

On diffusion driven oscillations in coupled dynamical systems

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Abstract

The paper deals with the problem of destabilization of diffusively coupled identical systems. It is shown that globally asymptotically stable systems being diffusively coupled, may exhibit oscillatory behavior. It is shown that if the diffusive medium consists of hyperbolically nonminimum phase systems and the diffusive factors exceed some threshold value, the origin of the overall system undergoes a Poincaré-Andronov-Hopf bifurcation resulting in oscillatory behavior.

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1 Introduction

Oscillations play an important role in nature. Every life form exhibits an oscillatory behavior at every level of biological organization with periods ranging from milliseconds (neurons) to seconds (cardiac cells), minutes (oscillatory enzymes), hours, days, weeks and even years (epidemiological processes and predator-prey interactions in ecology). Of course biological systems are very complex and difficult to model, nevertheless many attempts were made to explain the mechanism of how persistent oscillations occur in complex interconnected or distributed biological systems. As a survey of some theoretical and experimental works we refer to [Mosekilde, 1997] (see Chapter 9).

A recognizable starting point in related research activities is the paper by A. Turing [1952] who studied a possible mechanism for morphogenesis – the process of creation of forms and patterns in development of the embryo. A crucial role in the study is played by the diffusive medium which can be described in two ways – continuous, involving partial differential equations, or discrete, involving diffusion of enzymes (morphogenes in the terminology of Turing) between separate cells described by ordinary differential equations.

The diffusion which is very important for cooperative behavior of living cells, was usually considered as a smoothening or trivializing process. However it turns out that it can result in nontrivial oscillatory behavior in different systems. Some phenomena occurring in distributed systems written as PDEs are described in the paper by We-Ming Ni [1998] and references therein. In this paper we are concerned with oscillatory phenomena occurring in systems consisting of diffusively coupled subsystems described by ordinary differential equations. A motivation of our study is the paper by S. Smale [1976] who proposed an example of two 4th order diffusively coupled differential systems. Each system describes a mathematical cell and by itself is inert or dead in the sense that it is globally asymptotically stable. In interaction, however “the cellular system pulses (or expressed perhaps overdramatically, becomes alive!) in the sense that the concentrations of the enzymes in each cell will oscillate infinitely”. In his paper S. Smale posed a problem to find conditions under which globally asymptotically stable systems being diffusively coupled will oscillate. A solution to this problem based on a frequency domain formalism was proposed in [Tomberg & Yakubovich, 1989] where diffusive interaction of two Lur’e systems with scalar nonlinearity was studied. Our approach is based on the related concepts of passivity and minimum phaseness and the main contribution of the paper is that we show that the Turing instability occurs as a result of a Poincaré-Andronov-Hopf bifurcation. Conditions presented in the paper ensure the existence of oscillatory behavior for diffusively coupled systems with arbitrary topology of the interconnections. We show that this scenario occurs in diffusively coupled nonminimum phase systems. At the same time in [Pogromsky, 1998] it has been shown that diffusive coupling of minimum phase systems results in another interesting oscillatory phenomenon, namely synchronization. Therefore the results of this paper and [Pogromsky, 1998] allow a better understanding of oscillatory behavior occurring in

diffusively coupled systems.

The paper is organized as follows. First we give some relevant definitions and recall some useful results. The problem statement is presented in Section 3. Section 4 contains some results regarding boundedness and ultimate boundedness of solutions of interconnected systems. In Section 5 we will present conditions resulting in the Turing instability and an illustrative example will be considered in Section 6.

2 Basic definitions

The Euclidean norm in \mathbb{R}^n is denoted simply as $|\cdot|$, $|x|^2 = x^\top x$, where $^\top$ stands for the transpose operation.

A function $V : X \rightarrow \mathbb{R}^+$ defined on a subset X of \mathbb{R}^n , with $0 \in X$, is *positive definite* if $V(x) > 0$ for all $x \in X \setminus \{0\}$ and $V(0) = 0$. It is *radially unbounded* if $V(x) \rightarrow \infty$ as $|x| \rightarrow \infty$. If the quadratic form $x^\top P x$ with P is a symmetric matrix: $P = P^\top$ is positive definite then the matrix P is called positive definite. For positive definite matrices we use the notation $P > 0$; moreover $P > Q$ means that the matrix $P - Q$ is positive definite.

A matrix A which has all eigenvalues with negative real parts is called Hurwitz.

For matrices A and B the notation $A \otimes B$ (the Kronecker product) stands for the matrix composed of submatrices $A_{ij}B$, i.e.

$$A \otimes B = \begin{pmatrix} A_{11}B & A_{12}B & \cdots & A_{1n}B \\ A_{21}B & A_{22}B & \cdots & A_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1}B & A_{n2}B & \cdots & A_{nn}B \end{pmatrix} \quad (1)$$

where A_{ij} , $i, j = 1 \dots n$, stands for the ij -th entry of the $n \times n$ matrix A .

Let x_e be an equilibrium of the system of differential equations $\dot{x} = f(x)$, $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$. If the Jacobian $\partial f(x_e)/\partial x$ has no eigenvalues on the imaginary axis then the point x_e is called a *hyperbolic fixed point*.

In this paper we are interested in the study of oscillatory behavior of dynamical systems described by systems of autonomous ordinary differential equations. To define the concept of oscillatory system we first give a definition of an oscillatory function.

Definition 1 A scalar function $x : \mathbb{R}^1 \rightarrow \mathbb{R}^1$ is called *oscillatory in the sense of Yakubovich (or Y-oscillation)* for $t \rightarrow +\infty$ if $x(t)$ is bounded on \mathbb{R}^+ and

$$\overline{\lim}_{t \rightarrow +\infty} x(t) \geq \beta, \quad \underline{\lim}_{t \rightarrow +\infty} x(t) \leq \alpha$$

for some $\beta > \alpha$.

Similarly, definitions of Y-oscillation for $t \rightarrow -\infty$ and of (two-sided) Y-oscillation can be given.

Consider a general autonomous continuous-time systems:

$$\dot{x} = F(x), \quad (2)$$

where $x(t) \in \mathbb{R}^n$ is the n -dimensional state vector and the vector field F satisfies conditions guaranteeing existence and uniqueness of the solutions of (2) for all $t \geq 0$ and all initial conditions $x(0) = x_0 \in \mathbb{R}^n$.

Definition 2 *The system (2) is called Lagrange stable if each solution of (2) is bounded.*

Next we define a useful definition expressing ultimate boundedness of the solutions of (2). A system possessing such a property was called *dissipative* by N. Levinson [Levinson, 1944; Coddington & Levinson 1955]. Since other versions of the dissipativity concept are also in use both in nonlinear control theory and in other fields of science, we refer to the property in question as the dissipativity in the sense of Levinson, Levinson dissipativity or \mathcal{L} -dissipativity.

Definition 3 *The system (2) is called \mathcal{L} -dissipative if there exists an $R > 0$ such that*

$$\overline{\lim}_{t \rightarrow \infty} |x(t)| \leq R$$

for all initial conditions $x(0) = x_0 \in \mathbb{R}^n$.

In other words there exists a ball of radius R such that for any solution $x(t)$ there exists a time instant $t_1 = T(x_0) \geq 0$, $x_0 = x(0)$ such that for all $t \geq t_1$ we have $|x(t)| \leq R$. Obviously the ball can be replaced by any compact set in \mathbb{R}^n .

As one can notice all solutions of the \mathcal{L} -dissipative system are bounded and tend to some set which does not depend on the initial conditions $x(0)$. Any \mathcal{L} -dissipative system is Lagrange stable while the converse statement is not true.

Consider the following nonlinear system

$$\dot{x} = F(x), \quad x(t) \in \mathbb{R}^n, \quad (3)$$

$$y = h(x), \quad y(t) \in \mathbb{R}^l. \quad (4)$$

where as before F satisfies assumptions guaranteeing existence of a unique solution on the infinite time interval, and y in Eq. (4) represents the output of the dynamical system (3).

Definition 4 *The system (3), (4) is called Y-oscillatory with respect to a scalar output y if it is Lagrange stable and for almost all initial conditions*

$$\lim_{t \rightarrow +\infty} y(t) < \overline{\lim}_{t \rightarrow +\infty} y(t).$$

This definition is motivated by the definition of self-oscillatory systems given in [Yakubovich, 1973]. Compared to the definition given in [Yakubovich, 1973], instead of \mathcal{L} -dissipativity we require a milder condition of Lagrange stability. Thus, according to our definition, a simple frictionless pendulum defined on the cylindric manifold, is an Y-oscillatory system. Nevertheless the same pendulum defined on \mathbb{R}^2 is not Y-oscillatory because on \mathbb{R}^2 it is not Lagrange stable. However in what follows we are interested in oscillatory behavior of systems with ultimately bounded solutions and therefore the difference in the definitions is not significant for our study.

