, where Φ is given by (14). Let $P_j(-\Phi)$ denote the function of x obtained by replacing w^r with $(-1)^r \Phi^{(r)}(x)$ in the polynomial $P_j(w)$. It is clear that each $P_j(-\Phi)$ is absolutely continuous and of bounded variation in $(-\infty, \infty)$. It should also be noted that $P'_j(-\Phi)$ is square integrable with respect to Lebesgue measure.

In the following, for any function $K(x)$ of bounded variation in $(-\infty, \infty)$, we denote the c.f. of K by $\chi(t | K)$, i.e.,

$$(18) \quad \chi(t | K) = \int_{-\infty}^{\infty} e^{itx} dK(x)$$

for every real t . If K is absolutely continuous, χ is, of course, $(2\pi)^{\frac{1}{2}}$ times the Fourier transform of K' . The reader may refer to [8, Chapters I-III] for such elements of Fourier transform theory as are used in this paper.

LEMMA 3. For every j, t , and x

$$(19) \quad \chi(t | P_j(-\Phi)) = P_j(it) e^{-\frac{1}{2}t^2}$$

and

$$(20) \quad P'_j(-\Phi) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} e^{-itx} P_j(it) d\Phi(t).$$

PROOF. As is pointed out in [1, p. 49], we have

$$(21) \quad \chi(t | \Phi^{(r)}) = (-it)^r e^{-\frac{1}{2}t^2}$$

for $r = 0, 1, \dots$. Suppose, for given j , that $P_j(w) = \sum_{r=0}^N a_r w^r$, where the a_r and N are constants (depending on j). Then $P_j(-\Phi) = \sum_{r=0}^N a_r (-1)^r \Phi^{(r)}(x)$; hence the left side of (19) equals $\sum_0^N a_r (-1)^r \chi(t | \Phi^{(r)})$; (19) now follows from (21). The relation (20) follows from (19) by the inversion formula for the Fourier transform, since $dP_j(-\Phi) = P'_j(-\Phi) dx$, and $d\Phi(t) = (2\pi)^{-\frac{1}{2}} e^{-\frac{1}{2}t^2} dt$.

A probability d.f. $K(x)$ is said to satisfy condition (C) if

$$\limsup_{|t| \rightarrow \infty} |\chi(t | K)| < 1.$$

In the following lemma the F_j are arbitrary probability d.fs.

LEMMA 4. If F_1 satisfies (C), and if F_1 is absolutely continuous with respect to F_2 , then F_2 also satisfies (C).

PROOF. In this proof, for any probability d.f. K let K^* denote the symmetrized d.f. defined by $K^*(x) = \int_{-\infty}^{\infty} K(x+y) dK(y)$. We then have

$$\chi(t | K^*) = \int_{-\infty}^{\infty} \cos(tx) dK^* = |\chi(t | K)|^2$$

for all t .

Suppose, contrary to the lemma, that there exists a sequence $\{t_j : j = 1, 2, \dots\}$ such that $|t_j| \rightarrow \infty$ and $|\chi(t_j | F_2)| \rightarrow 1$ as $j \rightarrow \infty$. It then follows from the above paragraph with $K = F_2$ that $\int_{-\infty}^{\infty} \cos(t_j x) dF_2^* \rightarrow 1$. Hence $\cos(t_j x) \rightarrow 1$ in F_2^* -measure. Since F_2 -measure dominates F_1 -measure, it is easily seen that F_2^* -measure dominates F_1^* -measure. Consequently, $\cos(t_j x) \rightarrow 1$ in F_1^* -measure. It now follows from the above paragraph with $K = F_1$ that $|\chi(t_j | F_1)|^2 \rightarrow 1$ as $j \rightarrow \infty$, which is impossible. This completes the proof.

