On dilation operators in Besov spaces

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Abstract

We consider dilation operators $T_k : f \to f(2^k \cdot)$ in the framework of Besov spaces $B^s_{p,q}(\mathbb{R}^n)$ when $0 < p \leq 1$. If $s > n\big(\frac{1}{p} - 1\big)$, T_k is a bounded linear operator from $B^s_{p,q}(\mathbb{R}^n)$ into itself and there are optimal bounds for its norm. We study the situation on the line $s = n\big(\frac{1}{p} - 1\big)$, an open problem mentioned in [ET96]. It turns out that the results shed new light upon the diversity of different approaches to Besov spaces on this line, associated to definitions by differences, Fourier-analytical methods and subatomic decompositions.

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Introduction

In this article we consider dilation operators

$$T_k f(x) = f(2^k x), \quad x \in \mathbb{R}^n, \quad k \in \mathbb{N}, \tag{0.1}$$

which represent bounded operators from $B^s_{p,q}(\mathbb{R}^n)$ into itself. Their behaviour is well known when $s > \sigma_p = n \max\left(\frac{1}{p} - 1, 0\right)$. In this situation we have for $0 < p, q \leq \infty$,

$$\|T_k|\mathcal{L}(B^s_{p,q}(\mathbb{R}^n))\| \sim 2^{k(s-\frac{n}{p})}, \qquad s > \sigma_p,$$

cf. [ET96]. We study the dependence of the norm of T_k on k on the line $s = \sigma_p$, where $0 < p \leq 1$. In particular, we obtain for $0 < q \leq \infty$ that

$$\|T_k|\mathcal{L}(B^{\sigma_p}_{p,q}(\mathbb{R}^n))\| \sim 2^{k(\sigma_p-\frac{n}{p})} k^{1/q}.$$

The situation $s = 0$ was already investigated in [Vyb08, Sect. 3], where it was proved that for $0 < q \leq \infty$,

$$\|T_k|\mathcal{L}(B^0_{p,q}(\mathbb{R}^n))\| \sim 2^{-k\frac{n}{p}} \cdot \begin{cases} k^{\frac{1}{q}-\frac{1}{\max(p,q,2)}}, & \text{if } 1 < p < \infty, \\ k^{\frac{1}{q}}, & \text{if } p = 1 \text{ or } p = \infty. \end{cases}$$

We generalize the methods (and adapt the notation) used there.

As a by-product the new results for the dilation operators lead to new insights concerning the nature of the different approaches to Besov spaces – namely the classical ($\mathbf{B}^s_{p,q}$), the Fourier-analytical ($B^s_{p,q}$) and the subatomic approach ($\mathfrak{B}^s_{p,q}$) – on the line $s = \sigma_p$.
So far recent results by Hedberg, Netrusov [HN07] on atomic decompositions and by Triebel [Tri06, Sect. 9.2] on the reproducing formula prove coincidences

$$\mathbf{B}^s_{p,q}(\mathbb{R}^n) = \mathfrak{B}^s_{p,q}(\mathbb{R}^n), \qquad s > 0, \quad 0 < p, q \leq \infty, \tag{0.2}$$

and

$$B^s_{p,q}(\mathbb{R}^n) = \mathbf{B}^s_{p,q}(\mathbb{R}^n) = \mathfrak{B}^s_{p,q}(\mathbb{R}^n), \qquad s > \sigma_p, \quad 0 < p, q \leq \infty, \tag{0.3}$$

(in terms of equivalent norms). Furthermore, since for $s < n(\frac{1}{p} - 1)$ the δ-distribution belongs to $B^s_{p,q}(\mathbb{R}^n)$ – which is a singular distribution and cannot be interpreted as a function – we know as well that

$$B^s_{p,q}(\mathbb{R}^n) \neq \mathfrak{B}^s_{p,q}(\mathbb{R}^n), \qquad 0 < s < \sigma_p.$$

The situation on the line $s = \sigma_p$, $0 < p < 1$, so far remained an open problem. Our results yield, that in this case the Fourier-analytical approach and the classical approach do not coincide, i.e., for all $0 < q \leq \infty$,

$$B^{\sigma_p}_{p,q}(\mathbb{R}^n) \neq \mathbf{B}^{\sigma_p}_{p,q}(\mathbb{R}^n),$$

where $\mathbf{B}^{\sigma_p}_{p,q}(\mathbb{R}^n)$ can be replaced by $\mathfrak{B}^{\sigma_p}_{p,q}(\mathbb{R}^n)$ in view of (0.2).

