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ON DIFFERENTIABILITY OF SRB STATES FOR PARTIALLY HYPERBOLIC SYSTEMS.

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Abstract. Consider a one parameter family of diffeomorphisms f_ε such that f_0 is an Anosov element in a standard abelian Anosov action having sufficiently strong mixing properties. Let ν_ε be any u-Gibbs state for f_ε. We prove (Theorem 1) that if A is a C^∞ function then the map $A \to \nu_\varepsilon(A)$ is differentiable at $\varepsilon = 0$. This implies (Corollary 1) that the difference of Birkhoff averages of the perturbed and unperturbed systems is proportional to ε. We apply this result (Corollary 3) to show that if f_0 is a time one map of geodesic flow on a unit tangent bundle over a surface of negative curvature then a generic perturbation has a unique SRB measure with good statistical properties.

1. Introduction

This paper deals with the question of stability of stochastic behavior. Let us make few definitions. Let f be a smooth diffeomorphism of a smooth compact manifold M and let μ be an f-invariant measure. Define its *basin* as

$$\mathbb{B}(\mu) = \{x : \forall A \in C(M) \quad \lim_{n\to+\infty} \frac{1}{n}\sum_{j=0}^{n-1} A(f^j x) = \mu(A)\}.$$

Call μ an *SRB measure* if the Lebesgue measure of its basin is positive. The question of the existence of SRB measures and their dependence on parameters is one of the central questions in smooth ergodic theory. So far the only situation where this question is well understood is uniformly hyperbolic systems. Namely if the system satisfies a no-cycle condition (which prevents the phase portrait of the system from explosion) then it has a finite number of SRB states [55, 9] which depend continuously and even smoothly on f [3, 18, 31, 32, 48].

In the recent decades a lot of research was devoted to extension of this result beyond uniform hyperbolicity. Here the main directions of research are non-uniform hyperbolicity [41, 42, 60] and partial hyperbolicity [14, 28, 15, 57].

In [30] Jakobson proved the first result about the stability of SRB measures for non-uniformly hyperbolic systems. Namely, he proved that in the quadratic family $f_a(x) = 1 - ax^2$ there is a positive measure set of parameters near $a = 2$ such that f_a has an SRB measure which is absolutely continuous with respect to the Lebesgue measure. By now this result was generalized to a large class of non-uniformly hyperbolic systems (see reviews [25, 37, 57, 61]). In particular, it is known that one can choose a large set of parameters A such that the map $a \to \mu_{SRB}(f_a)$ is continuous on A, however it is essential to discard some parameters to get continuity (see [56]). Similar results are expected to hold in higher dimensional situations. In particular, one should expect discontinuous behavior of SRB measures in the Newhouse domain (see [39, 40, 57] for more discussion of this subject).

Another important development was [26] where the stable ergodicity of the time one map of the geodesic flow on a surface of constant negative curvature was proved (in other words, [26] shows that for small volume preserving perturbations of the time one map the volume is the only SRB measure). Currently this result is extended for a large class of partially hyperbolic systems (see [15] for a survey).This result demonstrated that for partially hyperbolic systems much better continuity properties can be expected. In fact, in [58, 1, 5, 21] several open sets of partially hyperbolic systems were constructed such that each diffeomorphism has a unique SRB measure and the dependence of this measure on parameters is continuous.

This paper is devoted to differentiability of SRB measures for partially hyperbolic systems. The question of differentiability plays important role in averaging, rigidity theory and statistical physics ([35, 31, 8, 50]) but not much is known beyond the uniformly hyperbolic case (in [17, 6] very interesting results about the differentiability of SRB measures for uniformly hyperbolic systems with singularities were obtained). In [48, 51] some explicit formulas for derivatives of SRB measures were proposed which should hold for a large class of dynamical systems, however the question of their applicability remains open.

Here we present a new method to prove differentiability. We illustrate this method on the example of abelian Anosov actions. Recall that an Anosov action is a partially hyperbolic system whose central directions is spanned by a symmetry group of the system. For an abelian action the symmetry group is abelian. (See Section 2 for more precise definitions and some examples.) We note, however, that nothing in our approach depends on a particular structure of abelian Anosov actions so it seems possible to extend our results to a more general setting. We restrict our attention to abelian Anosov actions because abelian

Anosov actions is one of the best studied classes of higher dimensional dynamical systems (because harmonic analysis provides us with effective tools to investigate their properties). For this reason we quite often can verify the conditions of our theorem for abelian Anosov actions. For other classes of systems, much less is known concerning the estimates needed for our approach to work, even though it is believed that they could be satisfied quite often.

It turns out more convenient to study differentiability properties of u-Gibbs states rather than SRB measures. To define those recall that a quite natural approach to constructing SRB measures is to study the iterations of the Lebesgue measure. For partially hyperbolic systems the image of any domain becomes elongated in the unstable directions. This implies ([43]) that the limiting measure should be absolutely continuous with respect to the unstable foliation (that is any set intersecting any unstable leaf by a set of zero Lebesgue leaf measure itself has measure zero) but it can be singular in the transverse direction. Invariant measures which are absolutely continuous with respect to the unstable foliation are called u-Gibbs states. U-Gibbs states always exist (in fact any limit point of Birkhoff averages of the Lebesgue measure is u-Gibbs), see [43]. Also, if μ_j are u-Gibbs for f_j and $f_j \to f$ in C^2-topology and $\mu_j \to \mu$ weakly then μ is u-Gibbs for f. Since these existence and continuity results fail for SRB measures the question of dependence on parameters might be easier for u-Gibbs states than for SRBs. The relation between u-Gibbs states and SRB measures is the following. If there is unique u-Gibbs state then it is also SRB measure. However if there are several u-Gibbs states then SRB measure need not exist and even if it does exist there might be extra u-Gibbs states which are not SRB. Therefore, in general, there are more u-Gibbs states than SRB measures. For this reason it is usually very difficult to prove the uniqueness of the u-Gibbs state. To get around this problem we do the following. Consider an one parameter family f_ε of diffeomorphisms such that f_0 is an Anosov element in an abelian Anosov action and let ν_ε be any u-Gibbs state for f_ε. We then ask if the map $\varepsilon \to \nu_\varepsilon(A)$ is differentiable at $\varepsilon = 0$ for sufficiently smooth functions A. Now following [48, 51] we can write down a formal expression for the derivative in terms of an infinite series of correlation functions. Our main result (Theorem 1) states that under the conditions needed to ensure the convergence of this formal series the sum indeed is the actual value of the derivative. This implies via results of [22] some non-trivial bounds for Birkhoff averages of the perturbed system for Lebesgue almost every point. See Section 2 for the proof of the main theorem.

The main difference between our approach and the previous work on differentiability [3, 4, 17, 18, 31, 32, 48, 6] is that we do not make any assumptions on the dynamics of the perturbed system. This extends greatly the range of applicability of our method. Also it allows proving Jakobson type results for partially hyperbolic systems. We illustrate this last point in Section 3 which deals with the question of existence of SRB measures for perturbed dynamics. Our setting is the following. Recently [54] considered the simplest example of an Anosov action of $\mathbb{Z} \times \mathbb{T}^1-$ a skew $\mathbb{T}^1$–extension of an Anosov diffeomorphisms of $\mathbb{T}^2$ and showed that small perturbations of these examples could lead to an open set of Bernoulli diffeomorphisms with non-zero Lyapunov exponents. This bifurcation is quite interesting since, among other things, it leads to explicit constructions of diffeomorphisms with many remarkable properties such as non-absolutely continuous central foliation or hyperbolic diffeomorphisms with countably many ergodic components etc. Here we consider the same bifurcation for the time one map of the geodesic flow on the unit circle bundle over a negatively curved surface. We show that our result implies that the diffeomorphisms with the properties described above appear in a generic one-parameter family of volume preserving diffeomorphisms passing through f. Also, using the results of [21] we show that generic perturbations lead to exponentially mixing diffeomorphisms satisfying the Central Limit Theorem. If the perturbations are not required to preserve the volume we show that generically there is only one SRB state after the perturbation and its basin covers all M up to a set of measure 0. An important ingredient of our approach is a second order expansion of the central Lyapunov exponent. Recently I received a preprint of David Ruelle ([52]) who obtains a similar expansion for the perturbation of the original Shub-Wilkinson example but he does not consider dissipative perturbations.

Let us finish with a note concerning the notation. In the proofs below K, C, C_1, C_2 etc. stand for positive constants and $\alpha, \gamma, \lambda, \theta$ stand for constants between 0 and 1. The precise value of constants without subscripts may change from entry to entry. Also, if $\alpha \geq 0$ and c are constants, we write $LHS \sim c\epsilon^\alpha$ to mean $LHS - c\varepsilon^\alpha = o(\varepsilon^\alpha)$.

2. Differentiation of u-Gibbs states.

2.1. **Preliminaries.**

Here we recall some facts about partially hyperbolic systems.

Let M be a C^∞ compact Riemannian manifold and $f : M \to M$ be a C^∞ partially hyperbolic diffeomorphism. This means that there are constants $C > 0$ and $\lambda_1, \lambda_2 < 1$ and a df–invariant splitting

$$TM = E_u \oplus E_c \oplus E_s$$

such that

$$\forall v \in E_s \quad \forall n > 0 \quad ||df^n(v)|| \leq C\lambda_1^n ||v||; \tag{1}$$

$$\forall v \in E_u \quad \forall n > 0 \quad ||df^{-n}(v)|| \leq C\lambda_1^n ||v|| \quad \text{and} \tag{2}$$

$$\forall x \quad \frac{||df^n|E_s||(x)}{||(df^n|E_c)^{-1}||^{-1}(x)} \leq C\lambda_2^n; \tag{3}$$

$$\forall x \quad \frac{||df^n|E_c||(x)}{||(df^n|E_u)^{-1}||^{-1}(x)} \leq C\lambda_2^n. \tag{4}$$

By [28], Theorem 4.1 there exists a continuous foliation with smooth leaves W^u called *the unstable foliation* which is tangent to E_u (the same is true for E_s but we will not use this in this section).

Let $d_u = \dim(E_u)$, $d = \dim(E_c)$. Denote $E_{cs} = E_c \oplus E_s$, $E_{cu} = E_c \oplus E_u$. We assume that E_c is tangent to the orbits of a C^∞ action of $\mathbb{R}^d$, $\mathbf{g}_t : M \to M$ such that $f\mathbf{g}_t = \mathbf{g}_t f$. We call $\mathbf{g}_t$ *an abelian Anosov action* and f *an Anosov element* of this action. We refer to [33, 34] for general discussion of abelian Anosov actions. Let $\{e_l\}_{l=1}^d$ be a standard frame in $\mathbb{R}^d$. Denote

$$e_l(x) = \left(\left[\frac{d\mathbf{g}}{dt}(x) \right] |_{t=0} \right) e_l.$$

Note that the definition of partial hyperbolicity is independent of the choice of Riemannian metric and we can choose a metric such that $\{e_l\}$ is orthonormal. Then $(df|E_c)$ is an isometry. Also, if $v \in E_s$ then the equality

$$(df^n)(d\mathbf{g}_t v) = (d\mathbf{g}_t)(df^n v)$$

shows that $d\mathbf{g}_t v \in E_s$. Thus $\mathbf{g}_t$ preserves E_s. Likewise it preserves E_u. Hence all the distributions E_s, E_c and E_u are smooth along the orbits of $\mathbf{g}$. However the transverse regularity of E_u and E_s is only Holder, that is, the functions $x \to E_s(x)$ and $x \to E_u(x)$ are only Holder continuous. (See [44] for the discussion of the optimal regularity of these distributions.)

Requiring that $\{e_l\}$ is orthonormal does not specify the metric completely. In fact, one can further modify the metric to make it adapted, that is, arrange that (1)–(4) hold with $C = 1$, possibly at the expense of replacing λ_1 and λ_2 by slightly larger numbers. (One way to do this is to consider

$$||v||_{new} = \sum_{j=0}^{N-1} ||df^j v||$$

for sufficiently large N.) We shall assume that the metric is adapted since it simplifies the proofs below.

We assume that f has a unique SRB measure ν whose basin has total Lebesgue measure. Since $x \in \mathbb{B}(\nu)$ implies that $\mathbf{g}_t x \in \mathbb{B}(\mathbf{g}_t \nu)$ and $\mathbf{g}_t$ preserves Lebesgue measure class it follows that ν is $\mathbf{g}_t$ invariant.

Moreover, we assume that f is *rapidly mixing.* This means the following. Let α be a positive constant. Let $C^\alpha_k(M)$ be the space of functions A such that for all $x \in M$, the function $t \to A(\mathbf{g}_t x) \in C^k(\mathbb{R}^d)$ and $[\partial_t^j(A(\mathbf{g}_t x))]|_{t=0} \in C^\alpha(M)$ for $0 \leq |j| \leq k$. Here j is a multiindex $(j_1, j_2 \dots j_d)$ and $\partial_t^j = \partial_{t_d}^{j_d} \dots \partial_{t_1}^{j_1}$, $|j| = \sum_{k=1}^d j_k$. Denote

$$||A||_{C^\alpha_k}(M) = \max_{o \leq |j| \leq k} \left|\left| \left[\partial_t^j(A(\mathbf{g}_t x))\right]_{t=0} \right|\right|_{C^\alpha(M)}.$$

Fix positive constants $\mathbf{r}, C_1, \alpha_1$. Call a set $S \subset W^u$ $(\mathbf{r}, C_1, \alpha_1)$–*regular* if $\mathrm{diam}(S) \leq \mathbf{r}$ and $\mathrm{mes}(\partial_\varepsilon S) < C_1 \varepsilon^{\alpha_1}$ where $\partial_\varepsilon S = \{y \in S : \mathrm{dist}(y, \partial S) < \varepsilon\}$. Let ρ be a probability density on S. Let $\ell_{S,\rho}$ denote the probability measure defined for $A \in C(M)$ by

$$\ell_{S,\rho}(A) = \int_S \rho(x) A(x) dx.$$

Call f rapidly mixing if $\forall \alpha > 0$ $\forall \mathbf{r}, C_1, \alpha_1, \alpha_2$ $\forall m$ there exist constants k and $C = C(m, \mathbf{r}, C_1, \alpha, \alpha_1, \alpha_2)$ such that for any $(\mathbf{r}, C_1, \alpha_1)$–regular set S, for any probability density $\rho \in C^{\alpha_2}(S)$, for any function $A \in C^\alpha_k(M)$, the following inequality holds

$$\left|\ell_{S,\rho}(A \circ f^N) - \nu(A)\right| \leq C(m) ||A||_{C^\alpha_k(M)} ||\rho||_{C^{\alpha_2}(S)} N^{-m}. \tag{5}$$

The condition that ρ is a density is needed only to ensure that $\ell_{S,\rho}$ is a probability measure. If ρ is any non-negative Holder continuous function on S which is not identically zero, denote $\bar{\rho} = \rho / \ell_{S,\rho}(1)$. Then $\ell_{S,\bar{\rho}}$ is a probability measure and $||\bar{\rho}||_{C^{\alpha_2}(S)} = ||\rho||_{C^{\alpha_2}(S)} / \ell_{S,\rho(1)}$. Hence one obtains from (5)

$$\ell_{S,\rho}(A \circ f^N) = \ell_{S,\rho}(1) \ell_{S,\bar{\rho}}(A \circ f^N) =$$

$$\nu(A) \ell_{S,\rho}(1) + O\left(||A||_{C^\alpha_k(M)} ||\rho||_{C^{\alpha_2}(S)} N^{-m}\right). \tag{6}$$

Now any real valued Holder function can be represented as a difference of two non-negative Holder functions (e.g. $\rho = \max(\rho, 0) - \max(-\rho, 0)$). Thus (6) holds for an arbitrary Holder function.

Let $f_\varepsilon : [-\varepsilon_0, \varepsilon_0] \times M \to M$ be a C^∞ one-parameter family of diffeomorphisms such that $f = f_0$ is an Anosov element in an abelian Anosov action (note that we do not require f_ε to be Anosov elements of any action for $\varepsilon \neq 0$). Then by [28], Theorem 2.15, for small ε the diffeomorphism f_ε is partially hyperbolic and its unstable foliation $W^u(f_\varepsilon)$ is close to $W^u(f)$. (We note that Theorems 7.1 and 7.2 of [28] imply that for small ε the distribution $E_c(f_\varepsilon)$ is uniquely integrable so there is a foliation $W^c(f_\varepsilon)$ with smooth leaves tangent to $E_c(f_\varepsilon)$. However the transverse smoothness of $W_c(f_\varepsilon)$ can be quite bad. In particular, [54] gives an example of a perturbation where $W_c(f_\varepsilon)$ is absolutely singular for small $\varepsilon \neq 0$, that is, there exists a set Ω_ε of total Lebesgue measure intersecting each leaf of $W^c(f_\varepsilon)$ by a set of zero leaf measure (see [53] for more discussion of this phenomenon). For this reason we will not use the unique integrability of $E_c(f_\varepsilon)$ in our analysis.)

We want to study the u-Gibbs measures for f_ε. Recall that if $g : M \to M$ is a partially hyperbolic diffeomorphism than u-Gibbs measures are defined as follows. Call (S, ρ) $(\mathbf{r}, C_1, C_2, \alpha_1, \alpha_2)$ regular pair if S is a $(\mathbf{r}, C_1, \alpha_1)$ regular set, ρ is a probability density on S and $||\rho||_{C^{\alpha_2}(S)} \leq C_2$. Let $\mathbf{E}(\mathbf{r}, C_1, C_2, \alpha_1, \alpha_2)$ denote the set of measures $\{\ell_{S,\rho}\}$ where (S, ρ) are $(\mathbf{r}, C_1, C_2, \alpha_1, \alpha_2)$ regular pairs. Let $\bar{\mathbf{E}}(\mathbf{r}, C_1, C_2, \alpha_1, \alpha_2)$ denote the closure of the convex hull of $\mathbf{E}(\mathbf{r}, C_1, C_2, \alpha_1, \alpha_2)$. Denote by $\bar{\mathbf{E}}_{inv}(\mathbf{r}, C_1, C_2, \alpha_1, \alpha_2)$ the set of g invariant elements of $\bar{\mathbf{E}}(\mathbf{r}, C_1, C_2, \alpha_1, \alpha_2)$. A g-invariant measure μ is called *u-Gibbs state* if there are constants $\mathbf{r}, C_1, C_2, \alpha_1, \alpha_2$ such that

$$\mu \in \bar{\mathbf{E}}_{inv}(\mathbf{r}, C_1, C_2, \alpha_1, \alpha_2).$$

Some technical properties of u-Gibbs measures are collected in Appendix A. We note that the *a priori* estimates on the regularity of u-Gibbs states given in Proposition 8 are crucial for our method.

Our interest in u-Gibbs states comes from the following.

If $\Phi_n(x)$ is a family of measurable functions let $\operatorname{ess\,lim\,sup}_{n\to\infty} \Phi_n(x)$ denote the infimum of all c such that the set $\limsup_{n\to\infty} \Phi_n(x) > c$ has zero Lebesgue measure. Define ess lim inf similarly.

Proposition 1. *([22], Proposition 11) Let $g : M \to M$ be a partially hyperbolic diffeomorphism.*

(a) For all $A \in C(M)$

$$[\operatorname{ess\,lim\,inf} \frac{1}{N}\sum_{n=1}^{N} A(g^n x), \operatorname{ess\,lim\,sup} \frac{1}{N}\sum_{n=1}^{N} A(g^n x)] \subset [\inf \mu(A), \sup \mu(A)]$$

where inf *and* sup *are taken over a set of all u-Gibbs measures.*

(b) In particular, if g has a unique u-Gibbs state μ then μ is an SRB state and $\mathbb{B}(\mu)$ has total Lebesgue measure in M.

2.2. **Statement of the results.** Let f be an Anosov element in an abelian Anosov action. Let $VC_k^\alpha(M)$ denote the set of the vector fields on M which are C^k along the orbits of $\mathbf{g}_t$ and the derivatives are α-Holder. Define $||\cdot||_{VC_k^\alpha(M)}$ similarly to $||\cdot||_{C_k^\alpha(M)}$. Let $C_{1,k}^\alpha(M)$ be the space of functions such that for any $VC_k^\alpha-$ vectorfield v the function $\partial_v A \in C_k^\alpha(M)$. Let

$$||A||_{C_{1,k}^\alpha} = \sup_{||v||_{VC_k^\alpha}\leq 1} ||\partial_v A||_{C_k^\alpha} + \sup_{x\in M} |A(x)|.$$

Our main result is the following.

Theorem 1. *Let f_ε be a C^∞ one parameter family of diffeomorphisms such that $f = f_0$ is a rapidly mixing Anosov element in an abelian Anosov action. Let ν be its u-Gibbs state (observe that by (5) the SRB measure is the only u-Gibbs state of f). Fix $\alpha > 0$. Then there exist k_0 and a linear functional $\omega : C_{1,k_0}^\alpha(M) \to \mathbb{R}$ such that if ν_ε is any u-Gibbs state for f_ε and $A \in C_{1,k_0}^\alpha(M)$ then*

$$\nu_\varepsilon(A) - \nu(A) = \varepsilon\omega(A) + o\left(\varepsilon ||A||_{C_{1,k_0}^\alpha(M)}\right). \tag{7}$$

Applying the results of [22] we obtain the following consequence.