We conclude this section with a description of the functions $S_1(x), S_2(x)$ which occur in the Euler-Maclaurin sum formulae, and which are required in the analysis of Case 2. It is convenient to define S_1 as follows:

$$(22) \quad S_1(x) = \frac{1}{2} - x \quad \text{for } 0 < x \leq 1; \quad S_1(x + 1) \equiv S_1(x).$$

For $j \geq 2, S_j$ may be defined as

$$(23) \quad S_j(x) = \begin{cases} \frac{1}{2^{j-1}} \sum_{r=1}^{\infty} \frac{\cos(2\pi r x)}{(\pi r)^j} & (j \text{ even}) \\ \frac{1}{2^{j-1}} \sum_{r=1}^{\infty} \frac{\sin(2\pi r x)}{(\pi r)^j} & (j \text{ odd}). \end{cases}$$

Each S_j is a bounded and periodic function; S_j is absolutely continuous for $j \geq 2$; and at each non-integral x we have

$$(24) \quad S'_1(x) = -1, \quad S'_{j+1}(x) = (-1)^j S_j(x) \quad (j = 1, 2, \dots).$$

3. I_n in Case 1. Suppose that the d.f. of X_1 satisfies (C). Since $Y_1 = X_1 - a$, it is plain that F , the d.f. of Y_1 , also satisfies (C). It is easily seen that F and G (the d.f. of Z_1) are absolutely continuous with respect to each other. It therefore follows from Lemma 4 with $F_1 = F$ and $F_2 = G$ that G also satisfies (C).

Let k be an arbitrary but fixed positive integer. It follows from the conclusion of the preceding paragraph by Cramér's theorem [1, p. 81] that $H_n(x) = K_n(x) + R_n(x)$, where

$$(25) \quad K_n(x) = \sum_{j=0}^k n^{-j} P_j(-\Phi)$$

and $R_n(x)$ is of the order $n^{-(k+1)/2}$ uniformly in x . It follows hence from (12) that

$$(26) \quad I_n = n^{\frac{1}{2}} \alpha \int_0^{\infty} e^{-n^{\frac{1}{2}} \alpha x} [K_n(x) - K_n(0)] dx + O(n^{-\frac{1}{2}k-\frac{1}{2}}).$$

We have

$$(27) \quad \chi(t | K_n) = \sum_{j=0}^k n^{-j} P_j(it) e^{-t^2}$$

by (19) and (25). Let $f_n(x) = \exp(-n^{\frac{1}{2}} \alpha x)$ for $x \geq 0$ and $f_n(x) = 0$ other-

wise. Then $\int_{-\infty}^{\infty} e^{itx} f_n(x) dx = 1/(n^{\frac{1}{2}}\alpha - it) = g_n(t)$ say. Consequently, by first using integration by parts and then Parseval's formula, it follows that

$$(28) \quad \begin{aligned} n^{\frac{1}{2}}\alpha \int_0^{\infty} e^{-n^{\frac{1}{2}}\alpha x} [K_n(x) - K_n(0)] dx &= \int_0^{\infty} e^{-n^{\frac{1}{2}}\alpha x} K_n'(x) dx \\ &= \int_{-\infty}^{\infty} f_n(x) K_n'(x) dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} \overline{g_n(t)} \chi(t | K_n) dt. \end{aligned}$$

It follows from (26), (27) and (28) that

$$(29) \quad \alpha(2\pi n)^{\frac{1}{2}} I_n = \int_{-\infty}^{\infty} \left(1 + \frac{it}{n^{\frac{1}{2}}\alpha}\right)^{-1} \left(\sum_{j=0}^k n^{-\frac{1}{2}j} P_j(it)\right) d\Phi(t) + O(n^{-\frac{1}{2}k}).$$

Define

$$(30) \quad \mu_{r,s} = \int_{-\infty}^{\infty} (it)^r P_s(it) d\Phi(t) \quad (r, s = 0, 1, 2, \dots).$$