Recalling Definition 1 of Y-oscillation given for an arbitrary scalar function (not necessary being a solution of some differential equation) we call the system with vector output y Y-oscillatory if it is Y-oscillatory with respect to at least one of the components of the vector y . If the system is Y-oscillatory with respect to the output $y \equiv x$ then it is referred to as just Y-oscillatory.

The following result can be proved similarly to the proof of Theorem 1.1, 3°, 5° [Tomberg & Yakubovich, 1989].

Theorem 1 *Assume that*

A1. *The equation $F(x) = 0$ has only isolated solutions $\bar{x}_j, j = 1, 2, \dots$*

A2. *The system (2) is \mathcal{L} -dissipative.*

A3. *\bar{x}_j are hyperbolic fixed points and each matrix $\frac{\partial F}{\partial x}(\bar{x}_j)$ has at least one eigenvalue with positive real part.*

Then the system (2) is Y-oscillatory.

The proof of this statement is based on the fact that the hyperbolicity implies the existence of a homeomorphism defined in a neighborhood of each equilibrium between solutions of the nonlinear system and its linearization (Hartman-Grobman theorem) [Hartman, 1964]. Therefore the set of initial conditions for which a solution tends to a fixed point is a local manifold. Then a compactness argument completes the proof.

In fact, in [Tomberg & Yakubovich, 1989] a more detailed result was proved. Namely, they established conditions ensuring oscillatory behavior with respect to a given output of the system written in Lur'e form. It is worth mentioning that the requirement of hyperbolicity can be relaxed. A. Fradkov [1998] pointed out that the statement of the theorem remains true if instead of hyperbolicity one imposes a weaker assumption that the system linearized about each equilibrium has at least one eigenvalue with positive real part. The proof of this statement is based on a compactness argument. However in this paper we are interested in the study of oscillatory behavior of structurally stable systems [Arnold, 1983] and therefore the requirement of hyperbolicity is natural in this case although not necessary. Moreover if one tries to relax the conditions of the theorem and imposes the assumption that each equilibrium point is unstable in the sense of Lyapunov, then, without additional conditions, the statement of theorem can not be true because the set of all initial conditions for which the corresponding solution tends to an unstable equilibrium can be of nonzero measure.

3 Problem Statement

Let \mathbb{P}^n be a closed nonnegative orthant in \mathbb{R}^n : $\mathbb{P}^n = \{x \in \mathbb{R}^n, x = \text{col}(x^{(1)}, \dots, x^{(n)}), x^{(j)} \geq 0, j = 1, \dots, n\}$. Let \mathcal{R} be a smooth mapping $\mathcal{R} : \mathbb{P}^n \rightarrow \mathbb{P}^n$, such that the system

$$\dot{x} = \mathcal{R}(x), \quad x(t) \in \mathbb{P}^n \quad (5)$$

is globally (in \mathbb{P}^n) asymptotically stable, that is there is a point $x_e \in \mathbb{P}^n$, such that $\mathcal{R}(x_e) = 0$, x_e is Lyapunov stable and all solutions starting from \mathbb{P}^n tend to x_e . The problem studied by S. Smale [Smale, 1976] is to find (if possible) a positive definite diagonal $n \times n$ matrix D such that the system on $\mathbb{P}^{2n} \subset \mathbb{R}^{2n}$

$$\begin{cases} \dot{x}_1 = \mathcal{R}(x_1) + D(x_2 - x_1) \\ \dot{x}_2 = \mathcal{R}(x_2) + D(x_1 - x_2) \end{cases} \quad (6)$$

has a nontrivial periodic solution with a trajectory attracting almost all other solutions of (6). The state vector for each system consists of the concentrations of the chemicals which are nonnegative. The dynamics (5) describes a chemical reaction between some chemicals (“morphogenes”) inside one cell. The system (6) describes a possible diffusive interaction between two cells and is referred to as the Turing equation. The paper by Smale posed a “sharp problem, namely to ‘axiomatize’ the properties necessary to bring about oscillation via diffusion. In the 2-cell case, just what properties does the pair (\mathcal{R}, D) need to possess to make the Turing interacting system oscillate? In the many cell case, how does the topology contribute?”

It was shown by Smale that to design a system (6) it is sufficient to design a system with the same properties but assuming that $\mathcal{R} : \mathbb{R}^n \rightarrow \mathbb{R}^n$. In this case we can assume that the unique equilibrium point of (5) is the origin.

First we give a definition of diffusive coupling of k identical systems. We will understand a diffusive medium as a system of ordinary differential equations consisting of interconnected identical systems. Each separate system has an input and output of the same dimension. The diffusive coupling is described by a static relation between all inputs and all outputs.

Definition 5 *Given the systems*

$$\begin{cases} \dot{x}_j = f(x_j) + Bu_j \\ y_j = Cx_j \end{cases} \quad (7)$$

where $j = 1, \dots, k$, $x_j(t) \in \mathbb{R}^n$ is the state of the j -th system, $u_j(t) \in \mathbb{R}^m$ is the input, $y_j(t) \in \mathbb{R}^m$ is the output of the j -th system, $f(0) = 0$ and B, C are constant matrices of corresponding dimension. We say that the systems (7) are diffusively coupled if the matrix CB is similar to a positive definite matrix and the systems (7) are interconnected by the following feedback

$$u_j = -\gamma_{j1}(y_j - y_1) - \gamma_{j2}(y_j - y_2) - \dots - \gamma_{jk}(y_j - y_k) \quad (8)$$

where $\gamma_{ij} = \gamma_{ji} \geq 0$ are constants such that $\sum_{j=1}^k \gamma_{ji} > 0$ for all $i = 1, \dots, k$.

Let us clarify this definition. To this end we rewrite (7) in a new coordinate system which is convenient for subsequent study. Notice that according to the definition the matrix B (as well as C) is of full rank: $\text{rank} B = \text{rank} C = m$, $m \leq n$. Therefore there exists a full rank $(n - m) \times n$ left annihilator N of B : $NB = 0$ and the null space of N is spanned by the columns of B . Consider the $n \times n$ matrix Φ : $\Phi = (N^\top, C^\top)^\top$. Let us prove that Φ is nonsingular. Suppose for some $x \in \mathbb{R}^n$ we have $\Phi x = 0$. Therefore $Nx = 0$ and hence x is a linear combination of the columns of the matrix B : $x = B\tilde{x}$ for some $\tilde{x} \in \mathbb{R}^m$. $\Phi x = 0$ implies also that $Cx = 0$, or in other words $CB\tilde{x} = 0$. However CB is nonsingular and therefore $\tilde{x} = 0$ and that implies in turn $x = 0$. So, the matrix Φ is nonsingular. Now consider the following linear coordinate transformation

$$\begin{pmatrix} z_j \\ y_j \end{pmatrix} = \begin{pmatrix} N \\ C \end{pmatrix} x_j, \quad z_j \in \mathbb{R}^{n-m}.$$

In the new coordinates the system (7) can be rewritten as

$$\begin{cases} \dot{z}_j = Nf(x_j) + NBu_j \\ \dot{y}_j = Cf(x_j) + CBu_j, \end{cases} \quad x_j = \Phi^{-1} \text{col}(z_j, y_j) \quad (9)$$

or in the following form

$$\begin{cases} \dot{z}_j = q(z_j, y_j) \\ \dot{y}_j = a(z_j, y_j) + CBu_j \end{cases} \quad (10)$$

where $z_j(t) \in \mathbb{R}^{n-m}$, $y_j(t) \in \mathbb{R}^m$, $q : \mathbb{R}^{n-m} \times \mathbb{R}^m \rightarrow \mathbb{R}^{n-m}$, $a : \mathbb{R}^{n-m} \times \mathbb{R}^m \rightarrow \mathbb{R}^m$. Now it is clear that under some linear transformation the system (10) closed by the feedback (8) can be written such that CB is a positive definite matrix (even more without loss of generality we may assume that CB is a diagonal matrix with positive entries). In control literature the representation (10) for the system (7) is usually referred to as the *normal form*. Moreover there are conditions under which the system has a normal form (locally or globally defined) in case when the input and output mappings are defined by not constant but state-dependent matrices (see [Byrnes & Isidori, 1991, Byrnes *et. al.*, 1991]).

When $\text{rank} B = n$ and $k = 2$ the closed loop system (10) coincides with the system (6). The problem under consideration is to find conditions on f, B, C, γ_{ji} such that before coupling all k systems are globally asymptotically stable, but being coupled the closed loop system becomes Y-oscillatory. As it is now easy to notice, the problem we consider in this paper, differs from the problem posed by Smale, namely we allow the matrix D to be similar to a positive semidefinite matrix. The result is that the systems (10) have some inherent dynamics given by the equations

$$\dot{z}_j = q(z_j, 0)$$

which in the control literature is usually referred to as *zero dynamics*. As we will see later this dynamics plays an important role in the stability analysis of the closed loop system. Moreover one of the results of this paper (see Theorem 3) is that if we solve the problem for positive semidefinite matrix D then we can solve the problem

for positive definite D and in this sense we can say that our problem is equivalent to Smale's problem.

