Dedicated to the memory of my father El Hadj Mowoum Moussa 1929-2004

On difference equations for orthogonal polynomials on nonuniform lattices

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Abstract

By the study of various properties of some divided-difference equations, we simplify the definition of classical orthogonal polynomials given by Atakishiyev, Rahman and Suslov (1995), then prove that orthogonal polynomials obtained by some modifications of the classical orthogonal polynomials on nonuniform lattices satisfy a single fourth-order linear homogeneous divided-difference equation with polynomial coefficients. Moreover, we factorize and solve explicitly these divided-difference equations. Also, we prove that the product of two functions, each of which satisfying a second-order linear homogeneous divided-difference equation is a solution of a fourth-order linear homogeneous divided-difference equation. This result holds in particular when the divided-difference operator is carefully replaced by the Askey-Wilson operator $\mathcal{D}_q$, following pioneering work by Alphonse Magnus (1988) connecting $\mathcal{D}_q$ and divided-difference operators. Finally, we propose a method to look for polynomial solutions of linear divided-difference equations with polynomial coefficients.

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1 Introduction

Let $x(s)$ be a function of the variable s and $y(x(s))$ a function of $x(s)$ satisfying a difference equation of hypergeometric type namely

$$\phi(x(s))\,\frac{\Delta}{\Delta x_{-1}(s)}\left[\frac{\nabla y(x(s))}{\nabla x(s)}\right]+\frac{\psi(x(s))}{2}\left[\frac{\Delta y(x(s))}{\Delta x(s)}+\frac{\nabla y(x(s))}{\nabla x(s)}\right]+\lambda\,y(x(s))=0, \qquad (1)$$

where ϕ and ψ are polynomials of degree at most 2 and 1 respectively; λ is a constant, Δ and ∇ are the forward and the backward operators

$$\Delta f(s)=f(s+1)-f(s),\ \nabla f(s)=f(s)-f(s-1). \qquad (2)$$

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Here the lattice $x_\mu(s)$ for a complex number μ is defined as

$$x_\mu(s) = x(s + \frac{\mu}{2}). \tag{3}$$

Atakishiyev, Rahman and Suslov [5] (see also [28]) proved that the divided-difference $\frac{\Delta y(x(s))}{\Delta x(s)}$ satisfies an equation of the same type as (1) if and only if $x(s)$ is a linear, a q-linear, a quadratic or a q-quadratic lattice; that is $x(s)$ is of the form

$$x(s) = \begin{cases} c_1\, q^{-s} + c_2\, q^s + c_3 & \text{if} \quad q \neq 1 \\ c_4\, s^2 + c_5\, s + c_6 & \text{if} \quad q = 1, \end{cases} \tag{4}$$

where $q \in \mathbb{C}$, and the c_i are arbitrary constants such that $(c_1, c_2) \neq (0, 0)$ and $(c_4, c_5) \neq (0, 0)$. Lattices of the form (4) with $c_1\, c_2 \neq 0$ or $c_4 \neq 0$ are called nonuniform lattices [5, 28].

Next they used this characterization to give a definition of classical orthogonal polynomials (in the broad sense of Hahn, and consistent with the latest definition proposed by Andrews and Askey [2]):

Definition 1 *A polynomial sequence (P_n) is classical if and only if:*

1. *(P_n) is orthogonal on a real interval $(x(a), x(b))$ with respect to the weight function $\rho(s)$ i.e.*

$$\begin{cases} \text{degree}(P_n) = n, \ n \geq 0, \\ \sum_{i=0}^{N} P_n(x(s_i))\, P_m(x(s_i))\, \rho(s_i)\, \nabla x_1(s_i) = k_n\, \delta_{n,m}, \quad k_n \neq 0, \, n, \, m \geq 0, \end{cases} \tag{5}$$

with

$$s_0 = a, s_{i+1} = s_i + 1, \ s_{N+1} = b, \ N \in \mathbb{N}_0 \cup \{\infty\} \tag{6}$$

for the discrete orthogonality or

$$\begin{cases} \text{degree}(P_n) = n, \ n \geq 0, \\ \int_C P_n(x(s))\, P_m(x(s))\, \rho(s)\, \nabla x_1(s)\, ds = k_n\, \delta_{n,m}, \quad k_n \neq 0, \, n, \, m \geq 0, \end{cases} \tag{7}$$

where C is a contour in the complex s-plane, for the continuous orthogonality;

2. *Any $P_n(x(s))$ satisfies a difference equation of the form (1) with $x(s)$ given by (4).*

3. *The weight ρ satisfies the Pearson-type difference equation*

$$\frac{\Delta}{\nabla x_1(s)} (\sigma(s)\, \rho(s)) = \psi(x(s))\, \rho(s), \tag{8}$$

where $\psi(s)$ is a polynomial of degree 1 in $x(s)$ and the function ϕ defined by

$$\phi(x(s)) = \sigma(s) + \frac{1}{2} \psi(x(s))\, \nabla x_1(s) \tag{9}$$

is a non-zero polynomial of degree at most 2 in the variable $x(s)$, with the border conditions

$$\begin{cases} \sigma(s)\, \rho(s)\, x^k(s - \frac{1}{2})\big|_{s=a,\, b} = 0, \ k = 0, \, 1, \, 2, \ldots, \\ \int_C \Delta \left[\sigma(s)\, \rho(s)\, x^k(s - \frac{1}{2})\right] ds = 0, \ k = 0, \, 1, \, 2, \ldots. \end{cases} \tag{10}$$

for the discrete orthogonality and the continuous orthogonality respectively.

This definition covers the q-Racah polynomials as well as the Askey-Wilson polynomials and their limiting and special cases. It covers also the very classical orthogonal polynomials– the classical orthogonal polynomials of a continuous variable (Jacobi, Laguerre and Hermite by virtue of the limiting procedure); the classical orthogonal polynomials of a discrete variable (Hahn, Meixner, Charlier and Krawtchouk) and the classical orthogonal polynomials of a q-discrete variable (Big q-Jacobi, ...).

Remark 1 *It should be noticed that in case of the continuous orthogonality, for the family $(P_n(x(s)))$ to be classical orthogonal polynomials in the real variable $x(s)$, it should be possible [5] to choose a contour C in such a way that the second relation of (7) can be expressed as a real orthogonality relation*

$$\int_a^b P_n(x)\, P_n(x)\, \rho(x)\, dx = k_n\, \delta_{n,m},\ k_n \neq 0,\ n,\ m \geq 0,$$

with $\rho(x) > 0,\ x \in (a, b)$.

According to this definition, for a family of polynomials (P_n) orthogonal with respect to a certain weight function ρ to be classical, the weight should satisfy a Pearson-type equation with some border conditions **and** any P_n should additionally satisfy an equation of type (1).

By studying various properties of the operators $\mathbb{D}_x$ and $\mathbb{S}_x$

$$\mathbb{D}_x f(x(s)) = \frac{\Delta f(x(s))}{\Delta x(s)},\ \mathbb{S}_x f(x(s)) = \frac{f(x(s+1)) + f(x(s))}{2}, \tag{11}$$

we prove that the second condition of Definition 1 is not necessary. Therefore, we get rid of this condition and obtain a definition which is similar to that of the very classical orthogonal polynomials. Next, we derive, factorize and solve the fourth-order divided-difference equation satisfied by the orthogonal polynomials obtained by modifications of classical orthogonal polynomials.

Section 2 is devoted to the study of the properties of the operators $\mathbb{D}_x$ and $\mathbb{S}_x$. In particular, we prove that the product of two functions, each of which satisfying a second-order linear homogeneous divided-difference equation is a solution of a fourth-order linear homogeneous divided-difference equation. We show how this result works when the divided-difference operator is replaced by the Askey-Wilson operator $\mathcal{D}_q$. In the third section we simplify Definition 1 and derive the expressions of the recurrence coefficients of classical orthogonal polynomials in terms of the polynomials ϕ and ψ in the same line as for the very classical orthogonal polynomials. The fourth section is devoted to the derivation, the factorization as well as the solution of the fourth-order divided-difference equations for **modifications** of classical orthogonal polynomials (see [11, 12, 13] for the cases of the very classical orthogonal polynomials). Section 5 gives some specializations and applications. In particular, we point out some situations for which the first associated of the Askey-Wilson (resp. the Racah) polynomials remains classical and express these in terms of the initial families. We also point out a method to look for polynomial solutions of higher order divided-difference equations with polynomial coefficients in the same line as the one of higher order differential equation with polynomial coefficients.

To complete the introduction, we would like to mention that here by modifications of orthogonal polynomials (P_n) we mean any family of orthogonal polynomials $(\tilde{P}_n)$ which is related to the initial family (P_n) by a relation of the form

$$\tilde{P}_n = I_{n,r,k}(x)\, P_{n+r}(x) + J_{n,r,k}(x)\, P^{(1)}_{n+r-1}(x), \tag{12}$$

where $P^{(1)}_{n+r-1}$ is defined in (13) and (14), r and k are nonnegative integers, $I_{n,r,k}(x)$ and $J_{n,r,k}(x)$ are polynomials in the variable x with the property that they **do not** depend on n for $n \geq k$, i.e.:

$$I_{n,r,k} := I_{r,k},\ \ J_{n,r,k} := J_{r,k} \neq 0,\ n \geq k.$$

Among these modifications are the rth associated orthogonal polynomials which are obtained by replacing n by $n+r$ in the recurrence coefficients β_n and γ_n of the three-term recurrence relation satisfied by the initial orthogonal polynomial sequence (P_n)

$$P_{n+1}(x) = (x - \beta_n)\, P_n(x) - \gamma_n\, P_{n-1}(x),\ n \geq 1,\ P_0(x) = 1,\ P_{-1}(x) = 0. \tag{13}$$

The new polynomial family obtained, which is denoted by $(P_n^{(r)})$, is orthogonal thanks to Favard's theorem [8] since it satisfies

$$P_{n+1}^{(r)}(x) = (x - \beta_{n+r})\, P_n^{(r)}(x) - \gamma_{n+r}\, P_{n-1}^{(r)}(x),\ n \geq 1,\ P_0^{(r)}(x) = 1,\ P_{-1}^{(r)}(x) = 0. \tag{14}$$

The relation

$$P_n^{(r)} = \frac{P_{r-1}}{\Gamma_{r-1}}\, P_{n+r-1}^{(1)} - \frac{P_{r-2}^{(1)}}{\Gamma_{r-1}}\, P_{n+r},\ n \geq 0,\ r \geq 1, \tag{15}$$

linking (P_n) and its rth associated compared to relation (12) yields

$$k = 0,\ \ I_{r,0} = -\frac{P_{r-2}^{(1)}}{\Gamma_{r-1}},\ \ J_{r,0} = \frac{P_{r-1}}{\Gamma_{r-1}},\ \text{ with } \Gamma_k = \prod_{j=1}^{k} \gamma_j,\ k \geq 1,\ \Gamma_0 := 1. \tag{16}$$

Other modifications of the three-term recurrence relation which lead to relations of the type (12) are the co-recursive and the generalized co-recursive orthogonal polynomials; the co-recursive associated and the generalized co-recursive associated orthogonal polynomials; the co-dilated and the generalized co-dilated orthogonal polynomials; the co-modified and the generalized co-modified orthogonal polynomials. Information about these families of orthogonal polynomials as well as the relations of type (12) they satisfy can be found in [11, 12, 13] and references therein.

2 Properties of some divided-difference operators

2.1 Properties of the quadratic and q-quadratic lattices

Let $x(s)$ be a lattice given by (4). Such lattice satisfies [5]

$$x(s+k) - x(s) = \gamma_k\, \nabla x_{k+1}(s), \tag{17}$$
$$\frac{x(s+k) + x(s)}{2} = \alpha_k\, x_k(s) + \beta_k, \tag{18}$$

for $k = 0,\ 1,\ \ldots$, with

$$\alpha_0 = 1,\ \alpha_1 = \alpha,\ \beta_0 = 0,\ \beta_1 = \beta, \gamma_0 = 0,\ \gamma_1 = 1, \tag{19}$$

where the sequences $(\alpha_k),\ (\beta_k),\ (\gamma_k)$ satisfy the following relations

$$\begin{aligned} \alpha_{k+1} - 2\,\alpha\,\alpha_k + \alpha_{k-1} &= 0, \\ \beta_{k+1} - 2\,\beta_k + \beta_{k-1} &= 2\,\beta\,\alpha_k, \\ \gamma_{k+1} - \gamma_{k-1} &= 2\,\alpha_k, \end{aligned} \tag{20}$$

for $k = 0,\ 1,\ \ldots$. The lattice $x(s)$ has also the property [28]

$$x(s+1)^2 + x(s)^2 = 2\, A_2(x_1(s)) = a_2\, x_1(s)^2 + a_1\, x_1(s) + a_0, \tag{21}$$

where the a_j are constants (to be found later in (36)), with $a_2 \neq 0$.

For practical reasons, it is important to express $x_k(s)$, $-4 \le k \le 4$, in terms of $x(s)$ and $x(s+1) = x_2(s)$ using (17) and (18). For this purpose, we consider Equation (17) for $k = 2,\ s = s-1$ and $k = 2,\ s = s - \frac{3}{2}$; and Equation (18) for $k = 1;\ s = s$, for $k = 1;\ s = s - \frac{1}{2}$ and for $k = 2,\ s = s-1$. Then we solve the system of five linear equations obtained for the unknowns

$$x_{-3}(s),\, x_{-1}(s),\, x_{-2}(s),\, x_1(s),\, x_3(s)$$

and obtain the expressions

$$\begin{aligned}
2\alpha\, x_{-3}(s) &= -(2\alpha-1)(2\alpha+1)\, x_2(s) + (4\alpha^2+2\alpha-1)(4\alpha^2-2\alpha-1)\, x(s) \\
&\quad +2\beta\,(2\alpha+1)(4\alpha^2+2\alpha-1); \\
x_{-2}(s) &= -x_2(s) + (4\alpha^2-2)\, x(s) + 4\beta\,(\alpha+1); \\
2\alpha\, x_{-1}(s) &= -x_2(s) + (2\alpha-1)(2\alpha+1)\, x(s) + 2\beta\,(2\alpha+1); \\
2\alpha\, x_1(s) &= x_2(s) + x(s) - 2\beta; \\
2\alpha\, x_3(s) &= (2\alpha-1)(2\alpha+1)\, x_2(s) - x(s) + 2\beta\,(2\alpha+1),
\end{aligned} \tag{22}$$

from which we deduce

$$\begin{aligned}
x_{-4}(s) &= -(4\alpha^2-2)\, x_2(s) + (4\alpha^2-1)\,(4\alpha^2-3)\, x(s) + 4\beta(\alpha+1)(4\alpha^2-1); \\
x_4(s) &= (4\alpha^2-2)\, x_2(s) - x(s) + 4\beta(\alpha+1).
\end{aligned} \tag{23}$$

2.2 The operators $\mathbb{D}_x$ and $\mathbb{S}_x$

The operators $\mathbb{D}_x$ and $\mathbb{S}_x$ defined in (11) transform any polynomial in $x(s)$ into a polynomial of the variable $x_1(s)$. More precisely we have (see also [28], p. 149):

Proposition 1 *If $P_n(x(s))$ is a polynomial of degree $n \ge 1$ in $x(s)$, then*

$$\mathbb{D}_x(P_n(x(s))) = q_{n-1}(x_1(s)),\ \ \mathbb{S}_x(P_n(x(s))) = r_n(x_1(s)), \tag{24}$$

where q_{n-1} and r_n are polynomials of degree $n-1$ and n respectively.

More generally,

$$\mathbb{D}_{x_\mu}(P_n(x_\mu(s))) = \tilde{q}_{n-1}(x_{\mu+1}(s)),\ \ \mathbb{S}_{x_\mu}(P_n(x_\mu(s))) = \tilde{r}_n(x_{\mu+1}(s)), \tag{25}$$

where $\tilde{q}_{n-1}$ and $\tilde{r}_n$ are polynomials of degree $n-1$ and n respectively.

Proof: First, for fixed $n \ge 1$, we write the relation

$$\Delta(f(s)\, g(s)) = \frac{f(s+1)+f(s)}{2}\Delta\, g(s) + \frac{g(s+1)+g(s)}{2}\Delta\, f(s),$$

for $f(s) = x^{n-1}(s)$ and $g(s) = x(s)$ and obtain using relations (18) for $k = 1$

$$\mathbb{D}_x\, x^n(s) = (\alpha\, x_1(s) + \beta)\, \mathbb{D}_x\, x^{n-1}(s) + \mathbb{S}_x\, x^{n-1}(s).$$

In the same way using in addition (21), we obtain by taking $f(s) = x^{n-1}(s)\, e^{i\pi s}$ and $g(s) = x(s)$

$$\mathbb{S}_x\, x^n(s) = (\alpha\, x_1(s) + \beta)\, \mathbb{S}_x\, x^{n-1}(s) + \left[A_2(x_1(s)) - (\alpha\, x_1(s) + \beta)^2\right]\, \mathbb{D}_x\, x^{n-1}(s).$$

From the two previous relations, it is easy to show by induction that $\mathbb{D}_x\, x^n(s)$ and $\mathbb{S}_x\, x^n(s)$ are polynomials of degree at most $n-1$ and n respectively in the variable $x_1(s)$. Next, we write

$$\mathbb{D}_x\, x^n(s) = \sum_{k=0}^{n-1} D_{n,k}\, x_1^k(s),\ \mathbb{S}_x\, x^n(s) = \sum_{k=0}^{n} S_{n,k}\, x_1^k(s), \tag{26}$$

and get from the previous equation a system of two recurrence relations in $D_{n,n-1}$ and $S_{n,n}$. Solving this system with the initial conditions $D_{1,0} = 1,\ D_{2,1} = 2\,\alpha,\ S_{0,0} = 1,\ S_{1,1} = \alpha$, one obtains that

$$D_{n,n-1} \neq 0,\ n \geq 1 \text{ and } S_{n,n} \neq 0,\ n \geq 0.$$

This proves the first assertion of the proposition. Equation (25) is obtained by replacing s by $s + \frac{\mu}{2}$ in (24). The coefficients $D_{n,k-1}$ and $S_{n,k}$ for $k = n,\ n-1$ and $n-2$ will be given explicitly later. □

2.2.1 The product and quotient rules for $\mathbb{D}_x$ and $\mathbb{S}_x$

Next, we state and prove product and quotient rules for the companion operators $\mathbb{D}_x$ and $\mathbb{S}_x$.

Theorem 1 *The following statements hold.*

1. *The operators $\mathbb{D}_x$ and $\mathbb{S}_x$ obey the following product rules:*

$$\mathbb{D}_x\left(f(x(s))g(x(s))\right) = \mathbb{S}_x f(x(s))\,\mathbb{D}_x g(x(s)) + \mathbb{D}_x f(x(s))\,\mathbb{S}_x g(x(s)), \tag{27}$$
$$\mathbb{S}_x\left(f(x(s))g(x(s))\right) = Q_2(x_1(s))\,\mathbb{D}_x f(x(s))\,\mathbb{D}_x g(x(s)) + \mathbb{S}_x f(x(s))\,\mathbb{S}_x g(x(s)), \tag{28}$$

where Q_2 is a polynomial of degree 2

$$Q_2(x_1(s)) = (\alpha^2 - 1)\, x_1^2(s) + 2\,\beta\,(\alpha+1)\, x_1(s) + \delta_x, \tag{29}$$

and δ_x is a constant depending on α, β and the initial values $x(0)$ and $x(1)$ of $x(s)$:

$$\delta_x = \frac{x^2(0) + x^2(1)}{4\alpha^2} - \frac{(2\alpha^2 - 1)}{2\alpha^2} x(0)\, x(1) - \frac{\beta\,(\alpha+1)}{\alpha^2}(x(0) + x(1)) + \frac{\beta^2\,(\alpha+1)^2}{\alpha^2}. \tag{30}$$

2. *The operators $\mathbb{D}_x$ and $\mathbb{S}_x$ also satisfy the quotient rules*

$$\mathbb{D}_x\left(\frac{f(x(s))}{g(x(s))}\right) = \frac{\mathbb{S}_x f(x(s))\,\mathbb{D}_x g(x(s)) - \mathbb{D}_x f(x(s))\,\mathbb{S}_x g(x(s))}{Q_2(x_1(s))\,[\mathbb{D}_x g(x(s))]^2 - [\mathbb{S}_x g(x(s))]^2}; \tag{31}$$
$$\mathbb{S}_x\left(\frac{f(x(s))}{g(x(s))}\right) = \frac{Q_2(x_1(s))\,\mathbb{D}_x f(x(s))\,\mathbb{D}_x g(x(s)) - \mathbb{S}_x f(x(s))\,\mathbb{S}_x g(x(s))}{Q_2(x_1(s))\,[\mathbb{D}_x g(x(s))]^2 - [\mathbb{S}_x g(x(s))]^2}, \tag{32}$$

provided that $g(x(s)) \neq 0,\ s \in (a,b)$.

3. *More generally, relations (27)-(32) remain valid if we replace x and x_1 by x_μ and $x_{\mu+1}$ respectively, $\mu \in \mathbb{C}$. In particular, the constant δ_x remains unchanged if we replace x in (30) by $x_k,\ k \in \mathbb{Z}$, i.e.,*

$$\delta_{x_k} = \delta_x := \delta,\ k \in \mathbb{Z}. \tag{33}$$

Proof: First, we solve the equations

$$\mathbb{D}_x f(x(s)) = \frac{f(x(s+1)) - f(x(s))}{x(s+1) - x(s)}, \ \mathbb{S}_x f(x(s)) = \frac{f(x(s+1)) + f(x(s))}{2},$$

in terms of $f(x(s+1))$ and $f(x(s))$. Then, we substitute this result for f and g in the equations

$$\begin{aligned}\mathbb{D}_x \left(f(x(s))\, g(x(s))\right) &= \frac{f(x(s+1))g(x(s+1)) - f(x(s))g(x(s))}{x(s+1) - x(s)},\\ \mathbb{S}_x \left(f(x(s))\, g(x(s))\right) &= \frac{f(x(s+1))g(x(s+1)) + f(x(s))g(x(s))}{2}\end{aligned}$$

and obtain respectively (27) and

$$\mathbb{S}_x \left(f(x(s))g(x(s))\right) = \frac{(x(s+1) - x(s))^2}{4} \mathbb{D}_x f(x(s))\, \mathbb{D}_x g(x(s)) + \mathbb{S}_x f(x(s))\, \mathbb{S}_x g(x(s)).$$

By taking $f(x(s)) = g(x(s)) = x(s)$ in the previous equation, we get

$$Q_2(x_1(s)) := \frac{(x(s+1) - x(s))^2}{4} = \mathbb{S}_x x^2(s) - \left(\mathbb{S}_x x(s)\right)^2.$$

By means of Proposition 1, $Q_2(x_1(s))$ is a polynomial of degree at most 2 in the variable $x_1(s)$. Hence,

$$(x(s+1) - x(s))^2 = \delta_2\, x_1^2(s) + \delta_1\, x_1(s) + \delta_0,$$

where δ_j are constants. Application of the operator $\mathbb{D}_{x_1}$ on both sides of the previous equation and use of Equations (17) and (18) for $k = 2$ produce

$$\begin{aligned}\delta_2\,(x_1(s+1) + x_1(s)) + \delta_1 &= \frac{(x(s+2) - 2\,x(s+1) + x(s))\,(x(s+2) - x(s))}{x_1(s+1) - x_1(s)}\\ &= 2\,\gamma_2\,[(\alpha_2 - 1)\,x(s+1) + \beta_2].\end{aligned}$$

The previous equation gives by means of (18) for $k = 1$ and s replaced by $s + \frac{1}{2}$

$$2\delta_2\,(\alpha\, x(s+1) + \beta) + \delta_1 = 2\,\gamma_2\,[(\alpha_2 - 1)\,x(s+1) + \beta_2].$$

Therefore,

$$\delta_2 = \frac{\gamma_2\,(\,\alpha_2 - 1)}{\alpha} = 4\,(\alpha^2 - 1),\ \ \delta_1 = 8\beta\,(\alpha + 1)$$

and

$$Q_2(x_1(s)) = \frac{(x(s+1) - x(s))^2}{4} = (\alpha^2 - 1)\,x_1^2(s) + 2\,\beta\,(\alpha + 1)\,x_1(s) + \delta_x. \tag{34}$$

Equation (30) is obtained by taking $s = 0$ in the previous equation and using (22).