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1 Besov spaces $B^s_{p,q}(\mathbb{R}^n)$

We use standard notation. Let $\mathbb{N}$ be the collection of all natural numbers and let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Let $\mathbb{R}^n$ be euclidean n-space, $n \in \mathbb{N}$, $\mathbb{C}$ the complex plane. The set of multi-indices $\beta = (\beta_1, \ldots, \beta_n)$, $\beta_i \in \mathbb{N}_0$, $i = 1, \ldots, n$, is denoted by $\mathbb{N}_0^n$, with $|\beta| = \beta_1 + \cdots + \beta_n$, as usual. Moreover, if $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{N}_0^n$ we put $x^\beta = x_1^{\beta_1} \cdots x_n^{\beta_n}$.
We use the equivalence '$\sim$' in

$$a_k \sim b_k \quad \text{or} \quad \varphi(x) \sim \psi(x)$$

always to mean that there are two positive numbers c_1 and c_2 such that

$$c_1\, a_k \leq b_k \leq c_2\, a_k \quad \text{or} \quad c_1\, \varphi(x) \leq \psi(x) \leq c_2\, \varphi(x)$$

for all admitted values of the discrete variable k or the continuous variable x, where $\{a_k\}_k$, $\{b_k\}_k$ are non-negative sequences and φ, ψ are non-negative functions. If $a \in \mathbb{R}$, then $a_+ := \max(a, 0)$ and $[a]$ denotes the integer part of a.
All unimportant positive constants will be denoted by c, occasionally with subscripts. For convenience, let both $\mathrm{d}x$ and $|\cdot|$ stand for the (n-dimensional) Lebesgue measure in the sequel. As we shall always deal with function spaces on $\mathbb{R}^n$, we may usually omit the '$\mathbb{R}^n$' from their notation for convenience.

Let $Q_{j,m}$ with $j \in \mathbb{N}_0$ and $m \in \mathbb{Z}^n$ denote a cube in $\mathbb{R}^n$ with sides parallel to the axes of coordinates, centered at $2^{-j}m$, and with side length 2^{-j+1}. For a cube Q in $\mathbb{R}^n$ and $r > 0$, we denote by rQ the cube in $\mathbb{R}^n$ concentric with Q and with side length r times the side length of Q.

Furthermore, when $0 < p \leq \infty$ the number σ_p is given by

$$\sigma_p = n \left(\frac{1}{p} - 1 \right)_+ . \tag{1.1}$$

The Fourier-analytical approach

The Schwartz space $\mathcal{S}(\mathbb{R}^n)$ and its dual $\mathcal{S}'(\mathbb{R}^n)$ of all complex-valued tempered distributions have their usual meaning here. Let $\varphi_0 = \varphi \in \mathcal{S}(\mathbb{R}^n)$ be such that

$$\operatorname{supp} \varphi \subset \{ y \in \mathbb{R}^n : \ |y| < 2 \} \quad \text{and} \quad \varphi(x) = 1 \quad \text{if} \quad |x| \leq 1 \,, \tag{1.2}$$

and for each $j \in \mathbb{N}$ let $\varphi_j(x) = \varphi(2^{-j}x) - \varphi(2^{-j+1}x)$. Then $\{\varphi_j\}_{j=0}^\infty$ forms a *smooth dyadic resolution of unity*. Given any $f \in \mathcal{S}'(\mathbb{R}^n)$, we denote by $\widehat{f}$ and $f^\vee$ its Fourier transform and its inverse Fourier transform, respectively. If $f \in \mathcal{S}'(\mathbb{R}^n)$, then the compact support of $\varphi_j \widehat{f}$ implies by the Paley-Wiener-Schwartz theorem that $(\varphi_j \widehat{f})^\vee$ is an entire analytic function on $\mathbb{R}^n$.