The zero dynamics system can be interpreted as following. It describes the dynamics of some chemicals with diffusive rates neglectable compared to the diffusive rates of other chemicals. On the other hand, this dynamics can not be neglected when modelling a chemical reaction inside one cell alone.

Geometrically a closed loop system consisting of diffusively coupled identical sub-systems can be illustrated by figure 1 where an example for the case of five cells is depicted. In the figure we showed the case when only the nearest neighbors are connected but this is not a limitation for our approach.

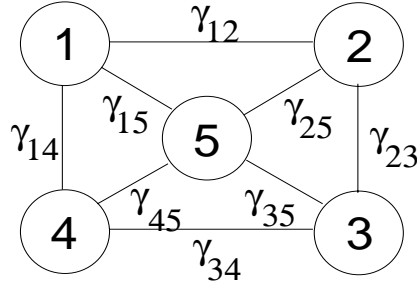


Figure 1: Possible diffusive coupling of 5 cells

4 Boundedness of solutions of interconnected systems

Consider the nonlinear time-invariant affine in the control system:

$$\begin{cases} \dot{x} = f(x) + g(x)u \\ y = h(x) \end{cases} \quad (11)$$

where $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^m$ is the input which is assumed to be a continuous and bounded function of time: $u \in \mathcal{C}^0 \cap \mathcal{L}_\infty$; $y(t) \in \mathbb{R}^m$ is the output; $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ and the columns of the matrix $g : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times m}$ are smooth vector fields, $f(0) = 0$; and $h : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is a smooth mapping, $h(0) = 0$.

Definition 6 *The internal dynamics of the system (11) consistent with the external constraint $y(t) \equiv 0$ for all $0 \leq t < T_{u,x_0}$, where T_{u,x_0} is the upper time limit for which the solution with initial condition $x(0) = x_0$ and corresponding input u exists, is called the zero dynamics of the system (11).*

Let us clarify this definition. Assume that we are looking for all initial conditions $x(0)$ for which there exists an input function $u(t)$ producing a solution $x(t)$ satisfying

the constraint $y(t) = h(x(t)) = 0$ for all time instants for which the corresponding solution exists. The dynamical system with solutions coinciding with the solutions of (11) closed by the input $u(t)$ resulting in zero output is called the zero dynamics system. If the system (11) possesses a zero dynamics system with hyperbolically stable zero solution it is referred to as *hyperbolically minimum phase*. If the origin is an hyperbolically unstable solution of the zero dynamics system for (11) then the system (11) is referred to as *hyperbolically nonminimum phase*. More about concepts of zero dynamics, minimumphaseness and related normal forms can be found in [Isidori, 1995; Byrnes & Isidori, 1991; Byrnes *et. al.*, 1991; Nijmeijer & Schaft, 1990]

Associated with the system (11) consider a real-valued function w defined on $\mathbb{R}^n \times \mathbb{R}^m$ called the *supply rate*. We assume that this function is well defined on any compact subset of the set $\mathbb{R}^n \times \mathbb{R}^m$.

Definition 7 *The system (11) with supply rate w is said to be \mathcal{C}^r -dissipative in the sense of Willems (or \mathcal{W} -dissipative) if there exists a \mathcal{C}^r -smooth, ($r \geq 0$) nonnegative function $V : \mathbb{R}^n \rightarrow \mathbb{R}^+$, called the storage function, such that the following dissipation inequality holds:*

$$V(x(t)) - V(x(0)) \leq \int_0^t w(x(s), u(s)) ds \quad (12)$$

for all $u \in \mathcal{C}^0 \cap \mathcal{L}_\infty$, $x(0) \in \mathbb{R}^n$, $0 \leq t < T_{u,x_0}$, where T_{u,x_0} is the upper time limit for which the solution corresponding to the input u and initial conditions $x(0) = x_0$ exists.

This definition has a clear physical interpretation if the storage function is understood as the total energy of the system. Then the left hand side of this inequality is the increment of the energy at time t and the dissipation inequality means that this increment should not exceed the integral of the supply rate.

Definition 8 *The system (11) is \mathcal{C}^r -passive if it is \mathcal{C}^r -dissipative with supply rate $w(x, u) = y^\top u$ and the storage function V satisfies $V(0) = 0$.*

The above definition is classical in nonlinear control theory but in this paper we also need some weakened version of the passivity property, namely semipassivity, introduced in [Pogromsky, 1998]:

Definition 9 *The system (11) is called \mathcal{C}^r -semipassive if it is \mathcal{C}^r -dissipative with supply rate $w(x, u) = y^\top u - H(x)$ where the scalar function H is nonnegative outside some ball:*

$$\exists \rho > 0, \forall |x| \geq \rho \implies H(x) \geq \varrho(|x|) \quad (13)$$

for some continuous nonnegative function ϱ defined for $|x| \geq \rho$. If the function H is positive outside some ball, i.e. (13) holds for some continuous positive function ϱ , then the system (11) is referred to as strictly semipassive.

Consider the following k systems:

$$\begin{cases} \dot{x}_j = f_j(x_j) + g_j(x_j)u_j \\ y_j = h_j(x_j) \end{cases} \quad (14)$$

where $j = 1, \dots, k$, $x_j(t) \in \mathbb{R}^{n_j}$ are the states, $u_j(t) \in \mathbb{R}^m$ are the inputs, $y_j(t) \in \mathbb{R}^m$ are the outputs and the functions f_j, g_j, h_j are smooth enough to ensure existence, at least on some time interval, of all solutions considered below.

Define the symmetric matrix Γ as

$$\Gamma = \begin{pmatrix} \sum_{i=1}^k \gamma_{1i} & -\gamma_{12} & -\gamma_{13} & \cdots & -\gamma_{1k} \\ -\gamma_{21} & \sum_{i=1}^k \gamma_{2i} & -\gamma_{23} & \cdots & -\gamma_{2k} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -\gamma_{k1} & -\gamma_{k2} & -\gamma_{k3} & \cdots & \sum_{i=1}^k \gamma_{ki} \end{pmatrix} \quad (15)$$

where $\gamma_{ij} = \gamma_{ji} \geq 0$. The matrix Γ is symmetric and therefore all its eigenvalues are real numbers. Moreover applying Gerschgorin's theorem about localization of eigenvalues (see, e.g. [Stewart & Sun, 1990]) one can see that all eigenvalues of Γ are nonnegative, that is the matrix Γ is positive semidefinite.

The following result gives conditions under which the solutions of the interconnected systems (14) are bounded.

Lemma 1 *Consider the systems (14) in closed loop with the following feedback*

$$u_j = -\gamma_{j1}(y'_j - y'_1) - \gamma_{j2}(y'_j - y'_2) - \dots - \gamma_{jk}(y'_j - y'_k) \quad (16)$$

where $y'_j = h'_j(x_j)$ and $h'_j : \mathbb{R}^{n_j} \rightarrow \mathbb{R}^m$ is a smooth mapping such that $h'_j(0) = 0$. Suppose that the systems (14) are C^r -semipassive ($r \geq 0$) with respect to input u_j and output $y_j = h_j(x_j)$ with radially unbounded storage functions $V_j : \mathbb{R}^{n_j} \rightarrow \mathbb{R}^+$ and functions H_j satisfying (13), $j = 1, \dots, k$. Assume the function

$$H(x) = \sum_{j=1}^k H_j(x_j) + y'^\top (I_m \otimes \Gamma) y'$$

with $x = \text{col}(x_1, \dots, x_k)$, $y = \text{col}(y_1, \dots, y_k)$, $y' = \text{col}(y'_1, \dots, y'_k)$, satisfies

$$\exists \rho > 0, \forall |x| \geq \rho \implies H(x) \geq \varrho(|x|) \quad (17)$$

for some continuous function ϱ defined for $|x| \geq \rho$. Then if the function ϱ is nonnegative then all solutions of the closed loop system (14), (16) exist for all $t \geq 0$ and are bounded, that is the system (14), (16) is Lagrange stable, moreover if the function ϱ is positive then the closed loop system (14), (16) is \mathcal{L} -dissipative.

Proof: First notice that due to smoothness of the right hand side of the closed loop system all solutions are unique and exist at least on some finite time interval.