To prove the second statement, we take $f(x(s)) = \frac{1}{g(x(s))}$ in (27) and (28) to get

$$\begin{aligned}\mathbb{D}_x g(x(s))\, \mathbb{S}_x \frac{1}{g(x(s))} + \mathbb{S}_x g(x(s))\, \mathbb{D}_x \frac{1}{g(x(s))} &= 0,\\ \mathbb{S}_x g(x(s))\, \mathbb{S}_x \frac{1}{g(x(s))} + Q_2(x_1(s))\, \mathbb{D}_x g(x(s))\, \mathbb{D}_x \frac{1}{g(x(s))} &= 1.\end{aligned}$$

The determinant of the previous system with respect to the unknowns

$$\mathbb{S}_x \frac{1}{g(x(s))} \ \text{ and } \ \mathbb{D}_x \frac{1}{g(x(s))}$$

is

$$Q_2(x_1(s))\,[\mathbb{D}_x g(x(s))]^2 - [\mathbb{S}_x g(x(s))]^2 = 4\,g(x(s))\,g(x(s+1)) \neq 0. \tag{35}$$

Hence,

$$\begin{aligned}
\mathbb{S}_x \frac{1}{g(x(s))} &= \frac{-\mathbb{S}_x g(x(s))}{Q_2(x_1(s))\,[\mathbb{D}_x g(x(s))]^2 - [\mathbb{S}_x g(x(s))]^2}, \\
\mathbb{D}_x \frac{1}{g(x(s))} &= \frac{\mathbb{D}_x g(x(s))}{Q_2(x_1(s))\,[\mathbb{D}_x g(x(s))]^2 - [\mathbb{S}_x g(x(s))]^2}.
\end{aligned}$$

Application of the product rules (27)-(28) to the product $f(x(s)) \times \frac{1}{g(x(s))}$ produces (31) and (32). The third statement of the theorem is proved by replacing s by $s+\frac{\mu}{2}$ in (27)-(32). Also, Equation (33) is obtained by direct computation using (22) and (30). □

From now on we denote δ_x by δ, i.e. $\delta := \delta_x$.

Remark 2 *The operators $\mathbb{D}_x$ and $\mathbb{S}_x$ appeared already in the works of Magnus [23, 24, 25]. In [23] Magnus gave also the product and quotient rules for a more general divided-difference operator. Theorem 1 is more specific to quadratic lattices and gives in detail the coefficients appearing in the product and quotient rules, in terms of the parameters α, β and δ of the lattice.*

Corollary 1 *As direct consequence of the previous theorem we have:*

1. *From the quotient rules (31) and (32), one observes that the operators $\mathbb{D}_x$ and $\mathbb{S}_x$ transform a rational function of the variable $x(s)$ into a rational function of the variable $x_1(s)$.*
2. *If we square both members of (18) for $k=1$ and combine with (34), we obtain*

$$\begin{aligned}
x^2(s+1) + x^2(s) &= 2\,(2\,\alpha^2 - 1)\,x_1^2(s) + 4\,\beta\,(2\,\alpha+1)\,x_1(s) + 2\,(\beta^2+\delta), && (36)\\
x(s)\,x(s+1) &= x_1^2(s) - 2\,\beta\,x_1(s) + \beta^2 - \delta. && (37)
\end{aligned}$$

Remark 3 *The parameters α, β and δ are the ingredients for the classical orthogonal polynomials. As will be shown later, the recurrence coefficients of the classical orthogonal polynomials are expressed explicitly in terms of these three parameters and the coefficients of the polynomials ϕ and ψ involved in the Pearson-type equation satisfied by the orthogonality weight function (see (8)).*

2.2.2 Consequences of the product and quotient rules

The product rules for $\mathbb{D}_x$ and $\mathbb{S}_x$ provide the recurrence relations for the coefficients $D_{n,k}$ and $S_{n,k}$.

Proposition 2 *The coefficients $D_{n,k}$ and $S_{n,k}$ of the expansions (26) satisfy*

$$\begin{aligned}
S_{n,k} &= -\alpha\,D_{n,k-1} - \beta\,D_{n,k} + D_{n+1,k}, \quad 0 \le k \le n, && (38)\\
S_{n+1,k} &= (\alpha^2-1)\,D_{n,k-2} + 2\,(\alpha+1)\,\beta\,D_{n,k-1} + \delta\,D_{n,k} \\
&\quad + \alpha\,S_{n,k-1} + \beta\,S_{n,k}, \quad 0 \le k \le n+1, && (39)
\end{aligned}$$

with the convention

$$D_{n,n} = D_{n,n+1} = S_{n,n+1} = D_{n,-1} = D_{n,-2} = S_{n,-1} = 0, \; n \ge 0. \tag{40}$$

Proof: Equations (38) and (39) are obtained from the expansion formulaes (26) and the product rules (27) and (28) for $f(x(s)) = x^n(s)$ and $g(x(s)) = x(s)$. □

Coefficients $D_{n,k}$ and $S_{n,k}$

Substitution of (38) in (39) reads

$$D_{n+2,k} - 2\,\alpha\,D_{n+1,k-1} + D_{n,k-2} = 2\,\beta\,D_{n+1,k} + (\delta - \beta^2)\,D_{n,k} + 2\,\beta\,D_{n,k-1}. \tag{41}$$

The previous equation for $k = n+1$ gives a second-order homogenous linear difference equation with constant coefficients

$$D_{n+2,n+1} - 2\,\alpha\,D_{n+1,n} + D_{n,n-1} = 0,$$

whose solution with the initial conditions $D_{0,-1} = 0,\ D_{1,0} = 1$ is

$$D_{n,n-1} = \begin{cases} n & \text{if} \quad \alpha = 1, \\ \frac{q^{\frac{n}{2}} - q^{-\frac{n}{2}}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}})} & \text{if} \quad \alpha = \frac{q^{\frac{1}{2}} + q^{-\frac{1}{2}}}{2}, \end{cases} \tag{42}$$

where the parameter q is the one appearing in (4).

$S_{n,n}$ is therefore deduced using (38) for $k = n$

$$S_{n,n} = \begin{cases} 1 & \text{if} \quad \alpha = 1, \\ \frac{q^{\frac{n}{2}} + q^{-\frac{n}{2}}}{2} & \text{if} \quad \alpha = \frac{q^{\frac{1}{2}} + q^{-\frac{1}{2}}}{2}. \end{cases} \tag{43}$$

The coefficients $D_{n,n-2}$ and $S_{n,n-1}$ are deduced by solving (41) for $k = n-1$ with the initial conditions $D_{0,-2} = D_{1,-1} = 0$ and using (42) and (43). Using the computer algebra software Maple 9 [27] we get with $p = q^2$

$$D_{n,n-2} \quad = \quad \begin{cases} \frac{1}{3}\beta\,n\,(n-1)\,(2\,n-1) & \text{if} \quad \alpha = 1, \\ \frac{2\,p\,\beta\,n\,(p^{1-n}+p^n)}{(p+1)\,(p-1)^2} - \frac{2\,p^2\,\beta\,(p^n - p^{-n})}{(p+1)\,(p-1)^3} & \text{if} \quad \alpha = \frac{q^{\frac{1}{2}} + q^{-\frac{1}{2}}}{2}; \end{cases} \tag{44}$$

$$S_{n,n-1} \quad = \quad \begin{cases} \beta\,n\,(2\,n-1) & \text{if} \quad \alpha = 1, \\ -\frac{\beta\,n\,(p^{1-n} - p^n)}{p-1} & \text{if} \quad \alpha = \frac{q^{\frac{1}{2}} + q^{-\frac{1}{2}}}{2}. \end{cases} \tag{45}$$

The coefficients $D_{n,n-3}$ and $S_{n,n-2}$ are obtained likewise using Equations (38)-(45) with the initial conditions $D_{1,-2} = D_{2,-1} = 0$

$$D_{n,n-3} = \begin{cases} \frac{1}{30} n\,(n-1)\,(n-2)\,(4\,\beta^2\,n^2 - 8\,\beta^2\,n + 5\,\delta + 3\,\beta^2), & \text{for} \quad \alpha = 1; \\ \left(\frac{p^{2-n}+p^n}{(p^2-1)^2} p\,n - \frac{p^n - p^{-n}}{(p^2-1)^3} 2\,p^3\right)\delta & \\ \quad + \left(\frac{p^n - p^{2-n}}{(p-1)^3\,(p+1)} 2\,p\,n^2 + \frac{p^{3-n} - 7\,p^{2-n} + 7\,p^{n+1} + p^n}{(p-1)^4\,(p+1)} p\,n \right. & \\ \quad\quad \left. + \frac{p^n - p^{-n}}{(p-1)^5\,(p+1)}\,6\,p^3\right)\beta^2, & \text{for} \quad \alpha = \frac{q^{\frac{1}{2}} + q^{-\frac{1}{2}}}{2}. \end{cases} \tag{46}$$

and

$$S_{n,n-2} = \begin{cases} \frac{1}{6}\,n\,(n-1)\,(4\,\beta^2\,n^2 - 8\,\beta^2\,n + 3\,\delta + 3\,\beta^2), & \text{for} \quad \alpha = 1; \\ \frac{p^n - p^{2-n}}{2\,(p^2-1)}\delta\,n + \left(\frac{p^n + p^{2-n}}{(p-1)^2}\,n^2 + \frac{p^n - 3\,p^{n+1} + 3\,p^{2-n} - p^{3-n}}{2\,(p-1)^3}\,n\right)\beta^2, & \text{for} \quad \alpha = \frac{q^{\frac{1}{2}} + q^{-\frac{1}{2}}}{2}. \end{cases} \tag{47}$$

The remaining coefficients $D_{n,k-1}$ and $S_{n,k},\ k = 1 \ldots n-3$ can be computed by following the same procedure.

Additional properties for $\mathbb{D}_x$ and $\mathbb{S}_x$

The product rules for $\mathbb{D}_x$ and $\mathbb{S}_x$ can also be used to write the expressions $\mathbb{D}_x x^n(s)$ and $\mathbb{S}_x x^n(s)$ in terms of the polynomials Q_1 and Q_2 and also in matrix form:

Theorem 2 *For any integer n, the following relations hold*

$$\mathbb{D}_x x^n(s) = \frac{\left(Q_1(x_1(s)) + \sqrt{Q_2(x_1(s))}\right)^n - \left(Q_1(x_1(s)) - \sqrt{Q_2(x_1(s))}\right)^n}{2\sqrt{Q_2(x_1(s))}}; \tag{48}$$

$$\mathbb{S}_x x^n(s) = \frac{\left(Q_1(x_1(s)) + \sqrt{Q_2(x_1(s))}\right)^n + \left(Q_1(x_1(s)) - \sqrt{Q_2(x_1(s))}\right)^n}{2}, \tag{49}$$

where $Q_1(x_1(s)) = \mathbb{S}_x\, x(s) = \alpha\, x_1(s) + \beta$ and $Q_2(x_1(s)$ given by (29).

Proof: Let n be a nonnegative integer. From (27) and (28) for $f(x(s)) = x(s)$ and $g(x(s)) = x^n(s)$, we obtain the relation

$$\begin{bmatrix} \mathbb{D}_x x^{n+1}(s) \\ \mathbb{S}_x x^{n+1}(s) \end{bmatrix} = \begin{bmatrix} Q_1(x_1(s)) & 1 \\ Q_2(x_1(s)) & Q_1(x_1(s)) \end{bmatrix} \begin{bmatrix} \mathbb{D}_x x^n(s) \\ \mathbb{S}_x x^n(s) \end{bmatrix}$$

from which we deduce

$$\begin{bmatrix} \mathbb{D}_x x^{n}(s) \\ \mathbb{S}_x x^{n}(s) \end{bmatrix} = \begin{bmatrix} Q_1(x_1(s)) & 1 \\ Q_2(x_1(s)) & Q_1(x_1(s)) \end{bmatrix}^n \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Since the matrix involved in the previous equation is invertible for its determinant being different from zero, we perform linear algebra calculus to compute its nth power in terms of Q_1, Q_2 and n and deduce (48) and (49) for any nonnegative integer n.

Taking into account that relations (48) and (49) are satisfied for nonnegative integers, these relations for negative integers are obtained by using (31) and (32) for $f(x(s)) = 1$ and $g(x(s)) = x^n(s)$

$$\begin{aligned} \mathbb{D}_x x^{-n}(x) &= -\frac{\mathbb{D}_x x^n(s)}{Q_2(x_1(s))\,[\mathbb{D}_x x^n(s)]^2 - [\mathbb{S}_x x^n(s)]^2} \\ &= \frac{\left(Q_1(x_1(s)) + \sqrt{Q_2(x_1(s))}\right)^{-n} - \left(Q_1(x_1(s)) - \sqrt{Q_2(x_1(s))}\right)^{-n}}{2\sqrt{Q_2(x_1(s))}}; \end{aligned}$$

$$\begin{aligned} \mathbb{S}_x x^{-n}(x) &= \frac{\mathbb{S}_x x^n(s)}{Q_2(x_1(s))\,[\mathbb{D}_x x^n(s)]^2 - [\mathbb{S}_x x^n(s)]^2} \\ &= \frac{\left(Q_1(x_1(s)) + \sqrt{Q_2(x_1(s))}\right)^{-n} + \left(Q_1(x_1(s)) - \sqrt{Q_2(x_1(s))}\right)^{-n}}{2}. \end{aligned}$$

The proof is therefore complete. Notice however that (48) and (49) can also be obtained directly using the equations

$$\begin{aligned} Q_1(x_1(s)) &= \alpha\, x_1(s) + \beta = \frac{x(s+1) + x(s)}{2}; \\ Q_2(x_1(s)) &= (\alpha^2 - 1)\, x_1^2(s) + 2\,\beta\,(\alpha + 1)\, x_1(s) + \delta = \frac{(x(s+1) - x(s))^2}{4} \end{aligned}$$

to express $x(s)$ and $x(s+1)$ in terms of $Q_1(x_1(s))$ and $Q_2(x_1(s))$

$$x(s+1) = Q_1(x_1(s)) + \sqrt{Q_2(x_1(s))}, \quad x(s) = Q_1(x_1(s)) - \sqrt{Q_2(x_1(s))}.$$

□

The following theorem gives relations between the products of the operators $\mathbb{D}_x$ and $\mathbb{S}_x$

$$\mathbb{D}_{x_{-1}} \mathbb{D}_{x_{-2}},\ \mathbb{D}_{x_{-1}} \mathbb{S}_{x_{-2}},\ \mathbb{S}_{x_{-1}} \mathbb{D}_{x_{-2}},\ \mathbb{S}_{x_{-1}} \mathbb{S}_{x_{-2}}.$$

These relations happen to be very important in the next part of this work as well as in the characterization of the classical orthogonal polynomials [14].

Theorem 3 *The following relations hold:*

$$\mathbb{D}_{x_{-1}} \mathbb{S}_{x_{-2}} \mathbb{T}_{-2} = \alpha\, \mathbb{S}_{x_{-1}} \mathbb{D}_{x_{-2}} \mathbb{T}_{-2} + U_1(s)\, \mathbb{D}_{x_{-1}} \mathbb{D}_{x_{-2}} \mathbb{T}_{-2}; \tag{50}$$

$$\mathbb{S}_{x_{-1}} \mathbb{S}_{x_{-2}} \mathbb{T}_{-2} = U_1(s)\, \mathbb{S}_{x_{-1}} \mathbb{D}_{x_{-2}} \mathbb{T}_{-2} + \alpha\, U_2(s)\, \mathbb{D}_{x_{-1}} \mathbb{D}_{x_{-2}} \mathbb{T}_{-2} + \mathbb{I}, \tag{51}$$

where

$$\begin{aligned} U_1(s) &:= U_1(x(s)) = (\alpha^2 - 1)\, x(s) + \beta\, (\alpha + 1); \\ U_2(s) &:= U_2(x(s)) = Q_2(x(s)) = (\alpha^2 - 1)\, x^2(s) + 2\, \beta (\alpha + 1)\, x(s) + \delta. \end{aligned} \tag{52}$$

Here, the operator $\mathbb{T}_\mu$ which acts on the variable s is defined by

$$\mathbb{T}_\mu f(s) = f(s + \frac{\mu}{2}), \quad \text{or} \quad \mathbb{T}_\mu f(x(s)) = f(x_\mu(s)), \tag{53}$$

while $\mathbb{I}$ is the identity operator $\mathbb{I} f(s) = f(s)$.

Proof: We write

$$\mathbb{D}_{x_{-1}} \mathbb{S}_{x_{-2}} f(x_{-2}(s)) = B_1\, \mathbb{S}_{x_{-1}} \mathbb{D}_{x_{-2}} f(x_{-2}(s)) + B_2\, \mathbb{D}_{x_{-1}} \mathbb{D}_{x_{-2}} f(x_{-2}(s)) + B_3\, f(x(s));$$

$$\mathbb{S}_{x_{-1}} \mathbb{S}_{x_{-2}} f(x_{-2}(s)) = C_1\, \mathbb{S}_{x_{-1}} \mathbb{D}_{x_{-2}} f(x_{-2}(s)) + C_2\, \mathbb{D}_{x_{-1}} \mathbb{D}_{x_{-2}} f(x_{-2}(s)) + C_3\, f(x(s)),$$

and use (11) with x replaced by x_{-1} or x_{-2} to transform the previous equations into linear combinations of $f(x(s-1)),\ f(x(s))$ and $f(x(s+1))$. Then we equate the coefficients of $f(x(s-1)),\ f(x(s))$ and $f(x(s+1))$ in the resulting equations and obtain for each equation a system of three linear equations in terms of the unknowns B_i (respectively (C_i)). Solving these two systems, we obtain

$$B_1 = \frac{1}{2} \frac{x(s+1) - x(s-1)}{x(s+\frac{1}{2}) - x(s-\frac{1}{2})}, \quad B_2 = \frac{x(s+1) - 2\, x(s) + x(s-1)}{4}, \quad B_3 = 0,$$

and

$$\begin{aligned} C_1 &= \frac{x(s+1) - 2\, x(s) + x(s-1)}{4}, \quad C_3 = 1, \\ C_2 &= -\frac{1}{8}\, x(s-1)\, x(s+\frac{1}{2}) + \frac{1}{8}\, x(s-1)\, x(s-\frac{1}{2}) + \frac{1}{8}\, x(s+1)\, x(s+\frac{1}{2}) - \frac{1}{8}\, x(s+1)\, x(s-\frac{1}{2}). \end{aligned}$$

The previous equation combined with (22) and (49) yields

$$B_1 = \alpha,\ B_2 = U_1(s),\ B_3 = 0,\ C_1 = U_1(s),\ C_2 = \alpha\, U_2(s),\ C_3 = 1.$$

□

2.3 The operators $\mathbb{F}_x$ and $\mathbb{M}_x$

For the manipulation of the difference equations for orthogonal polynomials, it is sometimes more convenient to work with the operators $\mathbb{F}_x$ and $\mathbb{M}_x$

$$\mathbb{F}_x f(s) = \frac{\Delta}{\Delta x(s-\frac{1}{2})} \frac{\nabla f(s)}{\nabla x(s)}, \quad \mathbb{M}_x f(s) = \frac{1}{2}\left(\frac{\Delta f(s)}{\Delta x(s)} + \frac{\nabla f(s)}{\nabla x(s)}\right), \tag{54}$$

for they share the property of transforming any polynomial in the variable $x(s)$ into a polynomial of lower degree but in the **same** variable $x(s)$. This property is similar to the one of the usual derivative.

2.3.1 The product and quotient rules

Theorem 4 *The following properties hold:*

1. *If $P_n(x(s))$ is a polynomial of degree n in the variable $x(s)$, then $\mathbb{F}_x(P_n(x(s)))$ and $\mathbb{M}_x(P_n(x(s)))$ are polynomials of degree $n-2$ and $n-1$ respectively in $x(s)$.*
2. *$\mathbb{F}_x$ and $\mathbb{M}_x$ obey the product rules*

$$\mathbb{F}_x\mathbb{M}_x f(s) = (2\,\alpha^2 - 1)\,\mathbb{M}_x\mathbb{F}_x f(s) + 2\,\alpha\, U_1(s)\,\mathbb{F}_x\mathbb{F}_x f(s); \tag{55}$$

$$\mathbb{M}_x\mathbb{M}_x f(s) = \alpha\,\mathbb{F}_x f(s) + 2\,\alpha\, U_1(s)\,\mathbb{M}_x\mathbb{F}_x f(s) + (2\,\alpha^2 - 1)\, U_2(s)\,\mathbb{F}_x\mathbb{F}_x f(s), \tag{56}$$

where the expression $\mathbb{F}_x\mathbb{M}_x f(s)$ refers to $\mathbb{F}_x\left(\mathbb{M}_x f(s)\right)$ and

$$U_1(s) = (\alpha+1)\,[(\alpha-1)\,x(s)+\beta],\ U_2(s) = (\alpha^2-1)\,x^2(s) + 2\,\beta\,(\alpha+1)\,x(s) + \delta. \tag{57}$$

Proof: From the definition of $\mathbb{D}_x$, $\mathbb{S}_x$, $\mathbb{F}_x$ and $\mathbb{M}_x$ (see (11) and (54)) we have

$$\mathbb{F}_x(f(x(s))) = \frac{\Delta}{\Delta x(s-\frac{1}{2})} \frac{\nabla f(x(s))}{\nabla x(s)} = \mathbb{D}_{x_{-1}}\mathbb{D}_{x_{-2}} f(x_{-2}(s)), \tag{58}$$

$$\mathbb{M}_x(f(x(s))) = \frac{1}{2}\left(\frac{\Delta f(x(s))}{\Delta x(s)} + \frac{\nabla f(x(s))}{\nabla x(s)}\right) = \mathbb{S}_{x_{-1}}\mathbb{D}_{x_{-2}} f(x_{-2}(s)). \tag{59}$$

The previous relations are equivalent to

$$\mathbb{F}_x = \mathbb{D}_{x_{-1}}\,\mathbb{D}_{x_{-2}}\,\mathbb{T}_{-2}, \quad \mathbb{M}_x = \mathbb{S}_{x_{-1}}\,\mathbb{D}_{x_{-2}}\,\mathbb{T}_{-2}. \tag{60}$$

Let $P_n(x(s))$ be a polynomial of degree n in $x(s)$. From (58) and Proposition 1, we deduce that $\mathbb{D}_{x_{-2}}$ transforms $P_n(x_{-2}(s))$ into a polynomial of degree $n-1$ in $x_{-1}(s)$ and $\mathbb{D}_{x_{-1}}$ transforms $\mathbb{D}_{x_{-2}}P_n(x_{-2}(s))$ into a polynomial of degree $n-2$ in $x_0(s) = x(s)$. Therefore, $\mathbb{F}_x P_n(x(s))$ is a polynomial of degree $n-2$ in $x(s)$. Similarly, we deduce that $\mathbb{M}_x P_n(x(s))$ is a polynomial of degree $n-1$ in $x(s)$.