Definition 1.1 *Let $s \in \mathbb{R}$, $0 < p \leq \infty$, $0 < q \leq \infty$, and $\{\varphi_j\}_j$ a smooth dyadic resolution of unity. The Besov space $B^s_{p,q}(\mathbb{R}^n)$ is the set of all distributions $f \in \mathcal{S}'(\mathbb{R}^n)$ such that*

$$\big\| f \,|B^s_{p,q}(\mathbb{R}^n) \big\| = \left(\sum_{j=0}^\infty 2^{jsq} \big\| (\varphi_j \widehat{f})^\vee |L_p(\mathbb{R}^n) \big\|^q \right)^{1/q} \tag{1.3}$$

is finite (with the usual modification if $q = \infty$).

Remark 1.2 The spaces $B^s_{p,q}(\mathbb{R}^n)$ are independent of the particular choice of the smooth dyadic resolution of unity $\{\varphi_j\}_j$ appearing in their definition. They are (quasi-)Banach spaces (Banach spaces for $p, q \geq 1$), and $\mathcal{S}(\mathbb{R}^n) \hookrightarrow B^s_{p,q}(\mathbb{R}^n) \hookrightarrow \mathcal{S}'(\mathbb{R}^n)$, where the first embedding is dense if $p < \infty$ and $q < \infty$. The theory of the spaces $B^s_{p,q}(\mathbb{R}^n)$ has been developed in detail in [Tri83] and [Tri92] (and continued and extended in the more recent monographs [Tri01], [Tri06]), but has a longer history already including many contributors; we do not further want to discuss this here.
Note that the spaces $B^s_{p,q}(\mathbb{R}^n)$ contain tempered distributions which can only be *interpreted* as regular distributions (functions) for sufficiently high smoothness. More precisely, we have

$$B^s_{p,q}(\mathbb{R}^n) \subset L_1^{\mathrm{loc}}(\mathbb{R}^n) \quad \text{if, and only if,} \quad \begin{cases} s > \sigma_p, & \text{for } 0 < p \leq \infty,\ 0 < q \leq \infty, \\ s = \sigma_p, & \text{for } 0 < p \leq 1,\ 0 < q \leq 1, \\ s = \sigma_p, & \text{for } 1 < p \leq \infty,\ 0 < q \leq \min(p, 2), \end{cases} \tag{1.4}$$

cf. [ST95, Thm. 3.3.2]. In particular, for $s < \sigma_p$ one cannot interpret $f \in B^s_{p,q}(\mathbb{R}^n)$ as a regular distribution in general as may be seen from the δ-distribution which belongs to all $B^s_{p,q}(\mathbb{R}^n)$ with $s < n(\frac{1}{p} - 1)$ since $\mathcal{F}\delta = c$, recall definition (1.3).

Local means and atomic decompositions

There are equivalent characterisations for the Besov spaces $B^s_{p,q}(\mathbb{R}^n)$ in terms of *local means* and *atomic decompositions*. We first sketch the approach via local means. For further details we refer to [BPT96], [BPT97], and [Tri06] with forerunners in [Tri92, Sect. 2.5.3].
Let $B = \{y \in \mathbb{R}^n : |y| < 1\}$ be the unit ball in $\mathbb{R}^n$ and let κ be a C^∞ function in $\mathbb{R}^n$ with supp $\kappa \subset B$. Then

$$k(t, f)(x) = \int_{\mathbb{R}^n} \kappa(y) f(x + ty) \mathrm{d}y = t^{-n} \int_{\mathbb{R}^n} \kappa\Big(\frac{y - x}{t}\Big) f(y) \mathrm{d}y \tag{1.5}$$

with $x \in \mathbb{R}^n$, and $t > 0$ are *local means* (appropriately interpreted for $f \in S'(\mathbb{R}^n)$). For given $s \in \mathbb{R}$ it is assumed that the kernel κ satisfies in addition for some $\varepsilon > 0$,

$$\kappa^\vee(\xi) \neq 0 \text{ if } 0 < |\xi| < \varepsilon \qquad \text{and} \qquad (\mathrm{D}^\alpha \kappa^\vee)(0) = 0 \text{ if } |\alpha| \leq s. \tag{1.6}$$