Corollary 1. *Under the condition of Theorem 1 the following statements hold for all functions $A \in C_{1,k_0}^\alpha$*

$$(a) \quad \lim_{\varepsilon\to 0} \frac{\operatorname{ess\,lim\,sup}_{n\to+\infty} \left| \frac{1}{n}\sum_{j=0}^{n-1} A(f_\varepsilon^j x) - \nu(A) - \varepsilon\omega(A)\right|}{\varepsilon} = 0.$$

(b) Let $n_\varepsilon \to \infty$ so that $n_\varepsilon \varepsilon^2 \to c$ where $c \geq 0$ is a constant. Let x be chosen uniformly with respect to the Lebesgue measure on M. Then for all functions $A \in C_{1,k_0}^\alpha(M)$ such that $\nu(A) = 0$

$$\frac{\sum_{j=0}^{n_\varepsilon-1} A(f_\varepsilon^j x)}{\sqrt{n_\varepsilon}}$$

converges weakly to a Gaussian random variable with mean $c\omega(A)$ and variance

$$D(A) = \sum_{j=-\infty}^{\infty} \nu((A\circ f^j)A).$$

In the next subsection we give some examples of systems satisfying the assumptions of Theorem 1. Subsection 2.4 describes the plan of the proof. The proof itself is carried out in subsections 2.5–2.8. Subsection

2.9 contains the proof of Corollary 1. Subsection 2.10 discusses the formula for the derivative in the case when the invariant foliations are smooth. An application of Theorem 1 to estimation of Lyapunov exponents is given in Section 3.

2.3. **Examples.** Here we give some examples where our theorem applies. We refer to [33, 34] for a general discussion of Anosov actions.

(A) f is an Anosov diffeomorphism ($E_c = 0$ so that $C^{\alpha}_{1,k}(M) = C^{1+\alpha}(M)$). In this case Theorem 1 is known, see [31, 32].

(B) f is a $\mathbb{T}^d$ extension of an Anosov diffeomorphism. In this case $M = N \times \mathbb{T}^d$ and $f(x, y) = (h(x), y + \tau(x))$, where $h : N \to N$ is an Anosov diffeomorphism and $\tau \in C^{\infty}(N, \mathbb{T}^d)$. In this case $\mathbf{g}_t(x, y) = (x, y+t)$. By [20] a generic extension is rapidly mixing. (Corollary 6.5 of [20] gives (5) for S coming from Markov partition and [22], Proposition 4 extends it to arbitrary regular S. See also Appendix A of the present paper.)

This class can be used to explain why the assumptions of Theorem 1 could not be simplified. Consider $M = \mathbb{T}^4 = \mathbb{T}^2 \times \mathbb{T} \times \mathbb{T}$ and let

$$f(x, y_1, y_2) = (h(x), y_1 + \alpha_1\tau(x), y_2 + \alpha_2\tau(x)) \tag{8}$$

where h is an Anosov diffeomorphism. Suppose that $\tau(x)$ is not cohomologous to constant. Recall the criterion for rapid mixing for skew extensions. Let $W = (x_0, x_1, \dots x_{n-1}, x_n = x_0)$ be a closed chain such that $x_{j+1} \in W^u(x_j, h) \bigcup W^s(x_j, h)$. Then for any $y \in \mathbb{T}^2$ there exists unique chain $((x_0, y_0), (x_1, y_1) \dots (x_n, y_n))$ such that $y_0 = y$ and $(x_{j+1}, y_{j+1}) \in W^*((x_j, y_j), f)$ if $x_{j+1} \in W^*(x_j, h)$. However this chain is not closed, namely there is a vector $g(W)$ such that $y_n = y_0 + g(W)$. Let $\Gamma_t(l_1, l_2)$ denote the set $\{g(W)\}$ for all chains W such that the number of legs $n(W) \leq l_1$ and for all j $d_{W^*}(x_{j+1}, x_j) \leq l_2$. Then ([20], Section 4) f is rapidly mixing if and only if $\Gamma_t(l_1, l_2)$ is Diophantine for large (l_1, l_2), that is there are constants D, σ such that for each $k \in \mathbb{Z}^d - 0$ there exists $g \in \Gamma_t(l_1, l_2)$ such that

$$|\exp(2\pi i(k, g)) - 1| > \frac{D}{||k||^{\sigma}}.$$

Let us apply this to our example (8). By the results of ([16], Sections 9 and 12) the fact that $\tau(x)$ is not cohomologous to constant implies that $\Gamma_t(l_1, l_2)$ is a set $\{s(\alpha_1, \alpha_2)\}_{|s| \leq a(l_1, l_2)}$ (note that $\Gamma_t(l_1, l_2)$ is symmetric about the origin since $g((x_n, x_{n-1}, \dots x_1, x_0)) = -g((x_0, x_1, \dots x_{n-1}, x_n))$). Thus $\Gamma_t(l_1, l_2)$ is Diophantine if for each non-zero pair $(k_1, k_2) \in \mathbb{Z}^2$

$$\mathrm{dist}(\{(k_1\alpha_1 + k_2\alpha_2)s\}_{|s| \leq a(l_1, l_2)}, \mathbb{Z}) \geq D||k||^{-\sigma}.$$

But this holds if and only if

$$|k_1\alpha_1 + k_2\alpha_2| \geq \frac{D}{a(l_1, l_2)||k||^\sigma}.$$

Thus f is rapidly mixing if and only if $\frac{\alpha_1}{\alpha_2}$ is Diophantine.

Consider the following one parameter family

$$f_\varepsilon(x, y_1, y_2) = (g(x), y_1 + (\alpha_1 + \varepsilon)\tau(x), y_2 + \alpha_2\tau(x)). \tag{9}$$

The following results are established in [13]. If $\frac{\alpha_1+\varepsilon}{\alpha_2} = \frac{m_1}{m_2} \in \mathbb{Q}$ then the ergodic u-Gibbs states are Lebesgue measures on $\mathbb{T}^2\times$ (a leaf of $\{y_1 + m_1 s, y_2 + m_2 s\}$). Since each of those has zero measure basin of attraction, it follows that f_ε has no SRB states. On the other hand if $\frac{\alpha_1+\varepsilon}{\alpha_2} \notin \mathbb{Q}$ then the Lebesgue measure is the only u-Gibbs state which is therefore SRB. If f_ε satisfied the conclusion of Theorem 1, then since Lebesgue measure is a common u-Gibbs state for all f_ε, we would have $\omega \equiv 0$. Now let $\frac{\alpha_1+\varepsilon}{\alpha_2} = \frac{m_1}{m_2}$,

$$A(x, y) = a(x) \exp[2\pi i(k_1 y_1 + k_2 y_2)].$$

Let μ_ε be an ergodic u-Gibbs state. Then

$$\left|\int A(x, y) d\mu_\varepsilon\right| = \begin{cases} \left|\int a(x)dx\right| & \text{if } k_1 m_1 + k_2 m_2 = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Decomposing arbitrary function on $\mathbb{T}^4$ as

$$A(x, y) = \sum_{k_1, k_2} a_{k_1,k_2}(x) \exp[2\pi i(k_1 y_1 + k_2 y_2)]$$

one easily checks the following facts for the family (9)

(1) If $\frac{\alpha_1}{\alpha_2}$ is Diophantine then for each $k \in \mathbb{N}$, for $A \in C^k(\mathbb{T}^4)$

$$\nu_\varepsilon(A) - \int A dx dy = O(\varepsilon^{N_k} ||A||_{C^k(\mathbb{T}^4)})$$

where $N_k \to \infty$ as $k \to \infty$.

(2) Given $k \in \mathbb{N}$ there are Diophantine (α_1, α_2) and sequences $\{\varepsilon_j\}$ such that $\frac{\alpha_1+\varepsilon_j}{\alpha_2} = \frac{m_{1,j}}{m_{2,j}}$, $\varepsilon_j \to 0$, and $A_j(x, y) = \exp[2\pi i(m_{2,j} y_1 - m_{1,j} y_2)]$ such that

$$\liminf_{j\to\infty} \frac{|\nu_{\varepsilon_j}(A_j) - \int A_j dx dy|}{|\varepsilon_j| ||A_j||_{C^k(\mathbb{T}^4)}} > 0$$

(3) If $\frac{\alpha_1}{\alpha_2}$ is not Diophantine than there exist sequences $\{\varepsilon_j\}$ such that $\frac{\alpha_1+\varepsilon_j}{\alpha_2} = \frac{m_{1,j}}{m_{2,j}}$, $\varepsilon_j \to 0$, and $A_j(x,y) = \exp[2\pi i(m_{2,j}y_1 - m_{1,j}y_2)]$ such that

$$\liminf_{j\to\infty} \frac{|\nu_{\varepsilon_j}(A_j) - \int A_j dxdy|}{|\varepsilon_j| ||A_j||_{C^j(M)}} > 0.$$

This example shows that

(1) U-Gibbs states for the perturbed system need not be unique under the conditions of Theorem 1.

Remark. *The recent work of Pugh, Shub and others (see e.g* [15]*) shows that the accessibility condition often implies good stochastic behavior for volume preserving systems. The example considered above does not have the accessibility property. It is an open question if accessibility implies the uniqueness of u-Gibbs state in the volume preserving context. The answer is probably negative but I do not know any counterexamples.*

(2) There is no uniform bound on k_0 in Theorem 1. In particular, (7) need not hold for $A \in C^{1+\delta}$.

Remark. *It follows from the proof of Theorem 1 that (7) does hold for $A \in C^{1+\delta}$ if f is exponentially mixing that is if the convergence in (5) is exponential for $A \in C^\alpha(M)$.*

(3) Some mixing condition is necessary in Theorem 1.

(C) f is a time one map of an Anosov flow ϕ_t. In this case $\mathbf{g}_t = \phi_t$. In this example also f is generically rapidly mixing [19, 10] (again the definition of rapid mixing in [19] is slightly different from the one given here but it is not difficult to check that the results of [19] imply the rapid mixing in the sense of the present article). If we perturb f among time one maps then rapid mixing is not needed (see [31, 32, 18]). For general perturbations our result is new.

(D) Let $M = G/\Gamma$ where G is connected semisimple Lie group without compact factors and Γ is an irreducible lattice in G. Suppose that G has a factor not locally isomorphic to $SO(n,1)$, $SU(n,1)$. Let $\mathbf{X}$ be the element of Lie algebra of G such that $\mathrm{Sp}(ad(\mathbf{X})) \bigcap i\mathbb{R} = \{0\}$ and the zero eigenspace consists of centralizers of $\mathbf{X}$. Let $f(x) = \exp(\mathbf{X})x$. In this case $\{\mathbf{g}_t\}$ is an identity component of the centralizer of $\exp(\mathbf{X})$. By ([36], Section A.7) f is rapidly mixing.

2.4. **Idea of the proof.** Here we begin with the proof of Theorem 1. Before giving the precise argument let us provide an informal description first.

In order to prove Theorem 1 we need to control integrals of the form

$$\int_S A(f_\varepsilon^n x)\rho(x)dx$$

where S is f_ε-regular and n is large. Let κ be a small constant. (The precise conditions on κ are given at the end of Subsection 2.6.) Let $N_\varepsilon = \varepsilon^{-\kappa}$. The proof consists of the following steps.

- Show that a good control of $\int_S A(f_\varepsilon^{N_\varepsilon} x)\rho(x)dx$ allows to get estimates on ν_ε.
- Compare $\int_S A(f_\varepsilon^{N_\varepsilon} x)\rho(x)dx$ with $\int_S A(f^{N_\varepsilon} x)\rho(x)dx$.
- Show that $\int_S A(f^{N_\varepsilon} x)\rho(x)dx - \nu(A)$ is small.

The most difficult part is the second one. In fact, because of the exponential instability, $f_\varepsilon^{N_\varepsilon} x$ and $f^{N_\varepsilon} x$ are far apart. However given x we can find another point $y \in W^u(f_\varepsilon, x)$ such that $f_\varepsilon^j x$ and $f^j y$ are close for $0 \leq j \leq N_\varepsilon$. (Since we require the shadowing of finite orbits the choice of y is not unique so we impose additional constrains to guarantee uniqueness.) Here the choice of the dependence of N_ε on ε plays a critical role. On one hand we want to make N_ε large so that $\ell_{S,\rho}(A \circ f^N)$ and $\ell_{S,\rho}(A \circ f_\varepsilon^N)$ are close to their corresponding u-Gibbs states. On the other hand if N_ε is large then we have to shadow long pieces of orbits which becomes difficult. The choice of $N_\varepsilon = \varepsilon^{-\kappa}$ is a good compromise. In fact, the fastest divergence between $f_\varepsilon^N S$ and $f^N S$ takes place along the E_c–direction. For the unperturbed system $(df|E_c)$ is an isometry. So, the distance between $f_\varepsilon^j S$ and $f^j S$ grows at most linearly for $j \leq N_\varepsilon$ so at the moment N_ε they are not too far apart. On the other hand, rapid mixing ensures that $\ell_{S,\rho}(A \circ f^{N_\varepsilon})$ is quite close to $\nu(A)$.

Let us now make few technical remarks. First notice that if $x \in S$ then the domain of y shadowing x is some set S^* different from S. However since we need to control the integral not for one set S but for all regular sets this will cause little difficulty.

Another remark is that since $W^u(f_\varepsilon)$ is different from $W^u(f)$ f_ε-regular sets are not f-regular. However since $E_u(f_\varepsilon)$ is close to $E_u(f)$ f_ε-regular sets are uniformly transversal to $E_{cs}(f)$ and we shall show below that the estimate (6) holds also for those more general sets.

Finally we note that the difference between $\ell_{S,\rho}(A \circ f_\varepsilon^{N_\varepsilon})$ and $\ell_{S^*,\rho}(A \circ f^{N_\varepsilon})$ comes from two sources.

(1) $f_\varepsilon^N x$ is different from $f^N y$.

(2) The distortion of f_ε along the orbit of x is different from the distortion of f along the orbit of y.

We shall see that this will give rise to two parts of the derivative ω, the first part involving the derivatives of A and the second part involving A itself.

2.5. **Key estimates.** Here we present three main estimates (Propositions 2–4) used in the proof of Theorem 1. In the next subsection we derive Theorem 1 from these estimates.

First of all we address the issue that (f_ε)–regular sets are not f-regular. By the definition of partial hyperbolicity (1)–(4) for small $\bar{\delta}$ a family of cones

$$\mathcal{K}_u(x) = \{v_u + v_{cs} : v_u \in E_u, v_{cs} \in E_{cs}, ||v_{cs}|| \leq \bar{\delta}||v_u||\} \tag{10}$$

is df invariant, that is $df(\mathcal{K}_u(x)) \subset \mathcal{K}_u(fx)$. By continuity for small ε

$$df_\varepsilon(\mathcal{K}_u(x)) \subset \mathcal{K}_u(f_\varepsilon(x)) \tag{11}$$

We call a set S $(\mathbf{r}, C_1, C_3, \alpha_1)$–admissible if there is an immersion ϕ from the standard unit d_u–dimensional disc D to M such that $||\phi||_{C^2(D)} \leq C_3$ and if $V = \phi(D)$ then $TV \in \mathcal{K}_u$ and $S \subset V$ and in the induced Riemannian structure on V mes$(S) > \mathbf{r}$ and mes$(\partial_\varepsilon S) \leq C_1\varepsilon^{\alpha_1}$. Then by continuous dependence of the unstable foliation on parameters ([28], Corollary 2.12) there exist C_3, ε_0 such that for $\varepsilon \leq \varepsilon_0$ any $(\mathbf{r}, C_1, \alpha_1)$-regular set for f_ε is $(\mathbf{r}, C_1, C_3, \alpha_1)$-admissible for f. We need to extend the estimate (6) to admissible sets.

Proposition 2. *Let f be as in Theorem 1.*

(a) Estimate (6) holds if S is $(\mathbf{r}, C_1, C_3, \alpha_1)$-admissible.

(b) For all $\mathbf{r}, C_1, C_3, \alpha_1, \alpha_2, p$ there exist $k = k(p)$ and m_0 such that for all $(\mathbf{r}, C_1, C_3, \alpha_1)$-admissible S for all natural numbers N, m such that $m \geq m_0$, $N \geq 2m$ for all densities $\rho \in C^{\alpha_2}(S)$ for all functions $a : f^{N-m}S \to \mathbb{R}$ such that $a \in C^{\alpha_2}(f^{N-m}S)$ for all functions $A \in C^\alpha_k(M)$ such that $\nu(A) = 0$ the following estimate holds

$$\left| \int_S \rho(y) a(f^{N-m}y) A(f^N y) dy \right| \leq$$

$$||A||_{C^\alpha_k(M)} ||a||_{C^{\alpha_2}(f^{N-m}S)} ||\rho||_{C^{\alpha_2}(S)} m^{-p}.$$

(c) For all $\mathbf{r}, C_1, C_3, \alpha_1, \alpha_2, \epsilon$ there exist k, N_0 such that for all natural numbers N, N_1 such that

$$N \geq N_0, \quad N_0 \leq N_1 \leq \frac{N}{2}$$

for all $(\mathbf{r}, C_1, C_3, \alpha_1)$-admissible S for all probability densities $\rho \in C^{\alpha_2}(S)$ for all functions $A, a \in C^{\alpha}_k(M)$ such that $\nu(A) = 0$ the following estimate holds

$$\left|\sum_{j=1}^{N_1} \int_S \rho(y) a(f^{N-j}y) A(f^N y) dy - \sum_{j=1}^{\infty} \nu((a \circ f^{-j}) A)\right| \leq$$

$$\epsilon ||a||_{C^{\alpha}_k(M)} ||A||_{C^{\alpha}_k(M)} ||\rho||_{C^{\alpha_2}(S)}.$$

The proof of this proposition is given in Appendix A.

Remark. *The restrictions $N \geq 2m, N \geq 2N_1$ in parts (b) and (c) are not optimal. However they suffice for the proof of Theorem 1.*

We now formulate the shadowing result needed in the proof of Theorem 1. As it was mentioned in subsection 2.1 E_s need not be smooth. Let E_{as} be a smooth distribution which is C^0-close to E_s. More precisely, we want E_{as} to be transversal to E_{cu} and satisfy (12) below. Denote $E_{ac} = E_c \oplus E_{as}$. Let π_u, π_c and π_{as} be projections to E_u, E_c and E_{as} respectively along the sum of the complementary subspaces. Let $\Gamma_* = \pi_* df$, $* \in \{u, c, as\}$ and

$$\Gamma^j_*(y) = \Gamma_*(f^{j-1}y) \dots \Gamma_*(fy) \Gamma_*(y).$$

Now, if E_{as} is sufficently close to E_s, then since E_s is df–invariant and satisfies $||df|E_s|| \leq \lambda_1$ we have

$$||\Gamma_{as}|| \leq \tilde{\lambda}_1 < 1 \tag{12}$$

for some $\tilde{\lambda}_1 < 1$ close to λ_1. Let

$$X = \left(\frac{df_\varepsilon}{d\varepsilon} \circ f_\varepsilon^{-1}\right)|_{\varepsilon=0}. \tag{13}$$

Denote $X^* = \pi_* X$,

$$V(x) = \sum_{j=0}^{\infty} \Gamma^j_{as}(f^{-j}x) X^{as}(f^{-j}x)$$

and let $a_l(x)$ be the functions such that $X^c + \Gamma_c V = \sum_l a_l(x) e_l$.

More generally if S is a submanifold in M, $\dim(S) = d_u$, $TS \subset \mathcal{K}_u$ we can define $\pi_{as}(y, S)$ to be projection to E_{as} along $TS(y) \oplus E_c$. Let $\Gamma_{as}(y, S) = \pi_{as}(fy, fS) df$,

$$\Gamma^j_{as}(x, S) = \Gamma_{as}(f^{j-1}y, f^{j-1}S) \dots \Gamma_{as}(fy, fS) \Gamma_{as}(y, S).$$

If S also satisfies

$$T(f^{-k}S) \subset \mathcal{K}_u \quad \text{for } k = 1, 2 \dots j \tag{14}$$

then we can define for $n \le j$

$$V_n(x,S)=\sum_{k=0}^{n}\Gamma^k_{as}(f^{-k}x,f^{-k}S)\pi_{as}(f^{-k}x,f^{-k}S)X.$$

Thus $V(x)=\lim_{n\to\infty}V_n(x,W^u)$.

Some useful properties of V are collected in Appendix B.

Since Theorem 1 is trivial for $\varepsilon=0$ we can assume that $\varepsilon\neq 0$. To fix our notaion we suppose that $\varepsilon>0$. Denote

$$N_\varepsilon=\varepsilon^{-\kappa} \tag{15}$$

where κ is a small constant to be chosen later (see 31). Let

$$\bar N_\varepsilon=\frac{2N_\varepsilon}{3}. \tag{16}$$

Fix a small constant δ. Let S be a $(\mathbf{r},C_1,\alpha_1)$-regular set for f_ε. Let $\bar S$ denote the δ neighborhood of S in $W^u(f_\varepsilon,S)$ and let $\bar{\bar S}$ denote the 2δ neighborhood of S. In subsection 2.7 we prove the following statement.

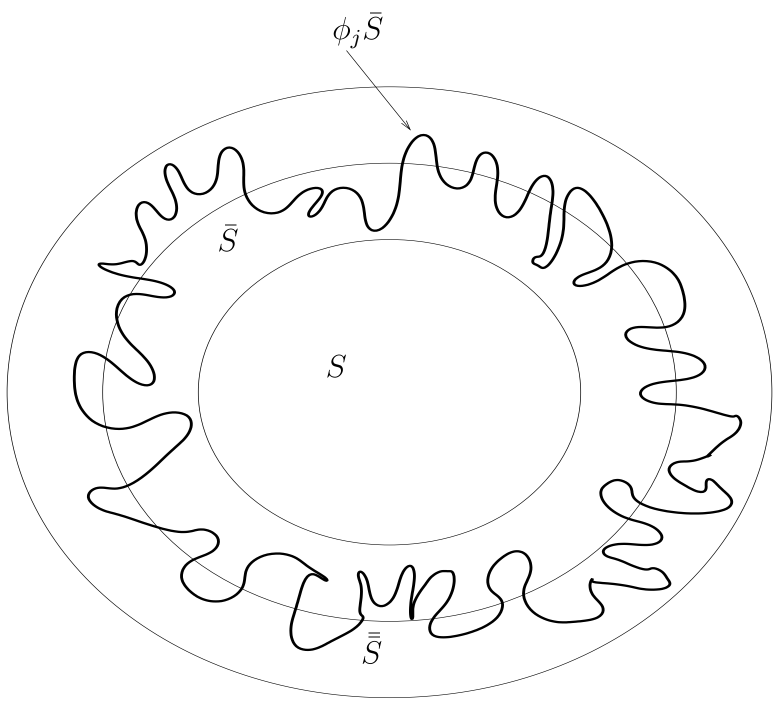

FIGURE 1. The sets $\phi_j^{\pm1}\bar S$ etc. can be quite wiggly. However Proposition 3 gives sufficient control on the boundary of $\phi_j^{-1}S$. Namely, $\phi_j^{-1}S$ is regular.