Before starting the proof of the second statement, one should keep in mind the relations

$$\mathbb{T}_\mu \mathbb{D}_x = \mathbb{D}_{x_\mu}, \quad \mathbb{T}_\mu \mathbb{S}_x = \mathbb{S}_{x_\mu}. \tag{61}$$

In the first step, we combine (60) with (61) to get

$$\begin{aligned}
\mathbb{F}_x\,\mathbb{M}_x &= \mathbb{D}_{x_{-1}}\,\mathbb{D}_{x_{-2}}\,\mathbb{T}_{-2}\left[\mathbb{S}_{x_{-1}}\,\mathbb{D}_{x_{-2}}\,\mathbb{T}_{-2}\right] \\
&= \mathbb{D}_{x_{-1}}\,\mathbb{D}_{x_{-2}}\left[\mathbb{S}_{x_{-3}}\,\mathbb{D}_{x_{-4}}\,\mathbb{T}_{-4}\right] \\
&= \mathbb{D}_{x_{-1}}\left[\mathbb{D}_{x_{-2}}\,\mathbb{S}_{x_{-3}}\right]\mathbb{D}_{x_{-4}}\,\mathbb{T}_{-4}. \\
&= \mathbb{D}_{x_{-1}}\left[\mathbb{D}_{x_{-2}}\,\mathbb{S}_{x_{-3}}\,\mathbb{T}_{-3}\right]\mathbb{T}_3\left[\mathbb{D}_{x_{-4}}\,\mathbb{T}_{-4}\right].
\end{aligned}$$

Use of (50) (and later on (27)) transform the previous equation into

$$
\begin{aligned}
\mathbb{F}_x\,\mathbb{M}_x &= \mathbb{D}_{x_{-1}}\left[\alpha\,\mathbb{S}_{x_{-2}}\,\mathbb{D}_{x_{-3}}\mathbb{T}_{-3} + U_1(x_{-1}(s))\,\mathbb{D}_{x_{-2}}\,\mathbb{D}_{x_{-3}}\mathbb{T}_{-3}\right]\,\mathbb{T}_3\,\mathbb{D}_{x_{-4}}\,\mathbb{T}_{-4} \\
&= \alpha\,\mathbb{D}_{x_{-1}}\,\mathbb{S}_{x_{-2}}\,\mathbb{D}_{x_{-3}}\,\mathbb{D}_{x_{-4}}\,\mathbb{T}_{-4} + \mathbb{D}_{x_{-1}}\left[U_1(x_{-1}(s))\,\mathbb{D}_{x_{-2}}\,\mathbb{D}_{x_{-3}}\mathbb{T}_{-3}\right]\,\mathbb{T}_3\,\mathbb{D}_{x_{-4}}\,\mathbb{T}_{-4} \\
&= \alpha\left[\alpha\,\mathbb{S}_{x_{-1}}\,\mathbb{D}_{x_{-2}}\mathbb{T}_{-2} + U_1(x(s))\,\mathbb{D}_{x_{-1}}\,\mathbb{D}_{x_{-2}}\mathbb{T}_{-2}\right]\mathbb{T}_2\,\mathbb{D}_{x_{-3}}\,\mathbb{D}_{x_{-4}}\,\mathbb{T}_{-4} \\
&\qquad + \mathbb{D}_{x_{-1}}\left[U_1(x_{-1}(s))\,\mathbb{D}_{x_{-2}}\,\mathbb{D}_{x_{-3}}\mathbb{T}_{-3}\right]\,\mathbb{T}_3\,\mathbb{D}_{x_{-4}}\,\mathbb{T}_{-4} \\
&= \left(2\,\alpha^2 - 1\right)\,\mathbb{M}_x\,\mathbb{F}_x + 2\,\alpha\,U_1(x(s))\,\mathbb{F}_x\,\mathbb{F}_x,
\end{aligned}
$$

taking into account that

$$\mathbb{D}_{x_{-1}}\,[U_1(x_{-1}(s))] = \alpha^2 - 1 \;\text{ and }\; \mathbb{S}_{x_{-1}}\,[U_1(x_{-1}(s))] = \alpha\,U_1(x(s).$$

Relation (56) is obtained in the same way using (28), (50) and (51). □

Theorem 5 *The following product and quotient rules hold:*

$$
\begin{aligned}
\mathbb{F}_x\,(fg) &= \mathbb{F}_x(f)\,g + f\,\mathbb{F}_x(g) + 2\,\alpha\,\mathbb{M}_x(f)\,\mathbb{M}_x(g) \\
&+ 2\,U_1\,[\mathbb{F}_x(f)\,\mathbb{M}_x(g) + \mathbb{M}_x(f)\,\mathbb{F}_x(g)] + 2\,\alpha\,U_2\,\mathbb{F}_x(f)\,\mathbb{F}_x(g);
\end{aligned} \tag{62}
$$

$$
\begin{aligned}
\mathbb{M}_x\,(fg) &= \mathbb{M}_x(f)\,g + f\,\mathbb{M}_x(g) + 2\,U_1\,\mathbb{M}_x(f)\,\mathbb{M}_x(g) \\
&+ 2\,\alpha\,U_2\,[\mathbb{F}_x(f)\,\mathbb{M}_x(g) + \mathbb{M}_x(f)\,\mathbb{F}_x(g)] + 2\,U_1\,U_2\,\mathbb{F}_x(f)\,\mathbb{F}_x(g);
\end{aligned} \tag{63}
$$

$$
\begin{aligned}
\mathbb{F}_x\left(\frac{f}{g}\right) &= \Big\{2\,\alpha\,f\,\left[\mathbb{M}_x(g)\right]^2 - 2\,\alpha\,\mathbb{M}_x(f)\,\mathbb{M}_x(g)\,g + \mathbb{F}_x(f)\,g^2 - f\,g\,\mathbb{F}_x(g) \\
&+ 2\,\alpha\,U_2\,\mathbb{F}_x(f)\,\mathbb{F}_x(g)\,g - 2\,\alpha\,U_2\,f\,\left[\mathbb{F}_x(g)\right]^2 \\
&+ 2\,U_1\,\mathbb{F}_x(f)\,\mathbb{M}_x(g)\,g - 2\,U_1\,\mathbb{M}_x(f)\,\mathbb{F}_x(g)\,g\;\Big\}\,/ \\
&\quad \Big\{U_2\,\left[U_1\,\mathbb{F}_x(g) + \alpha\,\mathbb{M}_x(g)\right]^2 - \left[g + 2\,U_1\,\mathbb{M}_x(g) + 2\,\alpha\,U_2\,\mathbb{F}_x(g)\right]^2\Big\}\;g;
\end{aligned} \tag{64}
$$

$$
\begin{aligned}
\mathbb{M}_x\left(\frac{f}{g}\right) &= \Big\{2\,U_1\,U_2\,f\,\left[\mathbb{F}_x(g)\right]^2 - 2\,U_1\,U_2\,\mathbb{F}_x(f)\,\mathbb{F}_x(g)\,g + \mathbb{M}_x(f)\,g^2 - f\,g\,\mathbb{M}_x(g) \\
&+ 2\,\alpha\,U_2\,\mathbb{M}_x(f)\,\mathbb{F}_x(g)\,g - 2\,\alpha\,U_2\,\mathbb{F}_x(f)\,\mathbb{M}_x(g)\,g \\
&+ 2\,U_1\,\mathbb{M}_x(f)\,\mathbb{M}_x(g)\,g - 2\,U_1\,f\,\left[\mathbb{M}_x(g)\right]^2\Big\}\,/ \\
&\quad \Big\{U_2\,\left[U_1\,\mathbb{F}_x(g) + \alpha\,\mathbb{M}_x(g)\right]^2 - \left[g + 2\,U_1\,\mathbb{M}_x(g) + 2\,\alpha\,U_2\,\mathbb{F}_x(g)\right]^2\Big\}\;g,
\end{aligned} \tag{65}
$$

with

$$f \equiv f(s), U_j \equiv U_j(s) \text{ and } g \equiv g(s) \neq 0,\; \forall s \in (a,b).$$

Proof: Use of (60) gives the equation

$$\mathbb{F}_x\,(f(x(s))\,g(x(s))) = \mathbb{D}_{x_{-1}}\,\mathbb{D}_{x_{-2}}\,(f(x_{-2}(s))\,g(x_{-2}(s)))$$

which using (27) and (28) is transformed into

$$
\begin{aligned}
& \mathbb{F}_x\left(f(x(s))\, g(x(s))\right) = \\
& \mathbb{D}_{x_{-1}} \left[\mathbb{D}_{x_{-2}} f(x_{-2}(s))\, \mathbb{S}_{x_{-2}}\, g(x_{-2}(s)) + \mathbb{S}_{x_{-2}} f(x_{-2}(s))\, \mathbb{D}_{x_{-2}}\, g(x_{-2}(s))\right] \\
= \; & \mathbb{D}_{x_{-1}} \mathbb{D}_{x_{-2}} f(x_{-2}(s))\, \mathbb{S}_{x_{-1}} \mathbb{S}_{x_{-2}}\, g(x_{-2}(s)) + \mathbb{S}_{x_{-1}} \mathbb{D}_{x_{-2}} f(x_{-2}(s))\, \mathbb{D}_{x_{-1}} \mathbb{S}_{x_{-2}}\, g(x_{-2}(s)) \\
+ \; & \mathbb{D}_{x_{-1}} \mathbb{S}_{x_{-2}} f(x_{-2}(s))\, \mathbb{S}_{x_{-1}} \mathbb{D}_{x_{-2}}\, g(x_{-2}(s)) + \mathbb{S}_{x_{-1}} \mathbb{S}_{x_{-2}} f(x_{-2}(s))\, \mathbb{D}_{x_{-1}} \mathbb{D}_{x_{-2}}\, g(x_{-2}(s)).
\end{aligned}
$$

Elimination of the products of the form $\mathbb{S}_{x_{-1}} \mathbb{S}_{x_{-2}} f(x_{-2}(s))$ and $\mathbb{D}_{x_{-1}} \mathbb{S}_{x_{-2}} f(x_{-2}(s))$ in the previous equation using (50) and (51) produces

$$
\begin{aligned}
\mathbb{F}_x\left(f(x(s))\, g(x(s))\right) &= \mathbb{F}_x(f)\ [U_1(s)\, \mathbb{M}_x(g) + \alpha\, U_2(s)\, \mathbb{F}_x(g) + g] + \mathbb{M}_x(f)\ [\alpha\, \mathbb{M}_x(g) + U_1(s)\, \mathbb{F}_x(g)] \\
&\quad + \mathbb{F}_x(g)\ [U_1(s)\, \mathbb{M}_x(f) + \alpha\, U_2(s)\, \mathbb{F}_x(f) + f] + \mathbb{M}_x(g)\ [\alpha\, \mathbb{M}_x(f) + U_1(s)\, \mathbb{F}_x(f)] \\
&= \mathbb{F}_x(f)\, g + f\, \mathbb{F}_x(g) + 2\, \alpha\, \mathbb{M}_x(f)\, \mathbb{M}_x(g) \\
&\quad + 2\, U_1\, [\mathbb{F}_x(f)\, \mathbb{M}_x(g) + \mathbb{M}_x(f)\, \mathbb{F}_x(g)] + 2\, \alpha\, U_2\, \mathbb{F}_x(f)\, \mathbb{F}_x(g).
\end{aligned}
$$

Relation (63) is derived in the same way.

The quotient rules (64) and (65) are derived by applying the product rules (62) and (63) to $f(s) \times \frac{1}{g(s)}$. In fact, we first express $\mathbb{F}_x\left(\frac{1}{g(s)}\right)$ and $\mathbb{M}_x\left(\frac{1}{g(s)}\right)$ for $g(s) \neq 0$, $\forall s \in (a, b)$, in terms of $g(s)$, $\mathbb{F}_x g(s)$ and $\mathbb{M}_x g(s)$. For this purpose, we take $f(s) = \frac{1}{g(s)}$ in (62) and (63) and get the linear system in $\mathbb{F}_x\left(\frac{1}{g}\right)$ and $\mathbb{M}_x\left(\frac{1}{g}\right)$

$$
\begin{aligned}
& [g + 2\, U_1\, \mathbb{M}_x(g) + 2\, \alpha\, U_2\, \mathbb{F}_x(g)]\, \mathbb{F}_x\left(\frac{1}{g}\right) + 2\, [U_1\, \mathbb{F}_x(g) + \alpha\, \mathbb{M}_x(g)]\, \mathbb{M}_x\left(\frac{1}{g}\right) = -\frac{1}{g}\mathbb{F}_x(g) \\
& 2U_2\ [U_1\, \mathbb{F}_x(g) + \alpha\, \mathbb{M}_x(g)]\, \mathbb{F}_x\left(\frac{1}{g}\right) + [g + 2\, U_1\, \mathbb{M}_x(g) + 2\, \alpha\, U_2\, \mathbb{F}_x(g)]\, \mathbb{M}_x\left(\frac{1}{g}\right) = -\frac{1}{g}\mathbb{M}_x(g)
\end{aligned}
$$

whose determinant is

$$
\begin{aligned}
[g + 2\, U_1\, \mathbb{M}_x(g) + 2\, \alpha\, U_2\, \mathbb{F}_x(g)]^2 - 4U_2\ [U_1\, \mathbb{F}_x(g) + \alpha\, \mathbb{M}_x(g)]^2 &= g(s-1)\, g(s+1) \\
&\neq 0,\ \forall s \in (a, b).
\end{aligned}
$$

Therefore, $\mathbb{F}_x\left(\frac{1}{g}\right)$ and $\mathbb{M}_x\left(\frac{1}{g}\right)$ are uniquely determined from the previous linear system, and quotient rules $\mathbb{F}_x\left(\frac{f}{g}\right)$ and $\mathbb{M}_x\left(\frac{f}{g}\right)$ are deduced by application of the product rules (62) and (63) to $f(s) \times \frac{1}{g(s)}$. □

2.3.2 Consequences of the product and quotient rules

The product rules provide the recurrence relation for the coefficients $F_{n,k}$ and $M_{n,k}$ of the expansion

$$
\mathbb{F}_x x^n(s) = \sum_{k=0}^{n-2} F_{n,k}\, x^k(s), \quad \mathbb{M}_x x^n(s) = \sum_{k=0}^{n-1} M_{n,k}\, x^k(s). \tag{66}
$$

Proposition 3 *The coefficients $F_{n,k}$ and $M_{n,k}$ satisfy*

$$F_{n+1,k} - 2\,\alpha\, M_{n,k} - 2\,\beta\,(\alpha+1)\, F_{n,k} + (1-2\,\alpha^2)\, F_{n,k-1} = 0,\ 0 \le k \le n-1, \tag{67}$$

$$\begin{aligned} &M_{n+1,k} - 2\,\beta\,(\alpha+1)\, M_{n,k} - 2\,\alpha\,(\alpha^2-1)\, F_{n,k-2} - 2\,\alpha\,\delta\, F_{n,k} \\ &\quad -4\,\alpha\,\beta\,(\alpha+1)\, F_{n,k-1} + (1-2\,\alpha^2)\, M_{n,k-1} - \delta_{n,k} = 0,\ 0 \le k \le n, \end{aligned} \tag{68}$$

where $\delta_{n,k}$ is the Kronecker symbol, with the convention

$$F_{n+1,n} = F_{n,n+1} = M_{n,n+1} = F_{n,n} = M_{n,n} = F_{n,-1} = F_{n,-2} = M_{n,-1} = 0,\ n \ge 0.$$

Proof: The proof is obtained using (66) and the product rules (62) and (63) for $f(s) = x^n(s),\ g(s) = x(s)$. □

The previous proposition allows to compute recurrently the coefficients $F_{n,k}$ and $M_{n,k}$. Furthermore, these coefficients can also be computed via the following relations in terms of the coefficients $D_{n,k}$ and $S_{n,k}$.

Proposition 4 *The coefficients $D_{n,k}$ and $S_{n,k}$ of the expansions (26) are related to the coefficients $F_{n,k}$ and $M_{n,k}$ of the expansions (66) by*

$$F_{n,k} = \sum_{j=k+1}^{n-1} D_{n,j}\, D_{j,k},\ 0 \le k \le n-2, \tag{69}$$

$$M_{n,k} = \sum_{j=k}^{n-1} D_{n,j}\, S_{j,k},\ 0 \le k \le n-1. \tag{70}$$

Proof: The proof follows from Equations (26), (58), (59) and (66). Notice that the coefficients $F_{n,n-2-k},\ M_{n,n-1-k},\ 0 \le k \le 2$ are computed explicitly in (97) and (98). □

As another consequence of the product rule, we state the following:

Proposition 5 *For any nonnegative integer n, the following relations hold*

$$\mathbb{F}_x\, x^n(s) = \sum_{k=0}^{n-1} \frac{x^k(s)}{2\sqrt{U_2(s)}} \left[\left(V_1(s) + 2\,\alpha\sqrt{U_2(s)}\right)^{n-1-k} - \left(V_1(s) - 2\,\alpha\sqrt{U_2(s)}\right)^{n-1-k}\right]; \tag{71}$$

$$\mathbb{M}_x\, x^n(s) = \sum_{k=0}^{n-1} \frac{x^k(s)}{2} \left[\left(V_1(s) + 2\,\alpha\sqrt{U_2(s)}\right)^{n-1-k} + \left(V_1(s) - 2\,\alpha\sqrt{U_2(s)}\right)^{n-1-k}\right]; \tag{72}$$

$$\mathbb{F}_x \frac{1}{x^n(s)} = -\sum_{k=1}^{n} \frac{x^{-k}(s)}{2\sqrt{U_2(s)}} \left[\left(V_1(s) + 2\,\alpha\sqrt{U_2(s)}\right)^{k-n-1} - \left(V_1(s) - 2\,\alpha\sqrt{U_2(s)}\right)^{k-n-1}\right]; \tag{73}$$

$$\mathbb{M}_x \frac{1}{x^n(s)} = -\sum_{k=1}^{n} \frac{x^{-k}(s)}{2} \left[\left(V_1(s) + 2\,\alpha\sqrt{U_2(s)}\right)^{k-n-1} + \left(V_1(s) - 2\,\alpha\sqrt{U_2(s)}\right)^{k-n-1}\right], \tag{74}$$

with $V_1(s) = x(s) + 2\,U_1(s)$ where $U_1(s)$ and $U_2(s)$ are given by (57).

Proof: Using the product rules (62), (63) for $f(x(s)) = x^n(s),\ g(x(s)) = x(s)$ (respectively $f(x(s)) = \frac{1}{x^n(s)},\ g(x(s)) = x(s)$), we obtain

$$\begin{bmatrix} \mathbb{F}_x x^n(s) \\ \mathbb{M}_x x^n(s) \end{bmatrix} = \begin{bmatrix} V_1(x(s)) & 2\,\alpha \\ 2\,\alpha\, U_2(x(s)) & V_1(x(s)) \end{bmatrix} \begin{bmatrix} \mathbb{F}_x x^{n-1}(s) \\ \mathbb{M}_x x^{n-1}(s) \end{bmatrix} + \begin{bmatrix} 0 \\ x^{n-1}(s) \end{bmatrix}$$

and

$$\begin{bmatrix} \mathbb{F}_x \frac{1}{x^{n-1}(s)} \\ \\ \mathbb{M}_x \frac{1}{x^{n-1}(s)} \end{bmatrix} = \begin{bmatrix} V_1(x(s)) & 2\,\alpha \\ \\ 2\,\alpha\, U_2(x(s)) & V_1(x(s)) \end{bmatrix} \begin{bmatrix} \mathbb{F}_x \frac{1}{x^{n}(s)} \\ \\ \mathbb{M}_x \frac{1}{x^{n}(s)} \end{bmatrix} + \begin{bmatrix} 0 \\ \\ \frac{1}{x^{n}(s)} \end{bmatrix}$$

respectively.

The changes of variables

$$\overrightarrow{W_n} = \begin{bmatrix} \mathbb{F}_x x^n(s) \\ \\ \mathbb{M}_x x^n(s) \end{bmatrix},\ \overrightarrow{X_n} = \begin{bmatrix} 0 \\ \\ x^n(s) \end{bmatrix},\ \overrightarrow{Y_n} = \begin{bmatrix} \mathbb{F}_x \frac{1}{x^n(s)} \\ \\ \mathbb{M}_x \frac{1}{x^n(s)} \end{bmatrix},\ \overrightarrow{Z_n} = \begin{bmatrix} 0 \\ \\ \frac{1}{x^n(s)} \end{bmatrix},\ A = \begin{bmatrix} V_1 & 2\,\alpha \\ \\ 2\,\alpha\, U_2 & V_1 \end{bmatrix}$$

transform the previous two equations into relations

$$\overrightarrow{W_n} = A\,\overrightarrow{W_{n-1}} + \overrightarrow{X_{n-1}},\quad \overrightarrow{Y_{n-1}} = A\,\overrightarrow{Y_n} + \overrightarrow{Z_n},$$

whose iteration taking into account that the matrix A is invertible gives

$$\overrightarrow{W_n} = \sum_{k=0}^{n-1} A^{n+1-k}\,\overrightarrow{X_k},\quad \overrightarrow{Y_n} = -\sum_{k=1}^{n} A^{k-n-1}\,\overrightarrow{Z_k}.$$

Equations (71)-(74) are obtained from the previous ones by computing the kth power of the matrix A in terms of its eigenvalues $V_1(s) + 2\,\alpha\,\sqrt{U_2(s)}$ and $V_1(s) - 2\,\alpha\,\sqrt{U_2(s)}$. □

As a fourth consequence of the product rules, we shall prove another important result concerning the linear divided-difference equation of higher order satisfied by products of functions each of them satisfying a linear divided-difference equation. For this purpose, we start with the following preliminaries.