Consider the function $V : \mathbb{R}^{n_1} \times \dots \times \mathbb{R}^{n_k} \rightarrow \mathbb{R}^+$

$$V(x) = V_1(x_1) + \dots + V_k(x_k).$$

Clearly, this function is radially unbounded. Along the solutions of the closed loop system this function satisfies the following integral inequality:

$$V(x(t)) - V(x(0)) \leq \int_0^t -H(x(s))ds \quad (18)$$

Consider the set $\Omega = \{x : |x| \leq \rho\}$. Clearly, Ω is compact. Fix a constant C such that the set $\Omega_1 = \{x : V(x) \leq C\}$ contains Ω : $\Omega \subset \Omega_1$. Such a constant always exists since the function V is positive. Moreover the set Ω_1 is compact because of the radial unboundedness of V . Since Ω_1 is compact there exists a closed ball of the radius $\rho_1, \rho_1 \geq \rho > 0$ which contains Ω_1 . Consider the functions W' and ϱ'' defined as follows

$$W'(x) = \begin{cases} 0, & \text{if } |x| \leq \rho_1 \\ V(x), & \text{otherwise} \end{cases} \quad (19)$$

$$\varrho''(|x|) = \begin{cases} 0, & \text{if } |x| \leq \rho_1 \\ \varrho(|x|), & \text{otherwise} \end{cases} \quad (20)$$

The functions W' and ϱ'' satisfy the following integral inequality

$$W'(x(t)) - W'(x(0)) \leq \int_0^t -\varrho''(|x(s)|)ds \leq 0, \quad (21)$$

which means that the quantity $W'(x(t))$ is bounded for all $t \geq 0$. Since the function W' is radially unbounded, one can conclude that $x(t)$ is bounded and therefore exists for all $t \geq 0$. The first part of the lemma is proved.

It has been proved that all the solutions of the closed loop system are bounded, that is any solution $x(t)$ has values from some compact set for all $t \geq 0$. A continuous scalar function defined on a compact set attains its maximal value in this set. Since the norm of the right hand side of the closed loop system is a scalar continuous function we conclude that the right hand side of the closed loop system is bounded, or, in other words, $\dot{x}(t)$ is bounded for all $t \geq 0$. The function $x(t)$ is differentiable with respect to time and, hence, continuous, its derivative is bounded and therefore the function $x(t)$ is uniformly continuous in t .

From (21) it follows that the integral

$$\int_0^t \varrho''(|x(s)|)ds$$

is finite for all $t \geq 0$ and all initial conditions $x(0)$. Since the function ϱ'' is nonnegative this integral converges as $t \rightarrow \infty$. Now, define a function $\varrho'''(|x|)$ as follows

$$\varrho'''(|x|) = \begin{cases} 0, & \text{if } |x| \leq \rho_1 \\ \varrho(|x|) - \varrho(\rho_1), & \text{otherwise} \end{cases} \quad (22)$$

The function ϱ''' is continuous in x and nonnegative. At the same time $x(t)$ is uniformly continuous in t , therefore $\varrho'''(|x(t)|)$ is uniformly continuous in t . Moreover the integral

$$\int_0^\infty \varrho'''(|x(t)|)dt$$

exists and is finite for all initial conditions $x(0)$. To complete the proof we show that $\overline{\lim_{t \rightarrow \infty}} |x(t)| \leq \rho_1$, or that $\varrho'''(|x(t)|) \rightarrow 0$ as $t \rightarrow \infty$. This statement immediately follows from Barbalat's lemma.

Lemma 2 (Barbalat) [Popov, 1973] *Consider the function $\psi : \mathbb{R}^1 \rightarrow \mathbb{R}^1$. If ψ is uniformly continuous and $\lim_{t \rightarrow \infty} \int_0^\infty \psi(s)ds$ exists and is finite then*

$$\lim_{t \rightarrow \infty} \psi(t) = 0$$

■

It is worth mentioning that if the functions V_j , $j = 1, \dots, k$ are differentiable, i.e. $V_j \in \mathcal{C}^1$ then, because of the semigroup property of dynamical systems, the dissipation inequality (18) can be rewritten in the equivalent infinitesimal form

$$\dot{V} \leq -H(x).$$

In other words for differentiable storage functions the result of the previous lemma follows from the Yoshizawa theorem [Yoshizawa, 1960].

Corollary 1 *Suppose that the systems (14) are \mathcal{C}^r -semipassive ($r \geq 0$) with radially unbounded storage functions $V_j : \mathbb{R}^{n_j} \rightarrow \mathbb{R}^+$. Then all solutions of the systems (14) in closed loop with the feedback*

$$u_j = -\gamma_{j1}(y_j - y_1) - \gamma_{j2}(y_j - y_2) - \dots - \gamma_{jk}(y_j - y_k) \quad (23)$$

with $\gamma_{ji} = \gamma_{ij} \geq 0$, exist for all $t \geq 0$ and are bounded, that is, the system (14), (23) is Lagrange stable. Moreover, if the systems (14) are \mathcal{C}^r -strictly semipassive ($r \geq 0$) with radially unbounded storage functions $V_j : \mathbb{R}^{n_j} \rightarrow \mathbb{R}^+$ then all solutions of the coupled system (14), (23) exist for all $t \geq 0$ and are ultimately bounded, that is, the system (14), (23) is \mathcal{L} -dissipative.

The statement follows from the positive semidefiniteness of the matrix Γ defined in (15), which means that (17) is satisfied for $\rho = \rho_1 + \dots + \rho_k$.

It should be noted that the conditions of Lagrange stability and \mathcal{L} -dissipativity for interconnected systems presented in this section require exact knowledge of the storage functions for each system. In most cases this function can be found as the total energy of the system, e.g. for some mechanical or electrical systems it can be easily determined. However, in general this requirement is not constructive. At the same time for nonlinear systems which are written in the form “linear part plus nonlinearity” (i.e. in Lur'e form) there exist constructive frequency-domain conditions for existence of such functions (see, e.g. [Yakubovich, 1973; Tomberg & Yakubovich, 1986; Leonov *et. al.*, 1996] and references therein). Nevertheless, to simplify the presentation we will not use the frequency domain approach because all the storage functions considered below can be easily found.

5 Diffusion driven oscillations

5.1 An infinite interval of instability

The purpose of this section is to give an explicit construction of diffusively coupled globally asymptotically stable systems that become oscillatory being interconnected. We assume that the topology of the interconnection is described by the matrix Γ defined in Eq. (15) with entries γ_{ij} as in Definition 5. In this subsection we consider the case when the zero dynamics for each system is nontrivial. This condition will be relaxed in the next subsection.

Let A be an $n \times n$ matrix, $n \geq 3$

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

where A_{11} is an $(n-m) \times (n-m)$ matrix, $1 \leq m \leq n-2$ and the other matrices are of corresponding dimensions. Let B and C be full rank $n \times m$ and $m \times n$ matrices such that the product CB is a positive definite matrix.

Theorem 2 *Assume that the following assumptions hold for matrices A , B and C as above.*

A1 *The matrix A is Hurwitz.*

A2 *The matrix A_{11} has an even nonzero number of eigenvalues with positive real parts.*

A3 *The matrix $(CB)^{-1}T$ where $T = (A_{22} - A_{21}A_{11}^{-1}A_{12})$ has no positive real eigenvalues.*

Then there exists a \mathcal{C}^1 -function $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$, $f(z, y) = \text{col}(q(z, y), a(z, y))$, where $z \in \mathbb{R}^{n-m}$, $y \in \mathbb{R}^m$, $q : \mathbb{R}^{n-m} \times \mathbb{R}^m \rightarrow \mathbb{R}^{n-m}$, $a : \mathbb{R}^{n-m} \times \mathbb{R}^m \rightarrow \mathbb{R}^m$, such that

$$A_{11} = \frac{\partial q}{\partial z}(0, 0), \quad A_{12} = \frac{\partial q}{\partial y}(0, 0), \quad A_{21} = \frac{\partial a}{\partial z}(0, 0), \quad A_{22} = \frac{\partial a}{\partial y}(0, 0)$$

Moreover we have that

1) *The system*

$$\begin{cases} \dot{z} = q(z, y) \\ \dot{y} = a(z, y) \end{cases} \quad (24)$$

has a unique globally asymptotically stable equilibrium at the origin.

2) *For all $\gamma_{ij} \geq 0$ the system consisting of k diffusively coupled systems*

$$\begin{cases} \dot{z}_j = q(z_j, y_j) \\ \dot{y}_j = a(z_j, y_j) + CBu_j \\ u_j = -\gamma_{j1}(y_j - y_1) - \gamma_{j2}(y_j - y_2) - \dots - \gamma_{jk}(y_j - y_k) \end{cases} \quad (25)$$

is \mathcal{L} -dissipative, has the origin as a unique equilibrium and there exists a positive number $\bar{\gamma}$ such that for all $\gamma > \bar{\gamma}$, where γ stands for the maximal eigenvalue of the matrix Γ (cf. (15)), the closed loop system is Y -oscillatory.