Proposition 3. *Suppose that κ in (15) is less than 1/3. Then, for small ε, the following holds. There exist a sequence of C^∞ diffeomorphisms $\phi_j : \bar{S} \to \bar{\bar{S}}$ and a sequence of C^∞ vectorfields Z_j defined on $f^j\bar{S}$ with values in E_{ac}, $0 \leq j \leq N_\varepsilon$ such that*

(a) $\phi_0 = \mathrm{id}, Z_0 = 0.$

(b) For all $k \leq N_\varepsilon$ the range of ϕ_k contains S. Moreover there are constants C_5, α_3, α_4 such that for all $0 \leq j \leq k$

$$d_{C^{\alpha_3}(f^j\bar{S}}(f^j\phi_j^{-1}\phi_k f^{-j}, \mathrm{id}) \leq C_5\varepsilon^{\alpha_4}.$$

In particular

$$d_{C^{\alpha_3}(\bar{S}}(\phi_k, \mathrm{id}) \leq C_5\varepsilon^{\alpha_4}.$$

(c) Let $\psi_j = \phi_{j-1}^{-1}\phi_j$. Then

$$d_{C^2(f^j\bar{S})}(f^j\psi_j f^{-j}, \mathrm{id}) \leq C_6\varepsilon. \tag{17}$$

(d) $f_\varepsilon^j\phi_j y = \exp_{f^j y}(Z_j(f^j y)).$

(e) Let $Z_j = Z_j^c + Z_j^{as}$ where $Z_j^* \in E_*$. Then

$$||Z_j^c||_{C^0(f^j\bar{S})} \leq C_7\varepsilon j \quad \text{and} \quad ||Z_j^{as}||_{C^0(f^j\bar{S})} \leq C_8\varepsilon.$$

(f) The first derivatives of $Z_j : f^j\bar{S} \to E_{ac}$ are bounded by $C_9\varepsilon$, the second derivatives of $Z_j : f^j\bar{S} \to E_{ac}$ are bounded by $C_{10}\varepsilon$.

(g) For all $\bar{N}_\varepsilon \leq j \leq N_\varepsilon$ the following holds.

$$\left\|Z_j^{as}(z) - \varepsilon V_j(z, f^j\bar{S})\right\|_{C^1(f^j\bar{S})} \leq C_{11}{N_\varepsilon}^2\varepsilon^2. \tag{18}$$

In particular

$$\left\|Z_{N_\varepsilon}^{as}(z) - \varepsilon V(z)\right\|_{C^0} \leq C_{11}{N_\varepsilon}^2\varepsilon^2. \tag{19}$$

(h)
$$\left\|Z_{N_\varepsilon}^c(f^{N_\varepsilon}y) - \varepsilon\sum_{j=\bar{N}_\varepsilon}^{N_\varepsilon} X^c(f^j y) - \varepsilon\sum_{j=\bar{N}_\varepsilon}^{N_\varepsilon} df^{N_\varepsilon-j-1}\Gamma_c V(f^j y) - Z_{\bar{N}_\varepsilon-1}^c(f^{\bar{N}_\varepsilon-1}y)\right\|_{C^0(\bar{S})} \leq C_{12}{N_\varepsilon}^3\varepsilon^2.$$

Let us make several remarks.

(i) (d) is the key part of Proposition 3 since it essentially says that not too long orbits of f_ε can be shadowed by orbits of f. Other parts give various technical estimates on ϕs and Zs.

(ii)Note that we do not claim that each ϕ_j is uniquely defined because for large j, $f^j\bar{S}$ can be very dense in M and there can be several ways to shadow the same point. However the sequence $\{\phi_j\}$ is uniquely defined if we ask that $f^j\phi_j y$ is close to $f^j\phi_{j-1}y$ (condition (c)). See section 2.7 for the proof of this fact.

(iii) Proposition 3 gives two estimates for Z_j^{as}. For large j we have a stronger bound (18) whereas for small j we have weaker estimates (e) and (f). This weaker bound suffices since the influence of past perturbations decays quickly by rapid mixing. On the other hand (18) can not hold for small j because S is a part of an unstable leaf of f_ε and we need to wait several iterations before its image under f^j becomes sufficiently close to unstable leaves of f to justify (18).

The next statement deals with jacobians of ϕ_js. Recall the definition of the canonical density on W^u. (The canonical density is the conditional density of any u-Gibbs measure. It can be obtained by taking an arbitrary density, iterating it from $-\infty$ to 0, and normalizing. See [43].) We now give a more explicit definition.

Let S be any subset of W^u. Consider the density ρ_S defined by the conditions

$$(I) \quad \forall y_1, y_2 \in S \quad \frac{\rho_S(y_1)}{\rho_S(y_2)} = \prod_{j=0}^{\infty} \frac{\det(df^{-1}|E_u)(f^{-j}y_1)}{\det(df^{-1}|E_u)(f^{-j}y_2)}.$$

(II) $\int_S \rho_S(y)dy = 1$.

Consider the volume form $d\Omega_S(y) = \rho_S(y)dy$. Clearly for two different sets S', S'' in the same W^u–leaf we have $\Omega_{S'} = \text{Const}\Omega_{S''}$. In particular, if Y is a vectorfield tangent to W^u the divergence

$$\text{div}_u^{can} Y = \frac{L_Y \Omega_S}{\Omega_S},$$

where L denotes the Lie derivative, is independent of S. In local coordinates one has

$$\text{div}_u^{can} Y = \sum_j \left(\frac{dY_j}{dx_j} + \frac{Y_j \frac{d\rho_S}{dx_j}}{\rho_S} \right).$$

In subsection 2.8 we prove the following statement.

Proposition 4. *(a)* $\forall j \ \forall x \ |\det(d(\psi_j))(x) - 1| \leq C_{13}\varepsilon$.

(b) $\forall j \ ||\det(d(f^j \psi_j f^{-j})) - 1||_{C^\alpha(f^j \bar{S})} \leq C_{14}\varepsilon$.

(c) There exists $\gamma_1 > 0$ *such that for all* $j > \bar{N}_\varepsilon$

$$\left|\det(d(\psi_j))(z) - 1 + \varepsilon \text{div}_u^{can} \left[X^u + \Gamma_u V\right](f^j z)\right| \leq C_{15}\varepsilon^{1+\gamma_1}.$$

2.6. **Proof of Theorem 1.** Here we deduce Theorem 1 from Propositions 2–4. Without loss of generality we can assume that α is so small that for all k

$$V \in VC_k^\alpha(M) \tag{20}$$

(see Lemma 7 (a)). We need the following consequence of Propositions 2–4.

Proposition 5. *If κ in (15) is small enough and k is large enough, the following holds. Let $\mathbf{r}, C_1, C_2$ be the constants from Proposition 8(b). Let S be a $(\mathbf{r}, C_1, 1)$ regular set for f_ε and let ρ be a probability density on S such that $||\rho||_{\mathrm{Lip}(S)} \leq C_2$. Then for all $A \in C^\alpha_{1,k}(M)$ such that $\nu(A) = 0$*

$$\int_S A(f_\varepsilon^{N_\varepsilon} x)\rho(x)dx = \varepsilon\omega(A) + o\left(\varepsilon ||A||_{C^\alpha_k(M)}\right)$$

where

$$\omega(A) = \nu(\partial_V A) + \sum_l \sum_{j=0}^\infty \nu((a_l \circ f^{-j})\partial_{e_l} A) - \sum_{j=0}^\infty \nu(([\mathrm{div}_u^{can}(X^u + \Gamma_u V)]\circ f^{-j})A).$$

Proof of Theorem 1. By Proposition 8(b) there exist constants $\mathbf{r}, C_1, C_2$ such that, for small ε, all u-Gibbs states for f_ε belong to $\bar{\mathbf{E}}(\mathbf{r}, C_1, C_2, 1, 1)$.

Let ν_ε be u-Gibbs for f_ε, $A \in C^\alpha_{1,k}(M)$. Without loss of generality we may assume that $\nu(A) = 0$.

There exist measures ζ_n on the set of $(\mathbf{r}, C_1, C_2, 1, 1)$-regular pairs such that

$$\nu_\varepsilon = \lim_{n\to\infty} \int_\alpha \ell_{S_\alpha, \rho_\alpha} d\zeta_n(\alpha).$$

Since ν_ε are f_ε invariant, we have

$$\nu_\varepsilon(A) = \lim_{n\to\infty} \int_\alpha \ell_{S_\alpha, \rho_\alpha}(A \circ f_\varepsilon^{N_\varepsilon}) d\zeta_n(\alpha).$$

Applying Proposition 5 to each (S_α, ρ_α) we obtain the statement Theorem 1 for functions of zero mean. To obtain the result for an arbitrary $A \in C^\alpha_{1,k}(M)$ consider $A - \nu(A)1$. □

Proof of Proposition 5. Consider the f_ε-unstable manifold containing S. Let A be a $C^\alpha_{1,k}(M)$-function such that $\nu(A) = 0$. Let ϕ_j be the sequence constructed in Proposition 3. Make the change of variables $y = \phi_{N_\varepsilon}^{-1} x$ (for $x \in S$, y is well defined by Proposition 3 (b)). Let $S^* = \phi_{N_\varepsilon}^{-1} S$. Then

$$\int_S A(f_\varepsilon^{N_\varepsilon} x)\rho(x)dx = \int_{S^*} A(\exp_{f^{N_\varepsilon} y} Z_{N_\varepsilon}(y))\rho(\phi_{N_\varepsilon} y)\frac{dx}{dy} dy.$$

Now

$$A(\exp_{f^{N_\varepsilon} y} Z_{N_\varepsilon}(y)) = A(f^{N_\varepsilon} y) + \left[A(\exp_{f^{N_\varepsilon} y} Z_{N_\varepsilon}(y)) - A(f^{N_\varepsilon} y)\right].$$

Since $C^{\alpha}_{1,k}(M) \subset C^{1+\alpha}(M)$, there exists a constant K such that for all $x \in M$ and for all sufficiently small v

$$|A(\exp_x v) - A(x) - \partial_v A| \le K||A||_{C^{\alpha}_{1,k}}||v||^{1+\alpha}.$$

Thus by Proposition 3 (e)

$$\left[A(\exp_{f^{N_\varepsilon}y} Z_{N_\varepsilon}(y)) - A(f^{N_\varepsilon}y)\right] = (\partial_{Z_{N_\varepsilon}} A)(f^{N_\varepsilon}y) + O\left(||A||_{C^{\alpha}_k(M)}(\varepsilon N_\varepsilon)^{1+\alpha}\right).$$

Let $Z^c_{\bar{N}_\varepsilon-1}(z) = \sum_l \hat{a}_{l,\varepsilon}(z)e_l(z)$. Then by Proposition 3(g) and (h)

$$\left[A(\exp_{f^{N_\varepsilon}y} Z_{N_\varepsilon}(y)) - A(f^{N_\varepsilon}y)\right] =$$

$$\varepsilon(\partial_V A)(f^{N_\varepsilon}y) + \varepsilon \sum_l \sum_{j=0}^{N_\varepsilon-\bar{N}_\varepsilon} a_l(f^{N_\varepsilon-j}y)(\partial_{e_l}A)(f^{N_\varepsilon}y) +$$

$$\sum_l \hat{a}_{l,\varepsilon}(f^{\bar{N}_\varepsilon-1}y)(\partial_{e_l}A)(f^{N_\varepsilon}y) + O(||A||_{C^{\alpha}_k(M)}\varepsilon^{1+\gamma_2})$$

where

$$\gamma_2 = \min(\alpha - (1+\alpha)\kappa, 1-3\kappa). \tag{21}$$

Now

$$y = \psi^{-1}_{N_\varepsilon} \circ \cdots \circ \psi^{-1}_2 \circ \psi^{-1}_1(x)$$

so

$$\frac{dx}{dy} = \prod_{j=1}^{N_\varepsilon} \det(d(\psi_j))(\phi_j^{-1}x) =$$

$$\prod_{j=0}^{N_\varepsilon-1} \left(1 + \left[\det(d(\psi_j))(\phi_j^{-1}x) - 1\right]\right).$$

By Proposition 4(a)

$$\frac{dx}{dy} = 1 + \sum_{j=0}^{N_\varepsilon-1} \left[\det(d(\psi_j))(\phi_j^{-1}x) - 1\right] + O(\varepsilon^2 N_\varepsilon).$$

Hence

$$\int_S A(f^{N_\varepsilon}_\varepsilon x)\rho(x)dx = \tag{22}$$

$$\int_{S^*} \left[A(f^{N_\varepsilon}y) + A(f^{N_\varepsilon}y)\left\{\sum_{j=\bar{N}_\varepsilon}^{N_\varepsilon} \left(\det(d(\psi_j))(\phi_j^{-1}x) - 1\right)\right\} + \right.$$

$$+A(f^{N_\varepsilon}y)\left\{\sum_{j=1}^{\bar{N}_\varepsilon-1} \left(\det(d(\psi_j))(\phi_j^{-1}x) - 1\right)\right\} + \varepsilon(\partial_V A)(f^{N_\varepsilon}y) +$$

$$\varepsilon \sum_l \sum_{j=0}^{\bar{N}_\varepsilon - 1} a_l(f^{N_\varepsilon - j} y)(\partial_{e_l} A)(f^{N_\varepsilon} y) + \sum_l \hat{a}_{l,\varepsilon}(f^{\bar{N}_\varepsilon - 1} y)(\partial_{e_l} A)(f^{N_\varepsilon} y)\Bigg] \rho(\phi_{N_\varepsilon} y) dy$$

$$+O(\varepsilon^{1+\bar{\gamma}}||A||_{C^\alpha_k(M)}) = I_\varepsilon + I\!I_\varepsilon + I\!I\!I_\varepsilon + I\!V_\varepsilon + V_\varepsilon + VI_\varepsilon + O\left(\varepsilon^{1+\bar{\gamma}}||A||_{C^\alpha_k(M)}\right),$$

where $\bar{\gamma} = \min(\gamma_1, \gamma_2)$.

By Proposition 3(b) there exist constants $\bar{C}_1, \bar{C}_2, \bar{C}_3, \bar{\mathbf{r}}, \bar{\alpha}_1, \bar{\alpha}_2$ such that S^* is $(\bar{\mathbf{r}}, \bar{C}_1, \bar{C}_3, \bar{\alpha}_1)$ admissible and $||\bar{\rho}||_{C^{\bar{\alpha}_2}(S^*)} \leq \bar{C}_2$, where $\bar{\rho}$ denotes $\rho \circ \phi_{N_\varepsilon}$. Recall also that by (20) for all k, $V \in VC^{\hat{\alpha}}_k(M)$, and for all k, l, $a_l \in C^{\hat{\alpha}}_k(M)$. These observations allow us to use Proposition 2 to estimate correlation functions containing $\bar{\rho}, V$ and a_l.

We now proceed to estimate $(I_\varepsilon) - (VI_\varepsilon)$. Take k so large that if $A \in C^\alpha_k(M)$ then Proposition 5(a) allows us to use (6) with $m = 2/\kappa$ and in Proposition 5(b) we can take $p = 2/\kappa$.

By Proposition 2(a)

$$I_\varepsilon = \nu(A) + O(\varepsilon^2 ||A||_{C^\alpha_k(M)}) = O(\varepsilon^2 ||A||_{C^\alpha_k(M)}). \tag{23}$$

Also Proposition 4(c) implies that

$$I\!I_\varepsilon = -\varepsilon \sum_{j=0}^{\bar{N}_\varepsilon - 1} A(f^{N_\varepsilon} y)[\mathrm{div}^{can}_u(\Gamma_u V + X^u)](f^{N_\varepsilon - j}\phi^{-1}_{N_\varepsilon - j} x)\bar{\rho}(y) dy + O(\varepsilon^{1+\gamma_1 - \kappa}||A||_{C^\alpha_k(M)}).$$

Observe that Proposition 3(b) implies in particular that

$$d(f^{N_\varepsilon - j}\phi^{-1}_{N_\varepsilon - j} x, f^{N_\varepsilon - j} y) \leq C_5 \varepsilon^{\alpha_4} \tag{24}$$

(since $f^{N_\varepsilon - j}\phi^{-1}_{N_\varepsilon - j} x = f^{N_\varepsilon - j}\phi^{-1}_{N_\varepsilon - j}\phi_{N_\varepsilon} f^{-(N_\varepsilon - j)}(f^{N_\varepsilon - j} y)$). Hence

$$I\!I_\varepsilon = -\varepsilon \sum_{j=0}^{\bar{N}_\varepsilon - 1} A(f^{N_\varepsilon} y)[\mathrm{div}^{can}_u(\Gamma_u V + X^u)](f^{N_\varepsilon - j} y)\bar{\rho}(y) dy + O(\varepsilon^{1+\gamma_3}||A||_{C^\alpha_k(M)}).$$

where $\gamma_3 = \max(\alpha_4, \gamma_1 - \kappa)$. So if

$$\kappa < \gamma_1 \tag{25}$$

then by Proposition 2(c)

$$I\!I_\varepsilon \sim -\varepsilon \sum_{j=0}^{\infty} \nu\left([\mathrm{div}^{can}_u(\Gamma_u V + X^u)](f^{-j} y) A(y)\right) \tag{26}$$

Also, (24), Propositions 2(b) and Proposition 4(b) give

$$I\!I\!I_\varepsilon = O(\varepsilon^2 ||A||_{C^\alpha_k(M)}) \tag{27}$$

Again by Proposition 2(a)

$$I\!V_\varepsilon \sim \varepsilon \nu(\partial_V A) \tag{28}$$

To estimate V_ε note that for all l $\nu(\partial_{e_l} A)=0$ since $\mathbf{g}_t$ preserves ν (see subsection 2.1). Thus Proposition 2(c) gives

$$V_\varepsilon \sim \varepsilon \sum_l \sum_{j=0}^{\infty} \nu(a_l(f^{-j}y)(\partial_{e_l}A)(y)) \tag{29}$$

It remains to estimate VI_ε. By Proposition 3 (parts (e) and (f)) the Lipschitz norm of $Z_{\bar N_\varepsilon}$ on $f^{\bar N_\varepsilon}\bar S$ is bounded by $C_{10}N_\varepsilon\varepsilon$. Hence by Proposition 2(b)

$$VI_\varepsilon = O(\varepsilon^2 ||A||_{C^\alpha_k(M)}) \tag{30}$$

Combining the estimates (23)–(30) we obtain the proposition. □

Now we can describe the choice of κ. It is governed by inequalities (21) and (25). Namely we need that

$$\kappa < \min\left(\frac{1}{3}, \frac{\alpha}{1+\alpha}, \gamma_1\right) \tag{31}$$

where γ_1 is the constant from Proposition 4(c).

2.7. **Shadowing.** Here we prove Proposition 3. The proof proceeds by induction. Namely we assume that (e) and (f) hold up to time j and deduce that (e) and (f) are satisfied for $j+1$ provided that constants C_7, C_8, C_9 and C_{10} satisfy certain inequlities. We then show that (e) and (f) imply the rest of Proposition 3.