Lemma 1 *Let $f(s)$ be a function of the variable $x(s)$ satisfying a second-order divided-difference equation*

$$\mathbb{F}_x f(s) + a_1(s)\,\mathbb{M}_x f(s) + a_0(s)\,f(s) = 0, \tag{75}$$

where a_0 and a_1 are given functions of $x(s)$.

Then the expressions $\mathbb{F}_x\mathbb{F}_x f(s)$, $\mathbb{M}_x\mathbb{F}_x f(s)$, $\mathbb{F}_x\mathbb{M}_x f(s)$, $\mathbb{M}_x\mathbb{M}_x f(s)$ and $\mathbb{F}_x f(s)$ can be written uniquely in the form

$$c_1(s)\,f(s) + c_2(s)\,\mathbb{M}_x f(s),$$

where $c_1(s)$ and $c_2(s)$ are functions of $a_0(s)$ and $a_1(s)$.

Proof: Assuming that $x(s+1) \neq x(s)$ for $s \in (a\,,b)$, Equation (75) is equivalent to

$$(\nabla x_1(s)\,a_1(s) + 2)\,f(s+1) + C_0(s)\,f(s) + (\nabla x_1(s)\,a_1(s) - 2)\,f(s-1) = 0, \tag{76}$$

where $C_0(s)$ is a function of $a_1(s)$, $a_0(s)$ and $x(s)$.

1. If $a_1(s) = \pm\frac{2}{\nabla x_1(s)}$, then the previous equation is equivalent to $\frac{f(s+1)}{f(s)} = C_1(s)$; therefore, $f(s+j)$, $j = 1\ldots$ is proportional to $f(s)$.

2. If $a_1(s) \neq \pm\frac{2}{\nabla x_1(s)}$, then from (76), we get

$$
\begin{aligned}
&f(s-2) = \\
&\frac{([-4+2\,a_1\,(s)\,\nabla x_1\,(s)-2\,a_1\,(s-1)\,\nabla x_1\,(s-1)+a_1\,(s-1)\,\nabla x_1\,(s-1)\,a_1\,(s)\,\nabla x_1\,(s)]\,f\,(s)}{(-2+a_1\,(s)\,\nabla x_1\,(s))\,(-2+a_1\,(s-1)\,\nabla x_1\,(s-1))} \\
&+\frac{(C_0\,(s-1)\,C_0\,(s))\,f\,(s)}{(-2+a_1\,(s)\,\nabla x_1\,(s))\,(-2+a_1\,(s-1)\,\nabla x_1\,(s-1))} \\
&+\frac{C_0\,(s-1)\,(2+a_1\,(s)\,\nabla x_1\,(s))\,f\,(s+1)}{(-2+a_1\,(s)\,\nabla x_1\,(s))\,(-2+a_1\,(s-1)\,\nabla x_1\,(s-1))}
\end{aligned}
\tag{77}
$$

$$
\begin{aligned}
f(s-1) &= \frac{(\nabla x_1(s)\,a_1(s)+2)\,f(s+1)+C_0(s)\,f(s)}{\nabla x_1(s)\,a_1(s)-2}; \\
f(s+2) &= \frac{(\Delta x_1(s)\,a_1(s+1)-2)\,f(s)-C_0(s+1)\,f(s+1)}{\Delta x_1(s)\,a_1(s+1)+2}.
\end{aligned}
$$

Then, we use the definition of the operators $\mathbb{F}_x$ and $\mathbb{M}_x$ of (54) to write $\mathbb{F}_x\mathbb{F}_x f(s)$, $\mathbb{M}_x\mathbb{F}_x f(s)$, $\mathbb{F}_x\mathbb{M}_x f(s)$ and $\mathbb{M}_x\mathbb{M}_x f(s)$ as linear combination of $f(s+j)$, $-2 \leq j \leq 2$.

Next, we use the previous equations to write the expressions $\mathbb{F}_x\mathbb{F}_x f(s)$, $\mathbb{M}_x\mathbb{F}_x f(s)$, $\mathbb{F}_x\mathbb{M}_x f(s)$ and $\mathbb{M}_x\mathbb{M}_x f(s)$ as linear combination of $f(s)$ and $f(s+1)$ only. Finally we use (75) and the following equation linking the operators $\mathbb{D}_x$, $\mathbb{F}_x$ and $\mathbb{M}_x$

$$f(s+1) = f(s) + \Delta x(s)\mathbb{M}_x f(s) + \frac{1}{2}\nabla x_1(s)\,\Delta x(s)\,\mathbb{F}_x f(s)$$

to convert the expressions $\mathbb{F}_x\mathbb{F}_x f(s)$, $\mathbb{M}_x\mathbb{F}_x f(s)$, $\mathbb{F}_x\mathbb{M}_x f(s)$ and $\mathbb{M}_x\mathbb{M}_x f(s)$ from the linear combination of $f(s)$ and $f(s+1)$ to the linear combination of $f(s)$ and $\mathbb{M}_x f(s)$.

□

Theorem 6 *Let $f(s)$ and $g(s)$ be two functions of the variable $x(s)$ satisfying respectively*

$$\mathbb{F}_x f(s) + a_1(s)\,\mathbb{M}_x f(s) + a_0(s)\,f(s) = 0,\ \mathbb{F}_x g(s) + b_1(s)\,\mathbb{M}_x g(s) + b_0(s)\,g(s) = 0, \tag{78}$$

where a_j and b_j are given functions of $x(s)$.

Then, the product $f(s)\,g(s)$ is a solution of a fourth-order divided-difference equation of the form

$$I_4(s)\,\mathbb{F}_x\mathbb{F}_x y(s) + I_3(s)\,\mathbb{M}_x\mathbb{F}_x y(s) + I_2(s)\,\mathbb{F}_x y(s) + I_1(s)\,\mathbb{M}_x y(s) + I_0(s)\,y(s) = 0$$

where I_j are functions of a_j and b_j. If the $a_j(s)$, $j = 0, 1$ and the $b_j(s)$, $j = 0, 1$ are rational functions in $x(s)$, then the coefficients $I_j(s)$, $j = 0 \ldots 4$ can be chosen to be polynomials in the variable $x(s)$.

Proof: We apply the identity operator as well as the operators $\mathbb{F}_x\mathbb{F}_x$, $\mathbb{M}_x\mathbb{F}_x$, $\mathbb{F}_x$ and $\mathbb{M}_x$ to the equation

$$y(s) = f(s)\,g(s),$$

and use the product rules (62) and (63) to get five equations whose right-hand sides are linear combinations of expressions of the form $p_1(s)\,p_2(s)$ with

$$p_j(s) \in \{\mathbb{F}_x\mathbb{F}_x h_j(s),\, \mathbb{M}_x\mathbb{F}_x h_j(s),\, \mathbb{F}_x\mathbb{M}_x h_j(s),\, \mathbb{M}_x\mathbb{M}_x h_j(s),\, \mathbb{F}_x h_j(s)\}\,,\ j = 1,\, 2,$$

with $h_1 = f$, $h_2 = g$. These right-hand sides are transformed by means of the previous lemma into linear combinations of

$$f(s)\, g(s),\; f(s)\, \mathbb{M}_x g(s),\; [\mathbb{M}_x f(s)]\, g(s) \text{ and } [\mathbb{M}_x f(s)]\, \mathbb{M}_x g(s).$$

Thus, these five equations can be written as

$$\begin{array}{rcl} X_{0,0} & = & y(s), \\ c_{2,1}(s)\, X_{0,0}(s) + c_{2,2}(s)\, X_{0,1}(s) + c_{2,3}(s)\, X_{1,0}(s) + c_{2,4}(s)\, X_{1,1}(s) & = & \mathbb{M}_x y(s), \\ c_{3,1}(s)\, X_{0,0}(s) + c_{3,2}(s)\, X_{0,1}(s) + c_{3,3}(s)\, X_{1,0}(s) + c_{3,4}(s)\, X_{1,1}(s) & = & \mathbb{F}_x y(s), \\ c_{4,1}(s)\, X_{0,0}(s) + c_{4,2}(s)\, X_{0,1}(s) + c_{4,3}(s)\, X_{1,0}(s) + c_{4,4}(s)\, X_{1,1}(s) & = & \mathbb{M}_x \mathbb{F}_x y(s), \\ c_{5,1}(s)\, X_{0,0}(s) + c_{5,2}(s)\, X_{0,1}(s) + c_{5,3}(s)\, X_{1,0}(s) + c_{5,4}(s)\, X_{1,1}(s) & = & \mathbb{F}_x \mathbb{F}_x y(s), \end{array} \tag{79}$$

with the notations

$$X_{0,0} = f(s)\, g(s),\; X_{0,1} = f(s)\, \mathbb{M}_x g(s),\; X_{1,0} = [\mathbb{M}_x f(s)]\; g(s),\; X_{1,1} = [\mathbb{M}_x f(s)]\; \mathbb{M}_x g(s)$$

where $c_{j,k}$ are functions of a_j and b_j. The system (79) contains 5 linear equations for 4 unknowns, namely $X_{j,k}(s),\; j,k = 0,1$. For the solutions of this system to exist, it is necessary for $y(s)$ to satisfy the equation

$$\begin{vmatrix} 1 & 0 & 0 & 0 & y(s) \\ c_{2,1}(s) & c_{2,2}(s) & c_{2,3}(s) & c_{2,4}(s) & \mathbb{M}_x y(s) \\ c_{3,1}(s) & c_{3,2}(s) & c_{3,3}(s) & c_{3,4}(s) & \mathbb{F}_x y(s) \\ c_{4,1}(s) & c_{4,2}(s) & c_{4,3}(s) & c_{4,4}(s) & \mathbb{M}_x \mathbb{F}_x y(s) \\ c_{5,1}(s) & c_{5,2}(s) & c_{5,3}(s) & c_{5,4}(s) & \mathbb{F}_x \mathbb{F}_x y(s) \end{vmatrix} = 0, \tag{80}$$

which is the fourth-order divided-difference equation desired. □

As consequence of this theorem, we claim the following:

Corollary 2 *If $f_j,\; j = 1, \dots n$ are functions of the variable $x(s)$ such that any f_j satisfies a linear divided-difference equation of order r_j involving only the operators $\mathbb{F}_x$ and $\mathbb{M}_x$, then the product $f = \prod_{j=1}^{n} f_j$ satisfies a divided-difference equation of order $r = \prod_{j=1}^{n} r_j$ involving only (at most) the operators*

$$\mathbb{M}_x^j\, \mathbb{F}_x^k,\; j = 0,\, 1,\; \text{and } 0 \le 2\,k + j \le r = \prod_{j=0}^{n} r_j.$$

3 Recurrence coefficients for classical orthogonal polynomials

3.1 From orthogonality to second-order difference equation

The definition of classical orthogonal polynomials given in [5] is not similar to those of the very classical orthogonal polynomials, because, according to this definition, for a family of polynomials to be classical, it should be orthogonal with respect to a weight function satisfying a Pearson-type equation; and, should in addition, satisfy a second-order difference equation. The requirement for P_n to satisfy a second-order difference equation, which in the case of the very classical orthogonal polynomials is a consequence of the orthogonality, is redundant. This condition can be omitted.

Let us remind that Atakishiyev, Rahman and Suslov [5], using the second-order difference equation (1) satisfied by an orthogonal family (P_n), the Pearson-type equation (8) satisfied by the orthogonality weight and the border conditions (10), obtained the orthogonality relation (5). Here, we prove the converse: The orthogonality relation plus the Pearson-type equation and the border conditions (all provided in Definition 1) lead to the second-order divided-difference equation of form (1).

Theorem 7 *Let (P_n) be a sequence of polynomials orthogonal with respect to a weight function ρ (see (5)) satisfying (8) and (10). Then, each P_n satisfies*

$$\phi(x(s))\,\mathbb{F}_x P_n(x(s)) + \psi(x(s))\,\mathbb{M}_x P_n(x(s)) + \lambda_n\,P_n(x(s)) = 0, \tag{81}$$

where λ_n is a constant term given by

$$\lambda_n = -D_{n,n-1}\,(\phi_2\,D_{n-1,n-2} + \psi_1\,S_{n-1,n-1}), \tag{82}$$

with

$$\phi(x(s)) = \phi_2\,x^2(s) + \phi_1\,x(s) + \phi_0,\ \ \psi(x(s)) = \psi_1\,x(s) + \psi_0. \tag{83}$$

In order to simplify the proof, we state and prove the following lemma

Lemma 2 *Under the hypothesis of the previous theorem, the following identities hold:*

$$\begin{aligned}\Delta\left[\sigma(s)\,\rho(s)\,\frac{\nabla P_n(x(s))}{\nabla x(s))}\right] P_m(x(s)) - \Delta\left[\sigma(s)\,\rho(s)\,\frac{\nabla P_m(x(s))}{\nabla x(s))}\right] P_n(x(s))\\ = \Delta\left\{\sigma(s)\,\rho(s)\,W(P_n(x(s)), P_m(x(s)))\right\};\end{aligned} \tag{84}$$

$$\begin{aligned}\Delta\left[\sigma(s)\,\rho(s)\,\frac{\nabla P_m(x(s))}{\nabla x(s))}\right]\\ = \left[\phi(x(s)\,\mathbb{F}_x P_m(x(s)) + \psi(x(s))\,\mathbb{M}_x P_m(x(s)\right]\,\rho(s)\,\nabla x_1(s),\end{aligned} \tag{85}$$

where

$$W(P_n(x(s)), P_m(x(s))) = P_n(x(s))\frac{\nabla P_m(x(s))}{\nabla x(s))} - P_m(x(s))\frac{\nabla P_n(x(s))}{\nabla x(s))} \tag{86}$$

is the discrete analog of the Wronskian.

The Wronskian $W(P_n(x(s)), P_m(x(s)))$ is a polynomial of degree at most $n+m-1$ in the variable $x_{-1}(s) = x(s-\frac{1}{2})$.

Proof: First, we observe that the Pearson-type equation (8) is equivalent to

$$\frac{\rho(s+1)}{\rho(s)} = \frac{\sigma(s) + \phi(s)\,\nabla x_1(s)}{\sigma(s+1)},\ \text{with}\ \phi(x(s)) \equiv \phi(s). \tag{87}$$

Using the relation

$$\Delta(f(s)\,g(s)) = f(s)\,\Delta g(s) + g(s+1)\,\Delta f(s) = f(s+1)\,\Delta g(s) + g(s)\,\Delta f(s), \tag{88}$$

we obtain for given $n,\ m \in \mathbb{N}$,

$$\begin{aligned}\Delta\left[\sigma(s)\,\rho(s)\,\frac{\nabla P_n(x(s))}{\nabla x(s))}\right] P_m(x(s)) - \Delta\left[\sigma(s)\,\rho(s)\,\frac{\nabla P_m(x(s))}{\nabla x(s))}\right] P_n(x(s))\\ = \Delta\left[\sigma(s)\,\rho(s)\,W(P_n(x(s)), P_m(x(s)))\right],\end{aligned}$$

where

$$W(P_n(x(s)), P_m(x(s))) = P_n(x(s))\frac{\nabla P_m(x(s))}{\nabla x(s))} - P_m(x(s))\frac{\nabla P_n(x(s))}{\nabla x(s))}. \tag{89}$$

The Wronskian $W(P_n(x(s)), P_m(x(s)))$ thanks to (88) can also be written as

$$W(P_n(x(s)), P_m(x(s))) = \qquad (90)$$
$$\mathbb{S}_{x_{-2}} P_n(x_{-2}(s))\, \mathbb{D}_{x_{-2}} P_m(x_{-2}(s)) - \mathbb{S}_{x_{-2}} P_m(x_{-2}(s))\, \mathbb{D}_{x_{-2}} P_n(x_{-2}(s)).$$

Therefore, from (25), we remark that $W(P_n(x(s)), P_m(x(s)))$ is a polynomial of degree at most $n+m-1$ in the variable $x_{-1}(s) = x(s-\frac{1}{2})$.

For the second identity, we use again (88), then (8) together with (87) and finally the identity (easy to obtain)

$$\frac{1}{2}\left[\frac{\Delta y(s)}{\Delta x(s)} + \frac{\nabla y(s)}{\nabla x(s)}\right] = \frac{\nabla y(s)}{\nabla x(s)} + \frac{1}{2}\nabla\left[\frac{\nabla y(s)}{\nabla x(s)}\right],$$

to get

$$\begin{aligned}
&\Delta\left[\sigma(s)\,\rho(s)\,\frac{\nabla P_m(x(s))}{\nabla x(s))}\right]\\
&= \Delta\left[\sigma(s)\,\rho(s)\right]\,\frac{\nabla P_m(x(s))}{\nabla x(s))} + \sigma(s+1)\,\rho(s+1)\Delta\frac{\nabla P_m(x(s))}{\Delta x(s))}\\
&= \psi(s)\,\nabla x_1(s)\,\rho(s)\frac{\nabla P_m(x(s))}{\nabla x(s))} + (\sigma(s)+\psi(s)\,\nabla x_1(s))\rho(s)\Delta\frac{\nabla P_m(x(s))}{\nabla x(s))}\\
&= \psi(s)\,\nabla x_1(s)\,\rho(s)\left[\mathbb{M}_x P_m(x(s)) - \frac{1}{2}\Delta\frac{\nabla P_m(x(s))}{\nabla x(s))}\right]\\
&\quad +(\sigma(s)+\psi(s)\,\nabla x_1(s))\rho(s)\Delta\frac{\nabla P_m(x(s))}{\nabla x(s))}\\
&= \left[\phi(x(s)\,\mathbb{F}_x P_m(x(s)) + \psi(x(s))\,\mathbb{M}_x P_m(x(s)\right]\,\rho(s)\,\nabla x_1(s),
\end{aligned}$$

with ϕ given by (9). □

Next we give the proof of Theorem 7.

Proof: We set $n,\, m \in \mathbb{N}$ and write

$$V_n(x(s)) = \phi(x(s))\,\mathbb{F}_x P_n(x(s)) + \psi(x(s))\,\mathbb{M}_x P_n(x(s),\ n \geq 1,\ V_0(x(s)) = 1.$$

We shall prove that the family (V_n) is also orthogonal with respect to the weight $\rho(s)$. First we prove that $\mathrm{degree}(V_n) = n,\ n \geq 1$. To do this, we assume that (P_n) is monic and use the expansions (66) of $\mathbb{F}_x x^n(s)$ and $\mathbb{M}_x x^n(s)$ to obtain that the leading coefficient of V_n, which we denote by h_n, is

$$\begin{aligned}
h_n &= \phi_2\, F_{n,n-2} + \psi_1\, M_{n,n-1} \\
&= D_{n,n-1}\,(\phi_2\, D_{n-1,n-2} + \psi_1\, S_{n-1,n-1}),\ n \geq 1,\ h_0 = 1,
\end{aligned} \qquad (91)$$

with the latter identity obtained thanks to Proposition 4.

Following [5], page 204, if we define the moments of the weight function ρ by

$$M_n = \sum_{s=a}^{b-1} [x(s) - x(a+n-1)]^{(n)}\,\rho(s)\,\nabla x_1(s)$$

or

$$M_n = \frac{1}{2\pi i}\int_C [x(s) - x(a+n-1)]^{(n)}\,\rho(s)\,\nabla x_1(s)\,ds$$

for the discrete or the continuous orthogonality respectively, with $\sigma(a) = 0$ and the generalized power of the lattice $x(s)$ defined as

$$[x_r(z) - x_r(s)]^{(k)} = \prod_{j=0}^{k-1}[x_r(z) - x_r(s-j)],$$

it turns out that M_n satisfies (see [5], Eqn. (6.8))

$$(\phi_2\, D_{n,n-1} + \psi_1\, S_{n,n}) M_{n+1} = -\psi_n(a)\, M_n,\ n \geq 0.$$

For all the moments $M_n,\ n \geq 0$ to exist, it is necessary to have

$$\phi_2\, D_{n,n-1} + \psi_1\, S_{n,n} \neq 0,\ n \geq 0.$$

Since $D_{n,n-1} \neq 0,\ n \geq 1$, we deduce that $h_n \neq 0,\ n \geq 1$ and $\text{degree}(V_n) = n,\ n \geq 1$.

Next, we assume without any loss of generality that $m \leq n$. Then, using (84) and (85), we get

$$\begin{aligned} & \sum_{i=0}^{N} V_n(x(s_i))\, P_m(x(s_i))\, \rho(s_i)\, \nabla x_1(s_i) \\ = \quad & \sum_{i=0}^{N} P_n(x(s_i))\, V_m(x(s_i))\, \rho(s_i)\, \nabla x_1(s_i) + \sigma(s_j)\, \rho(s_j)\, W(P_n(x(s_j)), P_m(x(s_j)))|_{j=0}^{N+1}\,. \end{aligned}$$

Finally, use of the previous relation, the orthogonality relation for (P_n) (5) and the border conditions (10) together with the fact that $W(P_n(x(s)), P_m(x(s)))$ is a polynomial in the variable $x(s-\frac{1}{2})$, gives

$$\sum_{i=0}^{N} V_n(x(s_i))\, P_m(x(s_i))\, \rho(s_i)\, \nabla x_1(s_i) = \sum_{i=0}^{N} P_n(x(s_i))\, V_m(x(s_i))\, \rho(s_i)\, \nabla x_1(s_i) = h_n\, k_n\, \delta_{n,m}.$$

The latter relation, combined with the fact that V_n is of degree n assures that the family (V_n) is orthogonal with respect to the weight function ρ. Hence, (P_n) and (V_n) are proportional since they are orthogonal with respect to the same weight; therefore there exists a constant term λ_n such that

$$V_n(x(s)) = -\lambda_n\, P_n(x(s)),\ n \geq 0.$$

Comparison of the leading terms in both members of the previous equation yields

$$\lambda_n = -D_{n,n-1}\, (\phi_2\, D_{n-1,n-2} + \psi_1\, S_{n-1,n-1}).$$

The proof using the continuous orthogonality is obtained in the same way. □

Definition 2 *We therefore propose as definition of classical orthogonal polynomials Definition 1 without the second condition.*

3.2 Parameters $\alpha,\ \beta,\ \delta,\ \phi,\ \psi$

Let $x(s)$ be a lattice of the form (4). It satisfies (17) and (18), and therefore, $x(s)$ depends only on the parameters $\alpha,\ \beta$ and δ with the latter given by (30).