The second condition is empty if $s < 0$. Furthermore, let κ_0 be a second C^∞ function in $\mathbb{R}^n$ with supp $\kappa_0 \subset B$ and $\kappa_0^\vee(0) \neq 0$. The meaning of $k_0(f, t)$ is defined in the same way as (1.5) with κ_0 instead of κ.
We have the following characterization in terms of local means, cf. [Tri06, Th. 1.10] and [Ryc99].

Theorem 1.3 *Let $0 < q \leq \infty$ and $s \in \mathbb{R}$. Let κ_0 and κ be the above kernels of local means. Then for $f \in S'(\mathbb{R}^n)$,*

$$\|k_0(1, f)|L_p(\mathbb{R}^n)\| + \left(\sum_{j=1}^\infty 2^{jsq} \|k(2^{-j}, f)|L_p(\mathbb{R}^n)\|^q\right)^{1/q} \tag{1.7}$$

is an equivalent (quasi-)norm in $B^s_{p,q}(\mathbb{R}^n)$.

Remark 1.4 We shall only need one part of Theorem 1.3, namely that $\|f|B^s_{p,q}(\mathbb{R}^n)\|$ can be estimated from below by (1.7). In that case some of the asumptions in (1.6) may be omitted. The inspection of the proof, cf. [Ryc99, Rem. 3], shows that if κ is a C^∞ function in $\mathbb{R}^n$ with

$$\text{supp } \kappa \subset B \qquad \text{and} \qquad \mathrm{D}^\alpha \kappa^\vee(0) = 0, \qquad |\alpha| \leq N,$$

where $N > s - 1$, then

$$\|f|B^s_{p,q}(\mathbb{R}^n)\| \geq c \left(\sum_{j=1}^\infty 2^{jsq} \|k(2^{-j}, f)|L_p(\mathbb{R}^n)\|^q\right)^{1/q}.$$

The following atomic characterization of function spaces of type $B^s_{p,q}(\mathbb{R}^n)$ is sometimes preferred (compared with the above Fourier-analytical approach), e.g. when establishing the lower bound for the dilation operators later on; we closely follow the presentation in [Tri97, Sect. 13].

Definition 1.5 *Let* $0 < p \leq \infty$, $0 < q \leq \infty$, *and* $\lambda = \{\lambda_{\nu,m} \in \mathbb{C} : \nu \in \mathbb{N}_0, m \in \mathbb{Z}^n\}$. *Then*

$$b_{p,q} = \left\{ \lambda : \|\lambda|b_{p,q}\| = \left(\sum_{\nu=0}^{\infty} \left(\sum_{m\in\mathbb{Z}^n} |\lambda_{\nu,m}|^p \right)^{q/p} \right)^{1/q} < \infty \right\}$$

(with the usual modification if $p = \infty$ *and/or* $q = \infty$*).*

Definition 1.6

(i) *Let* $K \in \mathbb{N}_0$ *and* $d > 1$. *A* K*-times differentiable complex-valued function* a *on* $\mathbb{R}^n$ (*continuous if* $K = 0$) *is called a* 1_K*-atom if*

$$\operatorname{supp} a \subset dQ_{0,m} \quad \textit{for some} \quad m \in \mathbb{Z}^n, \tag{1.8}$$

and

$$|\mathrm{D}^\alpha a(x)| \leq 1 \quad \textit{for} \quad |\alpha| \leq K.$$

(ii) *Let* $s \in \mathbb{R}$, $0 < p \leq \infty$, $K \in \mathbb{N}_0$, $L+1 \in \mathbb{N}_0$, *and* $d > 1$. *A* K*-times differentiable complex-valued function* a *on* $\mathbb{R}^n$ (*continuous if* $K = 0$) *is called an* $(s,p)_{K,L}$*-atom if for some* $\nu \in \mathbb{N}_0$

$$\operatorname{supp} a \subset dQ_{\nu,m} \quad \textit{for some} \quad m \in \mathbb{Z}^n, \tag{1.9}$$

$$|\mathrm{D}^\alpha a(x)| \leq 2^{-\nu(s-\frac{n}{p})+|\alpha|\nu} \quad \textit{for } |\alpha| \leq K, \tag{1.10}$$

and

$$\int_{\mathbb{R}^n} x^\beta a(x)\mathrm{d}x = 0 \quad \textit{if } |\beta| \leq L. \tag{1.11}$$