To begin the proof we note that (a), (c) and (d) describe ϕ_{j+1} uniquely provided that the estimates of part (e) hold up to time j. In fact take some small constant δ. Let D_j be the ball of radius δ around $f^{j+1}y$ in $f^{j+1}\bar S$ and let D'_j be the ball of radius $\delta/2$ around $f_\varepsilon^{j+1}\phi_j y$ in $f_\varepsilon^{j+1}\bar{\bar S}$. Since both tangent spaces to $f^{j+1}\bar S$ and $f_\varepsilon^{j+1}\bar{\bar S}$ belong to $\mathcal{K}_u$ (recall (11)), both D_j and D'_j are uniformly transverse to E_{ac}. Hence every point q in a small neighborhood of $f^{j+1}y$ has unique decomposition $q=\exp_z Y$ where $z\in D_j, Y\in E_{ac}$, $||Y||\le\delta$ and the map $\Psi_j : q\to(z,Y)$ satisfies the following. There is a constant C_{16} such that for all j

$$||\Psi_j||_{C^2} \le C_{16}. \tag{32}$$

Now, since by the inductive hypothesis (e) and (f) hold up to time j, we get

$$f_\varepsilon^{j+1}\phi_j y = f_\varepsilon(\exp_{f^j y} Z_j(f^j y)) = \exp_{f^{j+1}y}\tilde Z_j, \tag{33}$$

where

$$\left\|\tilde Z_j - \left(df(Z_j)+\varepsilon X(f^{j+1}y)\right)\right\|_{C^2(f^{j+1}\bar S)} \le K_0(C_7,C_8,C_9,C_{10})(\varepsilon(j+1))^2. \tag{34}$$

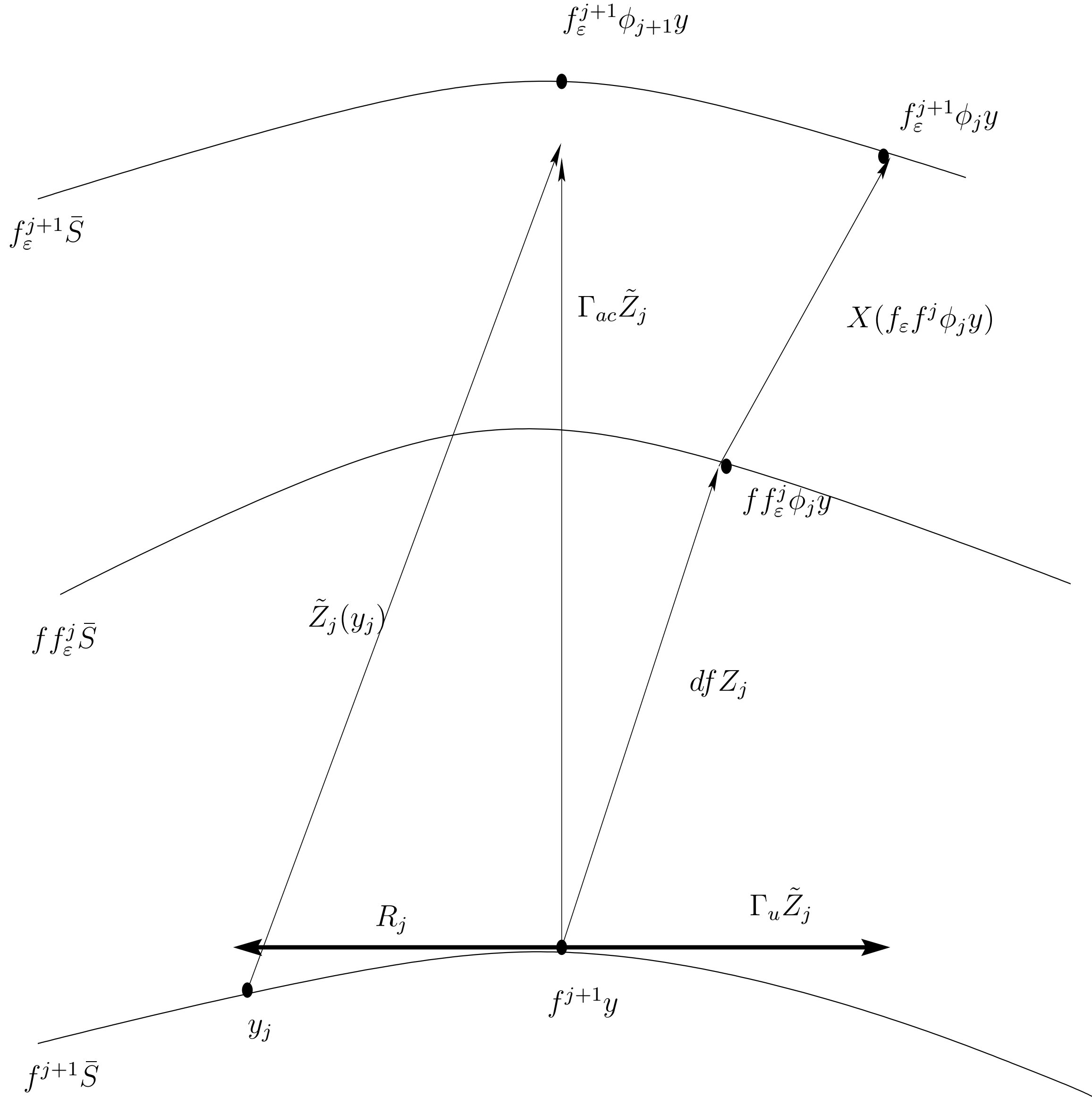

FIGURE 2. The proof of Proposition 3.

(K_0 depends also on other constants such as C_1–C_4, C_{16} etc. but we suppress this dependence here since those constants has been defined already whereas the existance of C_7–C_{10} satisfying the conclusions of Proposition 3 is not yet established, so here we must treat C_7–C_{10} as parameters.)

Hence $f^{j+1}\bar{S}$ is within distance $K_0(C_7, C_8, C_9, C_{10})(\varepsilon(j+1))^2$ from $f_\varepsilon\bar{\bar{S}}$ so ϕ_{j+1} is uniquely defined.

We need some notation. Let $y_j = f^{j+1}\psi_{j+1}y$. That is, y_j was shadowing at time j a point which is shadowed by $f^{j+1}y$ at the moment $j+1$.

Define R_j by

$$y_j = \widehat{\exp}_{(f^{j+1}y)}R_j.$$

where $\widehat{\exp}$ is computed using the induced Riemannian structure on $\hat{S}$.

Let $\pi_{u,j}$ denote the projection to $T(f^{j+1}\bar{S})$ along E_{ac} and

Lemma 1. *(a) (17) holds.*

(b) There exists a constant $\hat{K}_0(C_7 \dots C_{10})$ such that

$$||Z_{j+1} - \Gamma_{ac}(\tilde{Z}_j, f^{j+1}\bar{S})||_{C^2} \leq \hat{K}_0(C_7 \dots C_{10})(\varepsilon(j+1))^2, \tag{35}$$

and if Ω is a volume form on $f^{j+1}\bar{S}$

(36)

$$|\det{}_\Omega\left(d(f^{j+1}\psi_{j+1}f^{-(j+1)}) - 1 - \mathrm{div}_\Omega(-\pi_{u,j}\tilde{Z}_j)\right)| \leq \hat{K}_0(C_7 \dots C_{10})(\varepsilon(j+1))^2. [||\Omega||_{C^2}]$$

We shall also use (36) heavily in the next subsection. Now we shall work with (35). Using the fact that $\Gamma_{as}Z_j^c = 0$ we get

$$||Z_{j+1}^{as} - \Gamma_{as}Z_j^{as} - \varepsilon X^{as}||_{C^2(f^{j+1}\bar{S})} \leq \hat{K}_0(C_7 \dots C_{10})(\varepsilon(j+1))^2, \tag{37}$$

$$||Z_{j+1}^c - Z_j^c - \Gamma_c Z_j^{as} - \varepsilon X^c||_{C^2(f^{j+1}\bar{S})} \leq \hat{K}_0(C_7 \dots C_{10})(\varepsilon(j+1))^2. \tag{38}$$

(12) implies that Z_{j+1} satisfies (e) provided that Z_j satisfies (e) and

$$C_8 \geq \tilde{\lambda}_1 C_8 + ||X^{as}||_0 + \hat{K}_0(C_7 \dots C_{10})\varepsilon(j+1)^2, \tag{39}$$

$$C_7 \geq ||\Gamma_c||C_8 + ||X^c||_0 + \hat{K}_0(C_7 \dots C_{10})\varepsilon(j+1)^2. \tag{40}$$

The fact that Z_{j+1} also satisfies (f) is proven in Appendix C. Denote

$$U_{j+1} = Z_{j+1}^{as} - \Gamma_{as}Z_j^{as} - \varepsilon X^{as},$$

so that

$$Z_{j+1}^{as} = \Gamma_{as}Z_j^{as} - \varepsilon X^{as} + U_{j+1}, \tag{41}$$

Iterating (41) we get

$$Z_j^{as}(z) = \varepsilon V_j(z, f^j\bar{S}) + \sum_{k=0}^{j}(\Gamma_{as}^k(f^{-k}z, f^{j-k}\bar{S})U_{j-k})(f^{-k}z). \tag{42}$$

Observe that $\pi_{as}(\cdot, f^{j+1}\bar{S})U_{j-k} = U_{j-k}$, since $U_{j-k} \in E_{as}$. Therefore the second term in (42) equals $\tilde{V}_j(z, f^j\bar{S})$ where $\tilde{V}_j$ is the vectorfield defined in Lemma 7(c) with $\tilde{X}_k = U_{j-k}$. Now (18) follows by Lemma 7(c).

Note that, since $T\bar{S} \subset \mathcal{K}_u$, condition (3) implies that $T(f^j\bar{S})$ is exponentially close to E_u. Hence, if $j > \bar{N}_\varepsilon$, then by Lemma 7(d) we make exponentially small error replacing $V_j(z, f^j\bar{S})$ by $V_j(z, W^u(z))$. In other words,

$$||Z_j^{as}(z) - \varepsilon V_j(z)||_{C^0(f^j\bar{S})} \leq \text{Const}(N_\varepsilon \varepsilon)^2.$$

Now by Lemma 7(b)

$$||Z_j^{as}(z) - \varepsilon V(z)||_{C^0(f^j\bar{S})} \leq \text{Const}(N_\varepsilon \varepsilon)^2. \tag{43}$$

Now (19) follows from Lemma 7 (b). Substituting (43) into (38) we get

$$||Z_{j+1}^c - Z_j^c - \varepsilon\Gamma_c V(z) - \varepsilon X^c||_{C^0(f^{j+1}\bar{S})} \leq \hat{K}_0(C_7 \dots C_{10})(\varepsilon(j+1))^2.$$

Summation over j gives (h). Part (b) is proven in Appendix D. Since (b) also implies that $\phi_j(\bar{S}) \subset \bar{\bar{S}}$ this completes the proof of Proposition 3. □

Remark. *Let us now comment on the restrictions on C_7–C_{10}. First we take a small $\hat{\delta}$ and choose C_7, C_8 so that*

$$C_8 \geq \frac{||X^{as}||_0 + \hat{\delta}}{1 - \tilde{\lambda}_1},$$

$$C_7 \geq ||\Gamma_c||C_8 + ||X^c||_0 + \hat{\delta}.$$

Then we take C_9, C_{10} satisfying (104), (106) of Appendix C and then request that ε should be so small that (103), (105) are satisfied and

$$\varepsilon^{1-2\kappa} \leq \frac{\hat{\delta}}{\hat{K}_0(C_7 \dots C_{10})}.$$

2.8. **Distortion.** Here we prove Proposition 4. Our starting point is (36).

(b) follows from Proposition 3(c). Let us prove (a). Let y be the variable in $\bar{S}$, $z = f^j y$. Then we need to compute the jacobian of $f^j\psi_j f^{-j}$ with respect to the volume form $dy = \det(df^{-j}|T(f^j\bar{S}))(z)dz$. Take some point $\tilde{z}$ in the δ–neighborhood of z in $f^j\bar{S}$. Since the jacobian does not change if we multiply the volume form by a constant it suffices to compute the jacobian of $f^j\psi_j f^{-j}$ with respect to

$$\Omega_j(z) = \frac{\det(df^{-j}|Tf^j\bar{S})(z)}{\det(df^{-j}|Tf^j\bar{S})(\tilde{z})}dz.$$

More generally define

$$\Omega_{j,n}(z) = \frac{\det(df^{-n}|Tf^j\bar{S})(z)}{\det(df^{-n}|Tf^j\bar{S})(\tilde{z})}dz$$

so that

$$\Omega_j = \Omega_{j,j}.$$

Repeating the argument of Lemma 7 we obtain the following estimates

$$\forall n < j \quad ||\Omega_{j,n}||_{C^2(f^j \bar{S})} \leq C_{17}, \tag{44}$$

$$\forall m, n < j \quad ||\Omega_{j,n} - \Omega_{j,m}||_{C^2(f^j \bar{S})} \leq C_{18}\theta^{\min(m,n)} \tag{45}$$

for some $\theta < 1$.

Recall the definition of $\tilde{Z}_j$ ((33)). The fact that $\pi_{u,j}(Z_j^c) = 0$ and (34) imply

$$\pi_{u,j}(\tilde{Z}_j) = \pi_{u,j}\left[df(Z_j^{as}) + \varepsilon X\right] \tag{46}$$

and so Proposition 3 (e) and (f) give

$$\left\|(\pi_{u,j})\tilde{Z}_j/\varepsilon\right\|_{C^2} \leq C_{19}. \tag{47}$$

By Lemma 1(b)

$$\det(d\psi_{j-1}(y)) = 1 - \left[\operatorname{div}_{\Omega_j}(\pi_{u,j}\tilde{Z}_j)\right](f^j y) + O((\varepsilon(j+1))^2).$$

(44) and (47) prove part (a) of Proposition 4.

Let us prove (c). By Proposition 3(g), (36) and (46)

$$|\det(d\psi_j(y)) - 1| = -\varepsilon\left[\operatorname{div}_{\Omega_j}(\pi_{u,j}(V_j(\cdot, f^j\bar{S}) + X))\right](f^j y) + O((\varepsilon(j+1))^2).$$

By Lemma 7 (b), (d) and (45)

$$|\det(d\psi_j(y)) - 1| = -\varepsilon \operatorname{div}_{\Omega_{j,n}}(\pi_{u,j}(V_n + X)) + O((\varepsilon(j+1))^2 + \theta^n). \tag{48}$$

But V_n is a smooth vectorfield, namely, there is a constant K such that

$$||V_n + X||_{C^2(M)} \leq \mathrm{Const} K^n.$$

Now let $\tilde{S}_j$ be the f–unstable manifold of $\tilde{z}$. Then since $T\bar{S} \in \mathcal{K}_u$

$$\angle(T(f^j\bar{S}), E^u) \leq \mathrm{Const}\lambda_2^j \tag{49}$$

Let $p_{as,j}$ denote the map $f^j\bar{S} \to \tilde{S}_j$ such that $\exp_z^{-1}(p_{as,j}(z)) \in E_{ac}$. Then (49) implies

$$d_{C^3}(p_{as,j}, \text{inclusion}) \leq \mathrm{Const}\lambda_2^j.$$

Hence

$$||dp_{as,j}(V_n+X) - \Gamma_u(V_n+X)||_{C^2(\tilde{S}_j)} \leq \mathrm{Const}\lambda_2^j ||V_n+X||_{C^2(M)} \leq \mathrm{Const}\lambda_2^j K^n.$$

Now on $\tilde{S}_j$ we can consider volume forms

$$\Omega_{(n)} = \frac{\det(df^{-n}|E^u)(z)}{\det(df^{-n}|E^u)(\tilde{z})}dz = \prod_{k=0}^{n-1}\frac{\det(df^{-1}|E^u)(f^{-k}z)}{\det(df^{-1}|E^u)(f^{-k}\tilde{z})}dz,$$

$$\Omega_\infty = \prod_{k=0}^{\infty} \frac{\det(df^{-1}|E^u)(f^{-k}z)}{\det(df^{-1}|E^u)(f^{-k}\tilde{z})} dz.$$

Then

$$\left\|\Omega_{(n)} - \Omega_\infty\right\|_{C^1(\tilde{S}_j)} \leq \text{Const}\lambda_1^n.$$

In particular for any vectorfield Y on $\tilde{S}_j$

$$\left|\text{div}_{\Omega_{(n)}}(Y) - \text{div}_{\Omega_\infty}(Y)\right| \leq \text{Const}\lambda_1^n ||Y||_{C^1(\tilde{S}_j)}. \tag{50}$$

Note that Ω_∞ coincides with the canonical form on $\tilde{S}_j$ up to a constant factor and so

$$\text{div}_{\Omega_\infty} = \text{div}_u^{can}. \tag{51}$$

Also since $d(f^{-n}p_{as}z, f^{-n}z) \leq \text{Const}\tilde{K}^n \tilde{\lambda}_2^j$ we get

$$\left\|\Omega_{(n)} - dp_{as,j}\Omega_{j,n}\right\|_{C^2(\tilde{S}_j)} \leq \text{Const}\hat{K}^n \tilde{\lambda}_2^j$$

Thus

$$\left|\text{div}_{\Omega_{(n)}}[\Gamma_u(V_n + X)] - \text{div}_{\Omega_{j,n}}[\pi_{u,j}(V_n + X)]\right| \leq \tilde{K}^n \tilde{\lambda}_2^j. \tag{52}$$

Choose n so that

$$\max(K, \hat{K}, \tilde{K})^{-n} = \max(\lambda_1, \lambda_2, \tilde{\lambda}_2)^{j/2}.$$

Then (90), (48), (50), (51) and (52) imply part (c) of Proposition 4. □

2.9. **Proof of Corollary 1.**

Proof. (a) Follows from Theorem 1 and Proposition 1.

To prove (b) we use the following statement.

Proposition 6. *(*[22]*, Proposition 18) Let f_ε be a family of partially hyperbolic systems such that there exist numbers $\mathbf{r}, C_1, \alpha_1,$ a function space $\mathcal{B}$, a sequence $\{a(n)\}$ such that*

$$\sum_{n=1}^{\infty} a(n) < \infty$$

and a linear functional $\omega : \mathcal{B} \to \mathbb{R}$ such that for any S which is $(\mathbf{r}, C_1, \alpha_1)$-regular for f_ε and for all probability densities $\rho \in \text{Lip}(S)$ the following estimate holds

$$\left|\int_S A(f_\varepsilon^n x)\rho(x)dx - \nu(A) - \varepsilon\omega(A)\right| \leq ||A||_{\mathcal{B}} ||\rho||_{\text{Lip}(S)}(a(n) + o(\varepsilon)). \tag{53}$$

Let n_ε be a sequence such that $n_\varepsilon \to \infty$, $n_\varepsilon \varepsilon^2 \to c$ where $c \geq 0$ then if x is chosen according to Lebesgue measure then

$$\frac{1}{\sqrt{n_\varepsilon}} \sum_{j=0}^{n-1} \left[A(f_\varepsilon^j x) - \nu(A)\right]$$

converges weakly to a Gaussian random variable with mean $c\omega(A)$ and variance $D(A)$.

We want to verify (53) with $\mathcal{B} = C^\alpha_{1,k_0}(M)$ and $a(n) = \mathrm{Const}/n^2$. There are two cases.

(I) $n \geq N_\varepsilon$. The the result follows by Proposition 5 applied to $A - \nu(A)$.

(II) $n < N_\varepsilon$. We claim that in this case

$$|A(f_\varepsilon^n x)\rho(x)dx - \nu(A)| = O\left(\frac{1}{n^2}||A||_{C^\alpha_{1,k_0}(M)}||\rho||_{\mathrm{Lip}(S)}\right).$$

Indeed consider the decomposition (22) with N_ε replaced by n and $\bar{N}_\varepsilon$ replaced by $n/2$. Then all terms except the first one are of order $\varepsilon \ll \frac{1}{n^2}$ (recall (31)). However the first term is

$$\nu(A) + O\left(\frac{1}{n^2}||A||_{C^\alpha_{1,k_0}(M)}||\rho||_{\mathrm{Lip}(S)}\right)$$

by Proposition 2(a). □

2.10. **Case of $C^{1+\alpha}$ foliations.** If the invariant foliations are $C^{1+\alpha}$ we can simplify the expression for ω obtained in Proposition 5. In fact, in this case we can take $E^{ac} = E^{cs}$. Then $(\Gamma_s df)|(E^{as}) = (df|E^{as})$. Hence

$$V(x) = \sum_{j=0}^{\infty} df^j(X^s(f^{-j}(x))) \tag{54}$$

and $\Gamma_u V = 0$. Thus we obtain the following expression for $\omega(A)$.

Corollary 2. *If W^* are $C^{1+\alpha}$ foliations we have the following formula for ω*

$$\omega(A) = \nu(\partial_V \bar{A}) + \sum_l \sum_{j=0}^{\infty} \nu((a_l \circ f^{-j})\partial_{e_l}(\bar{A})) - \sum_{j=0}^{\infty} \nu(([\mathrm{div}_u^{can}(X^u)] \circ f^{-j})\bar{A})$$

where $V(x)$ is given by (54) and $\bar{A} = A - \nu(A)1$.

In case $E_c = 0$ this coincides with the expression obtained in [48].

Remark. *In case $W^* \notin C^1$ the comparison between the formula of Proposition 5 and that of* [48] *is more difficult. The problem is that since* $\mathrm{div}^u X^u$ *of* [48] *is only a distribution (note that unless W^* are smooth X^u have different meanings here and in* [48]*!) the convergence of the expression for ω given in* [48] *is not obvious. In fact this convergence is established in* [49] *by considering a suitable sequence of smooth approximations of W^s (and using absolute continuity of W^s). Thus the expression given in* [48] *corresponds to the limiting case of our formula when $E^{as} \to E^s$ in C^0-topology along a suitable sequence of approximations.*

3. Lyapunov exponents.

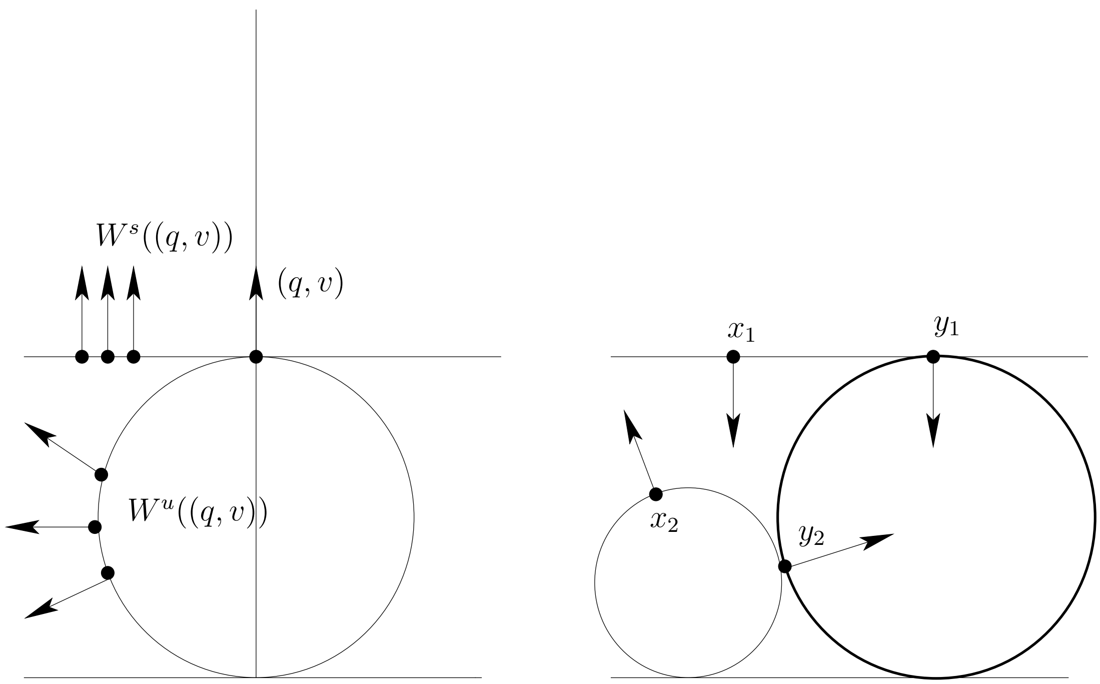

Figure 3. Two key properties of the geodesic flow used in the proof of Theorem 2:
(a) $E_u \oplus E_s$ is smooth (horocycles are perpendicular to geodesics);
(b) it has the three leg accessibility property (given two horocycles there exists a horocycle tangent to them).