First let us clarify the assumptions of the theorem. Assumption A1 allows one to find a smooth function $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that A is the Jacobian of f at zero and therefore the origin is a locally hyperbolically asymptotically stable equilibrium of the system $\dot{x} = f(x)$. Assumption A2 guarantees that the zero dynamics of the system $\dot{x} = f(x) + Bu$, $y = Cx$ is hyperbolically unstable at the origin. Assumption A3 is required in order to prove that losing stability the origin does not undergo a bifurcation resulting in the birth of additional equilibria. It is worth to mention that Assumption A3 cannot be satisfied if the matrix A_{11} has an odd number of eigenvalues with positive real parts (This follows from Schur's decomposition since Assumption A3 in this case contradicts the stability of the matrix A). Moreover as we will see further, to become an oscillatory system the origin should lose stability via a Poincaré-Andronov-Hopf bifurcation which is ensured by Assumptions A2 and A3.

We begin to prove the theorem with a preliminary result which immediately follows from the Inertia theorem due to O. Taussky [1961].

Lemma 3 *For any $n \times n$ matrix A which has no eigenvalues on the imaginary axis and for any negative definite matrix $N = N^\top < 0$ there exists an $n \times n$ symmetric nonsingular matrix P which satisfies*

$$A^\top P + PA = N \quad (26)$$

and the number of positive eigenvalues of P coincides with the number of eigenvalues of A with negative real parts and the number of negative eigenvalues of P coincides with the number of eigenvalues of A with positive real parts. Conversely, if (26) holds for some symmetric P and symmetric negative definite N then A has no eigenvalues on the imaginary axis.

Proof of Theorem 2: Let $x = \text{col}(z, y)$, $x \in \mathbb{R}^n$. Consider the function $\phi : \mathbb{R}^n \rightarrow \mathbb{R}^n$, defined as $\phi(x) = \text{col}(\phi_z(z), \phi_y(y))$, where $\phi_z : \mathbb{R}^n \rightarrow \mathbb{R}^{n-m}$, $\phi_y : \mathbb{R}^n \rightarrow \mathbb{R}^m$ and

$$\phi_z(z, y) = z(1 + |z|^2 + |y|^2), \quad \phi_y(z, y) = y(1 + |z|^2 + |y|^2)$$

Denote

$$\begin{aligned} q(z, y) &= A_{11}\phi_z(z, y) + A_{12}\phi_y(z, y) \\ a(z, y) &= A_{21}\phi_z(z, y) + A_{22}\phi_y(z, y) \end{aligned}$$

First we prove that the system

$$\dot{x} = f(x) \quad (27)$$

is globally asymptotically stable. Consider the following Lyapunov function candidate

$$V(x) = x^\top R x \quad (28)$$

where $R = R^\top > 0$ is a positive definite solution of the following Lyapunov equation

$$A^\top R + RA = -I_n$$

which is solvable because of Assumption A1. The time derivative of (28) with respect to (27) satisfies

$$\dot{V}(x) = -|x|^2(1 + |x|^2).$$

It is clear that \dot{V} is negative definite and hence the system (27) has a unique globally asymptotically stable equilibrium at the origin.

Now we should prove that the system consisting of k diffusively coupled systems (25) is Y-oscillatory for sufficiently large γ . To this end we will show that the closed loop system is \mathcal{L} -dissipative (step 1), it has a unique equilibrium for all $\gamma \geq 0$ (step 3) which loses its stability for sufficiently large γ (step 2).

Step 1. First let us prove that all solutions of the closed loop system are ultimately bounded for all $\gamma_{ij} \geq 0$. It is quite clear that the system

$$\dot{x}_j = f(x_j) + Bu \tag{29}$$

is strictly semipassive with respect to input u and output $h(x_j) = 2B^\top R x_j$. Indeed, differentiating (28) with respect to (29) we have

$$\dot{V} = -|x_j|^2(1 + |x_j|^2) + 2x_j^\top R B u = -|x_j|^2(1 + |x_j|^2) + h(x_j)^\top u$$

and therefore the dissipation inequality holds with $H(x_j) = |x_j|^2(1 + |x_j|^2) \geq 0$.

It is also easy to notice that the conditions of the Lemma 1 are satisfied since the growth rate of the function $|x_j|^2(1 + |x_j|^2)$ is as $|x_j|^4$ while the output $h(x_j)$ with respect to which each subsystem is semipassive is linear in x_j . Therefore the closed loop system is \mathcal{L} -dissipative.

Step 2. The next step is to prove that for sufficiently large γ the origin of the closed loop system loses its stability. According to Assumption A2 the matrix A_{11} has no eigenvalues on the imaginary axis and has eigenvalues with positive real parts. Therefore by virtue of Lemma 3 there exists a symmetric nonsingular indefinite or negative definite $(n - m) \times (n - m)$ matrix $P = P^\top$ which satisfies $A_{11}^\top P + P A_{11} < 0$.

The matrix Γ is symmetric and hence diagonalizable via a similarity transformation. Therefore using the block diagonal structure of the closed loop system its linearization about origin can be rewritten after some suitable linear coordinate transformation in the following form

$$\begin{cases} \dot{\hat{z}}_j = A_{11}\hat{z}_j + A_{12}\hat{y}_j \\ \dot{\hat{y}}_j = A_{21}\hat{z}_j + A_{22}\hat{y}_j - \hat{\gamma}_j C B \hat{y}_j \end{cases} \tag{30}$$

where $\hat{x}_j = \text{col}(\hat{z}_j, \hat{y}_j) \in \mathbb{R}^n$, $j = 1, \dots, k$ and $\hat{\gamma}_j$ stands for one of k eigenvalues of the matrix Γ . Indeed the linearized system can be rewritten in the form

$$\dot{x} = (I_k \otimes A)x - (\Gamma \otimes D)x \tag{31}$$

where $x \in \mathbb{R}^{kn}$, $x = \text{col}(z_1, y_1, \dots, z_k, y_k)$, D is the following $n \times n$ matrix

$$D = \begin{pmatrix} 0 & 0 \\ 0 & C B \end{pmatrix}$$

and the operation ‘ \otimes ’ is the Kronecker product (see Sec. 2). Now, using the coordinate transformation $\hat{x} = (M \otimes I_n)x$, where M is a nonsingular orthogonal matrix such that $\Gamma = M\Gamma_0 M^{-1}$, $\Gamma_0 = \text{diag}(\hat{\gamma}_1, \dots, \hat{\gamma}_k)$ we immediately obtain (30). Notice that since the matrix Γ is singular there is at least one zero among its eigenvalues.

Fix the largest eigenvalue $\hat{\gamma}_j = \gamma$ and the corresponding index j . We will study how the origin loses stability when the largest eigenvalue which will be considered as an increasing bifurcation parameter. Consider the following quadratic form

$$V(\hat{x}_j) = \hat{x}_j^\top Q \hat{x}_j$$

where

$$Q = \begin{pmatrix} P & 0 \\ 0 & I_m \end{pmatrix} \quad (32)$$

It is clear that Q is symmetric, nonsingular and indefinite. The time derivative of V with respect to solutions of the system (30) satisfies

$$\begin{aligned} \dot{V}(\hat{x}_j) &= \hat{z}_j^\top (A_{11}^\top P + P A_{11}) \hat{z}_j + 2 \hat{z}_j^\top P A_{12} \hat{y}_j \\ &\quad + 2 \hat{y}_j^\top A_{21} \hat{z}_j + \hat{y}_j^\top (A_{22}^\top + A_{22}) \hat{y}_j \\ &\quad - 2 \gamma \hat{y}_j^\top C B \hat{y}_j \\ &\leq 2 \left(-\delta |\hat{z}_j|^2 + \alpha_1 |\hat{z}_j| |\hat{y}_j| + \alpha_2 |\hat{y}_j|^2 - \gamma \beta |\hat{y}_j|^2 \right) \end{aligned} \quad (33)$$

where $\delta > 0$ is the smallest eigenvalue of the matrix $-(A_{11}^\top P + P A_{11})/2$, $\alpha_1 = |P A_{12} + A_{21}^\top|$, α_2 is the largest eigenvalue of the matrix $(A_{22}^\top + A_{22})/2$, $\beta > 0$ is the smallest eigenvalue of the matrix $C B$. Now it is clear that if

$$\gamma > \frac{\alpha_1^2}{4\delta} + \alpha_2 \quad (34)$$

then the time derivative of V becomes negative definite. Since the matrix Q is indefinite by virtue of Lemma 3 one can conclude that the subsystem (30) which corresponds to the largest eigenvalue of the matrix Γ has a hyperbolically unstable zero solution. Due to the block diagonal form of the system (30) this implies in turn that the linearized closed loop system has an unstable zero solution and therefore the closed loop system also has a hyperbolically unstable zero solution for sufficiently large γ . Moreover, if $\gamma_{ij} = \lambda > 0$ for all $i = 1, \dots, k, j = 1, \dots, k$ the eigenvalues of the matrix Γ are the following numbers $\{0, k\lambda, k\lambda, \dots, k\lambda\}$ and therefore as k increases the bound of the gain which ensures destabilization decays at least as k^{-1} .