It is convenient to write $a_{\nu,m}(x)$ instead of $a(x)$ if this atom is located at $Q_{\nu,m}$ according to (1.8) and (1.9). Assumption (1.11) is called a *moment condition*, where $L = -1$ means that there are no moment conditions. Furthermore, K denotes the smoothness of the atom, cf. (1.10). The atomic characterization of function spaces of type $B^s_{p,q}(\mathbb{R}^n)$ is given by the following result, cf. [Tri97, Thm. 13.8].

Theorem 1.7 *Let* $0 < p \leq \infty$, $0 < q \leq \infty$, *and* $s \in \mathbb{R}$. *Let* $K \in \mathbb{N}_0$ *and* $L+1 \in \mathbb{N}_0$ *with*

$$K \geq (1+[s])_+ \qquad \textit{and} \qquad L \geq \max(-1, [\sigma_p - s])$$

be fixed. Then $f \in S'(\mathbb{R}^n)$ *belongs to* $B^s_{p,q}(\mathbb{R}^n)$ *if, and only if, it can be represented as*

$$f = \sum_{\nu=0}^{\infty} \sum_{m\in\mathbb{Z}^n} \lambda_{\nu,m} a_{\nu,m}(x), \quad \textit{convergence being in} \quad S'(\mathbb{R}^n), \tag{1.12}$$

where the $a_{\nu,m}$ *are* 1_K*-atoms* ($\nu = 0$) *or* $(s,p)_{K,L}$*-atoms* ($\nu \in \mathbb{N}$) *with*

$$\operatorname{supp} a_{\nu,m} \subset dQ_{\nu,m}, \qquad \nu \in \mathbb{N}_0, \quad m \in \mathbb{Z}^n, \quad d > 1,$$

and $\lambda \in b_{p,q}$. *Furthermore,*

$$\inf \|\lambda|b_{p,q}\|,$$

where the infimum is taken over all admissible representations (1.12), *is an equivalent* (*quasi-*)*norm in* $B^s_{p,q}(\mathbb{R}^n)$.

2 Dilation operators

Theorem 2.1 *Let* $0 < p \leq 1$, $0 < q \leq \infty$, $k \in \mathbb{N}$, *and* T_k *be defined by* (0.1). *Then*

$$\|T_k|\mathcal{L}(B^{\sigma_p}_{p,q}(\mathbb{R}^n))\| \sim 2^{k(\sigma_p - \frac{n}{p})}k^{1/q} = 2^{-kn}k^{1/q}, \tag{2.1}$$

where the constants of equivalence do not depend on k.

Proof : Step 1. We give an estimate for the upper bounds of the dilation operators T_k similar to [Vyb08, Prop. 3.2]. Since the techniques used there even fail for $p = 1$, we need to find suitable substitutes when $0 < p \leq 1$.

Recall Definition 1.1, where in particular the dyadic resolution of unity was constructed such that

$$\varphi_j(x) = \varphi(2^{-j}x) - \varphi(2^{-j+1}x), \qquad j \in \mathbb{N}.$$

Elementary calculation yields

$$(\varphi_j(\xi)\widehat{f(2^k\cdot)}(\xi))^\vee(x) = 2^{-kn}(\varphi_j(\xi)\widehat{f}(2^{-k}\xi))^\vee(x) = (\varphi_j(2^k\xi)\widehat{f}(\xi))^\vee(2^k x). \tag{2.2}$$