3.1. **Statement of results.** Now we discuss an application of Theorem 1 to a problem of non-local bifurcation theory. Let $\mathbf{g}_t$ be a geodesic flow on M-a unit circle bundle of a compact negatively curved surface Q. We will denote the points of M by (q,v) where q is a point in Q and v is a unit tangent vector at q. Then $\varphi_t(q,v)=(q_t,v_t)$ where q_t is a point on the geodesic defined by (q,v) such that $d(q,q_t)=t$ and v_t is a tangent vector at q_t to this geodesic.

Let $f=\mathbf{g}_1$ and consider a C^∞ one-parameter family f_ε with $f_0 = f$. It is known ([11]) that the Lebesgue measure which we denote by ν is the unique u-Gibbs state for f and that $\ell_{S,\rho}(A\circ f^N)\to\nu(A)$ exponentially fast ([19]). We also make use of the following properties of f and its perturbations (see figure 3).

- The distribution $E_u\oplus E_s$ is C^∞. In fact, $E_u\oplus E_s$ is the kernel of the Poincare–Cartan form

$$\Theta(q,v)(\delta q,\delta v)=< v,\delta q > \tag{55}$$

  where $< \cdot,\cdot >$ is the scalar product on TQ.
- There exists a constant R such that for small ε for all $x_1,x_2\in M$ there exist points $y_1,y_2\in M$ such that

$$y_1\in W^u(x_1,f_\varepsilon),\quad y_2\in W^s(y_1,f_\varepsilon),\quad x_2\in W^u(y_2,f_\varepsilon) \tag{56}$$

  and $d(x_j,y_j)\le R$, $d(y_1,y_2)\le R$ where the distance is taken in the intristic metric of the corresponding leaves. (The proof of Lemma 2.1 in [59] gives (56) with W^u and W^s interchanged. To get the present statement note that f and f^{-1} are conjugated by the involution $i(q,v)=(q,-v)$.)
- ([27]) $E_u(x)$ and $E_s(x)$ are C^1. (In fact, Theorem 3.1 (2) of [29] gives that $E_u(x)$ and $E_s(x)$ are $C^{2-\delta}$ for any $\delta>0$ but we shall not use this.)

In case all f_ε preserve ν, the quantitative behavior of f_ε is given by the following result.

Proposition 7. *(See* [59, 15]*.) If all f_ε preserve ν then for small ε (f_ε,ν) is a K–system.*

It is not known if (f_ε,ν) is a Bernoulli shift. However [54] allows to construct some one parameter families f_ε for which (f_ε,m) is stably Bernoulli (that is for any g sufficiently close to f_ε, (g,m) is Bernoulli) for non-zero ε. We want to know how common is this phenomenon. To investigate this we, following [54], study how the Lyapunov exponents in the central direction change with parameter. Let

$$\lambda_c(\nu_\varepsilon)=\int \ln(df_\varepsilon|E_c)d\nu_\varepsilon.$$

Our first result is the following. Recall the definition of X (see (13)).

Theorem 2. *Let f_ε be any C^∞ one parameter family through f (volume preserving or not).*

(a) There is a quadratic form $c(X)$ such that if ν_ε is any u-Gibbs state for f_ε then

$$\lim_{\varepsilon\to 0}\frac{\lambda_c(\nu_\varepsilon)}{\varepsilon^2}=c(X).$$

(b) $c(X)$ is not identically zero even on the space of divergence free fields.

Since $c(X)$ is a quadratic form not identically equal to zero, the set of X such that $c(X)=0$ is a codimension one submanifold in the space of vectorfields. We say that $\{f_\varepsilon\}$ is generic if $c(X)\neq 0$.

The proof of Theorem 2 is given in Section 3.3. In Appendix E we exhibit a vector field X on M with $c(X)\neq 0$.

The proof of Theorem 2 relies on the following fact.

Theorem 3. *Let f be a partially hyperbolic diffeomorphism whose central direction is C^1. Let f_ε be a C^∞ one-parameter family such that $f_0=f$. Let $E_c(x,\varepsilon)$ be its central direction. Then the map $\varepsilon\to E_c(x,\varepsilon)$ is differentiable at 0. More precisely let $\hat{E}$ be any smooth distribution transverse to E_c. Then for small ε $E_c(x,\varepsilon)$ is the graph of a map $u_\varepsilon(x): E_c(x)\to\hat{E}(x)$, the map $\varepsilon\to u_\varepsilon(x)$ is differentiable at 0, and the derivative depends continuously on x.*

Remark. *It was already observed in* [2] *that if the map $x\to E_c(x)$ is not differentiable then in the family $f_\varepsilon=\varphi_\varepsilon\circ f\circ\varphi_\varepsilon^{-1}$ the map $\varepsilon\to E_c(x,\varepsilon)$ is not differentiable at 0.*

This theorem is proven in Section 3.2.

To apply Theorem 2 to the study of stochastic properties of f_ε we need the following auxiliary statement.

Lemma 2. *Under the conditions of Theorem 2 $W^s(f_\varepsilon)$ is topologically transitive for small ε.*

Proof. Iterating (56) backwards we see that given ε there is a chain (56) with $\mathrm{dist}(x_j,y_j)\leq\varepsilon$ (of course then $\mathrm{dist}(y_1,y_2)$ is large). In other words any two balls could be joined by a leaf of W^s. □

Let g be a partially hyperbolic diffeomorphism on a three dimensional manifold M. Call g mostly contracting (respectively, mostly expanding) if there exists a constant $\alpha>0$ such that any u-Gibbs state ν of g satisfies $\lambda_c(\nu)\leq-\alpha$ (respectively $\lambda_c(\nu)\geq\alpha$.) Mostly contracting and mostly expanding systems were studied in [1, 5, 21]. Using the results of these papers we derive the following consequence of Theorem 2.

Corollary 3. *Under the conditions of Theorem 2 for generic families, f_ε has a unique SRB measure for small ε, and the basin of this measure has total Lebesgue measure in M.*

Proof. Consider two cases.

(a) $c(X) < 0$ (mostly contracting case). The statement follows from [21]. ([21] considers mostly contracting systems which are dynamically coherent and u-convergent. In our case the dynamical coherence follows from the fact that $E_c(f)$ is C^1 ([28], Theorem 7.1 and 7.2) and u-convergence follows from (56) by [21], subsection 11(a)).

(b) $c(X) > 0$ (mostly expanding case.) By [1] there are at most finitely many SRB states whose basins cover all of M. Now let ν_ε^1 and ν_ε^2 be two SRB measures for f_ε. By [1], page 376, there exist 2-dimensional discs D_j transversal to E_s such that $\text{mes}(D_j - \mathbb{B}(\nu_\varepsilon^j)) = 0$. Write $\tilde{D}_j = D_j \bigcap \mathbb{B}(\nu_\varepsilon^j)$. By Lemma 2 there exist points $x_j \in D_j$ such that $x_2 \in W^s(x_1)$. Let $p_s : D_1 \to D_2$ be the stable holonomy. Then since p_s is absolutely continuous, $p_s(\tilde{D}_1) \bigcap \tilde{D}_2$ has positive measure. But

$$p_s(\tilde{D}_1) \bigcap \tilde{D}_2 \subset \mathbb{B}(\nu_\varepsilon^1) \bigcap \mathbb{B}(\nu_\varepsilon^2).$$

Hence $\mathbb{B}(\nu_\varepsilon^1) \bigcap \mathbb{B}(\nu_\varepsilon^2) \neq \emptyset$ and so $\nu_\varepsilon^1 = \nu_\varepsilon^2$. In other words, f_ε has a unique SRB measure ν_ε. But then by [1] $\text{mes}(M - \mathbb{B}(\nu_\varepsilon)) = 0$. □

For volume preserving families we have a stronger result.

Corollary 4. *For a generic family of volume preserving diffeomorphisms passing through f, either f_ε or f_ε^{-1} is mostly contracting for small ε.*

Proof. If $c(X) < 0$ then f_ε is mostly contracting. If $c(X) > 0$, let $\tilde{\nu}_\varepsilon$ be any u-Gibbs measure for f_ε^{-1}. Then since m is also u-Gibbs for f_ε^{-1} Theorem 2 gives

$$\lambda_c(\tilde{\nu}_\varepsilon, f_\varepsilon^{-1}) = \lambda_c(\nu, f_\varepsilon^{-1}) + o(\varepsilon^2) = -\lambda_c(\nu, f_\varepsilon) + o(\varepsilon^2) = -c(X)\varepsilon^2 + o(\varepsilon^2)$$

and so f_ε^{-1} is mostly contracting. □

Combining this with the properties of mostly contracting diffeomorphisms obtained in [21] we obtain

Corollary 5. *For a generic family of volume preserving diffeomorphisms passing through f, for small ε, (f_ε, ν) is Bernoulli, enjoys exponential mixing and satisfies the CLT. The same holds for g sufficiently close to f_ε.*

Remark. *It seems that results similar to* [21] *are also valid for mostly expanding systems, but I have not seen a proof of this fact. If it is so, the condition that f_ε preserve volume in Corollary 5 is redundant.*

3.2. **Variation of central direction.** Here we prove Theorem 3.

Proof. Let $\hat{E}$ be some distribution close to $E_s \oplus E_u$ (later on we assume that $\hat{E}$ is smooth but it is not necessary for the most of this subsection). By continuous dependence of the central distribution on parameters ([28], Corollary 2.12) $E_c(f_\varepsilon)$ is the graph of a map $u = u_\varepsilon : E_c(f) \to \hat{E}$. Suppose that df_ε has the following block form corresponding to the splitting $TM = \hat{E} \oplus E_c$:

$$df_\varepsilon = \begin{pmatrix} a_{nn}(x,\varepsilon) & a_{nc}(x,\varepsilon) \\ a_{cn}(x,\varepsilon) & a_{cc}(x,\varepsilon) \end{pmatrix}$$

Then $df_\varepsilon(e+u(e)) = a_{cc}(e) + a_{nc}(u) + a_{cn}(e) + a_{nn}(u)$. Let $L_\varepsilon(x)(e)$ be the map $E_c(x) \to E_c(f_\varepsilon(x))$ given by

$$L_\varepsilon(e) = \left[a_{cc} + a_{nc}u\right](e),$$

so that

$$df_\varepsilon(e+u(e)) = L_\varepsilon(e) + u(L_\varepsilon(e)).$$

Then

$$L_\varepsilon^{-1} = \left[1 + a_{cc}^{-1}a_{nc}u\right]^{-1} a_{cc}^{-1} = a_{cc}^{-1} - a_{cc}^{-1}a_{nc}ua_{cc}^{-1} + O(||u||^2).$$

Thus

$$u(f_\varepsilon x) = \left[a_{cn} + a_{nn}u(x)\right] L_\varepsilon^{-1} = a_{cn}a_{cc}^{-1} + a_{nn}ua_{cc}^{-1} - a_{cn}a_{cc}^{-1}a_{nc}u(x)a_{cc}^{-1} + O(||u||^2) =$$

$$\sigma_\varepsilon + (Q_\varepsilon u)(f_\varepsilon x) + O(||u||^2) \tag{57}$$

where $\sigma_\varepsilon = a_{cn}a_{cc}^{-1}$ and

$$\left[Q_\varepsilon(u)\right](x) = \left[a_{nn}ua_{cc}^{-1} - a_{cn}a_{cc}^{-1}a_{nc}ua_{cc}^{-1}\right] \circ (f_\varepsilon^{-1}x).$$

We now consider Q_ε as an operator on the space $\mathbb{L}(E_c, \hat{E})$ of continuous sections of the bundle over M whose fibers are linear maps $E_c \to \hat{E}$.

Lemma 3. *If $\hat{E}$ is sufficiently close to $E_u \oplus E_s$ then Q_ε is hyperbolic for small ε. More precisely, there exists a Q_ε invariant splitting*

$$\mathbb{L}(E_c, \hat{E}) = \mathbb{L}(E_c, \hat{E}_u) \oplus \mathbb{L}(E_c, \hat{E}_s) \tag{58}$$

such that

$$\forall u \in \mathbb{L}(E_c, \hat{E}_s) \quad ||Q_\varepsilon u|| \leq \tilde{\lambda}_2 ||u||, \tag{59}$$

$$\forall u \in \mathbb{L}(E_c, \hat{E}_u) \quad ||Q_\varepsilon^{-1} u|| \leq \tilde{\lambda}_2 ||u|| \tag{60}$$

where $\tilde{\lambda}_2$ is a constant close to λ_2.

Proof. Let

$$R = a_{nn} - a_{cn}a_{cc}^{-1}a_{nc}.$$

We claim that if $\hat{E}$ is close to $E_u \oplus E_s$ then R is hyperbolic in the sense that there is an R-invariant splitting $\hat{E} = \hat{E}_u \oplus \hat{E}_s$, such that

$$||R|\hat{E}_s|| \leq \tilde{\lambda}_1, \tag{61}$$

$$||R^{-1}|\hat{E}_u|| \leq \tilde{\lambda}_1 \tag{62}$$

where $\tilde{\lambda}_1$ is a constant close to λ_1. Let $\mathcal{K}_u$ be the cone defined by (10) and $\hat{\mathcal{K}}_u = \hat{E} \bigcap \mathcal{K}_u$. Then if ε is small and $\hat{E}$ is close to $E_u \oplus E_s$ then $\hat{\mathcal{K}}_u$ contains a d_u-dimensional hyperplane and is mapped by R inside itself. Let $\hat{E}_u = \bigcap_{n>0} R^n \hat{\mathcal{K}}(f_\varepsilon^{-n})$. Then $\hat{E}_u$ is R-invariant. Also $\hat{E}_u$ is close to E_u. Hence every vector in $\hat{E}_u$ is expanded by at least $\tilde{\lambda}_1^{-1}$. This implies (62). $\hat{E}_s$ is constructed similarly by considering R^{-1} instead of R. The fact that the splitting (58) satisfies (59) and (60) follow from the fact that $\hat{E}_*$ are close to E_* and (3), (4). □

By Lemma 3 the operator $1 - Q_\varepsilon$ is invertible so we obtain from (57)

$$u = [1 - Q_\varepsilon]^{-1} \sigma_\varepsilon + O(||u||^2).$$

Now choose $\hat{E}$ to be smooth. Then since E_c is C^1 the map $\varepsilon \to \sigma_\varepsilon$ is C^1 and since $\sigma_0 = 0$ it follows that $\sigma_\varepsilon(x) \sim \varepsilon \mathbf{b}(x) + o(\varepsilon)$ for a continuous map $\mathbf{b}(x)$. Hence

$$u = \varepsilon(1 - Q_\varepsilon)^{-1}\mathbf{b} + o(\varepsilon) + O(||u||^2) \tag{63}$$

Therefore

$$||u|| \leq \text{Const}(\varepsilon + ||u||^2).$$

The equation $s = \text{Const}(\varepsilon + s^2)$ has two roots, one of order ε and another of order 1. By continuity of the central direction $||u_\varepsilon|| \to 0$ as $\varepsilon \to 0$. This implies that $||u_\varepsilon||$ less than the first root so that $||u_\varepsilon|| = O(\varepsilon)$. This gives

$$u_\varepsilon \sim \varepsilon(1 - Q_\varepsilon)^{-1}\mathbf{b}.$$

Let now $\mathbf{b} = \mathbf{b}_s + \mathbf{b}_u$, where $\mathbf{b}_* \in \mathbb{L}(E_c, \hat{E}_*)$. Then

$$[1 - Q_\varepsilon]^{-1} \mathbf{b}_s = \sum_{j=0}^{\infty} Q_\varepsilon^j \mathbf{b}_s, \tag{64}$$

$$[1 - Q_\varepsilon]^{-1} \mathbf{b}_u = \sum_{j=1}^{\infty} Q_\varepsilon^{-j} \mathbf{b}_u \tag{65}$$

and both series converge uniformly. Now for each j the maps $(x, \varepsilon) \to [Q_\varepsilon^j \mathbf{b}_*](x)$ are continuous. Combining this observation with (64)–(65) we obtain

$$u_\varepsilon \sim \varepsilon[1 - Q_0]^{-1}\mathbf{b}. \tag{66}$$

This completes the proof of Theorem 3. □

Remark. *In the first version of this paper I required that $E_c \in C^{1+\delta}$ for some positive δ. I am grateful to the referee for pointing out that my proof works for C^1 case as well. Recently, results similar to Theorem 3 were established in* [52] *(in a slightly less general case) and* [45] *(in a more general case). The reader is espesially refered to* [45] *for a very detailed discussion of the partial differentiability of invariant splittings. I decided to keep my original proof to demonstrate the usefulness of approximating E_* by smooth distributions–the idea which also plays an important role in Section 2.*

3.3. **Proof of Theorem 2.** Choose vectors $e_* \in E_*$ such that $x \to e_*(x)$ are C^1, $\Theta(e_c) = 1$, and

$$d\Theta(e_s, e_u) = 1 \tag{67}$$

where Θ is the Poincare-Cartan form defined by (55). Define $\lambda(x)$ by $df(e_u(x)) = \lambda(x)e_u(fx)$. Then by (67) $df(e_s(x)) = \lambda^{-1}(x)e_s(fx)$. Then Let

$$f_\varepsilon = h_\varepsilon \circ f, \quad dh_\varepsilon = 1 + \varepsilon B_1 + \varepsilon^2 B_2 \dots.$$

Write

$$B_k e_l = \sum_m b_{lm}^{(k)} e_m.$$

By Theorem 3 we have

$$e_c(x, \varepsilon) = e_c(x) + \varepsilon\alpha(x)e_u(x) + \varepsilon\beta(x)e_s(x) + w_\varepsilon \tag{68}$$

where $w_\varepsilon \in E_u \oplus E_s$, $w_\varepsilon = o(\varepsilon)$.

Lemma 4. *The following asymptotic expansion holds*

$$\pi_c[df_\varepsilon(e_c(x, \varepsilon))] =$$

$$(1+\varepsilon b_{cc}^{(1)}(fx)+\varepsilon^2[b_{cc}^{(2)}(fx)+\alpha(x)\lambda(x)b_{uc}^{(1)}(fx)+\frac{\beta(x)}{\lambda(x)}b_{sc}^{(1)}(fx)]e_c(f_\varepsilon x)+o(\varepsilon^2)$$

where π_c denotes the projection to E_c along $E_u \oplus E_s$.

Proof.

$$dfe_c(x, \varepsilon) = e_c(fx) + \varepsilon\lambda(x)\alpha(x)e_u(x) + \varepsilon\frac{\beta(x)}{\lambda(x)}e_s(x) + df(w_\varepsilon).$$

Now since e_c and $E_u \oplus E_s$ are C^∞ we have

$$dh_\varepsilon(e_c) = 1 + \varepsilon b_{cc}^{(1)} + \varepsilon^2 b_{cc}^{(2)} + w'_\varepsilon + o(\varepsilon^2),$$

where $w'_\varepsilon \in E_u \oplus E_s$. Also since e_u and e_s are C^1

$$dh_\varepsilon(e_u) = \varepsilon b_{uc}^{(1)} e_c + w''_\varepsilon + o(\varepsilon)$$

and

$$dh_\varepsilon(e_s) = \varepsilon b_{sc}^{(1)} e_c + w'''_\varepsilon + o(\varepsilon)$$

where $w''_\varepsilon, w'''_\varepsilon \in E_u \oplus E_s$. Combining the last three estimates we obtain the statement of the lemma. □

Comparing Lemma 4 and (68) and using the df_ε invariance of $E_c(x,\varepsilon)$ we get

$$df_\varepsilon(e_c(x,\varepsilon)) =$$

$$\left(1 + \varepsilon b_{cc}^{(1)}(fx) + \varepsilon^2 [b_{cc}^{(2)}(fx) + \alpha(x)\lambda(x) b_{uc}^{(1)}(fx) + \frac{\beta(x)}{\lambda(x)} b_{sc}^{(1)}(fx)]\right) e_c(f_\varepsilon x, \varepsilon) + o(\varepsilon^2)$$

Therefore

$$\lambda_c(\nu_\varepsilon) = \int \ln df_\varepsilon(e_c(x,\varepsilon)) d\nu_\varepsilon(x) =$$

$$\varepsilon \nu_\varepsilon(b_{cc}^{(1)}(fx)) + \varepsilon^2 \nu_\varepsilon \left(b_{cc}^{(2)}(fx) + \alpha(x)\lambda(x) b_{uc}(fx) + \frac{\beta(x)}{\lambda(x)} b_{sc}(fx) - \frac{(b_{cc}^{(1)})^2}{2} \right) + o(\varepsilon^2).$$

Let L_X denote the Lie derivative with respect to X. Then

$$b_{cc}^{(1)} = \Theta(L_{e_c} X) = \partial_{e_c}(\Theta(X))$$

where the last equality holds since Θ is e_c invariant. By Theorem 1

$$\nu_\varepsilon(b_{cc}^{(1)}(fx)) = \nu(b_{cc}^{(1)}(fx)) + \varepsilon \omega(b_{cc}^{(1)}(fx)).$$

and $\nu(b_{cc}^{(1)}(fx)) = \nu(b_{cc}^{(1)}(x)) = \nu(\partial_{e_c}(\Theta(X))) = 0$ since e_c preserves ν. A similar computation shows that $\nu(b_{cc}^{(2)}(fx)) = 0$.