The classical orthogonal polynomials $(P_n(x(s)))$, solution of

$$\phi(x(s))\, \mathbb{F}_x P_n(x(s)) + \psi(x(s))\, \mathbb{M}_x P_n(x(s)) + \lambda_n\, P_n(x(s)) = 0, \tag{92}$$

depend only on the parameters $\alpha,\ \beta,\ \delta$ and the five coefficients of the polynomials ϕ and ψ. Equation (92) is the most general second-order difference equation satisfied by classical orthogonal polynomials. It contains the very classical orthogonal polynomials (special and limiting cases) as well as the classical orthogonal polynomials of quadratic and q-quadratic lattices.

In the following, we shall discuss special values of the parameters α, β, δ and the corresponding polynomial families.

First, computations using relations (17) and (18) show that $x(s)$ satisfies a second-order difference equation of the form [5]

$$x(s+1) - 2\,(2\,\alpha^2 - 1)\,x(s) + x(x-1) = 4\,\beta\,(\alpha+1),$$

whose solution is

$$x(s) = \begin{cases} C_1\,q^s + C_2\,q^{-s} + \frac{\beta}{1-\alpha}, & \alpha \neq 1, \\ 4\beta\,s^2 + C_3\,s + C_4, & \alpha = 1. \end{cases} \tag{93}$$

3.2.1 Case I: $\alpha = 1$

When $\alpha = 1$, the lattice $x(s) = 4\beta\,s^2 + C_3\,s + C_4$ is quadratic.

Case I.1: $\alpha = 1,\ \beta = C_3 = 0$

In this case, the lattice $x(s) = C_4$ is constant and from (30), $\delta_x = 0$. Therefore,

$$\mathbb{D}_x f(x(s)) = \lim_{x(s+1)\to x(s)} \frac{f(x(s+1)) - f(x(s))}{x(s+1) - x(s)} \equiv \frac{d}{dx} f(x),$$

and $\mathbb{F}_x$, $\mathbb{M}_x$ correspond to

$$\mathbb{F}_x \equiv \frac{d^2}{dx^2},\ \mathbb{M}_x \equiv \frac{d}{dx}.$$

Equation (92) reads

$$\phi(x)\,P_n''(x) + \psi(x)\,P_n'(x) + \lambda_n\,P_n(x) = 0.$$

Therefore, the case $\alpha = 1,\ \beta = \delta = 0$ corresponds to the classical orthogonal polynomials of a continuous variable [19] (Jacobi, Hermite, Laguerre and Bessel).

Case I.2: $\alpha = 1,\ C_3 = 1,\ \beta = C_4 = 0$

The lattice reads $x(s) = s$ from which one gets $\delta_x = \frac{1}{4}$. Equation (92) reduces to

$$\tilde{\phi}(s)\Delta\nabla P_n(s) + \psi(s)\,\Delta P_n(s) + \lambda_n\,P_n(s) = 0,$$

with $\tilde{\phi}(s) = \phi(s) - \frac{1}{2}\,\psi(s)$. Thus the case $\alpha = 1,\ \beta = 0,\ \delta_x = \frac{1}{4}$ corresponds to the classical orthogonal polynomials of a discrete variable on a linear lattice [19] (Hahn, Meixner, Charlier and Krawtchouk).

Case I.3: $\alpha = 1,\ \beta \neq 0$

Here, the lattice $x(s) = 4\,\beta\,s^2 + C_3\,s + C_4$ is quadratic and the corresponding polynomials are called classical orthogonal polynomials of a discrete variable on a quadratic lattice [19] (Wilson, the continuous dual Hahn, the continuous Hahn, the Meixner-Pollaczek, the Racah and the dual Hahn polynomials).

3.3 Case II: $\alpha = q^{\frac{1}{2}} + q^{-\frac{1}{2}},\ q \neq 0,\ q \neq 1$

In this case, it can be seen from (93) that the coefficient β is involved only in the constant term of the lattice. This constant can therefore be omitted without any loss of generality.

Case II.1: $\beta = 0,\ \delta = 0$

For $\beta = C_2 = 0$ and $C_1 = 1$, then $x(s) = q^s$ and one gets from (30) $\delta = 0$. The operators $\mathbb{D}_x$, $\mathbb{F}_x$ and $\mathbb{M}_x$ in this case read

$$\mathbb{D}_x = D_q,\ \mathbb{F}_x = D_q\, D_{\frac{1}{q}},\ \mathbb{M}_x = \frac{1}{2}\left[D_q + D_{\frac{1}{q}}\right],$$

where D_q is the Hahn operator [17] (also called Jackson derivative [18])

$$D_q f(x) = \frac{f(q\,x) - f(x)}{(q-1)\,x}.$$

Equation (92) is therefore equivalent to

$$\tilde{\phi}(x)\, D_q\, D_{\frac{1}{q}} P_n(x) + \psi(x)\, D_q P_n(x) + \lambda_n\, P_n(x) = 0,\quad x = q^s,$$

with $\tilde{\phi}(x) = q^2\,\phi(x) - \frac{1}{2}(q-1)\,x\,\psi(x)$. Thus the case $\alpha = q^{\frac{1}{2}} + q^{-\frac{1}{2}},\ \beta = \delta = 0$ corresponds to the q-classical orthogonal polynomials [19, 26]: The Big q-Jacobi, Big q-Laguerre, Little q-Jacobi, Little q-Laguerre (Wall), q-Laguerre, Alternative q-Charlier, Al-Salam-Carlitz I, Al-Salam-Carlitz II, Stieltjes-Wigert, Discrete q-Hermite, Discrete q^{-1}-Hermite II, q-Hahn, q-Meixner, Quantum q-Krawtchouk, q-Krawtchouk, Affine q-Krawtchouk, and q-Charlier polynomials).

Case II.2: $\beta = 0,\ \delta \neq 0$

When $C_1\, C_2 \neq 0$, then from (30), $\delta \neq 0$. The lattice $x(s) = C_1\, q^s + C_2\, q^{-s}$ is q-quadratic and the corresponding orthogonal polynomials are the classical orthogonal polynomials of a discrete variable on q-quadratic lattices [19] (The Askey-Wilson, the q-Racah, the continuous dual q-Hahn, the continuous q-Hahn, the dual q-Hahn, the Al-Salam Chihara, the q-Meixner-Pollaczek, the continuous q-Jacobi, the the continuous dual q-Krawtchouk, the continuous big q-Hermite, the continuous q-Laguerre, the continuous q-Hermite, the Wilson, the continuous dual Hahn, the continuous Hahn and the Meixner-Pollaczek polynomials).

To conclude this section, we give the parameters $\alpha,\ \beta,\ \delta$ as well as the polynomials ϕ and ψ for 14 families of classical orthogonal polynomials on nonuniform lattices out of 18 listed above. The four remaining families, namely, the Wilson, the continuous dual Hahn, the continuous Hahn and the Meixner-Pollaczek polynomials deal with the complex difference-derivative which is not included in this work and will be treated later separately. However, these families can be reached by limiting procedures from the Askey-Wilson polynomials for which we illustrate in the following lines the method we have used to obtain the parameters $\alpha,\ \beta,\ \delta$.

Part I: Cases of the q-quadratic lattices

1. **Askey-Wilson polynomials**

$$\frac{2^n a^n (abcd\, q^{n-1};q)_n}{(ab,ac,ad;q)_n}\, \tilde{p}_n(x;a,b,c,d|q) = {}_4\phi_3\left(\begin{matrix} q^{-n}, abcdq^{n-1}, ae^{i\theta}, ae^{-i\theta} \\ ab, ac, ad \end{matrix}\,\middle|\, q;q\right),\ x = \cos\theta.$$

$$\rho(x) := \rho(x;a,b,c,d|q) = \frac{1}{\sqrt{1-x^2}}\left|\frac{(e^{2i\theta};q)_\infty}{(ae^{i\theta}, be^{i\theta}, ce^{i\theta}, de^{i\theta};q)_\infty}\right|^2,\ x = \cos\theta.$$

If $a,\ b,\ c$ and d are real, or occur in complex conjugate pairs if complex, and if

$$\max(|a|,|b|,|c|,|d|) < 1,$$

then we have the following orthogonality relation

$$\frac{1}{2\pi}\int_{-1}^{1}\rho(x)\,\tilde{p}_n(x;a,b,c,d|q)\,\tilde{p}_m(x;a,b,c,d|q)\,dx = h_n\,\delta_{n,m},$$

where

$$h_n = \frac{\left[2^n\,(abcdq^{n-1};q)_n\right]^{-2}\,(abcdq^{n-1};q)_n\,(abcdq^{2n};q)_\infty}{(q^{n+1},abq^n,acq^n,adq^n,bcq^n,bdq^n,cdq^n;q)_\infty}.$$

For the lattice we write

$$x=\cos\theta=\frac{e^{i\theta}+e^{-i\theta}}{2}=\frac{q^s+q^{-s}}{2}:=x(s),\;\; q^s:=e^{i\theta}.$$

It follows from (18) that $\alpha_x=\frac{q^{\frac{1}{2}}+q^{-\frac{1}{2}}}{2}$, $\beta_x=0$, $C_5=C_6=\frac{1}{2}$ and $\delta_x=-\frac{(q-1)^2}{4q}$ by (30).

The Askey-Wilson operator

$$\mathcal{D}_q f(x) := \frac{\delta_q f(x)}{\delta_q x},\;\; x=\cos\theta,$$

with

$$\delta_q f(e^{i\theta}) = f(q^{\frac{1}{2}}e^{i\theta}) - f(q^{-\frac{1}{2}}e^{i\theta}),$$

can be written in terms of $\mathbb{D}_x$ as

$$\mathcal{D}_q f(x) = \mathbb{D}_{x_{-1}} f(x_{-1}(s)),\;\; x:=x(s)=\frac{q^s+q^{-s}}{2}. \tag{94}$$

More details about the Askey-Wilson operator are given in Subsection 5.1.4.

In order to compute the polynomials ϕ and ψ, we write the Pearson-type equation (8) in its equivalent form (using (9))

$$\frac{\rho(s+1)}{\rho(s)} = \frac{\phi(s)+\frac{1}{2}\psi(s)\nabla x_1(s)}{\phi(s+1)-\frac{1}{2}\psi(s+1)\Delta x_1(s)}$$

and use the previous expression of the Askey-Wilson weight $\rho(x)\equiv\rho(x(s))\equiv\rho(s)$ to get

$$\frac{(-1+aq^s)\,q^2\,(-1+bq^s)\,(-1+cq^s)\,(-1+dq^s)}{(q^sq-d)\,(q^sq-c)\,(q^sq-b)\,(q^sq-a)}.$$

Then we combine the last two equations and use the expansion (83) with $x(s)=\frac{q^s+q^{-s}}{2}$ to obtain a polynomial equation in q^s with coefficients depending linearly on those of the polynomials ϕ and ψ. Collection of different coefficients of the powers of q^s leads to a system of linear equations in the variables ϕ_2, ϕ_1, ϕ_0, ψ_1 and ψ_0. Solving this system, we obtain the polynomials ϕ and ψ:

$$\begin{aligned}\phi(s) &= 2\,(dcba+1)\,x^2(s) - (a+b+c+d+abc+abd+acd+bcd)\,x(s)\\ &\quad +ab+ac+ad+bc+bd+cd-abcd-1,\\ \psi(s) &= \frac{4\,(abcd-1)\,q^{\frac{1}{2}}\,x(s)}{q-1}+\frac{2\,(a+b+c+d-abc-abd-acd-bcd)\,q^{\frac{1}{2}}}{q-1}.\end{aligned}$$

By using the relations (24), (27), (58), (59) and (94), we have checked that the difference equation (Equation (3.1.6) in [19])

$$(1-q)^2\,\mathcal{D}_q\left[\rho(x;aq^{\frac{1}{2}},bq^{\frac{1}{2}},cq^{\frac{1}{2}},dq^{\frac{1}{2}},|q)\,\mathcal{D}_q\tilde{p}_n(x)\right]+\lambda_n\,\rho(x;a,b,c,d|q)\,\tilde{p}_n(x)=0,\;\; x=\cos\theta,$$

with $\tilde{p}_n(x;a,b,c,d|q) \equiv \tilde{p}_n(x)$, is equivalent to

$$\phi(x)\,\mathcal{D}_q^2\tilde{p}_n(x) + \psi(x)\,\mathcal{S}_q\mathcal{D}_q\tilde{p}_n(x) + \lambda_n\,\tilde{p}_n(x) = 0,$$

where the operator $\mathcal{S}_q$ is given in the next section by (130) and the constant λ_n by (104) with the polynomials $\phi(s)$ and $\psi(s)$ given as above.

The coefficients α, β, δ (relabelled α_x, β_x, δ_x in order to avoid confusion with the parameters bearing the same names involved in the definition) as well as the polynomials ϕ and ψ given below for the remaining families are obtained in the same way using the notations from [19]:

2. **q-Racah polynomials**

The lattice $x(s)$ and the coefficients α_x, β_x and δ_x are:

$$x(s) = q^{-s} + \gamma\delta q^{s+1},\ \alpha_x = \frac{q^{\frac{1}{2}} + q^{-\frac{1}{2}}}{2},\ \beta_x = 0,\ \delta_x = -(q-1)^2\gamma\delta.$$

Polynomials ϕ and ψ are:

$$\begin{aligned}\phi\,(x(s)) &= \left(q^2\alpha\,\beta + 1\right)x^2\,(s) - (q\beta\delta\,\gamma + q\alpha\,\gamma + q\alpha\,\beta\delta + \alpha\,\beta\,q + \alpha + \beta\delta + \gamma\delta + \gamma)\,q\,x\,(s)\\ &\quad -2\,\left(q^2\alpha\,\gamma\delta\,\beta - q\beta\delta\,\gamma - q\alpha\,\beta\delta - q\gamma^2\delta - q\alpha\,\gamma\,\delta - q\alpha\,\gamma - q\gamma\,\delta^2\beta + \gamma\delta\right)q,\\ \psi\,(x(s)) &= \frac{2\left(q^2\alpha\,\beta - 1\right)q^{\frac{1}{2}}x\,(s)}{q-1} - \frac{2q\,(q\beta\delta\,\gamma + q\alpha\,\gamma + q\alpha\,\beta\delta + \alpha\,\beta\,q - \alpha - \beta\delta - \gamma\delta - \gamma)\,q^{\frac{1}{2}}}{q-1}.\end{aligned}$$

3. **Continuous dual q-Hahn polynomials**

The lattice $x(s)$ and the coefficients α_x, β_x and δ_x are:

$$x(s) = \frac{q^s + q^{-s}}{2},\ \alpha_x = \frac{q^{\frac{1}{2}} + q^{-\frac{1}{2}}}{2},\ \beta_x = 0,\ \delta_x = -\frac{(q-1)^2}{4q}$$

Polynomials ϕ and ψ are:

$$\begin{aligned}\phi\,(x(s)) &= 2\,x^2\,(s) - (a+b+c+abc)\,x\,(s) + ab + ac + bc - 1,\\ \psi\,(x(s)) &= -\frac{4\,q^{\frac{1}{2}}\,x\,(s)}{q-1} + \frac{2\,(a+b+c-abc)\,q^{\frac{1}{2}}}{q-1}.\end{aligned}$$

4. **Continuous q-Hahn polynomials**

The lattice $x(s)$ and the coefficients α_x, β_x and δ_x are:

$$x(s) = \frac{e^{i\varphi}\,q^s + e^{-i\varphi}\,q^{-s}}{2},\ \alpha_x = \frac{q^{\frac{1}{2}} + q^{-\frac{1}{2}}}{2},\ \beta_x = 0,\ \delta_x = -\frac{(q-1)^2}{4q},\ e^{i\theta} := q^s.$$

Polynomials ϕ and ψ are:

$$\begin{aligned}\phi\,(x(s)) &= 2\,(dcba+1)\,x^2\,(s) - \frac{\left(d + dcb + at^2 + bt^2ad + cbat^2 + c + cda + bt^2\right)x\,(s)}{t}\\ &\quad + \frac{cat^2 + bt^2d - t^2cbad + cbt^2 + cd + t^2 + bt^4a + t^2ad}{t^2},\\ \psi\,(x(s)) &= \frac{4\,(-1+dcba)\,q^{\frac{1}{2}}x\,(s)}{q-1} - \frac{2q^{\frac{1}{2}}\left(-c - d + cda - bt^2 - at^2 + dcb + cbat^2 + bt^2ad\right)}{(q-1)\,t},\end{aligned}$$

with the notation $t = e^{i\varphi}$.

5. **Dual q-Hahn polynomials**

The lattice $x(s)$ and the coefficients $\alpha_x,\ \beta_x$ and δ_x are:

$$x(s) = q^{-s} + \gamma\delta q^{s+1},\ \alpha_x = \frac{q^{\frac{1}{2}} + q^{-\frac{1}{2}}}{2},\ \beta_x = 0,\ \delta_x = -(q-1)^2\gamma\delta.$$

Polynomials ϕ and ψ are :

$$\begin{aligned}\phi\left(x(s)\right) &= x^2\left(s\right) - \frac{\left(\gamma\, q + q^N q\gamma\,\delta + \gamma\, q^N q + 1\right) x\left(s\right)}{2q^N} + \frac{\gamma\,\left(q^N q\gamma\,\delta - \delta\, q^N + \delta + 1\right) q}{2q^N},\\ \psi\left(x(s)\right) &= -\frac{2q^{\frac{1}{2}}\, x\left(s\right)}{q-1} + \frac{\left(q^N q\gamma\,\delta + \gamma\, q^N q - \gamma\, q + 1\right) q^{\frac{1}{2}}}{\left(q-1\right) q^N}.\end{aligned}$$

6. **Al-Salam-Chihara polynomials**

The lattice $x(s)$ and the coefficients $\alpha_x,\ \beta_x$ and δ_x are:

$$x(s) = \frac{q^s + q^{-s}}{2},\ \alpha_x = \frac{q^{\frac{1}{2}} + q^{-\frac{1}{2}}}{2},\ \beta_x = 0,\ \delta_x = -\frac{(q-1)^2}{4q}.$$

Polynomials ϕ and ψ:

$$\phi\left(x(s)\right) = 2\, x^2\left(s\right) - \left(a+b\right) x\left(s\right) + ab - 1,\ \ \psi\left(x(s)\right) = -\frac{4q^{\frac{1}{2}}\, x\left(s\right)}{q-1} + \frac{2\left(a+b\right) q^{\frac{1}{2}}}{q-1}.$$

7. **q-Meixner-Pollaczek polynomials**

The lattice $x(s)$ and the coefficients $\alpha_x,\ \beta_x$ and δ_x are:

$$x(s) = \frac{e^{i\varphi}\, q^s + e^{-i\varphi}\, q^{-s}}{2},\ \alpha_x = \frac{q^{\frac{1}{2}} + q^{-\frac{1}{2}}}{2},\ \beta_x = 0,\ \delta_x = -\frac{(q-1)^2}{4q},\ e^{i\theta} := q^s.$$

Polynomials ϕ and ψ are:

$$\phi\left(x(s)\right) = 2\, x^2\left(s\right) - 2\, a\, \cos\varphi\, x\left(s\right) + a^2 - 1,\ \psi\left(x(s)\right) = -\frac{4\, q^{\frac{1}{2}}\, x\left(s\right)}{q-1} + \frac{4\, a\, q^{\frac{1}{2}}\, \cos\varphi}{q-1}.$$

8. **Continuous q-Jacobi polynomials**

The lattice $x(s)$ and the coefficients $\alpha_x,\ \beta_x$ and δ_x are:

$$x(s) = \frac{q^s + q^{-s}}{2},\ \alpha_x = \frac{q^{\frac{1}{2}} + q^{-\frac{1}{2}}}{2},\ \beta_x = 0,\ \delta_x = -\frac{(q-1)^2}{4q}$$

The coefficients of the polynomials ϕ and ψ are:

$$\begin{aligned}\phi_2 &= p^{2\,\alpha+2\,\beta+4} + 1,\ \ \phi_1 = \frac{1}{2}\,\left(p+1\right) p^{\frac{1}{2}}\left(p^{2\,\alpha+\beta+2} - p^{\alpha} - p^{\alpha+2\,\beta+2} + p^{\beta}\right),\\ \phi_0 &= -\frac{1}{2}\, p^{2\,\alpha+2\,\beta+4} - \frac{1}{2}\, p^{\alpha+\beta+3} + \frac{1}{2}\, p^{2\,\alpha+2} - p^{\alpha+\beta+2} + \frac{1}{2}\, p^{2\,\beta+2} - \frac{1}{2}\, p^{1+\alpha+\beta} - \frac{1}{2},\\ \psi_1 &= \frac{2\, p\left(p^{2\,\alpha+2\,\beta+4} - 1\right)}{\left(p-1\right)\left(p+1\right)},\ \ \psi_0 = -\frac{p^{\frac{3}{2}}\left(-p^{2\,\alpha+\beta+2} - p^{\alpha} + p^{\alpha+2\,\beta+2} + p^{\beta}\right)}{p-1},\end{aligned}$$

with $q=p^2$.