For convenience we assume $q < \infty$ in the sequel, but the counterpart for $q = \infty$ is obvious. From the definition of Besov spaces with $f(2^k x)$ in place of $f(x)$ we obtain

$$\begin{aligned}\|f(2^k\cdot)|B^{\sigma_p}_{p,q}\| &= \left(\sum_{j=0}^{\infty} 2^{j\sigma_p q}\|(\varphi_j(2^k\cdot)\widehat{f})^\vee(2^k\cdot)|L_p\|^q\right)^{1/q} \\ &= 2^{-k\frac{n}{p}}\left(\sum_{j=0}^{\infty} 2^{j\sigma_p q}\|(\varphi_j(2^k\cdot)\widehat{f})^\vee|L_p\|^q\right)^{1/q}.\end{aligned} \tag{2.3}$$

If $j \geq k+1$, then $\varphi_j(2^k x) = \varphi_{j-k}(x)$. This gives

$$\begin{aligned}&2^{-k\frac{n}{p}}\left(\sum_{j=k+1}^{\infty} 2^{j\sigma_p q}\|(\varphi_j(2^k\cdot)\widehat{f})^\vee|L_p\|^q\right)^{1/q} \\ =&2^{-k\frac{n}{p}}\left(\sum_{j=k+1}^{\infty} 2^{(j-k)\sigma_p q}2^{k\sigma_p q}\|(\varphi_{j-k}\widehat{f})^\vee|L_p\|^q\right)^{1/q} \\ =&2^{-k\frac{n}{p}+k\sigma_p}\left(\sum_{l=1}^{\infty} 2^{l\sigma_p q}\|(\varphi_l\widehat{f})^\vee|L_p\|^q\right)^{1/q} \\ \leq&c2^{-kn}\|f|B^{\sigma_p}_{p,q}\|.\end{aligned} \tag{2.4}$$

For the further calculations we make use of the following Fourier multiplier theorem, cf. [Tri83, Prop. 1.5.1],

$$\|(M\widehat{h})^\vee|L_p\| \leq c\|M^\vee|L_p\| \cdot \|h|L_p\|, \qquad \text{if } 0 < p \leq 1, \tag{2.5}$$

with $M^\vee \in S' \cap L_p$, and $\operatorname{supp}\widehat{h} \subset \Omega$, $\operatorname{supp} M \subset \Gamma$, where Ω and Γ are compact subsets of $\mathbb{R}^n$ (c does not depend on M and h, but may depend on Ω and Γ).
Of course, for $p = 1$, this is just the Hausdorff-Young inequality (which was also used in [Vyb08]).
We put $h = (\varphi_0\widehat{f})^\vee$, where $\operatorname{supp}\widehat{h} \subset \operatorname{supp}\varphi_0 = \Omega$.
If $j = 0$, we take $M_0 = \varphi_0(2^k\cdot)$, where $\operatorname{supp} M_0 \subset \operatorname{supp}\varphi_0 = \Gamma$, and calculate

$$\begin{aligned}2^{-k\frac{n}{p}}\|(\varphi_0(2^k\cdot)\widehat{f})^\vee|L_p\| &\leq c2^{-k\frac{n}{p}}\|\varphi_0(2^k\cdot)^\vee|L_p\| \cdot \|(\varphi_0\widehat{f})^\vee|L_p\|, \\ &= c2^{-k\frac{n}{p}}2^{k\sigma_p}\|\varphi_0{}^\vee|L_p\| \cdot \|(\varphi_0\widehat{f})^\vee|L_p\| \\ &= c'2^{k(\sigma_p-\frac{n}{p})}\|(\varphi_0\widehat{f})^\vee|L_p\| \\ &\leq c2^{k(\sigma_p-\frac{n}{p})}\|f|B^{\sigma_p}_{p,q}\| \\ &= c2^{-kn}\|f|B^{\sigma_p}_{p,q}\|.\end{aligned} \tag{2.6}$$