Since P_s is an even [odd] polynomial if s is even [odd], and since $\int_{-\infty}^{\infty} t^{2j+1} d\Phi(t) = 0$ for $j = 0, 1, 2, \dots$, it follows that each $\mu_{r,s}$ is a real constant, and that

$$(31) \quad \mu_{r,s} = 0 \quad \text{if } r + s \text{ is odd.}$$

Now, $(1 + itn^{-\frac{1}{2}}\alpha^{-1})^{-1} = \sum_{0 \leq r < k} (-itn^{-\frac{1}{2}}\alpha^{-1})^r + n^{-\frac{1}{2}k} t^k \omega_n(t)$, where $|\omega|$ is bounded in n and t . Since Φ has finite moments of all orders, it therefore follows from (29), (30) and (31) that

$$(32) \quad \alpha(2\pi n)^{\frac{1}{2}} I_n = \sum_{0 \leq j < \frac{1}{2}k} n^{-j} \left\{ \sum_{r+s=2j} \left(-\frac{1}{\alpha}\right)^r \mu_{r,s} \right\} + O(n^{-\frac{1}{2}k}).$$

Since $p_n = \rho^n I_n$, and since $\mu_{0,0} = 1$, it follows by replacing k with $2k + 2$ in (32) that (6) holds for any given k , with

$$(33) \quad b_n = \alpha^{-1}$$

and

$$(34) \quad c_{j,n} = \sum_{r+s=2j} \left(-\frac{1}{\alpha}\right)^r \mu_{r,s} \quad (j = 1, 2, \dots)$$

for every n . This establishes Theorem 2, and hence also Theorem 1, in Case 1.

It follows from (17) and (30) that the coefficients $\mu_{r,s}$ required to compute $c_{1,n}$ according to (34) are

$$(35) \quad \begin{aligned} \mu_{2,0} &= -1 \\ \mu_{1,1} &= 3\lambda_3 \\ \mu_{0,2} &= 3\lambda_4 - \frac{15}{2} \lambda_3^2 \end{aligned}$$

where the λ_j are given by (15). Similarly, $c_{2,n}$ can be computed from

$$\begin{aligned}
 \mu_{4,0} &= 3 \\
 \mu_{3,1} &= -15\lambda_3 \\
 (36) \quad \mu_{2,2} &= -15\lambda_4 + 105\lambda_3^2 \\
 \mu_{1,3} &= -15\lambda_5 + 105\lambda_3\lambda_4 - \frac{315}{2}\lambda_3^3 \\
 \mu_{0,4} &= -15\lambda_6 + 105\left(\frac{1}{2}\lambda_4^2 + \lambda_3\lambda_5\right) - \frac{945}{2}\lambda_3^2\lambda_4 + \frac{10395}{24}\lambda_3^4.
 \end{aligned}$$

We conclude this section with a remark concerning the role of Cramér's theorem [1, p. 81] in the preceding argument. Suppose that H_n is absolutely continuous, and that H'_n is square integrable over $(-\infty, \infty)$. It then follows, by integrating (12) by parts and using Parseval's formula, that

$$(29^*) \quad \alpha(2\pi n)^{\frac{1}{2}} I_n = \int_{-\infty}^{\infty} \left(1 + \frac{it}{n^{\frac{1}{2}}\alpha}\right)^{-1} \left\{ \left[\eta\left(\frac{it}{n^{\frac{1}{2}}}\right) \right]^n e^{\frac{1}{2}t^2} \right\} d\Phi(t)$$

where η is, as before, the m.g.f. of Z_1/σ . (The square integrability condition is imposed here for the validity of Parseval's formula, and can be replaced by others, e.g., that $(1 + t^2)^{-\frac{1}{2}} |\eta(it)|$ be integrable). According to (16), the function in curly brackets on the right side of (29*) can be expressed as $\sum_{j=0}^{\infty} n^{-\frac{1}{2}j} P_j(it)$. By comparing (29) and (29*) it is seen that, from a technical point of view, the role of Cramér's theorem in the present special case is to guarantee that when $\sum_{j=0}^{\infty} n^{-\frac{1}{2}j} P_j$ is replaced by $\sum_{j=0}^{k-1} n^{-\frac{1}{2}j} P_j$ on the right side of (29*), the error introduced is indeed of the order $n^{-\frac{1}{2}k}$. The same remark, but with (29*) replaced by a rather different formula for I_n , applies to the role of Esseen's theorem in the argument of the following section.