9. **Continuous Dual q-Krawtchouk polynomials**

The lattice $x(s)$ and the coefficients $\alpha_x,\ \beta_x$ and δ_x are:

$$x(s)=q^{-s}+cq^{s-N},\ \alpha_x=\frac{q^{\frac{1}{2}}+q^{-\frac{1}{2}}}{2},\ \beta_x=0,\ \delta_x=-c(q-1)^2\,q^{-1-N}.$$

Polynomials ϕ and ψ are:

$$\phi(x(s))=x^2(s)-(c+1)\,q^{-N}\,x(s)-2\,c\,(q^{-N}-q^{-2N}),\ \psi(x(s))=-\frac{2\,q^{\frac{1}{2}}}{q-1}x(s)+\frac{2(c+1)\,q^{\frac{1}{2}}}{(q-1)q^N}.$$

10. **Continuous big q-Hermite polynomials**

The lattice $x(s)$ and the coefficients $\alpha_x,\ \beta_x$ and δ_x are:

$$x(s)=\frac{q^s+q^{-s}}{2},\ \alpha_x=\frac{q^{\frac{1}{2}}+q^{-\frac{1}{2}}}{2},\ \beta_x=0,\ \delta_x=-\frac{(q-1)^2}{4q}.$$

Polynomials ϕ and ψ are:

$$\phi(x(s))=2\,x^2(s)-a\,x(s)-1,\ \psi(x(s))=-\frac{4\,q^{\frac{1}{2}}}{q-1}\,x(s)+\frac{2\,a\,q^{\frac{1}{2}}}{q-1}.$$

11. **Continuous q-Laguerre polynomials**

The lattice $x(s)$ and the coefficients $\alpha_x,\ \beta_x$ and δ_x are:

$$x(s)=\frac{q^s+q^{-s}}{2},\ \alpha_x=\frac{q^{\frac{1}{2}}+q^{-\frac{1}{2}}}{2},\ \beta_x=0,\ \delta_x=-\frac{(q-1)^2}{4q}.$$

Polynomials ϕ and ψ are:

$$\phi(x(s))=2\,x^2\,(s)-p^{\alpha+\frac{1}{2}}\,(p+1)\,x\,(s)+p^{2\alpha+2}-1,\ \psi\,(x(s))=-\frac{4p}{p^2-1}x\,(s)+\frac{2p^{\alpha+\frac{3}{2}}}{p-1}.$$

12. **Continuous q-Hermite polynomials**

The lattice $x(s)$ and the coefficients $\alpha_x,\ \beta_x$ and δ_x are:

$$x(s)=\frac{q^s+q^{-s}}{2},\ \alpha_x=\frac{q^{\frac{1}{2}}+q^{-\frac{1}{2}}}{2},\ \beta_x=0,\ \delta_x=-\frac{(q-1)^2}{4q}.$$

Polynomials ϕ and ψ are:

$$\phi(x(s))=2\,x^2(s)-1,\ \psi(x(s))=-\frac{4\,q^{\frac{1}{2}}}{q-1}\,x(s).$$

Part II: Cases of quadratic lattices

13. **Racah polynomials**

The lattice $x(s)$ and the coefficients $\alpha_x,\ \beta_x$ and δ_x are:

$$x(s)=s\,(s+\gamma+\delta+1),\ \alpha_x=1,\ \beta_x=\frac{1}{4},\ \delta_x=\frac{(\gamma+\delta+1)^2}{4}.$$

Polynomials ϕ and ψ are:

$$\begin{aligned}\phi\,(x(s)) &= 2\,x^2\,(s)+[-\beta\,\delta+\beta\,\gamma+2\,\gamma+\alpha\,\gamma+2\,\gamma\,\delta+4+\alpha\,\delta\\ &\quad +3\,\alpha+3\,\beta+2\,\beta\,\alpha+2\,\delta]\,x\,(s)+(1+\gamma)\,(1+\delta+\gamma)\,(\delta+\beta+1)\,(\alpha+1)\,,\\ \psi\,(x(s)) &= (2\,\alpha+2\,\beta+4)\,x\,(s)+2\,\,(1+\gamma)\,(\delta+\beta+1)\,(\alpha+1)\,.\end{aligned}$$

14. **Dual Hahn polynomials**

The lattice $x(s)$ and the coefficients $\alpha_x,\ \beta_x$ and δ_x are:

$$x(s)=s\,(s+\gamma+\delta+1),\ \alpha_x=1,\ \beta_x=\frac{1}{4},\ \delta_x=\frac{(\gamma+\delta+1)^2}{4}.$$

Polynomials ϕ and ψ are:

$$\phi\,(x(s))=(-1+2\,N+\delta-\gamma)\,x\,(s)+N\,(1+\gamma)\,(1+\delta+\gamma)\,,\ \psi\,(x(s))=-2\,x\,(s)+2\,N\,(1+\gamma)\,.$$

3.4 Three-term recurrence coefficients

The method we will use here to compute the recurrence coefficients is the same used for the very classical orthogonal polynomials, see e. g. [20].

We assume that (P_n) is a system of monic classical orthogonal polynomials such that each P_n satisfies the equation (1). Because of the orthogonality, (P_n) satisfies the three-term recurrence relation (13) which we recall here:

$$P_{n+1}(x)=(x-\beta_n)\,P_n(x)-\gamma_n\,P_{n-1}(x),\ n\geq 1,\ P_{-1}=0,\ P_0(x)=1.$$

First, we write

$$P_n(x(s))=\sum_{j=0}^{n}T_{n,n-j}\,x^j(s),\ \ T_{n,0}=1, \tag{95}$$

and substitute this expression into equation (1) (or (92)), then use the expansions (66) to get

$$\begin{aligned}&\left[\phi_2\,x^2(s)+\phi_1\,x(s)+\phi_0\right]\sum_{k=0}^{n-2}\left(\sum_{j=k+2}^{n}T_{n,n-j}\,F_{j,k}\right)\,x^k(s)+\\ &[\psi_1\,x(s)+\psi_0]\sum_{k=0}^{n-1}\left(\sum_{j=k+1}^{n}T_{n,n-j}\,M_{j,k}\right)\,x^k(s)+\lambda_n\,\sum_{k=0}^{n}T_{n,n-k}\,x^k(s)=0.\end{aligned}$$

Then we look for the coefficients of the monomials $x^n(s)$ in the previous equation and obtain

$$\lambda_n=-\phi_2\,F_{n,n-2}-\psi_1\,M_{n,n-1}. \tag{96}$$

Next, we collect the coefficients of the monomials $x^j(s),\ j = n-1,\ n-2$ to get the following linear system with respect to the unknowns $T_{n,1},\ T_{n,2}$

$$(\lambda_n + \phi_2 F_{n-1,n-3} + \psi_1 M_{n-1,n-2})\, T_{n,1} + \phi_2 F_{n,n-3} + \phi_1 F_{n,n-2} + \psi_1 M_{n,n-2} + \psi_0 M_{n,n-1} = 0,$$

$$\begin{aligned}&(\psi_0 M_{n-1,n-2} + \psi_1 M_{n-1,n-3} + \phi_1 F_{n-1,n-3} + \phi_2 F_{n-1,n-4})\, T_{n,1}\\ &\quad + (\phi_2 F_{n-2,n-4} + \psi_1 M_{n-2,n-3} + \lambda_n)\, T_{n,2}\\ &\qquad + \psi_0 M_{n,n-2} + \psi_1 M_{n,n-3} + \phi_1 F_{n,n-3} + \phi_2 F_{n,n-4} + \phi_0 F_{n,n-2} = 0.\end{aligned}$$

Solving this system produces the coefficients $T_{n,1}$ and $T_{n,2}$.

Computations taking into account (42)-(47) and the relations

$$\begin{aligned}&M_{n,n-1} = D_{n,n-1} S_{n-1,n-1},\\ &M_{n,n-2} = D_{n,n-1} S_{n-1,n-2} + D_{n,n-2} S_{n-2,n-2},\\ &M_{n,n-3} = D_{n,n-1} S_{n-1,n-3} + D_{n,n-2} S_{n-2,n-3} + D_{n,n-3} S_{n-3,n-3},\\ &F_{n,n-2} = D_{n,n-1} D_{n-1,n-2},\\ &F_{n,n-3} = D_{n,n-1} D_{n-1,n-3} + D_{n,n-2} D_{n-2,n-3},\\ &F_{n,n-4} = D_{n,n-1} D_{n-1,n-4} + D_{n,n-2} D_{n-2,n-4} + D_{n,n-3} D_{n-3,n-4},\end{aligned}$$

obtained from (69) and (70), give the following expressions for the coefficients $F_{n,k-1}$ and $M_{n,k}$ for $k = n,\ n-1$ and $n-3$.

First case: $\alpha_x = 1$:

$$\begin{aligned}F_{n,n-2} &= n\,(n-1)\,,\ F_{n,n-3} = \frac{4}{3}\,\beta_x\, n\,(n-2)\,(n-1)^2\,,\\ F_{n,n-4} &= \frac{1}{45}\, n\,(n-1)\,(n-2)\,(n-3)\left(32\,{\beta_x}^2 n^2 - 96\,{\beta_x}^2 n + 52\,{\beta_x}^2 + 15\,\delta_x\right),\\ M_{n,n-1} &= n,\\ M_{n,n-2} &= \frac{2}{3}\,\beta_x\, n\,(n-1)\,(4\,n-5)\,,\\ M_{n,n-3} &= \frac{2}{15}\, n\,(n-1)\,(n-2)\left(16\,{\beta_x}^2 n^2 - 52\,{\beta_x}^2 n + 32\,{\beta_x}^2 + 5\,\delta_x\right).\end{aligned} \tag{97}$$

Second case: $\alpha_x = \frac{q^{\frac{1}{2}} + q^{-\frac{1}{2}}}{2},\ \beta_x = 0$:

$$\begin{aligned}F_{n,n-2} &= \frac{(q^n-1)(q^n-q)q^{\frac{1}{2}}}{(q-1)^2\, q^n},\\ F_{n,n-3} &= 0,\\ F_{n,n-4} &= \left(\frac{(q+1)(q^{2n}-q^3)\, n}{(q-1)^3\, q^{n+\frac{1}{2}}} - \frac{(q^n-1)(3q^{n+1}+q^n-q^3-3q^2)q^{\frac{1}{2}}}{(q-1)^4\, q^n}\right)\delta_x,\\ M_{n,n-1} &= \frac{(q^n-1)(q^n+q)}{2\,(q-1)q^n},\\ M_{n,n-2} &= 0,\\ M_{n,n-3} &= \frac{\delta_x}{2}\left(\frac{(q+1)(q^{2n}+q^3)\, n}{(q-1)^2\, q^{n+1}} - \frac{(q+1)(q^n-1)(q^n+q^2)}{(q-1)^3\, q^n}\right).\end{aligned} \tag{98}$$

The coefficients β_n and γ_n are deduced from $T_{n,1}$ and $T_{n,2}$ using the following relations between $\beta_n,\ \gamma_n$ and the $T_{n,i}$ [8, 9]

$$T_{n+1,1} = T_{n,1} - \beta_n,\ T_{n+1,2} = T_{n,2} - \beta_n\, T_{n,1} - \gamma_n,$$

and the previous expressions of $F_{n,k}$ and $M_{n,k}$. We can now state the following explicit results obtained with the help of Maple [27]:

Theorem 8 *The coefficients β_n and γ_n of the recurrence coefficients (13) of the classical orthogonal polynomials satisfying (92) are given explicitly by:*

Case 1: $\alpha_x = 1$**:**

$$\begin{aligned}\beta_n &= -\frac{4n\left(2\,\psi_1 n - \psi_1 - 2\,\phi_2 n + 2\,\phi_2 n^2\right)(\psi_1 + \phi_2 n - \phi_2)}{(2\,\phi_2 n + \psi_1)(2\,\phi_2(n-1) + \psi_1)}\beta_x \\ &\quad -\frac{-2\,\psi_0\phi_2 + \psi_1\psi_0 - 2\,\phi_1 n\phi_2 + 2\,\phi_1\psi_1 n + 2\,\phi_1 n^2\phi_2}{(2\,\phi_2 n + \psi_1)(2\,\phi_2(n-1) + \psi_1)};\end{aligned} \tag{99}$$

$$\begin{aligned}\gamma_n &= -\frac{n(n-1)(\psi_1 + \phi_2 n - \phi_2)(\psi_1 + \phi_2 n - 2\,\phi_2)}{(2\,\phi_2 n - \phi_2 + \psi_1)(2\,\phi_2 n - 3\,\phi_2 + \psi_1)}\delta_x \\ &\quad + \frac{16(n-1)^3(\psi_1 + \phi_2 n - 2\,\phi_2)(\psi_1 + \phi_2 n - \phi_2)^3 n}{(2\,\phi_2 n - 3\,\phi_2 + \psi_1)(2\,\phi_2 n - \phi_2 + \psi_1)(2\,\phi_2 n - 2\,\phi_2 + \psi_1)^2}\beta_x^2 \\ &\quad + \left(2\,\phi_1 n^2\phi_2 - 4\,\phi_1 n\phi_2 + 2\,\phi_1\psi_1 n + 2\,\phi_2\phi_1 - 2\,\phi_1\psi_1 + \psi_1\psi_0\right)\times \\ &\quad \frac{4(n-1)(\psi_1 + \phi_2 n - \phi_2)(\psi_1 + \phi_2 n - 2\,\phi_2) n}{(2\,\phi_2 n - 3\,\phi_2 + \psi_1)(2\,\phi_2 n - \phi_2 + \psi_1)(2\,\phi_2 n - 2\,\phi_2 + \psi_1)^2}\beta_x \\ &\quad - \frac{(\psi_1 + \phi_2 n - 2\,\phi_2) n}{(2\,\phi_2 n - 3\,\phi_2 + \psi_1)(2\,\phi_2 n - \phi_2 + \psi_1)(2\,\phi_2 n - 2\,\phi_2 + \psi_1)^2}\times \\ &\quad \left(4\,\phi_0{\phi_2}^2 n^2 - 8\,\phi_0{\phi_2}^2 n + 4\,\phi_0{\phi_2}^2 - {\phi_1}^2 n^2\phi_2 + 2\,{\phi_1}^2 n\phi_2 + 4\,\phi_0\phi_2 n\psi_1 + {\psi_0}^2\phi_2\right. \\ &\quad \left. -4\,\phi_0\phi_2\psi_1 - {\phi_1}^2\phi_2 - {\phi_1}^2 n\psi_1 + {\phi_1}^2\psi_1 + \phi_0{\psi_1}^2 - \psi_0\phi_1\psi_1\right),\end{aligned} \tag{100}$$

Case 2: *For $\alpha_x = \frac{p+p^{-1}}{2}$ and $\beta_x = 0$, with the notations*

$$p = q^{\frac{1}{2}},\ \ q \neq 0,\ 1,\ \text{and } \zeta = p^n,$$

$$\begin{aligned}\beta_n &= \left[\left(-2\,\phi_1 p^3 + \psi_0 p^4 - 2\,\psi_0 p^2 + \psi_0 - 2\,p\phi_1\right)\left(\psi_1 p^2 + 2\,\phi_2 p - \psi_1\right)\zeta^6\right. \\ &- \left(p^2+1\right)\left(p^6\psi_1\psi_0 + 2\,p^5\phi_1\psi_1 - 2\,p^5\psi_0\phi_2 - 4\,p^4\phi_2\phi_1 - p^4\psi_1\psi_0\right. \\ &\qquad \left. +4\,p^3\psi_0\phi_2 - 4\,p^3\psi_1\phi_1 - 4\,p^2\phi_1\phi_2 - p^2\psi_1\psi_0 + 2\,p\phi_1\psi_1 - 2\,\psi_0\phi_2 p + \psi_1\psi_0\right)\zeta^4 \\ &- \left. p^2\left(-2\,\phi_1 p^3 + \psi_0 p^4 - 2\,\psi_0 p^2 + \psi_0 - 2\,p\phi_1\right)\left(\psi_1 p^2 - 2\,\phi_2 p - \psi_1\right)\zeta^2\right] / \\ &\quad \left\{\left(\left(\psi_1 p^2 + 2\,\phi_2 p - \psi_1\right)\zeta^4 + \psi_1 p^2 - 2\,\phi_2 p - \psi_1\right)\times\right. \\ &\qquad \left.\left(\left(\psi_1 p^2 + 2\,\phi_2 p - \psi_1\right)\zeta^4 + p^4\left(\psi_1 p^2 - 2\,\phi_2 p - \psi_1\right)\right)\right\},\end{aligned} \tag{101}$$

$$\begin{aligned}\gamma_n &= \left\{-2\left(-\psi_1 p^4 - \psi_1\zeta^2 + \psi_1 p^6 + \psi_1 p^2\zeta^2 + 2\,\phi_2 p\,\zeta^2 - 2\,\phi_2 p^5\right)\times\right. \\ &\quad \left(\zeta^2-1\right)p^3\zeta^2\left(p^2-1\right)^2\left(-\psi_1 p^4 - \psi_1\zeta^4 + \psi_1 p^6 + \psi_1\zeta^4 p^2 + 2\,\phi_2 p\,\zeta^4 - 2\,\phi_2 p^5\right)^2\phi_0 \\ &\quad + \left(-\psi_1 p^4 - \psi_1\zeta^2 + \psi_1 p^6 + \psi_1 p^2\zeta^2 + 2\,\phi_2 p\,\zeta^2 - 2\,\phi_2 p^5\right)\left(\zeta^2-1\right)p^2\left(p^2-\zeta^2\right)\times \\ &\quad \left(-2\,\phi_2 p^3 + 2\,\phi_2 p\,\zeta^2 - \psi_1 p^2 + \psi_1 p^4 - \psi_1\zeta^2 + \psi_1 p^2\zeta^2\right)\times \\ &\quad \left(-\psi_1 p^4 - \psi_1\zeta^4 + \psi_1 p^6 + \psi_1\zeta^4 p^2 + 2\,\phi_2 p\,\zeta^4 - 2\,\phi_2 p^5\right)^2\delta_x \\ &\quad + \left(-\psi_1 p^4 - \psi_1\zeta^2 + \psi_1 p^6 + \psi_1 p^2\zeta^2 + 2\,\phi_2 p\,\zeta^2 - 2\,\phi_2 p^5\right)\times \\ &\quad \left(\zeta^2-1\right)p^4\zeta^4\left(p^2-1\right)^2\left(\psi_0 p^4 - 2\,\phi_1 p^3 - \psi_0 p^2 + \zeta^2\psi_0 p^2 + 2\,\phi_1\zeta^2 p - \zeta^2\psi_0\right)\times \\ &\quad \left(p^6\psi_1\psi_0 - 2\,p^5\psi_0\phi_2 + 2\,p^5\phi_1\psi_1 - p^4\psi_1\zeta^2\psi_0 - 2\,p^4\psi_1\psi_0 - 4\,p^4\phi_2\phi_1 - 2\,p^3\zeta^2\phi_2\psi_0 + 2\,p^3\zeta^2\phi_1\psi_1\right. \\ &\qquad +2\,p^3\psi_0\phi_2 - 2\,p^3\psi_1\phi_1 + 2\,p^2\psi_1\zeta^2\psi_0 \\ &\qquad\qquad \left.\left. +4\,p^2\zeta^2\phi_1\phi_2 + p^2\psi_1\psi_0 + 2\,p\,\zeta^2\phi_2\psi_0 - 2\,p\,\zeta^2\phi_1\psi_1 - \psi_1\zeta^2\psi_0\right)\right\} /\end{aligned} \tag{102}$$

$$\Big\{(p^2-1)^2\,\big(\left(\psi_1p^2+2\,\phi_2p-\psi_1\right)\zeta^4+p^6\left(\psi_1p^2-2\,\phi_2p-\psi_1\right)\big)\times \\ \big(\left(\psi_1p^2+2\,\phi_2p-\psi_1\right)\zeta^4+p^2\left(\psi_1p^2-2\,\phi_2p-\psi_1\right)\big)\times \\ \big(\left(\psi_1p^2+2\,\phi_2p-\psi_1\right)\zeta^4+p^4\left(\psi_1p^2-2\,\phi_2p-\psi_1\right)\big)^2\Big\}.$$

The coefficient λ_n of (92) is given by

$$\lambda_n=-n\left(\psi_1+(n-1)\,\phi_2\right), \tag{103}$$

for $\alpha_x=1$ and

$$\lambda_n=\frac{\left(-\psi_1p^2-2\,\phi_2p+\psi_1\right)\zeta^4+\left(2\,\psi_1p^2+2\,\phi_2p-\psi_1+2\,\phi_2p^3-\psi_1p^4\right)\zeta^2+p^2\left(\psi_1p^2-2\,\phi_2p-\psi_1\right)}{2\,(p^2-1)^2\,\zeta^2} \tag{104}$$

for $\alpha_x=\frac{p+p^{-1}}{2}$, with $p=q^{\frac{1}{2}}$ and $\zeta=p^n$.

Remark 4 *1. By selecting specific values of the parameters α_x, β_x and δ_x (as indicated in Subsection 3.2) in the previous theorem, we recover after appropriate changes in ϕ and ψ the coefficients β_n and γ_n of the very classical orthogonal polynomials. The coefficients β_n and γ_n of the very classical orthogonal polynomials were given explicitly by W. Koepf and D. Schmersau [20, 21] using a similar approach.*

2. The relations (99)-(102) are important tools for computer algebra since they provide in 4 relations the recurrence coefficients of all classical orthogonal polynomials [19].

4 Fourth-order divided-difference equations for O.P.

4.1 Functions of the second kind

In connection with the classical orthogonal polynomials (P_n), there is an important function called function of second kind denoted Q_n and connected to P_n in the following way [31]

$$Q_n(x(z))=\frac{1}{\rho(z)}\sum_{s=a}^{b-1}\frac{P_n(x(s))\,\rho(s)\,\nabla x_1(s)}{x(s)-x(z)},\quad z\neq a,\,a+1,\,\ldots,\,b-1. \tag{105}$$

Also, the first associated $P_n^{(1)}$ of P_n, defined by the recurrence relation (14) for $r=1$ is related to P_n by

$$P_n^{(1)}(x(z))=\sum_{s=a}^{b-1}\frac{\left[P_{n+1}(x(s))-P_{n+1}(x(z))\right]\rho(s)\,\nabla x_1(s)}{x(s)-x(z)},\quad z\neq a,\,a+1,\,\ldots,\,b-1, \tag{106}$$

for the discrete orthogonality.

The following properties [31] will be used in the next subsection.

Theorem 9 *The function Q_n obeys:*

1. *$\forall n\in\mathbb{N}_0$, P_n and Q_n are two linearly independent solutions of (92).*
2. *P_n and Q_n are the two linearly independent solutions of the recurrence relation*

$$X_{n+1}(x(s))=(x(s)-\beta_n)\,X_n(x(s))-\gamma_n\,X_{n-1}(x(s)),\ n\geq 1.$$

3. *P_n and Q_n and the first associated $P_n^{(1)}$ are related from (105) and (106) by*

$$Q_n(x(s))=P_n(x(s))\,Q_0(x(s))+\frac{\gamma_0}{\rho(s)}\,P_{n-1}^{(1)}(x(s)),\ n\geq 1,\ s\neq a,\,a+1,\,\ldots,\,b-1. \tag{107}$$

4. *Q_n fulfils the following asymptotic property*

$$Q_n(x(s)) = -\frac{\prod_{i=0}^{n} \gamma_i}{\rho(s)\, x^{n+1}(s)} \left[1 + O\left(\frac{1}{x(s)}\right)\right], \quad x(s) \to \infty, \tag{108}$$

with

$$\gamma_0 = \sum_{s=0}^{N} \rho(s)\, \nabla x_1(s) \text{ or } \gamma_0 = \int_C \rho(s)\, \nabla x_1(s)\, ds,$$

for the discrete and the continuous orthogonality respectively (see definition 1).

4.2 Intermediate relations

To derive the fourth-order difference equations for the modifications of the classical orthogonal polynomials, we shall start with the following intermediate result.