Therefore $\lambda_c(\nu_\varepsilon) \sim c(X)\varepsilon^2$ where

$$c(X) = \nu \left(\alpha(x)\lambda(x) b_{uc}^{(1)}(fx) + \frac{\beta(x)}{\lambda(x)} b_{sc}^{(1)}(fx) - \frac{(b_{cc}^{(1)})^2}{2} \right) + \omega(b_{cc}^{(1)}). \tag{69}$$

(Note in particular that for volume preserving families the last term vanishes since $\omega = 0$.)

To complete the proof of Theorem 2 it remains to exhibit a volume preserving vectorfield X such that $c(X) \neq 0$. This is done in Appendix E. □

Appendix A. Approximation of u-Gibbs states by measures on unstable leaves.

A.1. **Regularity of densities.** Recall that a measure μ is called absolutely continuous with respect to a foliation $\mathcal{F}$ with smooth leaves if any set Ω such that for each x $\mathcal{F}(x)\bigcap\Omega$ has leafwise measure zero has $\mu(\Omega)=0$. An equivalent way to say this is the following. Let U be a flowbox for $\mathcal{F}$ and $V\subset U$ be a transversal to $\mathcal{F}$. Let $\mathcal{F}(x,U)$ denote the connected component of $\mathcal{F}(x)\bigcap U$. Then μ is absolutely continuous if for each U the restriction of μ on U can be written as

$$\mu_U=\int_V d\mathbf{m}(x)\int_{\mathcal{F}(x,U)}\rho(y,x)dy.$$

Let $g: M\to M$ be a partially hyperbolic diffeomorphism. Let W^u be the unstable foliation of g. observe that elements of $\bar{\mathbf{E}}(\mathbf{r},C_1,C_2,\alpha_1,\alpha_2)$ are absolutely continuous with respect to W^u. Indeed let U be a flowbox for W^u and $V\subset U$ be a transversal. Given (S,ρ) such that S is $(\mathbf{r},C_1,\alpha_1)$ regular and ρ is a probability density on S satisfying $||\rho||_{C^{\alpha_2}(S)}\leq C_2$ let $\tilde{\ell}_{S,\rho}=C_2\ell_{W^u(x,U),1}$ if S intersects some leave $W^u(x,U)$ and $\tilde{\ell}_{S,\rho}=0$ otherwise. Consider a measure $\mu\in\bar{\mathbf{E}}(\mathbf{r},C_1,C_2,\alpha_1,\alpha_2)$. Then $\mu=\lim_{n\to\infty}\mu_n$ where

$$\mu_n=\int_\alpha \ell_{S_\alpha,\rho_\alpha}d\lambda(\alpha).$$

Let

$$\tilde{\mu}_n=\int_\alpha \tilde{\ell}_{S_\alpha,\rho_\alpha}d\lambda(\alpha).$$

Then $\tilde{\mu}_n$ is a measure on U

$$\tilde{\mu}(1)\leq C_2\max_x \mathrm{mes}(W^u(x,U))$$

and $\tilde{\mu}_n-(\mu_n)_U$ is a measure. By passing to a subsequance we can assume that $\tilde{\mu}_n\to\tilde{\mu}$. Now each $\tilde{\mu}_n$ can be written as

$$\tilde{\mu}_n=\int_V d\zeta_n(x)\int_{W^u(x,U)}dy.$$

It follows that ζ_n converge to some measure ζ and

$$\tilde{\mu}=\int_V d\zeta(x)\int_{W^u(x,U)}dy.$$

Thus $\tilde{\mu}$ is absolutely continuous with respect to W^u. Since both μ are $\tilde{\mu}-\mu$ are measures, they are absolutely continuous as claimed.

The main result of this section is the following.

Proposition 8. *Let $g: M \to M$ be a partially hyperbolic diffeomorphism.*

(a) There exist constants $\mathbf{r}, C_1, C_2$ such that for any bounded set $S \subset W^u$ such that $\mathrm{mes}(\partial S) = 0$ and $\forall \rho \in L^1(S)$ $\forall \varepsilon > 0$ $\exists n_0$ such that $\forall n > n_0$ $\exists c_{j,n}, S_{j,n}, \rho_{j,n}$ such that

- $c_{j,n} > 0$
- *$S_{j,n}$ are $(\mathbf{r}, C_1, 1)$–regular*
- *$\rho_{j,n}$ is a probability density on $S_{j,n}$ and $||\rho_{j,n}||_{\mathrm{Lip}(S_{j,n})} \leq C_2$*
- *for all $k \geq 0$, for any $A \in C^0(M)$*

$$|\ell_{S,\rho}(A \circ g^{n+q}) - \sum_j c_{j,n} \ell_{S_{j,n},\rho_{j,n}}(A \circ g^q)| \leq \varepsilon ||A||_{C^0(M)}. \tag{70}$$

Moreover the constants $\mathbf{r}, C_1, C_2$ can be chosen uniformly for all $\tilde{g}$ in a small C^2 neighborhood of g. Also, for all $\bar{\mathbf{r}}, \bar{C}_1, \bar{C}_2, \bar{\alpha}_1, \bar{\alpha}_2$ there exists C_4 such that (70) holds with $n_0 = C_4|\ln \epsilon|$ for all $\tilde{g}$ C^2–near g and for all pairs (S, ρ) such that S is $(\bar{\mathbf{r}}, \bar{C}_1, \bar{\alpha}_1)$–regular and $||\rho||_{C^{\bar{\alpha}_2}(S)} \leq \bar{C}_2$

(b) There exist constants $\mathbf{r}, C_1, C_2$ such that any absolutely continuous g invariant measure belongs to $\bar{\mathbf{E}}_{inv}(\mathbf{r}, C_1, C_2, 1, 1)$. These constants can be chosen uniformly for all $\tilde{g}$ in a small C^2 neighborhood of g.

Remark. *The definition of u-Gibbs mesures given in our paper appears to be sligtly different from the one given in* [43] *(g-invariant absolutely continuous measures whose conditional densities are canoniacal ones, defined in Section 2.5), however they are equivalent. Indeed since any g invariant measure μ satisfies $\mu(A) = \frac{1}{N}\sum_{n=0}^{N-1} \mu(A \circ g^n)$, elements of $\bar{\mathbf{E}}_{inv}(\mathbf{r}, C_1, C_2, \alpha_1, \alpha_2)$ can be approximated by convex combinations of measures of the form $\frac{1}{N}\sum_{n=0}^{N-1} \ell_{S,\rho}(A \circ g^n)$. Therefore elements of $\bar{\mathbf{E}}_{inv}(\mathbf{r}, C_1, C_2, \alpha_1, \alpha_2)$ are u-Gibbs in the sense of* [43]*, by Theorem 4 of* [43]*. Conversely any u-Gibbs measure in the sense of* [43] *is also u-Gibbs according to our definition by Proposition 8(b).*

The proof of this proposition is similar to the arguments of [38, 43, 46, 47]. However, we give the proof below since the part (b) playing a crucial role in our analysis is not stated explicitely in the above mentioned papers.

Proof of Proposition 8. We follow [23]. Let $r > 0$ be fixed. Let W be a leaf of $W^u(g)$ and $\mathcal{S}$ be a maximal r-separated set in W. Given $p \in \mathcal{S}$ define its Dirichlet cell $D(p)$ by

$$D(p) = D_{\mathcal{S}}(p) = \{x \in W : d(p, x) = \min_{q \in \mathcal{S}} d(q, x)\}.$$

Then $B(p, r/2) \subset D(p) \subset B(p, r)$ and $\partial D(p)$ is contained in a union of finitely many sets

$$L(p, q_j) = \{x \in W : d(p, x) = d(x, q_j)\}$$

where $q_j \in \mathcal{S}$ are centers of the cells adjacent to $D(p)$. Now if $D(p)$ and $D(p)$ are adjecent, that is, if there is a point $x \in D(p) \bigcap D(q_j)$ then $d(p, q_j) \le d(x, p) + d(x, q_j) \le 2r$ so the number of cells adjacent to $D(p)$ is bounded by the maximal cardinality of an $r/2$–separated set in an unstable ball of radius $2r$. Fix some coordinate system $x = (x^l)$ near p. Then if r is sufficiently small the part of $L(p, q_j)$ inside $B(x, r)$ is close to a piece of hyperplane

$$\sum_l (x^l - p^l)^2 = \sum_l (x^l - q_j^l)^2.$$

These facts imply that there exist constants $\mathbf{r}, C_1$ such that for any $W, \mathcal{S}, p$, $D_{\mathcal{S}}(p)$ is $(\mathbf{r}, C_1, 1)$–regular. Now let S be any subset of W. Let

$$I(S) = \{p \in \mathcal{S} \bigcap S : d(p, \partial S) \ge r\}, \quad \tilde{S} = \bigcup_{p \in I(S)} D(p).$$

Then

$$\{x \in S : d(x, \partial S) \ge 2r\} \subset \tilde{S} \subset S.$$

We use these approximations to prove Proposition 8. Let (S, ρ) be as in that Proposition. Since Lipschitz functions are dense in $L^1(S)$ we can assume that $\rho \in \mathrm{Lip}(S)$.

Consider the decomposition

$$\widetilde{g^{n_0} S} = \bigcup_{p \in I_{n_0}} D(p).$$

where $I_{n_0} = I(g^{n_0} S)$. Let

$$I'_{n_0} = \left\{ p : \forall y \in D(p) \quad \rho(g^{-n_0} y) \ge \frac{\epsilon}{4 \mathrm{mes}(S)} \right\} \tag{71}$$

$$I''_{n_0} = I_{n_0} - I'_{n_0}.$$

Let $c(p) = \rho(g^{-n_0} p) \mathrm{mes}(g^{-n_0} D(p))$. Observe that by (71), $c(p)$ is positive for $p \in I'_{n_0}$. The change of variables $y = g^{n_0} x$ gives

$$\int_S A(g^{n_0+q} x) \rho(x) dx =$$

$$\sum_{I'_{n_0}} c(p) \int_{D(p)} \frac{\rho(g^{-n_0} p)}{c(p)} A(g^q y) \det(dg^{-n_0} | E_u)(y) dy +$$

$$\sum_{I'_{n_0}} c(p) \int_{D(p)} \frac{\rho(g^{-n_0}y) - \rho(g^{-n_0}p)}{c(p)} A(g^q y) \det(dg^{-n_0}|E_u)(y)dy+$$

$$\sum_{I''_{n_0}} \int_{g^{-n_0}D(p)} \rho(x)A(g^{n_0+q}x)dx + \int_{S-g^{-n_0}\widetilde{g^{n_0}S}} \rho(x)A(g^{n_0+q}x)dx =$$

$$\sum_{I'_{n_0}} c(p)\ell\left(D(p), \frac{\rho(g^{-n_0}p)}{c(p)} \det(dg^{-n_0}|E_u)\right)(A \circ g^q)+$$

$$\text{Remainder}_I + \text{Remainder}_{II} + \text{Remainder}_{III}.$$

We claim that the set of triples

$$\left\{\left(c(p), D(p), \frac{\rho(g^{-n_0}p)\det(dg^{-n_0}|E^u)(y)}{c(p)}\right)\right\}_{p\in I'_{n_0}}$$

satisfies the conditions of Proposition 8. First we check the regularity of the densities.

Lemma 5. *The Lipschitz norm of*

$$\frac{\rho(g^{-n_0}p)\det(dg^{-n_0}|E^u)(y)}{c(p)}$$

is uniformly bounded.

Proof.

$$\left|\ln\det(dg^{-n_0}|E^u)(y_1) - \ln\det(dg^{-n_0}|E^u)(y_2)\right| \le$$

$$\text{Const}\sum_{j=0}^{n_0-1} \left|\ln\det(dg^{-1}|E^u)(g^{-j}y_1) - \ln\det(dg^{-n_0}|E^u)(g^{-j}y_2)\right| \le$$

$$\text{Const}\sum_{j=0}^{n_0-1} \lambda_1^j d(y_1,y_2) \le \text{Const}d(y_1,y_2).$$

Hence there exists a constant K, independent of p, such that for all $y_0, y \in D(p)$

$$\frac{1}{K} \le \frac{\det(dg^{-n_0}|E_u)(y_0)\rho(g^{-n_0}p)}{\det(dg^{-n_0}|E_u)(y)\rho(g^{-n_0}p)} \le K.$$

Integrating y_0 over $D(p)$ we get

$$\frac{1}{K} \le \frac{\rho(g^{-n_0}p)\det(dg^{-n_0}|E^u)(y)}{c(p)} \le K.$$

Combining this with the uniform Lipschitz continuity of $\ln\det(dg^{-n_0}|E_u)$ we obtain the statement of the lemma. □

We now estimate remaining terms.

$$|\text{Remainder}_{III}| \leq ||A||_{C^0(M)}||\rho||_{Lip(S)}\text{mes}(g^{-n_0}(g^{n_0}S - \widetilde{g^{n_0}S})).$$

We have

$$g^{n_0}S - \widetilde{g^{n_0}S} \subset \partial_{2r}g^{n_0}S.$$

So

$$g^{-n_0}(g^{n_0}S - \widetilde{g^{n_0}S}) \subset \partial_{2r\lambda_1^{n_0}}g^{n_0}S$$

(recall that we assume that (1)–(4) hold with $C = 1$.) The condition $\text{mes}(\partial S) = 0$ implies that

$$\text{mes}(\partial_{2r\lambda_1^{n_0}}g^{n_0}S) \to 0 \quad \text{as} \quad n_0 \to \infty. \tag{72}$$

On the other hand

$$|\rho(g^{-n_0}y) - \rho(g^{-n_0}p)| \leq ||\rho||_{\text{Lip}(S)}d(g^{-n_0}y, g^{-n_0}p) \leq$$
$$||\rho||_{\text{Lip}(S)}\lambda_1^{n_0}\text{diam}(D(p)) \leq 2||\rho||_{\text{Lip}(S)}\lambda_1^{n_0}r$$

and so if $p \in I'_{n_0}$ then

$$\int_{D(p)} \frac{\rho(g^{n_0}y) - \rho(g^{-n_0}p)}{c(p)} A(g^q y) \det(dg^{-n_0}|E_u)(y)dy =$$

$$\int_{D(p)} \frac{\rho(g^{n_0}y) - \rho(g^{-n_0}p)}{c(p)} A(g^q y) \det(dg^{-n_0}|E_u)(y)\frac{\rho(g^{-n_0}p)}{\rho(g^{-n_0}p)}dy \leq$$

$$\frac{4\text{mes}(S)}{\epsilon}||A||_{C^0(M)}\times 2||\rho||_{\text{Lip}(S)}\lambda_1^{n_0}r\times\int_{D(p)} \frac{\det(dg^{-n_0}|E_u)(y)\rho(g^{-n_0}p)}{c(p)}dy \leq$$

$$\frac{8\text{mes}(S)||A||_{C^0(M)}}{\epsilon}||\rho||_{\text{Lip}(S)}\lambda_1^{n_0}r \tag{73}$$

Hence

$$|\text{Remainder}_I| \leq 8\frac{\text{mes}(S)||A||_{C^0(M)}}{\epsilon}||\rho||_{\text{Lip}(S)}\lambda_1^{n_0}r\sum_{I_{n_0}} c(p).$$

Now if n_0 is large enough then for all $y \in D(p)$, $\rho(p) \leq 2\rho(y)$. Hence

$$\sum_{I_{n_0}} c(p) \leq \sum_{I_{n_0}} 2\int_{g^{-n_0}D(p)} \rho(x)dx \leq 2\int_S \rho(x)dx = 2.$$

Combining the last two estimates we get

$$|\text{Remainder}_I| \leq 16\frac{\text{mes}(S)||A||_{C^0(M)}}{\epsilon}||\rho||_{\text{Lip}(S)}\lambda_1^{n_0}r \to 0 \tag{74}$$

as $n_0 \to \infty$.

We now again use the fact that if y', y'' belong to the same $D(p)$ then

$$\left|\rho(g^{-n_0}y') - \rho(g^{-n_0}y'')\right| \leq 2||\rho||_{\text{Lip}(S)}r\lambda_1^{n_0}$$

So if n_0 is so large that

$$||\rho||_{\mathrm{Lip}} r\lambda_1^{n_0} \leq \frac{\epsilon}{4\mathrm{mes}(S)} \tag{75}$$

then $\forall p \in I''_{n_0}$ $\forall x \in g^{-n_0}D(p)$ we have $\rho(x) < \frac{\epsilon}{2\mathrm{mes}(S)}$. Hence

$$|\mathrm{Remainder}_{\mathit{II}}| \leq ||A||_{C^0(M)} \frac{\epsilon}{2\mathrm{mes}(S)} \mathrm{mes}(S) = \frac{\epsilon}{2}||A||_{C^0(M)}.$$

This completes the proof of (70). Since we have used (72), (74) and (75) to estimate $\mathrm{Remainder}_{\mathit{III}}$, $\mathrm{Remainder}_I$ and $\mathrm{Remainder}_{\mathit{II}}$ respectively, the uniformities claimed in the part (a) also follow. (In the proof of (74) we used the fact that ρ is Lipschitz. If it is only Holder, we get $||\rho||_{C^{\bar{\alpha}_2}(S)}\lambda_1^{n_0\alpha}$ instead of $||\rho||_{\mathrm{Lip}(S)}\lambda_1^{n_0}$.) This completes the proof of the (a).

To prove (b) let μ have conditional densities on unstable leaves. Let $M = \bigcup_{j=1}^s U_j$ be a partition of M into domains with piecewise smooth boundaries. Let V_j be transversals to E_u in U_j. Denote $W_j^u(x) = W^u(x, U_j)$. We can write

$$\mu_{U_j} = \int_{V_j} d\mathbf{m}_j(x) \int_{W_j^u(x)} \rho_j(y,x)dy$$

for some measure $\mathbf{m}_j$ on V_j. In other words

$$\mu = \sum_j \int_{V_j} d\mathbf{m}_j(x) \ell_{W_j^u(x),\rho_j(x)}.$$

Since Lipschitz functions are dense in L^1, given ε there exist Lipshitz $\rho_{j,\varepsilon}$ such that

$$||\rho_j - \rho_{j,\varepsilon}||_{L^1(d\mathbf{m}_j dy)} \leq \varepsilon.$$

Hence for any $A \in C^0(M)$

$$\left| \mu(A) - \sum_j \int_{V_j} d\mathbf{m}_j(x) \ell_{W_j^u(x),\rho_{j,\varepsilon}(x)}(A) \right| \leq \varepsilon s ||A||_{C^0(M)}.$$

Since μ is g invariant then also for all n

$$|\mu(A) - \mu_{n,\varepsilon}(A)| \leq \varepsilon s ||A||_{C^0(M)}.$$

where

$$\mu_{n,\varepsilon}(A) = \sum_j \int_{V_j} d\mathbf{m}_j(x) \ell_{W_j^u(x),\rho_{j,\varepsilon}(x)}(A \circ g^n).$$

Let μ_ε be a limit point of $\mu_{n,\varepsilon}$, $n \to \infty$. Note that $\ell_{W_j^u(x),\rho_{j,\varepsilon}(x)}$ are not probabilities, however, they satisfy

$$\ell_{W_j^u(x),\rho_{j,\varepsilon}(x)}(1) \leq ||\rho_{j,\varepsilon}||_{C^0(U_j)} \max_x \mathrm{mes}(W_j^u(x)).$$

Therefore we can apply part (a) to each $\ell_{W^u_j(x),\rho_{j,\varepsilon}(x)}$ to conclude that $\mu_\varepsilon \in \bar{\mathbf{E}}(\mathbf{r}, C_1, C_2, 1, 1)$. Letting $\varepsilon \to 0$ we get $\mu \in \bar{\mathbf{E}}(\mathbf{r}, C_1, C_2, 1, 1)$ as claimed. $\square$

A.2. **Proof of Proposition 2.** Similarly to Proposition 8 we can prove the following.

Lemma 6. *$\forall \mathbf{r}, C_1, C_3, \alpha_1, \alpha_2$ there exist $\bar{\mathbf{r}}, \bar{C}_1, \bar{C}_2, \bar{C}_3$ $\theta_1 < 1, K > 0$ such that for any $(\mathbf{r}, C_1, C_3, \alpha_1)$–admissible set S for any probability density $\rho \in C^{\alpha_2}(S)$ for all $n > 0$ there exist c_j, S_j, ρ_j such that*

- *$c_j > 0$;*
- *S_j are $(\bar{\mathbf{r}}, \bar{C}_1, \bar{C}_3, 1)$–admissible;*
- *ρ_j is a probability density on S_j and $||\rho_j||_{\mathrm{Lip}(S_j)} \le \bar{C}_2$ and for all $q \ge 0$*

$$\left| \ell_{S,\rho}(A \circ f^{n+q}) - \sum_j c_j \ell_{S_j,\rho_j}(A \circ f^q) \right| \le K\theta_1^n ||A||_{C^0(M)} ||\rho||_{C^{\alpha_2}(S)}.$$

In other words, Lemma 6 says that the estimates of Proposition 8(a) hold for admissible sets as well ($\epsilon = K\theta_1^n$ here corresponds to $n_0 = C_4|\ln \epsilon|$ in Proposition 8). We now use this lemma to prove Proposition 2.