It is worth mentioning that the estimate (34) can be quite conservative and can be sharpened. However what is crucial for our study is that there exists a threshold value for γ which ensures instability for all $\gamma \geq \bar{\gamma}$.

Step 3. To complete the proof we need to show that for all $\gamma_{ij} \geq 0$ the closed loop system has a unique equilibrium point. The closed loop system can be rewritten in the form

$$\dot{x} = (I_k \otimes A)\psi(x) - (\Gamma \otimes D)x \quad (35)$$

where $\psi(x) = \text{col}(x_1(1 + |x_1|^2), \dots, x_k(1 + |x_k|^2))$, $x = \text{col}(x_1, \dots, x_k) \in \mathbb{R}^{kn}$.

Notice that if the matrix

$$S \otimes A - \Gamma \otimes D \quad (36)$$

for all $\gamma_{ij} \geq 0$ and arbitrary diagonal $k \times k$ matrix S with entries greater or equal to one, is nonsingular then the system (35) has a unique equilibrium. We can think of the diagonal entries of S as replacements for $1 + |x_j|^2$ and use the inequality $1 + |x_j|^2 \geq 1$. The matrix (36) is nonsingular if and only if the following matrix is nonsingular as well

$$I_k \otimes A - S^{-1/2} \Gamma S^{-1/2} \otimes D \quad (37)$$

The matrix $S^{-1/2} \Gamma S^{-1/2}$ is symmetric and positive semidefinite and therefore it is diagonalizable via orthogonal transformation. Let M be an orthogonal matrix such that $\Gamma_0 = M S^{-1/2} \Gamma S^{-1/2} M^\top$ is a diagonal matrix. Notice that

$$(M \otimes I_n)(I_k \otimes A - S^{-1/2} \Gamma S^{-1/2} \otimes D)(M^\top \otimes I_n) = I_k \otimes A - \Gamma_0 \otimes D$$

Since $|S^{-1/2}| \leq 1$ all eigenvalues of Γ_0 are nonnegative and less or equal to γ . Thus the matrix (37) is nonsingular for all γ if and only if the matrix

$$A_\mu = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} - \mu CB \end{pmatrix} \quad (38)$$

is nonsingular for all nonnegative μ . The matrix A is nonsingular by hypotheses. In view of Assumption A2 the matrix A_{11} is nonsingular, therefore

$$\det A_\mu = \det A_{11} \det(T - \mu CB),$$

with $T = A_{22} - A_{21} A_{11}^{-1} A_{12}$. Notice that $\det(T - \mu CB) \neq 0$ for positive μ if the matrix $(CB)^{-1}T$ has no positive real eigenvalues which is the case due to Assumption A3. Therefore the closed loop system for arbitrary $\gamma_{ij} \geq 0$ has a unique equilibrium at the origin. ■

Notice that we required so far that $n \geq 3$. Actually this is in a sense also necessary as one can see next.

Proposition 1 *Consider a locally Lipschitz continuous function $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$, $f(x) = -f(-x)$ such that*

- i. *The system $\dot{x} = f(x)$ has a globally hyperbolically stable zero solution.*
- ii. *The closed loop system*

$$\begin{cases} \dot{x}_1 = f(x_1) + B(x_2 - x_1) \\ \dot{x}_2 = f(x_2) + B(x_1 - x_2) \end{cases} \quad (39)$$

where B is similar to a positive semidefinite $n \times n$ matrix, is Y -oscillatory and has a unique zero equilibrium.

Then necessarily $n \geq 3$.

Proof: Consider the set $\Omega = \{x_1, x_2 \in \mathbb{R}^n : x_1 + x_2 = 0\}$. Notice that the hyperbolic stability of the zero equilibrium of the system $\dot{x} = f(x)$ implies that the set Ω (clearly, it is invariant) is at least locally attractive and attracts solutions starting from a nonzero measure set. Assume that $n = 2$ (the case $n = 1$ is trivial). Then Ω is a two dimensional plane containing origin. The closed loop system is Y-oscillatory and according to the Poincaré-Bendixon theorem it contains a closed trajectory which corresponds to some nontrivial periodic solution of the closed loop system with initial conditions from Ω . The index of the closed orbit is $+1$ and equals to the sum of the indices of the fixed points within it. Therefore the index of the unique equilibrium of the closed loop system (39) with the dynamics constrained to the set Ω is $+1$. It is not difficult to see that the dynamics of the closed loop system on the invariant set $x_1 = -x_2$ is described by the following equation

$$\dot{x} = f(x) - 2Bx$$

where x is written in some coordinate system on Ω . Let $A = \partial f(0)/\partial x$. The matrix A is Hurwitz and therefore $\text{tr}A < 0$. Since B is positive semidefinite, $\text{tr}B > 0$ and hence $\text{tr}(A - 2B) < 0$. The index of the zero equilibrium is $+1$, therefore it is either a sink, or a source or a center. Since $\text{tr}(A - 2B) < 0$ it must be a sink, or in other words, it is locally asymptotically stable. However this contradicts the fact that the closed loop system is Y-oscillatory. Therefore $n \geq 3$. ■

It is worth mentioning that the assumption $f(x) = -f(-x)$ is not restrictive in the general case. Indeed, suppose $B = \gamma B_0$, where $\gamma > 0$ and let us study how the origin loses its stability when γ increases. From the proof of the previous theorem it follows that the origin loses stability when some of the eigenvalues of the matrix $A - 2\gamma B_0$ cross the imaginary axis. This follows from the fact that Ω is a locally invariant manifold, i.e. in the vicinity of the origin. The origin can lose stability via one of the following bifurcations: saddle-node, pitchfork, transcritical or Poincaré-Andronov-Hopf. We will not consider singularities of codimension greater than 1 because in this case they can be avoided by arbitrary small continuous perturbations in the right hand side. As a result of the first two bifurcations, the system gains an additional asymptotically stable equilibrium and we have a contradiction. If the origin undergoes a transcritical bifurcation, to retain all trajectories bounded the system must gain some minimal compact invariant set distinct from the origin exactly when γ is equal to the bifurcation value. This situation is not possible in the general case since, according to Thom's transversality theorem (see [Arnold, 1983]), for a parametrized smooth vector field defined on a compact set all bifurcation points are isolated. Note that here "in the general case" means that such vector fields form a dense set in the set of all parametrized smooth vector fields. Therefore to become an oscillatory system its origin should lose stability via a Poincaré-Andronov-Hopf bifurcation. Now $n = 2$ contradicts $\text{tr}(A - 2\gamma B_0) < 0$.

5.2 Finite interval of instability

As we have seen the analysis of oscillatory diffusively coupled systems consists of two parts: analysis of boundedness of solutions of interconnected systems and local stability analysis of existing equilibria. A solution of the first problem can be obtained with the results of Section 4. To solve the second problem in the previous theorem we made a crucial assumption about instability of the zero dynamics (assumption A2). So, one may wonder how restrictive this assumption is for the analysis of diffusively coupled systems. At first glance this assumption is quite restrictive. Indeed, in the example proposed by Smale the matrix CB has rank n and therefore the system (25) has trivial zero dynamics. It motivates our subsequent study. First we show that Smale's understanding of diffusive coupling is equivalent to our definition. Our definition includes Smale's problem as a special case, therefore to show equivalence between them we need to prove that a solution to the problem posed in this paper implies existence of solution to the Smale problem. This fact will be established using the next theorem. Second we will relax the conditions imposed on the zero dynamics system.