Theorem 10 *Let (P_n) be a sequence of classical orthogonal polynomials satisfying the second-order difference equation (92) with the corresponding orthogonality weight satisfying the Pearson-type equation (8) and the border conditions (10) where the functions $\sigma(s)$ and the polynomials $\phi(s)$ and $\psi(s)$ are related by (9). Then, the first associated $(P_n^{(1)})$ of (P_n) satisfies*

$$\phi(s)\, [A_1(s)\, \mathbb{F}_x + B_1(s)\, \mathbb{M}_x + C_1(n)]\, P_{n-1}^{(1)}(x(s)) = 2\,\eta\, [D_1(s)\, \mathbb{M}_x + E_1(s,n)]\, P_n(s), \tag{109}$$

with

$$\begin{aligned} A_1(s) &= -\phi(s) - 2\alpha^2\, \mathrm{U}_2(s)\, \phi'' + 2\,\alpha\, U_1(s)\, \psi(s) - 2\, U_1(s)\, \mathbb{M}_x \phi(s) \\ &\qquad + 4\,\alpha\, U_1^2(s)\, \psi' + 2\alpha\, (2\alpha^2 - 1)\, U_2(s)\, \psi', \\ B_1(s) &= (2\,\alpha^2 - 1)\, \psi(s) - 2\,\alpha\, \mathbb{M}_x \phi(s) + 2\, (4\alpha^2 - 1)\, U_1(s)\, \psi' - 2\,\alpha\, U_1(s)\, \phi'', \\ C_1(n) &= (2\,\alpha^2 - 1)\, \psi' - \alpha\, \phi'' - \lambda_n, \\ D_1(s) &= \alpha\, \phi(s) - U_1(s)\, \psi(s), \\ E_1(s,n) &= -\lambda_n\, U_1(s), \end{aligned} \tag{110}$$

where the polynomials U_1 and U_2 are defined by (57),

$$\eta = \left(\alpha_x\, \psi' - \frac{\phi''}{2}\right) \gamma_0 = (\alpha_x\, \psi_1 - \phi_2)\, \gamma_0, \tag{111}$$

and

$$\gamma_0 = \sum_{s=0}^{N} \rho(s)\, \nabla x_1(s) \text{ or } \gamma_0 = \int_C \rho(s)\, \nabla x_1(s)\, ds \tag{112}$$

for discrete and continuous orthogonality respectively.

The proof of the theorem will use the following lemma.

Lemma 3 *The function of second kind $Q_n(x(s))$ defined by (105) satisfies:*

$$(\phi(s) - \frac{1}{2}\psi(s)\, \nabla x_1(s))\, \rho(s)\, \frac{\nabla Q_0(x(s))}{\nabla x(s)} = \eta, \ \forall s \in (a,b), \ s \neq a,\, a+1,\, \ldots,\, b-1; \tag{113}$$

$$\rho(s)\, \mathbb{M}_x Q_0(x(s) = \frac{\eta\, \phi(s)}{\phi^2(s) - U_2(s)\, \psi^2(s)}, \tag{114}$$

where η is given by (111).

Proof: Since Q_n is a solution of (92), taking into account the fact that $\lambda_0 = 0$ from (96), we have

$$\phi(s)\,\mathbb{F}_x Q_0(x(s)) + \tau(s)\,\mathbb{M}_x Q_0(x(s)) = 0 \Leftrightarrow \Delta\,\left[(\phi(s) - \frac{1}{2}\psi(s)\,\nabla x_1(s))\,\rho(s)\,\frac{\nabla Q_0(x(s))}{\nabla x(s)}\right] = 0.$$

Therefore, the left-hand side of (113) is a periodic function of period 1. This combined with the asymptotic behavior obtained using (108)

$$(\phi(s) - \frac{1}{2}\psi(s)\,\nabla x_1(s))\,\rho(s)\,\frac{\nabla Q_0(x(s))}{\nabla x(s)} = \eta\,\left[1 + O\left(\frac{1}{x(s)}\right)\right],\;\; x(s) \to \infty,$$

allows to deduce that the left-hand side of the previous equation is the constant η.

To derive Equation (114), we use an equivalent form of (113)

$$\frac{\nabla Q_0(x(s))}{\nabla x(s)} = \frac{\eta}{\rho(s)\,(\phi(s) - \frac{1}{2}\psi(s)\,\nabla x_1(s))}$$

and obtain

$$\begin{aligned}
\rho(s)\,\mathbb{M}_x Q_0(x(s)) &= \frac{\rho(s)}{2}\left(\frac{\Delta Q_0(x(s))}{\Delta x(s)} + \frac{\nabla Q_0(x(s))}{\nabla x(s)}\right) \\
&= \frac{1}{2}\left(\frac{\eta\,\rho(s)}{\rho(s+1)\,[\phi(s+1) - \frac{1}{2}\psi(s+1)\,\Delta x_1(s)]} + \frac{\eta}{\phi(s) - \frac{1}{2}\psi(s)\,\nabla x_1(s)}\right).
\end{aligned}$$

Next, we use an equivalent form of the Pearson-type equation (8)

$$\frac{\rho(s+1)}{\rho(s)} = \frac{\phi(s) + \frac{1}{2}\psi(s)\,\nabla x_1(s)}{\phi(s+1) - \frac{1}{2}\psi(s+1)\,\Delta x_1(s)} \tag{115}$$

to get

$$\begin{aligned}
\rho(s)\,\mathbb{M}_x Q_0(x(s)) &= \frac{1}{2}\left(\frac{\eta}{\phi(s) + \frac{1}{2}\psi(s)\,\nabla x_1(s)} + \frac{\eta}{\phi(s) - \frac{1}{2}\psi(s)\,\nabla x_1(s)}\right) \\
&= \frac{\eta\,\phi(s)}{\phi^2(s) - \frac{1}{4}\,[\psi(s)\,\nabla x_1(s)]^2} \\
&= \frac{\eta\,\phi(s)}{\phi^2(s) - \psi^2(s)\,U_2(s)},
\end{aligned}$$

since from Equation (34),

$$\frac{[\nabla x_1(s)]^2}{4} = Q(x(s)) = U_2(s).$$

□

Let us now give the proof of Theorem 10.

Proof: In the first step, we use relation (107) and the fact that Q_n is a solution of (92) to get

$$(\phi(s)\,\mathbb{F}_x + \psi(s)\,\mathbb{M}_x + \lambda_n)\,\left[P_n(x(s))\,Q_0(x(s)) + \frac{1}{\rho(s)}\,P_{n-1}^{(1)}(x(s))\right] = 0. \tag{116}$$

Using the product rules (62) and (63) for $\mathbb{F}_x$ and $\mathbb{M}_x$ and Equation (92) in order to eliminate all occurrences of $\mathbb{F}_x P_n(x(s))$ and $\mathbb{F}_x Q_0(x(s))$ we transform the previous equation into an equation whose left-hand side is a linear combination of

$$\mathbb{F}_x P_{n-1}^{(1)}(x(s)),\; \mathbb{M}_x P_{n-1}^{(1)}(x(s)),\; P_{n-1}^{(1)}(x(s)),\; \mathbb{M}_x P_n(x(s)) \text{ and } P_n(x(s)).$$

The coefficients of this linear combination are functions of

$$x(s),\ \phi(s),\ \psi(s),\ U_1(s),\ U_2(s),\ \mathbb{M}_x Q_0(x(s)),\ \mathbb{F}_x\rho(s),\ \mathbb{M}_x\rho(s),\ \text{and}\ \rho(s).$$

The expression $\mathbb{M}_x Q_0(x(s))$ is eliminated thanks to (114). It remains now to express $\mathbb{F}_x\rho(s)$ and $\mathbb{M}_x\rho(s)$ in terms of $\rho(s)$ times rational functions of $x(s)$.

For this aim, we use (115) to eliminate all occurrences of $\rho(s-1)$ and $\rho(s+1)$ in the equations

$$\begin{aligned} \mathbb{F}_x\rho(s) &= \frac{\Delta}{\Delta x(s-\frac{1}{2})}\frac{\nabla\rho(s)}{\nabla x(s)}, \\ \mathbb{M}_x\rho(s) &= \frac{1}{2}\left(\frac{\Delta\rho(s)}{\Delta x(s)}+\frac{\nabla\rho(s)}{\nabla x(s)}\right), \end{aligned}$$

and obtain

$$\frac{\mathbb{F}_x\rho(s)}{\rho(s)}\ \text{and}\ \frac{\mathbb{M}_x\rho(s)}{\rho(s)}$$

as rational functions of the two variables $x(s)$ and $x(s+1)$ whose coefficients depend only on those of the polynomials $\phi(s)$ and $\psi(s)$ ((22) and (23) have been used as well). Then, assuming $\alpha \neq 1$, we combine (57) and the relation (which is easily deduced from (34) using (22))

$$U_2(s) = \frac{\left[x(s+1)+(1-2\alpha^2)x(s)-2\beta(\alpha+1)\right]^2}{4\,\alpha^2} = (\alpha^2-1)\,x^2(s)+2\,\beta\,(\alpha+1)\,x(s)+\delta,$$

to get the system

$$\begin{aligned} U_1(s) &= (\alpha+1)[(\alpha-1)x(s)+\beta], \\ 2\,u\,\alpha\,\sqrt{U_2(s)} &= x(s+1)+(1-2\alpha^2)x(s)-2\beta(\alpha+1), \end{aligned}$$

where $u=\pm1$. Solving this system in terms of the unknowns $x(s)$ and $x(s+1)$ yields

$$\begin{cases} x\,(s) = \frac{U_1(s)}{(\alpha-1)(1+\alpha)} - \frac{\beta}{\alpha-1}, \\ x\,(s+1) = \frac{\left(2\,\alpha^2-1\right)U_1(s)}{(\alpha-1)(1+\alpha)} + 2\,u\alpha\,\sqrt{U_2\,(s)} - \frac{\beta}{\alpha-1}. \end{cases} \tag{117}$$

Computations with Maple 9 [27] using the previous equation allow to express

$$\frac{\mathbb{F}_x\rho(s)}{\rho(s)}\ \text{and}\ \frac{\mathbb{M}_x\rho(s)}{\rho(s)},$$

as rational functions of the variable $x(s)$ depending only on *integer* powers of the functions

$$U_1(s),\ U_2(s),\ \mathbb{F}_x\phi(s),\ \mathbb{M}_x\phi(s), \phi(s),\ \mathbb{M}_x\psi(s)\ \text{and}\ \psi(s).$$

Summing up, we obtain Equation (109).

Notice that if $\alpha = 1$, the result obtained is still valid since the singularity $\frac{1}{\alpha-1}$ appearing in (117) will be automatically cancelled in the expressions of

$$\frac{\mathbb{F}_x\rho(s)}{\rho(s)}\ \text{and}\ \frac{\mathbb{M}_x\rho(s)}{\rho(s)}$$

after the computations.

□

4.3 Fourth-order difference equations for modifications of the classical orthogonal polynomials

Theorem 10 is the key to the following fundamental result.

Theorem 11 *Let (P_n) be a classical orthogonal polynomial system satisfying (92) and $(\tilde{P}_n)$ the orthogonal polynomial system related to (P_n) by*

$$\tilde{P}_n(x(s)) = I_{n,r,k}(s)\, P_{n+r}(x(s)) + J_{n,r,k}(s)\, P^{(1)}_{n+r-1}(x(s)), \tag{118}$$

where r and k are nonnegative integers, $I_{n,r,k}(s)$ and $J_{n,r,k}(s)$ are polynomials in the variable $x(s)$ but not depending on n for $n \geq k$, i.e.

$$I_{n,r,k}(s) := I_{r,k}(s), \;\; J_{n,r,k}(s) := J_{r,k}(s) \neq 0 \;\text{ for }\; n \geq k. \tag{119}$$

Then, each $\tilde{P}_n$ satisfies a fourth-order difference equation of the factorized form

$$\begin{aligned}\mathbb{G}_{n,r,k}(y(x(s)) = \left[\tilde{A}_{n,r,k}(s)\, \mathbb{F}_x + \tilde{B}_{n,r,k}(s)\, \mathbb{M}_x + \tilde{C}_{n,r,k}(s)\right] \times \\ [A_{r,k}(s)\, \mathbb{F}_x + B_{r,k}(s)\, \mathbb{M}_x + C_{n,r,k}(s)]\; y(s) = 0,\; n \geq k,\end{aligned} \tag{120}$$

where $A_{r,k}(s)$, $B_{r,k}(s)$, $C_{n,r,k}(s)$, $\tilde{A}_{n,r,k}(s)$, $\tilde{B}_{n,r,k}(s)$ and $\tilde{C}_{n,r,k}(s)$ are polynomials in the variable $x(s)$ whose degree does not depend on n.

This fourth-order difference equation can also be written as

$$\begin{aligned}\mathbb{G}_{n,r,k}(y(x(s)) \;&=\; [G_4(s;n,r,k)\, \mathbb{F}_x\mathbb{F}_x \,+ G_3(s;n,r,k)\, \mathbb{M}_x\mathbb{F}_x + G_2(s;n,r,k)\, \mathbb{F}_x \\ &+\; G_1(s;n,r,k)\, \mathbb{M}_x + G_0(s;n,r,k)]\; y(s) = 0,\; n \geq k,\end{aligned} \tag{121}$$

where $G_j(s;n,r,k)$, $j = 0 \ldots 4$ are polynomials in the variable $x(s)$ whose degree does not depend on n.

Proof: First, we use Equation (109) for $n = n + r$ and (118) to get

$$\begin{aligned}&\phi(s)\, [A_1(s)\, \mathbb{F}_x + B_1(s)\, \mathbb{M}_x + C_1(n+r)] \times \\ &\qquad \left[\frac{\tilde{P}_n(x(s))}{J_{r,k}(s)} - \frac{I_{r,k}(s)}{J_{r,k}(s)}\, P_{n+r}(x(s))\right] \\ &= \qquad 2\,\eta\; [D_1(s)\, \mathbb{M}_x + E_1(s, n+r)]\, P_{n+r}(s),\; n \geq k.\end{aligned} \tag{122}$$

Next, we combine the previous equation, the quotient rules (64)-(65) and Equation (92) (in order to eliminate $\mathbb{F}_x P_{n+r}(x(s))$) to obtain

$$\begin{aligned}&[A_{r,k}(s)\, \mathbb{F}_x + B_{r,k}(s)\, \mathbb{M}_x + C_{n,r,k}(s)]\, \tilde{P}_n(x(s)) \\ &\qquad = [D_{n,r,k}(s)\, \mathbb{M}_x + E_{n,r,k}(s)]\, P_{n+r}(x(s)),\; n \geq k,\end{aligned} \tag{123}$$

where the coefficients $A_{r,k}(s)$, $B_{r,k}(s)$, $C_{n,r,k}(s)$, $D_{n,r,k}(s)$ and $E_{n,r,k}(s)$ are polynomials, and functions of the polynomials $I_{r,k}(s)$, $J_{r,k}(s)$, $A_1(s)$, $B_1(s)$, $C_1(n)$, $D_1(s)$ and $E_1(s, n+r)$.

If we write

$$\tilde{Q}_{n+r}(x(s)) = [D_{n,r,k}(s)\, \mathbb{M}_x + E_{n,r,k}(s)]\, P_{n+r}(x(s)),\; n \geq k, \tag{124}$$

then using the result of Lemma 1 and the fact that P_{n+r} satisfies (109) for $n = n + r$, we get

$$\mathbb{M}_x\tilde{Q}_{n+r}(x(s)) \;=\; [G_{n,r,k}(s)\, \mathbb{M}_x + H_{n,r,k}(s)]\; P_{n+r}(x(s)),\; n \geq k, \tag{125}$$

$$\mathbb{F}_x\tilde{Q}_{n+r}(x(s)) \;=\; \left[\tilde{G}_{n,r,k}(s)\, \mathbb{M}_x + \tilde{H}_{n,r,k}(s)\right]\; P_{n+r}(x(s)),\; n \geq k, \tag{126}$$

where $G_{n,r,k}(s)$, $G_{n,r,k}(s)$, $\tilde{G}_{n,r,k}(s)$ and $\tilde{H}_{n,r,k}(s)$ are rational functions of the variable $x(s)$. Equations (124)-(126) which can be seen as a system of three linear equations with respect to the unknowns $\mathbb{M}_x P_{n+r}(x(s))$ and $P_{n+r}(x(s))$ produce the second-order difference equation for $\tilde{Q}_{n+r}(x(s))$

$$\begin{vmatrix} D_{n,r,k}(s) & E_{n,r,k}(s) & \tilde{Q}_{n+r}(x(s)) \\ G_{n,r,k}(s) & H_{n,r,k}(s) & \mathbb{M}_x\tilde{Q}_{n+r}(x(s)) \\ \tilde{G}_{n,r,k}(s) & \tilde{H}_{n,r,k}(s) & \mathbb{F}_x\tilde{Q}_{n+r}(x(s)) \end{vmatrix} = 0,\ n \geq k.$$

The previous equation after cancellation of the common denominator can be brought into the form

$$\left[\tilde{A}_{n,r,k}(s)\,\mathbb{F}_x + \tilde{B}_{n,r,k}(s)\,\mathbb{M}_x + \tilde{C}_{n,r,k}(s)\right]\tilde{Q}_{n+r}(x(s)) = 0,\ n \geq k,$$

where $\tilde{A}_{n,r,k}(s)$, $\tilde{B}_{n,r,k}(s)$ and $\tilde{C}_{n,r,k}(s)$ are polynomials in the variable $x(s)$ whose degree does not depend on n. Combination of (123) and the previous equation produces Equation (120). Finally, (121) is derived from (120) by simultaneous application of the product rules (55)-(56) and (62)-(63). □

4.4 Solutions of the fourth-order difference equations

In the following, we solve the fourth-order difference equation satisfied by the modifications of classical orthogonal polynomials in terms of the polynomials P_n and its corresponding function of second kind Q_n.

Theorem 12 *Under the hypothesis of Theorem 11, we have: The four linearly independent solutions of the difference equation (120)*

$$\mathbb{G}_{n,r,k}(y(s) = 0,\ n \geq k,$$

are

$$\begin{aligned} S_1(s;n,r,k) &= \rho(s)\,J_{r,k}(s)\,P_{n+r}(x(s)), \\ S_2(s;n,r,k) &= \rho(s)\,J_{r,k}(s)\,Q_{n+r}(x(s)), \\ S_3(s;n,r,k) &= \left[I_{r,k}(s) - \gamma_0^{-1}\rho(s)\,Q_0(x(s))\,J_{r,k}(s)\right]P_{n+r}(x(s)), \\ S_4(s;n,r,k) &= \left[I_{r,k}(s) - \gamma_0^{-1}\rho(s)\,Q_0(x(s))\,J_{r,k}(s)\right]Q_{n+r}(x(s)). \end{aligned}$$

Proof: In the first step, we observe that because of the factorization in Equation (120), any solution of the equation

$$[A_{r,k}(s)\,\mathbb{F}_x + B_{r,k}(s)\,\mathbb{M}_x + C_{n,r,k}(s)]\,y(s) = 0,\ n \geq k \tag{127}$$

is also solution of (120).

In the second step, we also observe from the procedure we have used to construct Equation (120) that (using (116) and (122))

$$\begin{aligned} [A_{r,k}(s)\,\mathbb{F}_x + B_{r,k}(s)\,\mathbb{M}_x &+ C_{n,r,k}(s)]\,y(s) = 0 \\ &\Updownarrow \\ [A_1(s)\,\mathbb{F}_x + B_1(s)\,\mathbb{M}_x &+ C_1(n+r)]\left\{\frac{y(s)}{J_{r,k}(s)}\right\} = 0,\ n \geq k \\ &\Updownarrow \\ [\phi(s)\mathbb{F}_x + \psi(s)\mathbb{M}_x &+ \lambda_{n+r}]\left\{\frac{y(s)}{\rho(s)\,J_{r,k}(s)}\right\} = 0,\ n \geq k. \end{aligned}$$

Therefore, $S_1(s;n,r,k)$ and $S_2(s;n,r,k)$ are solutions of (127) and therefore of (120).

In the third step, we use (118) and the relation

$$P_n^{(r)}(x(s)) = \frac{\rho(s)[P_{r-1}(x(s))\,Q_{n+r}(x(s)) - Q_{r-1}(x(s))\,P_{n+r}(x(s))]}{\gamma_0\,\Gamma_{r-1}}$$

obtained from the fact that P_{n+r}, Q_{n+r} and $P_n^{(r)}$ satisfy the second-order recurrence relation (see Theorem 9 and also [12]) to get

$$\tilde{P}_n(x(s)) = S_3(s;n,r,k) + \gamma_0^{-1}\,S_2(s;n,r,k).$$

Here, Γ_{r-1} and γ_0 are given respectively by (16) and (112). Then, since $\tilde{P}_n(x(s))$ and $S_2(s;n,r,k)$ are both solutions of (120), it turns out from the previous equation that $S_3(s;n,r,k)$ is also solution of (120). Also, $S_4(s;n,r,k)$ is another solution of (120) because it is obtained by replacing P_{n+r} by Q_{n+r} in the expression of $S_3(s;n,r,k)$ and the functions P_{n+r} and Q_{n+r} satisfy the same second-order difference equation, namely (92) for $n = n+r$. We complete the proof by observing that the four solutions are linearly independent since by means of (108), they enjoy different asymptotic behavior. □

For any modification of the classical orthogonal polynomials leading to a relation of type (12), we get explicit expressions of the functions $S_j(s;n,r,k)$, $j = 1 \ldots 4$ in terms of $\rho(s)$, $P_n(x(s))$ and $Q_n(x(s))$. As can be seen from the previous theorem, these solutions have the same structure for the difference equations satisfied by modifications of all classical orthogonal polynomials. These solutions were given explicitly in our previous works for the modifications of the very classical orthogonal polynomials. We therefore refer to these three papers [11, 12, 13].

It should be mentioned that the fourth-order difference equation satisfied by the Laguerre-Hahn polynomials orthogonal on special nonuniform lattices was derived in [7]. This result which is based on the properties of the formal Stieltjes function of the corresponding functional [23] covers the modifications of the classical orthogonal polynomials. However, it does not deal with factorization nor with the solutions of the difference equation derived. Our approach, which uses the operators $\mathbb{F}_x$, $\mathbb{M}_x$, the Pearson-type equation for the orthogonality weight and the second-order divided difference equation satisfied by the initial polynomials allows us to factorize and solve the difference equations obtained and constitute a natural extension of the results obtained for the very classical orthogonal polynomials [11, 12, 13].

Corollary 3 *The four linearly independent solutions of the fourth-order divided-difference equations satisfied by the rth associated classical orthogonal polynomials are*

$$\begin{aligned}
S_1(s;n,r,0) &= \rho(s)P_{r-1}(x(s)\,P_{n+r}(x(s));\\
S_2(s;n,r,0) &= \rho(s)Q_{r-1}(x(s)\,P_{n+r}(x(s));\\
S_3(s;n,r,0) &= \rho(s)P_{r-1}(x(s)\,Q_{n+r}(x(s));\\
S_4(s;n,r,0) &= \rho(s)Q_{r-1}(x(s)\,Q_{n+r}(x(s)).
\end{aligned}$$

This is obtained by combining the previous theorem, (15), (16) and (107).