4. I_n in Case 2. Suppose that X_1 is a lattice variable. Let d be the maximum span of X_1 , i.e., $d > 0$ is the g.c.d. of the differences between consecutive possible values of X_1 . Let x_0 be the number such that $a \leq x_0 < a + d$, and such that the possible values of X_1 are included in the set $\{x_0 + rd : r = 0, \pm 1, \pm 2, \dots\}$. Let

$$(37) \quad \beta = d/\sigma, \quad \gamma = \tau d, \quad \kappa = (x_0 - a)/d$$

It should be noted that $0 \leq \kappa < 1$. For each n , let

$$(38) \quad \theta_n = n\kappa - [n\kappa], \quad 0 \leq \theta_n < 1,$$

where $[x]$ denotes the greatest integer contained in x .

Let k be an arbitrary but fixed positive integer. It follows from Esseen's theorem for the lattice case [7, p. 61] that $H_n(x) = K_n(x) + L_n(x) + R_n(x)$, where $K_n(x)$ is given by (25), R_n is of the order $n^{-(k+1)/2}$ uniformly in x , and L_n

is defined as follows. For any $j = 1, 2, \dots$ let $h_j = 1$ if $j \equiv 1$ or $2 \pmod{4}$ and $h_j = -1$ if $j \equiv 0$ or $3 \pmod{4}$. Then

$$(39) \quad L_n(x) = \sum_{j=1}^k n^{-\frac{1}{2}j} h_j \beta^j S_j(n^{\frac{1}{2}}\beta^{-1}x - \theta_n) K_n^{(j)}(x) = \sum_{j=1}^k M_{j,n}(x) \text{ say,}$$

where $K_n^{(j)}$ is the j th derivative of K_n . It follows hence from (12) that

$$(40) \quad \begin{aligned} I_n &= n^{\frac{1}{2}}\alpha \int_0^\infty e^{-n^{\frac{1}{2}}\alpha x} [K_n(x) - K_n(0)] dx \\ &\quad + \sum_{j=1}^k n^{\frac{1}{2}}\alpha \int_0^\infty e^{-n^{\frac{1}{2}}\alpha x} [M_{j,n}(x) - M_{j,n}(0)] dx + O(n^{-\frac{1}{2}k-1}). \end{aligned}$$

The first term on the right side of (40) is (cf., (28)) equal to $\int_0^\infty e^{-n^{\frac{1}{2}}\alpha x} K_n^{(1)}(x) dx$. We observe next that, for $j \geq 2$,

$$(41) \quad \begin{aligned} n^{\frac{1}{2}}\alpha \int_0^\infty e^{-n^{\frac{1}{2}}\alpha x} [M_{j,n}(x) - M_{j,n}(0)] dx &= \int_0^\infty e^{-n^{\frac{1}{2}}\alpha x} M'_{j,n}(x) dx \\ &= n^{-\frac{1}{2}j} h_j \beta^j \int_0^\infty e^{-n^{\frac{1}{2}}\alpha x} [S_j(y_n) K_n^{(j+1)}(x) \\ &\quad + (-1)^{j-1} n^{\frac{1}{2}}\beta^{-1} S_{j-1}(y_n) K_n^{(j)}(x)] dx \\ &= n^{-\frac{1}{2}j} h_j \beta^j \int_0^\infty e^{-n^{\frac{1}{2}}\alpha x} S_j(y_n) K_n^{(j+1)}(x) dx \\ &\quad - n^{-\frac{1}{2}(j-1)} h_{j-1} \beta^{j-1} \int_0^\infty e^{-n^{\frac{1}{2}}\alpha x} S_{j-1}(y_n) K_n^{(j)}(x) dx \\ &= N_{j,n} - N_{j-1,n} \text{ (say)}. \end{aligned}$$