Proof. (a) We assume that ρ is probability density to simplify the notation. Note that in this case the pair (S, ρ) satisfies the conditions of Lemma 6. Let $\mathcal{T}(S, \delta)$ denote a d-dimensional tube of radius δ about S

$$\mathcal{T}(S, \delta) = \bigcup_{|t| \le \delta} \mathbf{g}_t S.$$

Denote

$$\zeta(\delta) = \frac{\mathrm{mes}(S)}{\mathrm{mes}(\mathcal{T}(S, \delta))} = \delta^{-d}\mathrm{Vol}(\text{Unit ball in } \mathbb{R}^d)^{-1} \tag{76}$$

Applying Lemma 6 we get

$$\ell_{S,\rho}(A \circ f^N) = \sum_j c_j \ell_{S_j,\rho_j}(A \circ f^{N/2}) + O(\theta_1^{N/2} ||A||_{C^0(M)} ||\rho||_{C^{\alpha_2}(S)}). \tag{77}$$

The idea of the proof of (a) is the following. Since each $S_j \subset f^{N/2}S$, (4) implies that S_js can be well aproximated by the leaves of W^u, and so we can approximate the integrals over S_j by the integrals over pieces of unstable manifolds. To establish these approximations it is convinient to work with narrow tubes.

Since $(f|E_c)$ is an isometry we have

$$\ell_{S_j,\rho_j}(A \circ f^{N/2}) = \tag{78}$$

$$\zeta(N^{-m})\iint_{\mathcal{T}(S_j,N^{-m})} A(f^{N/2}\mathbf{g}_t y)\rho_j(y)dydt + O(N^{-m}||A||_{C_1^\alpha(M)}||\rho||_{C^{\alpha_2}(S)}).$$

Take arbitrary $y_j \in S_j$ and let $p_s : M \to W^{cu}(y_j)$, $p_{cs} : M \to W^u(y_j)$, denote the stable (respectively, the center-stable) holonomy. By (4) the angle between the tangent space TS_j and E_u satisfies

$$\angle(TS_j, E_u) \leq \text{Const}\lambda_2^{N/2}. \tag{79}$$

Since $\mathbf{g}_t$ preserves the partially hyperbolic splitting, we get from this that for any $\delta > 0$

$$\angle(T(\mathcal{T}(S_j,\delta), E_{cu}) \leq \text{Const}\lambda_2^{N/2}. \tag{80}$$

(79) and (80) imply

$$\forall y \in S_j \quad d(y, p_{cs}(y)) \leq \text{Const}\lambda_2^{N/2},$$

$$\forall y \in \mathcal{T}(S_j, N^{-m}) \quad d(y, p_s(y)) \leq \text{Const}\lambda_2^{N/2}. \tag{81}$$

The last two line imply in particular

$$\forall y \in S_j \quad d(p_s(y), p_{cs}(y)) \leq \text{Const}\lambda_2^{N/2}, \tag{82}$$

Now by [14], Section 3 (Theorem 3.1 and its proof) p_s is absolutely continuous with Holder jacobian. Denoting this jacobian by $\mathbf{j}$ and changing varaibles in (78) $(y,t) \to (z,\tau)$ where $z \in W^u(y_j)$, $\mathbf{g}_t y = \mathbf{g}_\tau z$ we get

$$\iint_{\mathcal{T}(S_j,N^{-m})} A(f^{N/2}\mathbf{g}_t y)\rho_j(y)dydt = \tag{83}$$

$$\iint_{p_s\mathcal{T}(S_j,N^{-m})} A(f^{N/2}p_s^{-1}\mathbf{g}_\tau z)\rho_j(p_s^{-1}\mathbf{g}_\tau z)\mathbf{j}(p_s^{-1}p_s^{-1}\mathbf{g}_\tau z)dzd\tau.$$

Now $p_s\mathcal{T}(S_j,N^{-m}) = \mathcal{T}(p_s S_j, N^{-m})$ since $\mathbf{g}_t$ commutes with f. Combining this with (82) we obtain

$$\mathcal{T}(p_{cs}S_j, N^{-m}-\text{Const}\lambda_2^{N/2}) \subset p_s\mathcal{T}(S_j,N^{-m}) \subset \mathcal{T}(p_{cs}S_j, N^{-m}+\text{Const}\lambda_2^{N/2}).$$

Therefore

$$\iint_{p_s\mathcal{T}(S_j,N^{-m})} A(f^{N/2}p_s^{-1}\mathbf{g}_\tau z)\rho_j(p_s^{-1}\mathbf{g}_\tau z)\mathbf{j}(p_s^{-1}p_s^{-1}\mathbf{g}_\tau z)dzd\tau = \tag{84}$$

$$\iint_{\tilde{\mathcal{T}}_j} A(f^{N/2}p_s^{-1}\mathbf{g}_\tau z)\rho_j(p_s^{-1}\mathbf{g}_\tau z)\mathbf{j}(p_s^{-1}p_s^{-1}\mathbf{g}_\tau z)dzd\tau + O(\lambda_2^{N/2}||A||_{C^0(M)}||\rho||_{C^{\alpha_2}(S)})$$

where $\tilde{\mathcal{T}}_j$ denotes $\mathcal{T}(p_{cs}S_j, N^{-m} - \text{Const}\lambda_2^{N/2})$. By (2)

$$d(f^{N/2}p_s^{-1}\mathbf{g}_\tau z, f^{N/2}\mathbf{g}_\tau z) \leq \lambda_2^{N/2} d(p_s^{-1}\mathbf{g}_\tau z, \mathbf{g}_\tau z) \leq \text{Const}\lambda_2^{N/2}.$$

So

$$\iint_{\tilde{\mathcal{T}}_j} A(f^{N/2} p_s^{-1} \mathbf{g}_\tau z)\rho_j(p_s^{-1}\mathbf{g}_\tau z)\mathbf{j}(p_s^{-1}p_s^{-1}\mathbf{g}_\tau z)dzd\tau = \tag{85}$$

$$\iint_{\tilde{\mathcal{T}}_j} A(f^{N/2}\mathbf{g}_\tau z)\rho_j(p_s^{-1}\mathbf{g}_\tau z)\mathbf{j}(p_s^{-1}p_s^{-1}\mathbf{g}_\tau z)dzd\tau + O(\lambda_2^{\alpha N/2}||A||_{C^\alpha(M)}||\rho||_{C^{\alpha_2}(S)}).$$

Denote $\bar\rho_j(z,\tau) = \rho_j(p_s^{-1}\mathbf{g}_\tau z)\mathbf{j}(p_s^{-1}p_s^{-1}\mathbf{g}_\tau z)$. Since p_s is Holder, there are constants $\tilde{\mathbf{r}}, \tilde C_1, \tilde C_2, \tilde\alpha_1, \tilde\alpha_2$ such that $p_{cs}(S_j)$ is $(\tilde{\mathbf{r}}, \tilde C_1, \tilde\alpha_1)$-regular and $||(\bar\rho_j)\circ p_s^{-1}||_{C^{\tilde\alpha_2}(\mathcal{T}_j)} \leq \tilde C_2 ||\rho||_{C^{\alpha_2}(S)}$. Hence (6) implies that if k is sufficiently large then for each τ

$$\int_{p_{cs}S_j} A(f^{N/2}\mathbf{g}_\tau z)\bar\rho_j(z,\tau)dz = O(N^{-m}||A||_{C^\alpha_k(M)}||\rho||_{C^{\alpha_2}(S)}).$$

Integrating over τ we obtain

$$\iint_{\tilde{\mathcal{T}}_j} A(f^{N/2}\mathbf{g}_\tau z)\bar\rho_j(z,\tau)dzd\tau = O\left(\zeta(N^{-m})N^{-m}||A||_{C^\alpha_k(M)}||\rho||_{C^{\alpha_2}(S)}\right). \tag{86}$$

Combining (78) with (83)–(86) we get

$$\ell_{S_j,\rho_j}(A\circ f^{N/2}) = O\left(N^{-m}||A||_{C^\alpha_k(M)}||\rho||_{C^{\alpha_2}(S)}\right).$$

Hence by (77)

$$\ell_{S,\rho}(A\circ F^{N/2}) = O\left(N^{-m}\sum_j c_j + \theta_1^{N/2}\right)||A||_{C^\alpha_k(M)}||\rho||_{C^{\alpha_2}(S)}).$$

Applying (77) to $A\equiv 1$ we get

$$\sum_j c_j = 1 + O(\theta_1^{N/2}||A||_{C^0(M)}||\rho||_{C^{\alpha_2}(S)}).$$

The last two formulas prove (a).

(b) By Lemma 6 with $n = N-m$

$$\ell_{S,\rho}((a\circ f^{N-m})(A\circ f^N)) =$$

$$\sum_j c_j\ell_{S_j,\rho_j}((A\circ f^m)a) + O(\theta_1^{N-m}||A||_{C^\alpha_k(M)}||a||_{C^{\alpha_2}(f^{N-m}S)}||\rho||_{C^{\alpha_2}(S)}) =$$

$$\sum_j c_j\ell_{S_j,\rho_j}((A\circ f^m)a) + O(\theta_1^{m}||A||_{C^\alpha_k(M)}||a||_{C^{\alpha_2}(f^{N-m}S)}||\rho||_{C^{\alpha_2}(S)}).$$

(the last equality here uses the assumption that $N\geq 2m$). On the other hand by part (a), given p there exists $k=k(p)$ such that for all $A\in C^\alpha_k(M)$ we have

$$\ell_{S_j,\rho_j}((A\circ f^m)a) = \ell_{S_j,a\rho_j}(A\circ f^m) = O\left(m^{-(p+1)}||A||_{C^\alpha_k(M)}||a\rho_j||_{C^{\alpha_2}(f^{N-m}S)}\right)$$

Therefore

$$\ell_{S,\rho}((a\circ f^{N-m})(A\circ f^N)) =$$

(87)
$$\sum_j c_j O\left(m^{-(p+1)}||A||_{C^\alpha_k}\big|\,|a\rho_j||_{C^{\alpha_2}(f^{N-m}S)}\right)+$$

$$O\left(\theta_1^m||A||_{C^\alpha_k(M)}||a||_{C^{\alpha_2}(f^{N-m}S)}||\rho||_{C^{\alpha_2}(S)}\right)$$

Applying Lemma 6 to $A\equiv 1$ we get

(88)
$$\sum_j c_j = 1+O(\theta_1^m||\rho||_{C^{\alpha_2}(S)})$$

Combining (87) and (88) we get

$$\left|\ell_{S,\rho}\left((A\circ f^N)(a\circ f^{N-m})\right)\right| \leq$$

$$\mathrm{Const}(m^{-(p+1)}+\theta_1^m)||A||_{C^\alpha_k(M)}||a||_{C^{\alpha_2}(f^{N-m}S)}||\rho||_{C^{\alpha_2}(S)}.$$

If m is sufficiently large the first factor is less than m^{-p} This proves (b).

To prove (c) break

$$\sum_{j=1}^{N_1}\int_S \rho(y)a(f^{N-j}y)A(f^N y)dy$$

into two parts so that, in the first, summation is over j from 1 to m, and in the second, summation is from m to N_1. By part (b) given p there exists $k_1(p)$ such that for all $k\geq k_1(p)$ for all $A\in C^\alpha_k(M)$ we have

$$\sum_{j=m}^{N_1}||A||_{C^\alpha_k(M)}||a||_{C^\alpha_k(M)}||\rho||_{C^{\alpha_2}(S)}j^{-p}\leq \mathrm{Const}||A||_{C^\alpha_k(M)}||a||_{C^\alpha_k(M)}||\rho||_{C^{\alpha_2}(S)}m^{-(p-1)}.$$

On the other hand for fixed j

$$\int \rho(y)a(f^{N-j}y)A(f^N y)dy =$$

$$\nu((a\circ f^{-j})A)+O(N^{-p}||(a\circ f^{-j})A||_{C^\alpha_k(M)}||\rho||_{C^{\alpha_2}(S)}) =$$

$$\nu((a\circ f^{-j})A)+O(N^{-p}||A||_{C^\alpha_k(M)}||a||_{C^\alpha_k(M)}||\rho||_{C^{\alpha_2}(S)}K^j).$$

for some $K>1$. Take $m=\frac{\ln N_0}{\ln K}$ then for all $j\leq m$, $K^j\leq N_0\leq N$ and so

$$\sum_{j=1}^{m}\int_S \rho(y)a(f^{N-j}y)A(f^N y)dy =$$

$$\sum_{j=1}^{m}\nu((a\circ f^{-j})A)+O\left(||A||_{C^\alpha_k(M)}||a||_{C^\alpha_k(M)}||\rho||_{C^{\alpha_2}(S)}N^{-(p-1)}\right).$$

By Proposition 8 there exist $\mathbf{r}', C_1', C_2'$ such that $\nu \in E_{inv}(\mathbf{r}', C_1', C_2', 1, 1)$. Therefore by part (b) given p there exists $k_2(p)$ such that for all $k \geq k_2(p)$ for all functions $a, A \in C_k^\alpha(M)$ we have

$$\sum_{j=m+1}^{\infty} \nu((a \circ f^{-j})A) \leq \text{Const}||A||_{C_k^\alpha(M)}||a||_{C_k^\alpha(M)}||\rho||_{C^{\alpha_2}(S)}m^{-(p-1)}.$$

Hence

$$\sum_{j=1}^{m} \nu((a \circ f^{-j})A) =$$

$$\sum_{j=1}^{\infty} \nu((a \circ f^{-j})A) + O\left(||A||_{C_k^\alpha(M)}||a||_{C_k^\alpha(M)}||\rho||_{C^{\alpha_2}(S)}m^{-(p-1)}\right)$$

and so

$$\sum_j \int_S \rho(y)a(f^{N-j}y)A(f^N y)dy =$$

$$\sum_{j=1}^{\infty} \nu((a \circ f^{-j})A) + O\left(||A||_{C_k^\alpha(M)}||a||_{C_k^\alpha(M)}||\rho||_{C^{\alpha_2}(S)}\left(N^{-(p-1)} + m^{-(p-1)}\right)\right)$$

which proves (c) . □

Appendix B. Properties of V.

The main result of this section is the following.

Lemma 7. *(a) There exists α such that for all k, $V \in VC_k^\alpha(M)$. In particular, for all k, l, $a_l \in C_k^\alpha(M)$.*

(b) There exists $\theta < 1$ such that if S satisfies (14) then for all numbers $m, n \leq j$

$$||V_n(S)||_{C^2(S)} \leq \text{Const}, \tag{89}$$

$$||V_n(S) - V_m(S)||_{C^2(S)} \leq \text{Const}\theta^{\min(m,n)}. \tag{90}$$

(c) If S satisfies (14) and if $\tilde{X}_k$ are vectorfields on $f^{-k}S$, $k = 0 \dots j$ and $n \leq j$ define

$$\tilde{V}_n(x, S) = \sum_{k=0}^{n} \Gamma_{as}^k(f^{-k}x, f^{-k}S)\pi_{as}(f^{-k}x, f^{-k}S)\tilde{X}_k.$$

Then

$$||\tilde{V}_n(S)||_{C^1(S)} \leq \text{Const} \max_{0 \leq k \leq n} ||\tilde{X}_k||_{C^1(f^{-k}S)}$$

(d) There exists $\hat{\alpha} > 0$ such that the following holds. If S_1 and S_2 are two submanifolds satisfying (14) such that there exists a map $\zeta : S_1 \to S_2$ such that $d_{C^2}(\zeta, \mathrm{id}) = \sigma$ then for all $n \le j$

$$||V_n(S_2) - d\zeta V_n(S_1)||_{C^1(S_2)} \le \text{Const}\left[\sigma^{\hat{\alpha}} + \theta^n\right].$$

Remark. *Estimates similar to (b)–(d) hold also for higher norms $C^k(S)$ but we only formulate the bounds we use.*

Proof. (a) $V(x)$ satisfies the equation

$$\left[\Gamma_{as}V + (X^{as} \circ f)\right](x) = V(fx). \tag{91}$$

Γ_{as} is a contraction of TM. So we can apply the Invariant Section Theorem ([28], Theorem 3.2) which asserts that if $\mathcal{E} \to \mathcal{X}$ is a fiber bundle over a manifold $\mathcal{X}$ and $H : \mathcal{E} \to \mathcal{E}$ is a C^r bundle map covering a C^r map $h : \mathcal{X} \to \mathcal{X}$ which contracts the fibers then there exists a unique H–invariant section and this section is C^r provided that

$$L^r\lambda < 1 \tag{92}$$

where λ is the fiber contraction rate and L is the Lipschitz constant of h^{-1}. In particular, the invariant section is always Holder continuous for some exponent α. Now apply the Invariant Section Theorem to M viewed as the disjoint union of $\mathbf{g}$-orbits. Since f acts isometrically on each orbit we get $L = 1$ in (92). Therefore, V is C^∞ restricted to any orbit of $\mathbf{g}$. Thus it remains to check that the derivatives of V along central directions are Holder continuous. Take $l \in \{1, 2, \dots d\}$. Let $U(x) = (L_{e_l}V)(x)$ where L denotes the Lie derivative. Since f commutes with $\mathbf{g}$, U satisfies

$$\left[\Gamma_{as}U + L_{e_l}\left(X^{as} \circ f\right) + (L_{e_l}\Gamma_{as})V\right](x) = U(fx).$$

which is an equation of the same type as (91). Thus U is Holder continuous with the same Holder exponent as the one guaranteed for V. Continuing by induction we obtain that all derivatives are Holder.

(b) We prove (89). (90) is similar. Note that since $TS \subset \mathcal{K}_u$ then for all $y \in TS$ $\|\Gamma_{as}(y, S)\| \le \tilde{\lambda}_1$ for some $\tilde{\lambda}_1 < 1$ close to λ_1. Hence $\|\Gamma_{as}^k(y, S)\| \le \tilde{\lambda}_1^k$ for all $k \le j$. This implies that $\|V_n(S)\|_{C^0} \le$ Const. Now

$$DV_n(S) = \sum_{k=0}^{n}\left[D\left(\Gamma_{as}(f^{-k}S) \circ f^{-k}\right)\right]\left[\left(\pi_{as}(f^{-k}S)X\right) \circ f^{-k}\right] +$$

$$\sum_{k=0}^{n}\left[\left(\Gamma_{as}^k(f^{-k}S)\right) \circ f^{-k}\right] D(\pi_{as}(f^{-k}S)X)(df^{-k}).$$

However

$$\|D\left(\Gamma_{as}(f^{-k}S)\circ f^{-k}\right)\|_{C^0}\leq \text{Const}k\tilde{\lambda}_1^k \tag{93}$$

since the LHS is a sum of k terms and each term is bounded by $\text{Const}\tilde{\lambda}_1^k$. On the other hand $\|(df^{-k})\|\leq\tilde{\lambda}_1^k$ since $T(f^{-l}S)\subset\mathcal{K}_u$ for all $l\leq j$. Therefore

$$\|DV_n(S)\|_{C^0}\leq \text{Const}.$$

The estimate $\|D^2V_n(S)\|_{C^0}\leq$ Const is obtained similarly using the bound

$$\|D^2(\Gamma^k_{as}(f^{-k}S)\circ f^{-k})\|_{C^0}\leq \text{Const}k^2\tilde{\lambda}_1^k$$

which can be derived similar to (93).

The proof of (c) is similar to (b).

(d) Using the estimate

$$D_{C^2(S_1)}(f^p\circ\zeta, f^p)\leq \text{Const}K^p\sigma$$

and the bound (90) we see that for each $m\leq n$ the following inequality holds.

$$\|V_n(S_1)-d\zeta V_n(S_2)\|\leq$$

$$\text{Const}\left[\sum_{p=1}^m K^p\sigma+||V_n(S_2)-V_m(S_2)||+||V_n(S_1)-V_m(S_1)||\right]\leq$$

$$\text{Const}(K^m+\theta^m).$$

Now if $K^n\sigma\leq\theta^n$ then we choose $m=n$ and (d) follows. Otherwise choose m so that $K^m\sigma\sim\theta^m$, that is $m=\frac{|\ln\sigma|}{\ln(K/\theta)}$. □

Appendix C. Derivatives of Z_n.

Here we verify claim (f) of Proposition 3. Along the way we prove Lemma 1.

Choose coordinate systems (ξ_j,η_j) around f^jy so that

- f^jS is given by $\eta_j=0$;
- $\eta=(\omega,t)$ and if z_1 has coordinates $(\xi,\omega,0)$ and z_2 has coordinates $(\xi,0,0)$ then $\exp^{-1}_{z_2}(z_1)\in E_{as}$;
- $\mathbf{g}_t$ is given by $\mathbf{g}_t(\xi,\omega,t')=(\xi,\omega,t+t')$.