Remark 5 *The fourth-order divided-difference equation for the rth associated classical orthogonal polynomials of a discrete variable on a nonuniform lattice given in Theorem 11 for nonnegative integer r is valid if r is a positive real number. The four linearly independent solutions given in the previous corollary are still valid but one should keep in mind that P_r and Q_r in this cases represent the two linearly independent solutions of*

$$\phi(x(s))\,\mathbb{F}_x y(x(s)) + \psi(x(s))\,\mathbb{M}_x y(x(s)) + \lambda_r\,y(x(s)) = 0$$

for the real number r [31]. This extension can be deduced following the method used for the rth associated classical orthogonal polynomials of a continuous variable (see [12], Theorem 8).

5 Specializations and applications

In this section, we mainly investigate the results obtained for specific values of the parameters α_x, β_x and δ_x of the lattice $x(s)$, namely those leading to the very classical orthogonal polynomials as already mentioned in Subsection 3.2. We also give some applications.

5.1 Specializations

5.1.1 The classical orthogonal polynomials of a continuous variable

For

$$\alpha_x = 1,\ \beta_x = 0 \text{ and } \delta_x = 0,$$

we have

$$\mathbb{M}_x = \frac{d}{dx},\ \mathbb{F}_x = \frac{d^2}{dx^2} \text{ and } U_1(s) = U_2(s) = 0.$$

Therefore, Relation (109) reads

$$\left[\phi(x)\,\frac{d^2}{dx^2} + (2\,\phi'(x) - \psi(s))\frac{d}{dx} + \lambda_n + \phi'' - \psi'\right] P_{n-1}^{(1)}(x) = (\phi'' - 2\,\psi')\,P_n'(x).$$

This relation, which was first derived by Ronveaux [29], is the key to the derivation and solutions of the fourth-order differential equations satisfied by modifications of the classical orthogonal polynomials of a continuous variable. For more details, we refer to the paper [12] which appears now to be a particular case of the results obtained in the framework of this work.

5.1.2 The classical orthogonal polynomials of a discrete variable on a linear lattice

For

$$\alpha_x = 1,\ \beta_x = 0 \text{ and } \delta_x = \frac{1}{4},$$

we have

$$x(s) = s,\ \mathbb{M}_x = \frac{1}{2}(\Delta + \nabla),\ U_1(s) = 0,\ \ \mathbb{F}_x = \Delta\nabla,\ \ \text{and } U_2(s) = \frac{1}{4}.$$

Therefore, Relation (109) reads

$$\begin{aligned}\big\{(2\,\phi(s) + \phi'' - \psi')\Delta\nabla \quad &+ \quad (\psi(s) + [\Delta + \nabla]\phi(s))[\Delta + \nabla] + \lambda_n + \phi'' + \psi'\big\}\,P_{n-1}^{(1)}(x) \\ &= \quad (\phi'' - 2\,\psi')\,\Delta\nabla\,P_n(x).\end{aligned}$$

The previous relation, due to Atakishiyev, Ronveaux and Wolf [6], has been used to derive the fourth-order difference equations satisfied by the modifications of classical orthogonal polynomials of a discrete variable on a linear lattice. We refer to the paper [11] for details about these equations as well as their solutions.

5.1.3 The q-classical orthogonal polynomials

For

$$\alpha_x = \frac{q^{\frac{1}{2}} + q^{\frac{-1}{2}}}{2},\ \beta_x = \delta_x = 0,\ \text{with } q \neq 0,\, 1,$$

we have

$$x(s) = q^{\pm s},\ \mathbb{F}_x = q^2\,D_q\,D_{\frac{1}{q}},\ \mathbb{M}_x = \frac{1}{2}(D_q + D_{\frac{1}{q}}),\ U_1(s) = (\alpha_x^2 - 1)\,x(s),\ \text{and } U_2(s) = (\alpha_x^2 - 1)\,x^2(s)$$

and the corresponding families are the q-classical orthogonal polynomials.

For $x(s) = q^s$, the relation (109) is equivalent to Equation (16) in [13] with the polynomial ϕ replaced by

$$q^2\,\phi(x(s)) - \frac{1}{2}(q-1)\,x(s)\,\psi(x(s)).$$

This relation which is due to Foupouagnigni, Ronveaux and Koepf [10] allowed in [13] to derive and solve the fourth-order q-difference equation satisfied by the modifications of the q-classical orthogonal polynomials.

5.1.4 The Askey-Wilson operator

Relations between $\mathbb{D}_x$, $\mathbb{S}_x$ and the Askey-Wilson operator $\mathcal{D}_q$

We establish relations between the operators $\mathbb{D}_x$, $\mathbb{S}_x$ and the Askey-Wilson operator $\mathcal{D}_q$ and then use these relations to state Theorems 1, 2 and 3 for the special case of the Askey-Wilson operator.

The Askey-Wilson operator $\mathcal{D}_q$ is defined as [4]

$$\mathcal{D}_q f(x) = \frac{\delta_q f(x)}{\delta_q x},\ x = \cos\theta,$$

with

$$\delta_q f\left(e^{i\theta}\right) = f\left(q^{\frac{1}{2}} e^{i\theta}\right) - f\left(q^{-\frac{1}{2}} e^{i\theta}\right).$$

The relation with $\mathbb{D}_x$ and $\mathbb{S}_x$

For $x = \cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}$, we set $e^{i\theta} = q^s$ and obtain $x := x(s) = \frac{q^s+q^{-s}}{2}$. The coefficients α, β and δ corresponding to this lattice are

$$\alpha = \frac{q^{\frac{1}{2}} + q^{-\frac{1}{2}}}{2},\ \beta = 0,\ \delta = -\frac{(q-1)^2}{4\,q}. \tag{128}$$

The operators $\mathbb{D}_x$ and $\mathcal{D}_q$ are related by

$$\mathcal{D}_q f(x) = \frac{f(x(s+\frac{1}{2})) - f(x(s-\frac{1}{2}))}{x(s+\frac{1}{2}) - (x(s-\frac{1}{2})} = \mathbb{D}_{x_{-1}} f(x_{-1}(s)). \tag{129}$$

As was the case for $\mathbb{D}_x$ and $\mathbb{S}_x$, there is a need to define the companion operator for $\mathcal{D}_q$, namely the operator $\mathcal{S}_q$

$$\mathcal{S}_q f(x) = \frac{\mathcal{E}_q^+ f(x) + \mathcal{E}_q^- f(x)}{2},\ x = \cos\theta, \tag{130}$$

with

$$\mathcal{E}_q^+ f\left(e^{i\theta}\right) = f\left(q^{\frac{1}{2}} e^{i\theta}\right),\ \mathcal{E}_q^- f\left(e^{i\theta}\right) = f\left(q^{-\frac{1}{2}} e^{i\theta}\right).$$

This operator is related to $\mathbb{S}_x$ in the following way

$$\mathcal{S}_q f(x) = \mathbb{S}_{x_{-1}} f(x_{-1}(s)). \tag{131}$$

The operators equivalent to $\mathbb{F}_x$ and $\mathbb{M}_x$ in this case are

$$\mathcal{F}_q = \mathcal{D}_q^2,\ \ \mathcal{M}_q = \mathcal{S}_q\,\mathcal{D}_q. \tag{132}$$

Iteration of (129) and (131) give

$$\begin{aligned} \mathcal{D}_q^2 f(x) &= \mathbb{D}_{x_{-1}}\mathbb{D}_{x_{-2}} f(x_{-2}(s)); \\ \mathcal{S}_q^2 f(x) &= \mathbb{S}_{x_{-1}}\mathbb{S}_{x_{-2}} f(x_{-2}(s)); \\ \mathcal{S}_q\mathcal{D}_q f(x) &= \mathbb{S}_{x_{-1}}\mathbb{D}_{x_{-2}} f(x_{-2}(s)); \\ \mathcal{D}_q\mathcal{S}_q f(x) &= \mathbb{D}_{x_{-1}}\mathbb{S}_{x_{-2}} f(x_{-2}(s)). \end{aligned} \tag{133}$$

Using (129) and (131), one remarks that the operator $\mathcal{D}_q$ (respectively $\mathcal{S}_q$) transforms a polynomial of degree n in the variable x into a polynomial of degree $n-1$ in x (respectively of degree n in x). Moreover, we have the following theorems.

Theorem 13 *The following statements hold.*

1. *The operators $\mathcal{D}_q$ and $\mathcal{S}_q$ obey the following product rules:*

$$\mathcal{D}_q(f(x)\,g(x)) = \mathcal{S}_q f(x)\,\mathcal{D}_q g(x) + \mathcal{D}_q f(x)\,\mathcal{S}_q g(x); \tag{134}$$

$$\mathcal{S}_q(f(x)\,g(x)) = \mathcal{S}_q f(x)\,\mathcal{S}_q g(x) + V_2(x)\mathcal{D}_q f(x)\,\mathcal{D}_q g(x), \tag{135}$$

where

$$V_2(x) = \frac{(q-1)^2}{4\,q}\,(x^2-1). \tag{136}$$

2. *The operators $\mathcal{D}_q$ and $\mathcal{S}_q$ also satisfy the quotient rules:*

$$\mathcal{D}_q\left(\frac{f(x)}{g(x)}\right) = \frac{\mathcal{S}_q f(x)\,\mathcal{D}_q g(x) - \mathcal{D}_q f(x)\,\mathcal{S}_q g(x)}{V_2(x)\;[\mathcal{D}_q g(x)]^2 - [\mathcal{S}_q g(x)]^2}; \tag{137}$$

$$\mathcal{S}_q\left(\frac{f(x)}{g(x)}\right) = \frac{V_2(x)\mathcal{D}_q f(x)\,\mathcal{D}_q g(x) - \mathcal{S}_q f(x)\,\mathcal{S}_q g(x)}{V_2(x)\;[\mathcal{D}_q g(x)]^2 - [\mathcal{S}_q g(x)]^2}. \tag{138}$$

Remark 6 *For any integer n, the following relations hold*

$$\mathcal{D}_q x^n = \frac{\left(\alpha\,x + \sqrt{(\alpha^2-1)(x^2-1)}\right)^n - \left(\alpha\,x - \sqrt{(\alpha^2-1)(x^2-1)}\right)^n}{2\,\sqrt{(\alpha^2-1)(x^2-1)}}; \tag{139}$$

$$\mathcal{S}_q x^n = \frac{\left(\alpha\,x + \sqrt{(\alpha^2-1)(x^2-1)}\right)^n + \left(\alpha\,x - \sqrt{(\alpha^2-1)(x^2-1)}\right)^n}{2}, \tag{140}$$

where α is given by (128).

Theorem 14 *The following relations hold:*

$$\mathcal{D}_q\,\mathcal{S}_q = \alpha\,\mathcal{S}_q\,\mathcal{D}_q + \alpha\,x\,\mathcal{D}_q\,\mathcal{D}_q; \tag{141}$$

$$\mathcal{S}_q\,\mathcal{S}_q = \alpha\,x\,\mathcal{S}_q\,\mathcal{D}_q + \alpha\,(\alpha^2-1)\,(x^2-1)\,\mathcal{D}_q\,\mathcal{D}_q + \mathbb{I}; \tag{142}$$

$$\mathcal{F}_q\,\mathcal{M}_q = (2\,\alpha^2-1)\,\mathcal{M}_q\,\mathcal{F}_q + 2\,\alpha\,(\alpha^2-1)\,x\,\mathcal{F}_q\,\mathcal{F}_q; \tag{143}$$

$$\mathcal{M}_q\,\mathcal{M}_q = \alpha\,\mathcal{F}_q + 2\,\alpha\,(\alpha^2-1)\,x\,\mathcal{M}_q\,\mathcal{F}_q + (2\,\alpha^2-1)\,V_2(x)\,\mathcal{F}_q\,\mathcal{F}_q, \tag{144}$$

where α and $V_2(x)$ given by (128) and (135) respectively.

Remark 7 *Theorem 13 and Proposition 6 for the operators $\mathcal{F}_q$ and $\mathcal{M}_q$ (see (132)) are deduced from Theorem 5 and Proposition 5 by taking $x(s) = x,\ \beta = 0,\ \alpha = \frac{q^{\frac{1}{2}}+q^{-\frac{1}{2}}}{2}$, with the operators $\mathbb{F}_x$, $\mathbb{M}_x$ replaced respectively by $\mathcal{F}_q$ and $\mathcal{M}_q$.*

Theorem 15 *Let $f(x)$ and $g(x)$ be two functions of the variable x satisfying respectively*

$$\mathcal{D}_q^2 f(x) + a_1(x)\,\mathcal{S}_q\mathcal{D}_q f(x) + a_0(x)\,f(x) = 0,\ \ \mathcal{D}_q^2 f(x) + b_1(x)\,\mathcal{S}_q\mathcal{D}_q f(x) + b_0(x)\,g(x) = 0, \tag{145}$$

where a_j and b_j are given functions of x.

Then, the product $f(x)\,g(x)$ is a solution of a fourth-order divided-difference equation of the form

$$I_4(x)\,\mathcal{D}_q^4 y(x) + I_3(x)\,\mathcal{S}_q\,\mathcal{D}_q^3 y(x) + I_2(x)\,\mathcal{D}_q^2 y(x) + I_1(x)\,\mathcal{S}_q\,\mathcal{D}_q y(x) + I_0(x)\,y(x) = 0$$

where I_j are functions of a_j and b_j. If the $a_j(x),\ j = 0,\ 1$ and the $b_j(x),\ j = 0,\ 1$ are rational functions of x, then the coefficients $I_j(x),\ j = 0\ \ldots\ 4$ can be chosen to be polynomials in the variable x.

More generally, if $f_j,\ j = 1, \ldots n$ are functions of the variable x such that any f_j satisfies a linear divided-difference equation of order r_j involving only the operators $\mathcal{D}_q^2$ and $\mathcal{S}_q\mathcal{D}_q$, then the product $f = \prod_{j=1}^{n} f_j$ satisfies a divided-difference equation of order $r = \prod_{j=1}^{n} r_j$ involving only (at most) the operators

$$\mathcal{D}_q^{2i} \text{ and } \mathcal{S}_q\,\mathcal{D}_q^{2j-1},\ \text{with } 0 \le 2\,i \le r,\ \text{and } 0 \le 2\,j-1 \le 2\,j \le r = \prod_{k=1}^{n} r_k.$$

As consequence of the previous theorem we state the following.

Corollary 4 *If $P_n(x)$ is a solution of a second-order divided-difference equation of hypergeometric form*

$$\phi(x)\,\mathcal{D}_q^2\,y(x)+\psi(x)\,\mathcal{S}_q\,\mathcal{D}_q\,y(x)+\lambda_n\,y(x)=0,$$

where ϕ and ψ are polynomials of degree at most 2 and one respectively, then the product $P_n(x)\,P_r(x)$ satisfies a fourth-order divided-difference equation

$$I_4(n,r,x)\,\mathcal{D}_q^4y(x)+I_3(n,r,x)\,\mathcal{S}_q\,\mathcal{D}_q^3y(x)+I_2(n,r,x)\,\mathcal{D}_q^2y(x)+I_1(n,r,x)\,\mathcal{S}_q\,\mathcal{D}_qy(x)+I_0(n,r,x)\,y(x)=0,$$

where the $I_j(n,r,x)$ are polynomials whose degree doesn't depend on n. Computations using computer algebra software can allow to find explicitly the coefficients $I_j(n,r,x)$ in terms of ϕ, ψ and λ_n.

5.2 Applications

5.2.1 Special cases of classical orthogonal polynomials

The relation (109) allows to observe that when $\eta=0$, the first associated $P_{n-1}^{(1)}$ satisfies the second-order difference equation

$$[A_1(s)\,\mathbb{F}_x+B_1(s)\,\mathbb{M}_x+C_1(n)]\,y(s)=0,$$

where the coefficients A_1, B_1 and C_1 are those of (109). It turns out that the previous equation is of hypergeometric type. Under certain conditions (on the parameters involved in the definition), the first associated of the Askey-Wilson, the q-Racah and the Racah polynomials, remains classical. This property is not true in general because the first associated of classical orthogonal polynomials is in general not classical but belongs rather to the so-called Laguerre-Hahn class [22]-[24]. Similar results exist for the very classical orthogonal polynomials [10, 29, 30]

Theorem 16 *We have the following.*

1. *For $a\,b\,c\,d=q$, the first associated of the Askey-Wilson polynomials remains classical and is related to the Askey-Wilson polynomials by*

$$\tilde{p}_n^{(1)}\left(x(s);a,b,c,\frac{q}{a\,b\,c}|q\right)=u^n\,\tilde{p}_n\left(u\,x(s);\frac{u\,q}{a},\frac{u\,q}{b},\frac{u\,q}{c},u\,a\,b\,c|q\right), \tag{146}$$

   with $u=\pm1$.

2. *For $\alpha\,\beta\,q=1$, the first associated of the q-Racah polynomials remains classical and is related to the q-Racah polynomials by*

$$\tilde{R}_n^{(1)}\left(x(s);\alpha,\frac{1}{q\,\alpha},\gamma,\delta|q\right)=(\gamma\,\delta)^n\,\tilde{R}_n\left(\frac{x(s)}{\gamma\,\delta};\frac{1}{\alpha},\alpha\,q,\frac{1}{\gamma},\frac{1}{\delta}|q\right). \tag{147}$$

3. *For $\alpha+\beta=-1$, the first associated of the Racah polynomials remains classical and is related to the Racah polynomials by*

$$\tilde{R}_n^{(1)}\left(x(s);\alpha,-1-\alpha,\gamma,\delta\right)=\tilde{R}_n\left(x(s)+\gamma(\alpha+1);\delta-\alpha,1+\alpha-\delta,-\gamma,\delta\right). \tag{148}$$

Proof: For the Askey-Wilson polynomials, one obtains by direct computation using (111) and the data given in Section 3.2.3 that

$$\eta=\left(\alpha_x\,\psi'-\frac{\phi''}{2}\right)=\frac{4(q-abcd)}{1-q}$$

with $\alpha_x=\frac{q^{\frac{1}{2}}+q^{-\frac{1}{2}}}{2}$ and

$$\begin{aligned}\gamma_{n+1}\left(a,b,c,\frac{q}{a\,b\,c}|q\right)&=\gamma_n\left(\frac{u\,q}{a},\frac{u\,q}{b},\frac{u\,q}{c},u\,a\,b\,c|q\right);\\ \beta_{n+1}\left(a,b,c,\frac{q}{a\,b\,c}|q\right)&=u\,\beta_n\left(\frac{u\,q}{a},\frac{u\,q}{b},\frac{u\,q}{c},u\,a\,b\,c|q\right).\end{aligned}$$

The Relation (146) is obtained using the last two relations and the fact that $\tilde{p}_{n+1}$ and $\tilde{p}_n^{(1)}$ satisfy the same three-term recurrence relation. The proof of the identities (147) and (148) is obtained in the same way since

$$\begin{aligned}
\eta = \frac{2q(1-\alpha\beta q}{1-q},\ \alpha_x &= \frac{q^{\frac{1}{2}}+q^{-\frac{1}{2}}}{2};\\
\gamma_{n+1}\left(\alpha, \frac{1}{q\,\alpha}, \gamma, \delta\right) &= \gamma^2\,\delta^2\,\gamma_n\left(\frac{1}{\alpha}, \alpha q, \frac{1}{\gamma}, \frac{1}{\delta}|q\right);\\
\beta_{n+1}\left(\alpha, \frac{1}{q\,\alpha}, \gamma, \delta\right) &= \gamma\,\delta\,\beta_n\left(\frac{1}{\alpha}, \alpha q, \frac{1}{\gamma}, \frac{1}{\delta}|q\right)
\end{aligned}$$

and

$$\begin{aligned}
\eta = 2(\alpha+\beta+1),\ \alpha_x &= 1;\\
\gamma_{n+1}(\alpha, -1-\alpha, \gamma, \delta) &= \gamma_n(\delta-\alpha, 1+\alpha-\delta, -\gamma, \delta);\\
\beta_{n+1}(\alpha, -1-\alpha, \gamma, \delta) &= \gamma_n(\delta-\alpha, 1+\alpha-\delta, -\gamma, \delta) - \gamma(\alpha+1)
\end{aligned}$$

for the q-Racah and the Racah polynomials, respectively. □

Notice that relations such as (146)-(148) exist for very classical orthogonal polynomials. They were given in references [15, 16], [3], [10] for Jacobi, Hahn and Big-q-Jacobi (and also Little-q-Jacobi) respectively.

5.2.2 Polynomial solutions of some difference equations

The product rules (55)-(56) and (62)-(63) can be used to look for polynomial solutions of difference equations with polynomial coefficients involving only the operators $\mathbb{F}_x$ and $\mathbb{M}_x$. For example, if $A(s)$, $B(s)$, $C(s)$ and $D(s)$ are polynomials in the variable $x(s)$, then the polynomial solution of the equation (if it exists)

$$A(s)\,\mathbb{F}_x y(s) + B(s)\,\mathbb{M}_x y(s) + C(s)\,y(s) = D(s)$$

can be found by writing

$$y(s) = \sum_{k=0}^{n} a_{n,k}\,x^k(s),$$

and solving the linear system obtained with respect to the unknowns $a_{n,k}$ in terms of the coefficients of the polynomials $A(s)$, $B(s)$, $C(s)$ and $D(s)$. This in the same way as for the usual differential equation since the operators $\mathbb{F}_x$ and $\mathbb{M}_x$ transform a polynomial in the variable $x(s)$ into a polynomial of the same variable. Using the quotient rules, one can look for rational solutions of some difference equations with polynomial coefficients in the same way. This approach can be used to look in general for analytic solutions (here we mean solutions which can be expanded in power series in terms of the variable $x(s)$) of difference equations with polynomial coefficients.

5.2.3 Steps forward towards the characterization of some classes of OP

In this work, we have derived diverse results for classical orthogonal polynomials in the same line as for those of the very classical orthogonal polynomials. In the sequel, we have obtained many intermediate results such as the product and quotient rules, the coefficients $D_{n,k}$, $S_{n,k}$, $F_{n,k}$, $M_{n,k}$, β_n, $T_{n,1}$, $T_{n,2}$, β_n and γ_n. These coefficients can be used for the implementation of codes in computer algebra relative to the classical orthogonal polynomials on quadratic and q-quadratic lattices. They could also be used for the complete characterization of the classical orthogonal polynomials. As illustration, for the very classical orthogonal polynomials, the ratio $\frac{T_{n,k}}{T_{n,k-1}}$ is a rational function of n **or** q^n depending on whether the variable is continuous, linear or q-linear respectively. This is an equivalent characterization property for the very classical orthogonal polynomials [1, 17]. But from the results obtained here, we observe that the ratio $\frac{T_{n,2}}{T_{n,1}}$ is a rational function of n **and** q^n (**at the same time**) for the q-quadratic lattice and for the basis $(x^n(s))$. This is an indication for the complexity for the formulation of the characterization theorems for the classical (not very classical) orthogonal polynomials. Summing up, we can conclude that the results obtained in this work constitute some important steps forward towards the complete characterization of the classical orthogonal polynomials as well as that of the semi-classical and the Laguerre-Hahn orthogonal polynomials of a discrete variable on nonuniform lattices [24, 23].

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