In (41), we have put $n^{\frac{1}{2}}\beta^{-1}x - \theta_n = y_n$, and used integration by parts, (24), and the identity $(-1)^j h_j = h_{j-1}$. In order to evaluate the contribution of $M_{1,n}$ to the right side of (40), suppose for the moment that $0 < \theta_n < 1$, and let

$$(42) \quad \zeta_0 = 0, \quad \zeta_r = (r - 1 + \theta_n)\beta/n^{\frac{1}{2}} \quad (r = 1, 2, \dots).$$

Let A_r denote the open interval (ζ_r, ζ_{r+1}) . Then $S_1(y_n)$ is linear in x over each A_r (cf. (22)), and its derivative there equals $-n^{\frac{1}{2}}\beta^{-1}$. By writing $\int_0^\infty = \sum_{r=0}^\infty \int_{A_r}$, and applying integration by parts to \int_{A_r} , it follows without difficulty that

$$(43) \quad \begin{aligned} n^{\frac{1}{2}}\alpha \int_0^\infty e^{-n^{\frac{1}{2}}\alpha x} M_{1,n}(x) dx &= - \int_0^\infty e^{-n^{\frac{1}{2}}\alpha x} K_n^{(1)}(x) dx \\ &\quad + N_{1,n} + M_{1,n}(0) + \beta n^{-\frac{1}{2}} \sum_{r=1}^\infty e^{-\gamma(r-1+\theta_n)} K_n^{(1)}(\zeta_r), \end{aligned}$$

where $\gamma = \alpha\beta = \tau d$ (cf. (37)). Now, $S_1(x)$ is a left-continuous function of x . It follows hence that, for given n , the left and right sides of (43) are right-

continuous in θ_n . Since (43) holds for each θ_n in $(0, 1)$, we conclude that (43) is valid for $\theta_n = 0$ also.

Since S_k and $K_n^{(k+1)}$ are bounded functions, it is plain from the definition of $N_{j,n}$ (cf. (41)) that $N_{k,n}$ is of the order $n^{-\frac{1}{2}k-\frac{1}{2}}$. It therefore follows from (40), (41) and (43) that

$$(44) \quad I_n = \beta n^{-\frac{1}{2}} \sum_{r=1}^{\infty} e^{-\gamma(r-1+\theta_n)} K_n^{(1)}(\zeta_r) + O(n^{-\frac{1}{2}k-\frac{1}{2}}).$$

Now, according to (20) and (25),

$$(45) \quad K_n^{(1)}(\zeta_r) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} e^{-it\zeta_r} \left(\sum_{j=0}^k n^{-\frac{1}{2}j} P_j(it) \right) d\Phi(t)$$

for every r . Let us write

$$(46) \quad z = e^{-\gamma}, \quad b_n = [\beta/(1-z)]z^{\theta_n}.$$

It follows from (44) and (45) that

$$(47) \quad b_n^{-1} (2\pi n)^{\frac{1}{2}} I_n = \int_{-\infty}^{\infty} \frac{(1-z) \exp[-it\beta\theta_n/n^{\frac{1}{2}}]}{(1-z \exp[-it\beta/n^{\frac{1}{2}}])} \cdot \left(\sum_0^k n^{-\frac{1}{2}j} P_j(it) \right) d\Phi(t) + O(n^{-\frac{1}{2}k}).$$

For any θ and any $j = 0, 1, 2, \dots$ let

$$(48) \quad l_j(\theta) = \frac{1}{j!} \left\{ \frac{d^j}{dw^j} \left(\frac{(1-z)e^{-\theta w}}{(1-ze^{-w})} \right) \right\}_{w=0}.$$