In this coordinate system $f^j_\varepsilon S$ is given by $\eta_j=H_j(\xi_j)$. Let $H_j=(H^\omega_j,H^t_j)$. We assume that

$$\left|H^\omega_j\right|\leq L_0\varepsilon\quad\left|H^t_j\right|\leq L_0\varepsilon j,$$

where $L_0=\max(C_7,C_8)$,

$$\left\|\frac{dH_j}{d\xi_j}\right\|\leq C_9\varepsilon \tag{94}$$

$$\left\| \frac{d^2 H_j}{d\xi_j^2} \right\| \le C_{10}\varepsilon \tag{95}$$

and show that H_{j+1} also satisfies (94) and (95) provided that $1 \ll C_9 \ll C_{10}$. Let f_ε be given by

$$\xi_{j+1} = F_j(\xi_j, \eta_j, \varepsilon), \quad \eta_{j+1} = G_j(\xi_j, \eta_j, \varepsilon).$$

Then $f_\varepsilon^{j+1} S$ is given by

$$\xi_{j+1} = F_j(\xi_j, H_j(\xi_j), \varepsilon), \quad \eta_{j+1} = G_j(\xi_j, H_j(\xi_j), \varepsilon).$$

Thus

$$\frac{d\eta_{j+1}}{d\xi_{j+1}} = \left[\frac{\partial G}{\partial \xi}(\xi_j, H_j(\xi_j), \varepsilon) + \frac{\partial G}{\partial \eta}(\xi_j, H_j(\xi_j), \varepsilon)\frac{dH_j}{d\xi_j}\right] \times$$
$$\left(\left[\frac{\partial F}{\partial \xi}(\xi_j, H_j(\xi_j), \varepsilon) + \frac{\partial F}{\partial \eta}(\xi_j, H_j(\xi_j), \varepsilon)\frac{dH_j}{d\xi_j}\right]^{-1}\right).$$

Now if E_{ac} is sufficiently close to E_{cs} then we have

$$\left\|\frac{\partial G}{\partial \eta}\right\| \le (1+\delta), \quad \left\|\frac{\partial F}{\partial \eta}\right\| \le \delta, \quad \left\|\left(\frac{\partial F}{\partial \xi}\right)^{-1}\right\| \le \lambda < 1. \tag{96}$$

Proof of Lemma 1. Denote

$$\mathbf{F}(\xi) = F(\xi, 0, 0). \tag{97}$$

Introduce ρ_j, ζ_j such that

$$F(\rho_j, H(\rho_j), \varepsilon) = \xi_{j+1} \quad \mathbf{F}(\zeta_j) = \xi_{j+1}.$$

Lemma 8. *There exists a constant $K(L_0, C_9, C_{10})$ such that the map $\zeta_j \to \rho_j$ is $K\varepsilon(j+1)$ close to* id *in C^2 topology.*

Proof. By implicit function estimate it suffices to prove that the map $\rho_j \to \zeta_j$ is $\tilde{K}\varepsilon(j+1)$ close to id. We have

$$\xi_{j+1} = \mathbf{F}(\rho_j) + \frac{\partial F}{\partial \eta}(\rho_j, 0, 0)H_j + \frac{\partial F}{\partial \varepsilon}(\rho_j, 0, 0)\varepsilon + F_2(\rho_j, H_j, \epsilon) \tag{98}$$

where F_2 is quadratic in (H_j, ε). Therefore

$$||\mathbf{F}(\zeta_j) - \mathbf{F}(\rho_j)|| \le \hat{K}(L_0)\varepsilon(j+1)$$

Since $\mathbf{F}$ expands distances

$$||\zeta_j - \rho_j|| \le \hat{K}(L_0)\varepsilon(j+1) \tag{99}$$

Now from the identity

$$\zeta_j = \mathbf{F}^{-1}(F(\rho_j, H(\rho_j), \varepsilon))$$

we get

$$\frac{d\zeta_j}{d\rho_j} = D\mathbf{F}^{-1}(\xi_{j+1})\frac{\partial F}{\partial \xi} + D\mathbf{F}^{-1}(\xi_{j+1})\frac{\partial F}{\partial \eta}\frac{dH_j}{d\xi_j}. \tag{100}$$

The second term is bounded by $\text{Const}(L_0, C_9, C_{10})\varepsilon j$ in C^1-norm due to the factor $\frac{dH_j}{d\xi_j}$. On the other hand

$$\frac{\partial F}{\partial \xi}(\rho_j, H(\rho_j), \varepsilon) = \frac{\partial F}{\partial \xi}(\zeta_j, 0, 0) + \tilde{F},$$

where $\tilde{F}$ denotes the terms which are at least linear in $\zeta_j - \rho_j$, $H(\rho_j)$ or ε. By (99) all these terms can be bounded by $\text{Const}(L_0, C_9, C_{10})\varepsilon$. Since $\frac{\partial F}{\partial \xi}(\zeta_j, 0, 0) = D\mathbf{F}(\zeta_j)$ we have

$$\frac{d\zeta_j}{d\rho_j} = 1 + D\mathbf{F}^{-1}(\xi_{j+1})\tilde{F} + O(\varepsilon(j+1)).$$

This gives requred bound in C^1-norm. In particular,

$$\left\| \frac{d}{d\zeta_j}(\rho_j - \zeta_j) \right\| \leq \text{Const}\varepsilon(j+1).$$

Togather with (95) this implies

$$\left| \frac{d}{d\zeta_j}\tilde{F} \right| \leq \text{Const}\varepsilon(j+1)$$

and the lemma follows. □

Since $\mathbf{F}(\rho_j) = f^{j+1}\phi_{j+1}f^{-(j+1)}\mathbf{F}(\zeta_j)$ (17) follows.

Now $H_{j+1}(\xi_{j+1}) = G_j(\rho_j, H(\rho_j), \varepsilon)$. Since $G(\rho_j, 0, 0) = 0$ we get

$$H_{j+1}(\xi_{j+1}) = \frac{\partial G}{\partial \eta}H(\rho_j) + \frac{\partial G}{\partial \varepsilon}\varepsilon + G_2$$

where G_2 is quadratic in $(H(\rho_j), \varepsilon)$. By (94), (95) the first two derivatives of G_2 with respect to ρ can be estimated by $\text{Const}\varepsilon^2$. So by Lemma 8

(101)
$$\left\| H_{j+1}(\xi_{j+1}) - \left(\frac{\partial G}{\partial \eta}H(\rho_j) + \frac{\partial G}{\partial \varepsilon}\varepsilon\right)\right\|_{C^2} \leq \text{Const}(L_0, C_9, C_{10})(j+1)^2\varepsilon^2$$

Recall that

$$H_{j+1}(\xi_{j+1}) = \exp_{(\xi_{j+1},0)} Z_{j+1}, \quad (\zeta_j, H(\zeta_j)) = \exp_{(\zeta_j,0)} Z_j.$$

Comparing (101) with (33) we get (35).

Now observe that by (97), (98) and Lemma 8
(102)

$$\zeta_j - \rho_j = D\mathbf{F}^{-1}(\zeta_j)\left[\frac{\partial F}{\partial \eta}H_j(\zeta_j,0,0) + \frac{\partial F}{\partial \varepsilon}(\zeta_j,0,0)\varepsilon\right] + O((j+1)^2\varepsilon^2).$$

Combining (100) and Lemma 8 we get

$$\frac{d\zeta_j}{d\rho_j} =$$

$$D\mathbf{F}^{-1}(\xi_{j+1})\left[D\mathbf{F}(\rho_j) + +\frac{\partial^2 F}{\partial\xi\partial\eta}H + \frac{\partial^2 F}{\partial\xi\partial\varepsilon}\varepsilon + \frac{\partial F}{\partial\eta}(\rho_j,0,0)\frac{dH}{d\xi}(\zeta_j)\right] + O((j+1)^2\varepsilon^2).$$

By (102)

$$D\mathbf{F}(\rho_j) =$$

$$D\mathbf{F}(\zeta_j) + \left[\frac{d}{d\xi}D\mathbf{F}(\zeta_j)\right]D\mathbf{F}^{-1}(\zeta_j)\left[\frac{\partial F}{\partial\eta}H_j(\zeta_j,0,0) + \frac{\partial F}{\partial\varepsilon}(\zeta_j,0,0)\varepsilon\right] + O((j+1)^2\varepsilon^2).$$

Combining the last two equations we get

$$\frac{d\zeta_j}{d\rho_j} = \frac{d}{d\xi}\left\{D\mathbf{F}^{-1}\left[\frac{\partial F}{\partial\eta}H + \frac{\partial F}{\partial\varepsilon}\varepsilon\right]\right\} + O((j+1)^2\varepsilon^2).$$

Now let $\tilde\Omega$ be a volume form on $f^j\bar S$, $d\tilde\Omega = \lambda(\xi)d\xi$. To compute jacobian of $\frac{d\xi_j}{d\rho_j}$ with respect to $\tilde\Omega$ we need to take into account the difference between $\lambda(\zeta_j)$ and $\lambda(\rho_j)$. This gives

$$j(\zeta_j(\rho_j),\tilde\Omega) = 1 + \mathrm{div}_{\tilde\Omega}\left\{D\mathbf{F}^{-1}\left(\frac{\partial F}{\partial\eta}H + \frac{\partial F}{\partial\varepsilon}\varepsilon\right)\right\} + O\left((j+1)^2\varepsilon^2\left[||\tilde\Omega||_{C^2} + 1\right]\right).$$

Now

$$D\mathbf{F}^{-1}\left[\frac{\partial F}{\partial\eta}H + \frac{\partial F}{\partial\varepsilon}\varepsilon\right]$$

differs by quadratic terms from

$$df^{-1}(\pi_{u,j}(tZ_j) + \varepsilon X)((\xi_{j+1},0))$$

(this difference is due to the fact that $\widehat{\exp}_y v = v+$ qadratic terms). Since the derivatives of quadratic in $H_j(\xi_j)$ terms remain quadratic by (94), (95) we get

$$j(\zeta_j(\rho_j),\tilde\Omega) = 1 + \mathrm{div}_{\tilde\Omega}\left\{df^{-1}(\pi_{u,j}(tZ_j) + \varepsilon X)\right\} + O((j+1)^2\varepsilon^2).$$

Hence

$$j(\rho_j(\zeta_j),\tilde\Omega) = 1 - \mathrm{div}_{\tilde\Omega}\left\{df^{-1}(\pi_{u,j}(tZ_j) + \varepsilon X)\right\} + O((j+1)^2\varepsilon^2).$$

Since $\rho_j = f^j\psi_{j+1}f^{-j}\zeta_j$ we get

$$j(f^j\psi_{j+1}f^{-j},\tilde\Omega) = 1 - \mathrm{div}_{\tilde\Omega}\left\{df^{-1}(\pi_{u,j}(tZ_j) + \varepsilon X)\right\} + O((j+1)^2\varepsilon^2).$$

By functorality

$$j(f^{j+1}\psi_{j+1}f^{-(j+1)}, f\tilde{\Omega}) = 1-\mathrm{div}_{f\tilde{\Omega}}\left\{df^{-1}(\pi_{u,j}(tZ_j)+\varepsilon X)\right\}+O((j+1)^2\varepsilon^2).$$

Denoting $\Omega = f\tilde{\Omega}$ we obtain (36). □

We now want to estimate $\frac{\partial G}{\partial \xi}$. Note that $G(\xi,\eta,0)$ corresponds to f. Using that f moves f^jS to $f^{j+1}S$ and that f commutes with $\mathbf{g}_t$ we get $G(\xi,0,t,0) = t + G(\xi,0,0,0) = t$. Thus $\frac{dG}{d\xi}(\xi,0,t,0) = 0$. Therefore there is a constant K_1 such that

$$\left\|\frac{\partial G}{\partial \xi}\right\| \leq K_1(\varepsilon + |\omega|).$$

Combining this with the estimates for other partial derivatives ((96)) we obtain

$$\left\|\frac{dH_{j+1}}{d\xi_{j+1}}\right\| \leq \lambda\frac{[K_1(1+L_0)+\delta C_9]\varepsilon}{1-(\delta C_9\varepsilon/\lambda)}.$$

Take some constant $\tilde{\lambda}$ such that $\lambda < \tilde{\lambda} < 1$. If ε is sufficiently small then

$$\frac{\lambda}{1-\delta C_9\varepsilon/\lambda} \leq \tilde{\lambda} \tag{103}$$

Hence if

$$C_9 \geq \frac{K_1(1+L_0)}{1-\delta\tilde{\lambda}} \tag{104}$$

then

$$\left\|\frac{dH_{j+1}}{d\xi_{j+1}}\right\| \leq C_9.$$

We now estimate the second derivative. If Q is a quadratic form and R is a linear map let $Q*R$ denote the form $(Q*R)(v,v) = Q(Rv,Rv)$. Then

$$\frac{d^2H_{j+1}}{d\xi_{j+1}^2} = \frac{dG}{d\eta}\left(\frac{d^2H_j}{d\xi_j^2} * \left[\frac{\partial F_j}{\partial \xi_j} + \frac{\partial F_j}{\partial \eta_j}\frac{dH_j}{d\xi_j}\right]^{-1}\right) +$$

$$\left[\frac{\partial G_j}{\partial \xi_j} + \frac{\partial G_j}{\partial \eta_j}\frac{dH_j}{d\xi_j}\right]\left[\frac{\partial F_j}{\partial \xi_j} + \frac{\partial F_j}{\partial \eta_j}\frac{dH_j}{d\xi_j}\right]^{-1}\frac{\partial F_j}{\partial \eta_j}\left(\frac{d^2H_j}{d\xi_j^2} * \left[\frac{\partial F_j}{\partial \xi_j} + \frac{\partial F_j}{\partial \eta_j}\frac{dH_j}{d\xi_j}\right]^{-1}\right)+R_j$$

where R_j denote the sum of terms which do not contain $\frac{dH_j}{d\xi_j}$. Using the first derivative estimates we obtain $||R_j|| \leq K_2\varepsilon$, where $K_2 = K_2(L_0, C_9)$. Hence

$$\left\|\frac{dH_{j+1}}{d\xi_{j+1}}\right\| \leq \left[K_2 + C_{10}(\delta\tilde{\lambda}^2 + \varepsilon C_9)\right]\varepsilon.$$

If ε is sufficiently small then

$$\delta\tilde{\lambda}^2 > \varepsilon C_9 \tag{105}$$

and so

$$\left\|\frac{dH_{j+1}}{d\xi_{j+1}}\right\| \leq \left[K_2 + 2\delta\tilde{\lambda}^2 C_{10}\right]\varepsilon.$$

Hence if

$$C_{10} > \frac{K_2}{1-2\delta\tilde{\lambda}^2} \tag{106}$$

then $\left\|\frac{dH_{j+1}}{d\xi_{j+1}}\right\| \leq C_{10}$.

Appendix D. *Shape of the image of ϕ_j.*

Here we prove part (b) of Proposition 3. In view of part (c) of that Proposition we may assume that $j < k$. We have

$$f^j\phi_j^{-1}\phi_k f^{-j} = f^j \circ \psi_{j+1} \circ \psi_{j+2} \circ \cdots \circ \psi_k \circ f^{-j} =$$

$$(f^j\psi_{j+1}f^{-j}) \circ (f^j\psi_{j+2}f^{-j}) \circ \cdots \circ (f^j\psi_k f^{-j})$$

so it is enough to estimate the C^{α_3}-distance between $f^j\psi_l f^{-j}$ and id. Now $\forall y \in f^j\bar{S}$

$$d(f^j\psi_l f^{-j}y, y) \leq \lambda_1^{l-j} d(f^l\psi_l f^{-l}(f^{l-j}y, f^l - jy) \leq \text{Const}\lambda_1^{l-j}\varepsilon$$

where the last estimate follows from Proposition 3(c). In particular for any pair y', y'' such that $d(y', y'') \geq \lambda_1^{(l-j)/(2\alpha_3)}$

$$d(f^j\psi_l f^{-j}y', f^j\psi_l f^{-j}y'') \leq \text{Const}\lambda_1^{(l-j)/2} d^{\alpha_3}(y', y'').$$

On the other hand if $d(y', y'') < \lambda_1^{(l-j)/(2\alpha_3)}$ then

$$d(f^j\psi_l f^{-j}y', f^j\psi_l f^{-j}y'') \leq \lambda_1^{l-j} d((f^l\psi_l f^{-l})f^{l-j}y', (f^l\psi_l f^{-l})f^{l-j}y'') \leq$$

$$\text{Const}\lambda_1^{l-j} d(f^{l-j}y', f^{l-j}y'') \leq \text{Const}K^{l-j} d(y', y'')$$

for some $K > 0$. Since $d(y', y'') < \lambda_1^{(l-j)/(2\alpha_3)}$ we have the estimate

$$K^{l-j}d(y', y'') = \left(K^{l-j}d^{1-\alpha_3}(y', y'')\right) d^{\alpha_3}(y', y'') \leq (K\lambda^{((1/\alpha_3)-1)/2})^{l-j} d^{\alpha_3}(y', y'').$$

Now choose α_3 so that

$$\lambda^{\frac{(1/\alpha_3)-1}{2}} < \frac{1}{K}.$$

Then $d_{C^{\alpha_3}}(f^j\psi_l f^{-j}, \text{id})$ tends to 0 exponentially fast in $l - j$ which implies the statement of the lemma. $\square$

Appendix E. Non-vanishing of c.

E.1. **An explicit formula.** Here we prove that $c \not\equiv 0$. Our computations are similar to [54, 24]. We begin by simplifying the expression for the shift of the central fiber (formula (66)) in case f is the time one map of the geodesic flow. In this case E_u and E_s are $C^{2-\delta}$ so we can take $\hat{E} = E_u \oplus E_s$. Then $\mathbb{L}(E_c, \hat{E}) = \mathbb{L}(E_c, E_u) \oplus \mathbb{L}(E_c, E_s)$ is the hyperbolic splitting for Q_0. Write Q_* for the restriction of Q_0 to $\mathbb{L}(E_c, E_*)$. Since $\dim E_c = 1$ we identify L_* with E_* by identifying $l : E_c \to E_*$ with $l(e_c)$. Also in our case

$$a_{cc}(x, 0) = 1, \quad a_{cn}(x, 0) = 0, \quad a_{nc}(x, 0) = 0.$$

Combining these formulas we get

$$[(1 - Q_s)^{-1} v](x) = \sum_{j=0}^{\infty} (df^j | E_s) v(f^{-j} x),$$

$$[(1 - Q_u)^{-1} v](x) = -\sum_{j=1}^{\infty} (df^{-j} | E_u) v(f^j x).$$

Finally $\mathbf{b} = b^{(1)}_{cu} e_u + b^{(1)}_{cs} e_s$. Combining this with (64)–(66) we get $e_c(x, \varepsilon) = e_c + \varepsilon v(x, \varepsilon)$ where

(107)
$$v(x, \varepsilon) \sim \varepsilon \left[\sum_{j=0}^{\infty} (df^j | E_s) b^{(1)}_{cs} (f^{-j} x) e_s - \sum_{j=1}^{\infty} (df^{-j} | E_u) b^{(1)}_{cu} (f^j x) e_u \right].$$

E.2. **A perturbation.** In order to show that $c(X)$ is not identically 0 we exhibit a one-parameter family X_δ of vector fields such that $\limsup_{\delta \to 0} \frac{c(X_\delta)}{\delta^3} < 0$. Our vector field will be supported in the union of two sets of size δ : the $\delta-$ ball centered at some non-periodic point $x_0 \in M$ and the image $f(B(x_0, \delta))$. We choose coordinate system (z_1, z_2, z_3) near x_0 so that

(1) $d\nu = dz_1 dz_2 dz_3$.
(2) $E_s(x_0) = \frac{\partial}{\partial z_1}$, $E_c(x_0) = \frac{\partial}{\partial z_2}$, $E_u(x_0) = \frac{\partial}{\partial z_3}$.

Let X_δ on $B(x_0, \delta)$ be given by

$$X_\delta(z) = \left(z_2 \xi\left(\frac{z_1^2 + z_2^2 + z_3^2}{\delta}\right), -z_1 \xi\left(\frac{z_1^2 + z_2^2 + z_3^2}{\delta}\right), 0 \right)$$

where $\xi : \mathbb{R} \to \mathbb{R}$ is a function of compact support. Define X_δ on $f B(x_0, \delta)$ by $X_\delta(fx) = df(X_\delta)$ and let $X_\delta = 0$ elsewhere. Then

$||X_\delta||_{C^1} \leq$ Const and the contribution to c in (69) comes only from the points in supp(X_δ). Thus

$$|c(X_\delta)| \leq \text{Const}\delta^3.$$

Also the contribution of $\alpha b_{uc}^{(1)}$ tends to zero since $b_{uc}^{(1)} \to 0$ because near x_0, E_u is close to $\frac{\partial}{\partial z_3}$ but z_3-component of X_δ is 0. Using (107) and the fact that, for any fixed $j > 0$, $b_{cs}^{(1)}(f^{-j}x) = 0$ for $x \in B(x_0, \delta)$ if δ is small enough we get

$$\beta(x) \sim b_{cs}^{(1)}(x) \sim \partial_{z_1}(-z_1\xi) = -(\xi + 2z_1^2\xi').$$

Therefore

$$\beta(x)b_{cs}^{(1)}(fx) \sim -\partial_{z_1}(z_1\xi)\partial_{z_2}(z_2\xi) = \\ -\left[\xi^2 + 2(z_1^2+z_2^2)\xi'\xi + 4z_1^2z_2^2(\xi')^2\right].$$

Since the contribution of $-(b_{cc}^{(1)})^2$ is non-positive we obtain from (69)

$$\limsup_{\delta\to 0} \frac{c(X_\delta)}{\delta^3} \leq \lim_{\delta\to 0} \frac{\nu\left(\frac{b_{cs}^{(1)}(fx)\beta(x)}{\lambda(x)}\right)}{\delta^3} = \\ -\frac{1}{\lambda(x_0)} \iiint \left[\xi^2 + 2(z_1^2+z_2^2)\xi'\xi + 4z_1^2z_2^2(\xi')^2\right] dz_1dz_2dz_3$$

where in the last equality we have used that $\lambda(x) \sim \lambda(x_0)$ on $B(x_0, \delta)$.

Integrating the second term by parts in the radial direction we get

$$-\iiint 2(z_1^2+z_2^2)\xi\xi' dz_1dz_2dz_3 = \iiint \frac{z_1^2+z_2^2}{z_1^2+z_2^2+z_3^2}\xi^2 dz_1dz_2dz_3$$

and thus

$$\limsup_{\delta\to 0} \frac{c(X_\delta)}{\delta^3} \leq -\frac{1}{\lambda(x_0)} \iiint \left[\frac{z_3^2}{z_1^2+z_2^2+z_3^2}\xi^2 + 4z_1^2z_2^2(\xi')^2\right] dz_1dz_2dz_3 < 0. \quad \square$$

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