Theorem 3 *Assume that there exists an $n \times n$ matrix D such that D is singular and is similar to a positive semidefinite matrix and the zero equilibrium of the system $\dot{x} = (A - D)x$ is hyperbolically unstable. Then there exists a positive definite matrix B such that $\dot{x} = (A - B)x$ is hyperbolically unstable and the number of positive eigenvalues of $A - B$ coincides with the number of positive eigenvalues of $A - D$.*

Proof: Without loss of generality we can assume that

$$D = \begin{pmatrix} 0 & 0 \\ 0 & D_1 \end{pmatrix}$$

where D_1 is a matrix with positive diagonal entries. Then, using the hyperbolic instability of $A - D$ we can complete D to a full rank diagonal matrix replacing diagonal zeros with some sufficiently small positive numbers. Indeed from Lemma 3 it follows that there exists negative definite or nonsingular indefinite symmetric matrix P such that

$$(A - D)^\top P + P(A - D) - 2\varepsilon P < 0$$

for sufficiently small $\varepsilon > 0$. In other words, the matrix $A - B$, where $B = D + \varepsilon I_n$ has the same number of positive eigenvalues as $A - D$ for sufficiently small ε . ■

Now we can describe a possible way in which the systems become oscillatory via diffusion. First of all consider the case when each free system has some inherent unstable dynamics (with a linearization determined by the matrix A_{11} as in Theorem 2). Then, if the largest eigenvalue of the matrix Γ which describes the interconnection topology exceeds some threshold value, the origin of the system loses its stability. Two possible scenarios can occur in general. If for this bifurcation value the linearized system has a zero eigenvalue (the Jacobi matrix is singular) then as a result of

bifurcation the system possesses additional stable equilibrium(ia) and no oscillations occur. The second scenario takes place if for the critical value of γ the Jacobi matrix is nonsingular, which corresponds to a pair(s) of purely nonzero imaginary eigenvalues of the linearized system. Then according to the Implicit Function Theorem the additional equilibrium cannot bifurcate from the origin and therefore the diffusively coupled systems exhibit oscillatory behavior. Notice that the condition that the zero dynamics has only an even number of eigenvalues with positive real parts is a sufficient condition that the Jacobi matrix of the linearized system is nonsingular for all $\gamma > 0$, i.e. when γ belongs to infinite interval. If we are interested in the interconnection analysis when γ lies in a certain range, this condition can be relaxed. However, it is crucial that the zero dynamics cannot have *one* real positive eigenvalue because in this case the origin cannot lose stability via a Poincaré-Andronov-Hopf bifurcation.

Now we present a result guaranteeing existence of oscillatory behavior in the diffusively coupled systems in case when the largest eigenvalue $\gamma > 0$ of the matrix Γ is not greater than some limit value γ_0 .

Theorem 4 *Suppose that $n \times m$ and $m \times n$, $m \leq n$ matrices B and C are such that CB is similar to a positive definite matrix. Assume that*

A1. *The $n \times n$ matrix A is Hurwitz.*

A2. *There exists $0 < \gamma_1 \leq \gamma_0$ such that for all $\gamma \in [\gamma_1, \gamma_0]$ the matrix $A - \gamma BC$ has at least two eigenvalues with positive real parts and the matrix Γ has at least one eigenvalue in the region $[\gamma_1, \gamma_0]$.*

A3. *The matrix $A^{-1}BC$ has no real eigenvalues in the region $[\gamma_0^{-1}, \infty)$.*

Then there exist a function $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that

$$A = \frac{\partial f}{\partial x}(0)$$

and

1) *The system $\dot{x} = f(x)$ has a unique globally asymptotically stable equilibrium at the origin.*

2) *The system consisting of k diffusively coupled systems, $j = 1, \dots, k$*

$$\begin{cases} \dot{x}_j = f(x_j) + Bu_j \\ y_j = Cx_j \\ u_j = -\gamma_{j1}(y_j - y_1) - \gamma_{j2}(y_j - y_2) - \dots - \gamma_{jk}(y_j - y_k) \end{cases}$$

is \mathcal{L} -dissipative, has a unique equilibrium and is Y -oscillatory.

Proof: As an example of the function f take $f(x) = Ax(1 + |x|^2)$. The statement 1 and \mathcal{L} -dissipativity of the coupled systems can be proven in the same way as in the proof of Theorem 2. Assumption A2 implies instability of the origin of the coupled systems. According to Theorem 1 it is sufficient to show that the whole system has a unique equilibrium. Then similarly to the proof of Step 3 of Theorem 2 we need to show that the matrix $A - \gamma BC$ is nonsingular for all $\gamma \in [0, \gamma_0)$. This fact immediately follows from Assumption A3. ■

As one can see conditions imposed in Theorems 2 and 4 ensure that the trivial solution of the coupled systems undergoes a Poincaré-Andronov-Hopf bifurcation for some critical value of γ . In the proof the cubic type nonlinearity we chose, guarantees that this bifurcation is a global bifurcation. So one may wonder whether the cubic nonlinearity is crucial to guarantee oscillations. The answer is negative. As soon as the local behavior of the nonlinearity is described by the conditions of the theorems its global behavior should ensure 1) ultimate boundedness of solutions of coupled systems and 2) uniqueness of the system equilibrium. A solution to the first problem can be derived from the results of section 4.

When parameters change a parametrized system with a unique equilibrium can possess an extra equilibrium in two ways. It can bifurcate from the existing equilibrium and it can appear far from this equilibrium. The nonlinearity we chose guarantees only the first way which significantly simplifies the proofs and allows us to concentrate on the local behavior of the coupled systems. So in general the condition we presented are local and not sufficient, although they give a good understanding of the mechanism to bring about oscillations via diffusion. Now let us briefly sketch a way to solve problem 2 for general systems. A region in the state space where the graph of nonlinearity lies can be described for example by a set of linear constraints. Therefore for the class of nonlinearities satisfying these constraints the system has the unique equilibrium if and only if the set of linear strict and nonstrict inequalities has no solutions. This problem can be easily solved by linear programming.

It is worth mentioning that the methodology presented for analysis of diffusively coupled systems can be extended to the case of multiple equilibria. However, in this case the complexity to check all conditions increases with the number of equilibria. Nevertheless in such systems it is possible to observe a quite interesting phenomenon when the coupled systems exhibit chaotic oscillations [Kocarev & Janić, 1995].

6 An example

As an example we design a third order globally asymptotically stable system such that a diffusive interconnection between two such systems results in oscillatory behavior. One of the problems posed in [Smale, 1976] was to find an example of minimal order of such systems. S. Smale proposed an example of the 4th order. We will design an example of the 3rd order which is a minimal order for diffusively interconnected systems with oscillatory behavior and unique equilibrium. Therefore this example solves one of the problems posed in [Smale, 1976].

Choose any stable nonminimum phase transfer function of 3rd order and relative degree one with a numerator having two eigenvalues with positive real parts. Take for example

$$W(s) = \frac{s^2 - s + 1}{s^3 + 2s^2 + 2s + 1} \quad (40)$$

Its state space representation can be written in the following form

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases}$$

with

$$A = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ -4 & 2 & -3 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad C = (0 \ 0 \ 1).$$

Notice that the eigenvalues of the 2×2 left upper submatrix of the matrix A coincide with the zeros of the transfer function (40), moreover as required in Definition 5 the matrix CB , which is a scalar in this case, is positive definite.

Now consider the following system

$$\begin{cases} \dot{x}_1 = Ax_1(1 + |x_1|^2) + \gamma BC(x_2 - x_1) \\ \dot{x}_2 = Ax_2(1 + |x_2|^2) + \gamma BC(x_1 - x_2) \end{cases}$$

where $x_1, x_2 \in \mathbb{R}^3$ and γ is a positive number. It is not difficult to calculate that for $\gamma = 0.6512$ the origin of the system undergoes a Poincaré-Andronov-Hopf bifurcation which, according to Theorem 2, results in oscillatory behavior of the interconnected system for all $\gamma > 0.6512$. Computer simulations are presented in figures 2 (for $\gamma = 0.5$) and 3 (for $\gamma = 1$). In the example $CB = 1$, however, according to Theorem 3 it is possible to design a system with the same properties for $\text{rank}CB = 2$ or 3. In Fig. 4 we plotted how the eigenvalues of the linearization of the closed loop system change when γ varies from 0 to 2. It is seen that for some value of γ , $\gamma = 0.6512$, a pair of eigenvalues crosses the imaginary axis. The linearized system has 6 eigenvalues, three of them are changing with γ and the rest three remain unchanged because the closed loop system has the invariant set $x_1 = -x_2$ which does not depend on γ .

As a simple exercise we propose to prove that for $n = 3$ the matrices A, B, C can be chosen as the state representation of *arbitrary* single-input-single-output stable nonminimum phase system of relative degree 1 whose zero dynamics has only eigenvalues with positive real parts.