It then follows easily from (31) and (47) that

$$(49) \quad b_n^{-1} (2\pi n)^{\frac{1}{2}} I_n = \sum_{0 \leq j < k/2} n^{-j} \left\{ \sum_{r+s=2j} \beta^r l_r(\theta_n) \mu_{r,s} \right\} + O(n^{-\frac{1}{2}k}).$$

By replacing k with $2k + 2$ in (49) we see that (6) holds for any given k , with b_n given by (46), and

$$(50) \quad c_{j,n} = \sum_{r+s=2j} \beta^r l_r(\theta_n) \mu_{r,s}.$$

This establishes Theorem 2 in Case 2, and hence also the first part of Theorem 1. To complete the proof of Theorem 1 in Case 2, we see from (37), (38) that $P(X_1 + \dots + X_n = na) > 0$ implies $\theta_n = 0$. Consequently if $P(X_1 = a) > 0$ then $\theta_n = 0$ for every n , and hence $b_n = \beta/(1-z)$ for every n .

It may be worthwhile to note that in the present case b_n can be expressed as $\alpha^{-1}[\gamma e^{\gamma(1-\theta_n)}/(e^\gamma - 1)]$, which shows that, in general, b_n oscillates about the value α^{-1} (cf. (33)) as $n \rightarrow \infty$ through the sequence $1, 2, \dots$.

An alternative formula for the coefficients l_j required in (50) is

$$(51) \quad l_j(\theta) = (-1)^j \sum_{r+s=j} \frac{\theta^r}{r!s!} \left\{ (1-z) \left(z \frac{d}{dz} \right)^s (1-z)^{-1} \right\}.$$

From (51) it is easily seen that, with $u = z/(1 - z)$,

$$\begin{aligned}
 l_0 &\equiv 1 \\
 l_1 &= -(\theta + u) \\
 (52) \quad l_2 &= \frac{1}{2}\{(\theta + u)^2 + u(1 + u)\} \\
 l_3 &= -\frac{1}{6}\{(\theta + u)^3 + 3u(1 + u)\theta + u(1 + u)(1 + 5u)\} \\
 l_4 &= \frac{1}{24}\{(\theta + u)^4 + 6u(1 + u)\theta^2 + 4u(1 + u)(1 + 5u)\theta \\
 &\quad + 23u^4 + 36u^3 + 14u^2 + u\}.
 \end{aligned}$$

The coefficients $c_{1,n}$ and $c_{2,n}$ can be computed from (35), (36), (50) and (52). The formulae for b_n and $c_{1,n}$ with $\theta = 0$ agree with the results of [3].

5. I_n in Case 3. If X_1 is not a lattice variable, then neither is Z_1 . It follows hence from a theorem of Esseen [7, p. 49] that $H_n(x) = \Phi(x) + n^{-\frac{1}{2}}f(x) + n^{-\frac{3}{2}}r_n(x)$, where $f(x) = (\text{const.}) (1 - x^2) \exp(-\frac{1}{2}x^2)$, and $r_n(x) \rightarrow 0$ uniformly in x as $n \rightarrow \infty$. The contribution of $n^{-\frac{1}{2}}f$ to I_n is $n^{-\frac{1}{2}} \int_0^\infty e^{-n^{\frac{1}{2}}\alpha x} f'(x) dx$, which is easily seen to be of the order $n^{-\frac{3}{2}}$. It follows that

$$\begin{aligned}
 (53) \quad I_n &= n^{\frac{1}{2}}\alpha \int_0^\infty e^{-n^{\frac{1}{2}}\alpha x} [\Phi(x) - \Phi(0)] dx + o(n^{-\frac{1}{2}}) \\
 &= \int_0^\infty e^{-n^{\frac{1}{2}}\alpha x} \Phi'(x) dx + o(n^{-\frac{1}{2}}) \\
 &= e^{\frac{1}{2}n\alpha^2} [1 - \Phi(n^{\frac{1}{2}}\alpha)] + o(n^{-\frac{1}{2}}) \\
 &= (2\pi n)^{-\frac{1}{2}}\alpha^{-1} + o(n^{-\frac{1}{2}}).
 \end{aligned}$$