7 Conclusion

In the paper we presented conditions guaranteeing existence of oscillatory behavior in diffusively coupled systems while each isolated system is globally asymptotically stable. A motivation for our study was the problem posed by Smale [1976]. He wrote: “There is a paradoxical aspect to the example. One has two dead (mathematically dead) cells interacting by a diffusion process which has a tendency in itself to equalize the concentrations. Yet in interaction, a state continues to pulse indefinitely”. From a control theory point of view diffusion is an analog for negative feedback and it is well known that a negative feedback is not always stabilizing and it can even destabilize a stable system (e.g. nonminimum phase). Therefore, for control theorists the result should not be so surprising although some work should have to be done. Indeed,

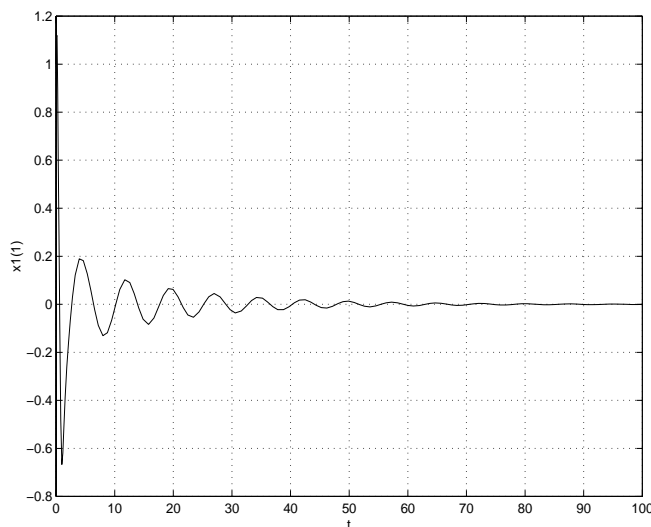


Figure 2: $\gamma = 0.5$

losing stability at the origin the diffusively coupled systems may gain an additional equilibrium and there need not to be oscillations. However if at the bifurcation point the Jacobian of the system is nonsingular, then according to the Implicit Function Theorem no additional equilibria bifurcate from the origin. Also additional conditions should be imposed to guarantee that the bifurcation resulting in oscillatory behavior is global. Moreover we have seen that to exhibit an oscillatory behavior after coupling, each system alone has an unstable dynamics consistent with some external state constraint. Thus if one accepted Smale's terminology that stability is death while instability is life he would say that each cell by itself can not be totally dead.

From the proof of Theorem 2 one can make some important observations. First it has been shown that if all γ_{ij} are equal to each other then the gain which ensures destabilization decreases with the growth of the number of interacting systems. Therefore the number of interacting systems can be considered as a bifurcation parameter. In other words, it is possible to create an oscillatory behavior in the colony of living cells if we add into the colony an additional cell which diffusively interacts to the other cells. Moreover in the diffusive medium consisting of diffusively coupled systems it is possible to observe locally generated spatial phenomena. Indeed, we have shown that the interconnected systems will exhibit an oscillatory behavior if the largest eigenvalue of the matrix Γ exceeds some threshold bifurcation value. Therefore the local changes in one parameter γ_{ij} which describes connection between i th and j th cells can result in propagation of oscillatory behavior through the whole diffusive medium. With this in mind it is interesting to notice that the real living cells have an ability to control diffusion between them. This control involves such processes as *facilitated diffusion*, *active transport* and transport through *coupled channel* (see [Raven & Johnson, 1992]). So, perhaps, an abstract mathematical approach will allow us to understand better what is going on in biological systems.

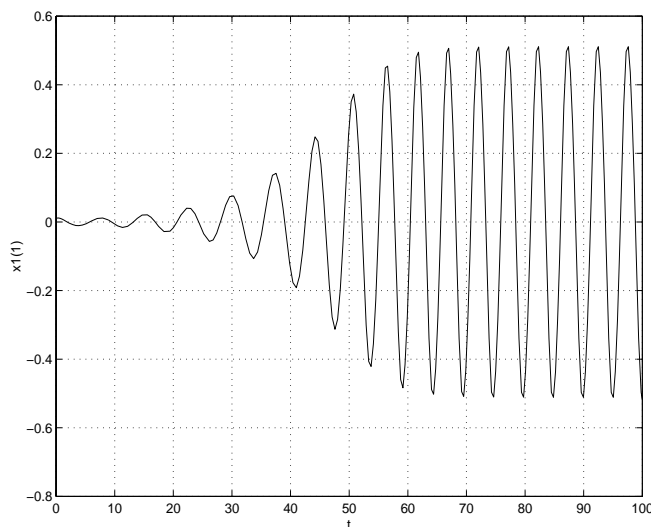


Figure 3: $\gamma = 1$

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References

- Arnold, V.I., [1983] *Geometrical Methods in the Theory of Ordinary Differential Equations*. (Springer-Verlag – New-York).
- Byrnes, C.I. and A.Isidori [1991] “Asymptotic stabilization of minimum- phase systems,” *IEEE Trans. on Automatic Control*, vol. **36**(10), pp. 1122-1137.
- Byrnes, C.I., A. Isidori and J.C. Willems [1991] “Passivity, feedback equivalence and the global stabilization of minimum phase nonlinear systems”, *IEEE Trans. Aut. Contr.*, vol. **36**(11), pp. 1228–1240.
- Coddington, E.A. and N. Levinson [1955] *The theory of ordinary differential equations*. (New York: McGraw Hill).
- Fradkov, A.L. [1998] Private communication.
- Hartman, P. [1964] *Ordinary differential equations*, (Wiley, New York).
- Isidori, A. [1995] *Nonlinear control systems*. 3rd edition. (Springer-Verlag, Berlin).
- Kocarev, L.M. and P.A. Janić [1995] “On Turing instability in two diffusively coupled systems” *IEEE Trans. Circ. Sys., I*, vol. **42**(10), pp. 779–784.

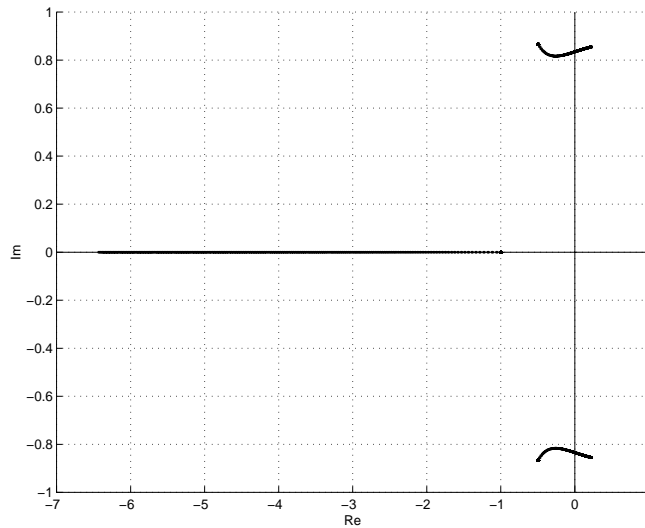


Figure 4: Eigenvalue loci of the linearized closed loop system.

Leonov, G.A., D.V. Ponomarenko and V.B. Smirnova [1996] *Frequency Methods for Nonlinear Analysis. Theory and Applications*, (World Scientific – Singapore).

Levinson, N. [1944] “Transformation theory of nonlinear differential equations of the second order”, *Ann. Math.*, vol. **45**(4), pp. 723–737.

Mosekilde, E. [1997] *Topics in Nonlinear Dynamics. Applications to Physics, Biology and Economic Systems*, (World Scientific – Singapore).

Nijmeijer, H. and A.J. Van der Schaft [1990] *Nonlinear Dynamical Control Systems*. (Springer-Verlag – Berlin).

Pogromsky, A. Yu. [1998] “Passivity based design of synchronizing systems” *Int. J. Bifurcation and Chaos*, vol. **8** (2).

Popov, V.M. [1973] *Hyperstability of control systems* (Springer-Verlag: New York), (in Romanian: 1966).

Raven, P.H. and G.B. Johnson [1992] *Biology*, Third edition, (Mosby Year Book–St. Louis).

Smale, S. [1976] “A mathematical model of two cells via Turing’s equation,” pp. 354–367, in Marsden, J.E. and M. McCracken (eds.) [1976] *The Hopf bifurcation and its applications*, Springer-Verlag, New York.

Stewart, G.W. and Ji-Guang Sun [1990] *Matrix Perturbation Theory*, Academic Press.

Taussky, O. [1961] “A generalization of a theorem of Lyapunov”, *SIAM J. Appl. Math.*, vol. **9**, pp. 640–643.

Tomberg, E.A. and V.A. Yakubovich [1989] “Self-oscillatory conditions in nonlinear systems”, *Siberian Math. Journal*, v. **30**(4), pp. 180–194, in Russian.

Turing, A. [1952] “The chemical basis of morphogenesis”, *Phylos. Trans. Royal Soc. (B)*, vol. **237**, pp. 37–72.

We Ming-Ni [1998] “Diffusion, cross-diffusion, and their spike-layer steady states” *Notices of the American Mathematical Society*, vol. **45**(1), pp. 9–18.

Yakubovich [1973] “Frequency-domain conditions of self-oscillations in nonlinear systems with one stationary nonlinearity,” *Siberian Math. Journal*, v. **14**(5), pp. 1100–1129, in Russian.

Yoshizawa, T. [1960] “Stability and boundedness of systems,” *Arch. Rational Mech. Anal.*, vol. **6**, pp. 409–421.