In (53), we have used integration by parts, a linear change of variable, and the leading term of the asymptotic formula [9, p. 179]

$$(54) \quad 1 - \Phi(x) = (2\pi)^{-\frac{1}{2}}e^{-\frac{1}{2}x^2}\{x^{-1} - x^{-3} + 3x^{-5} + O(x^{-7})\} \text{ as } x \rightarrow \infty.$$

It follows from (53) that (5) holds, with $b_n = \alpha^{-1}$ for every n . This completes the proof of Theorem 1.

Since $\Phi(x) + n^{-\frac{1}{2}}f(x) = K_n(x)$, where K_n is defined by (25) with $k = 1$, the conclusion of the preceding paragraph is also available from the argument of Section 3. We have used a direct calculation instead because this calculation suggests the form of the numerical approximations described in the following section.

6. Concluding remarks. Suppose, in a given case, and for given n and a , that it is required to compute the numerical value of p_n defined by (1). In this section we consider approximations of the form

$$(55) \quad q_n = \rho^n e^{\frac{1}{2}v_n^2}(1 - \Phi(v_n)),$$

where ρ and Φ are defined by (4) and (14), and v_n is a suitably chosen number.

We shall describe four choices of v_n , called v_n^* , $v_n^{(0)}$, $v_n^{(1)}$, and $v_n^{(2)}$. The resulting values of q_n are denoted by q_n^* , $q_n^{(0)}$, etc.

First consider

$$(56) \quad v_n^* = n^{\frac{1}{2}}\alpha$$

where α is given by (4), (8) and (9). This choice of v_n amounts (cf. (53)) to approximating I_n by replacing H_n with Φ on the right side of (12). It therefore follows from the Esseen-Berry theorem that we always have

$$(57) \quad |p_n - q_n^*| \leq 2C \frac{\rho^n E |Z_1|^3}{n^{\frac{1}{2}} \sigma^3}$$

where C is a universal constant. Wallace [10, p. 637] states that $C \leq 2.05$.

Next, consider

$$(58) \quad v_n^{(0)} = n^{\frac{1}{2}}/b_n$$

where b_n is defined by (33) in Cases 1 and 3, and by (46) in Case 2. (Of course, $q_n^{(0)} = q_n^*$ in Cases 1 and 3). Then $q_n^{(0)}$ satisfies (2), and the $o(1)$ term in (2) is known to be of the order n^{-1} in Cases 1 and 2. Finally, let $c_{j,n}$ be defined according to Section 4 in Cases 1 and 3, and according to Section 5 in Case 2. Define

$$(59) \quad v_n^{(1)} = v_n^{(0)} [1 - (b_n^2 + c_{1,n})/n]$$

if the expression within the square brackets is positive and $v_n^{(1)} = 0$ otherwise; and

$$(60) \quad v_n^{(2)} = v_n^{(1)} [1 + (b_n^4 + c_{1,n}^2 - b_n^2 c_{1,n} - c_{2,n})/n^2]$$

if the expression in square brackets is positive and $v_n^{(2)} = 0$ otherwise. Then $q_n^{(j)}$ also satisfies (2), and $o(1) = O(n^{-j-1})$ in Cases 1 and 2 ($j = 1, 2$). The stated theoretical properties of the approximations $q_n^{(j)}$ are easy consequences of (5), (6), (54), and (58).

Although (unlike q_n^*) the approximations $q_n^{(j)}$ are derived from asymptotic expansions corresponding to the case when $n \rightarrow \infty$ and a is held fixed, the usefulness of these approximations may be wider than is suggested by the derivation. Some evidence to this effect is provided by the fact that if X_1 is normally distributed then $p_n = q_n^{(0)} = q_n^{(1)} = q_n^{(2)}$ for every admissible a and every n .

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