Finally it remains to consider $1 \le j \le k$. This is the crucial step, leading to $k^{1/q}$. In this case $\varphi_j(x) = \bar\varphi(2^{-j}x)$, where $\bar\varphi = \varphi(x) - \varphi(2x)$. Hence

$$
\begin{aligned}
&2^{-k\frac{n}{p}} \left(\sum_{j=1}^{k} 2^{j\sigma_p q} \|(\varphi_j(2^k\cdot)\widehat{f})^\vee | L_p\|^q \right)^{1/q} \\
&\quad = 2^{-k\frac{n}{p}} \left(\sum_{j=1}^{k} 2^{j\sigma_p q} \|(\bar\varphi(2^{k-j}\cdot)\widehat{f})^\vee | L_p\|^q \right)^{1/q} \\
&\quad = 2^{-k\frac{n}{p}} \left(\sum_{j=1}^{k-1} 2^{j\sigma_p q} \|(\bar\varphi(2^{k-j}\cdot)\varphi_0\widehat{f})^\vee | L_p\|^q + 2^{k\sigma_p q}\|(\bar\varphi\widehat{f})^\vee | L_p\|^q \right)^{1/q} . \qquad (2.7)
\end{aligned}
$$

The term for $j = k$ in (2.7) needs some extra care. Using (2.5) where we set $M_k = \varphi_0(2\cdot)$, $\operatorname{supp} M_k \subset \operatorname{supp} \varphi_0 = \Gamma$ we obtain

$$
\begin{aligned}
2^{k\sigma_p q}\|(\bar\varphi\widehat{f})^\vee | L_p\|^q &= 2^{k\sigma_p q}\|(\varphi_0\widehat{f})^\vee - (\varphi_0(2\cdot)\widehat{f})^\vee | L_p\|^q \\
&\le c2^{k\sigma_p q}\left(\|(\varphi_0\widehat{f})^\vee | L_p\| + \|(\varphi_0(2\cdot)\varphi_0\widehat{f})^\vee | L_p\|\right)^q \\
&\le c' 2^{k\sigma_p q}\|(\varphi_0\widehat{f})^\vee | L_p\|^q \left(1 + \|\varphi_0^\vee(2\cdot) | L_p\|\right)^q \\
&= c_1 2^{k\sigma_p q}\|(\varphi_0\widehat{f})^\vee | L_p\|^q.
\end{aligned}
$$

This estimate can be incorporated into our further calculations. Now for $1 \le j \le k-1$ we use the multiplier theorem with $M_j = \bar\varphi(2^{k-j}\cdot)$, and observe that $\operatorname{supp} M_j \subset \{x : |2^{k-j}x| \le 2\} \subset \{x : |x| \le 2\} = \Gamma$. Now (2.7) yields

$$
\begin{aligned}
&2^{-k\frac{n}{p}} \left(\sum_{j=1}^{k} 2^{j\sigma_p q} \|(\varphi_j(2^k\cdot)\widehat{f})^\vee | L_p\|^q \right)^{1/q} \\
&\le c2^{-k\frac{n}{p}} \left(\sum_{j=1}^{k-1} 2^{j\sigma_p q} \|(\bar\varphi(2^{k-j}\cdot))^\vee | L_p\|^q \cdot \|(\varphi_0\widehat{f})^\vee | L_p\|^q + 2^{k\sigma_p q}\|(\varphi_0\widehat{f})^\vee | L_p\|^q \right)^{1/q} \\
&= c_2 2^{-k\frac{n}{p}} \cdot \|(\varphi_0\widehat{f})^\vee | L_p\| \left(\sum_{j=1}^{k-1} 2^{j\sigma_p q} \|(\bar\varphi(2^{k-j}\cdot))^\vee | L_p\|^q + 2^{k\sigma_p q} \right)^{1/q} \\
&\le c_2 2^{-k\frac{n}{p}} \cdot \|f | B^{\sigma_p}_{pq}\| \left(\sum_{j=1}^{k-1} 2^{j\sigma_p q} \|(\bar\varphi(2^{k-j}\cdot))^\vee | L_p\|^q + 2^{k\sigma_p q} \right)^{1/q} \\
&\le c_2 2^{-k\frac{n}{p}} \cdot \|f | B^{\sigma_p}_{pq}\| \left(\sum_{j=1}^{k-1} 2^{j(\frac{n}{p}-n)q} \|2^{(j-k)n}\bar\varphi^\vee(2^{j-k}\cdot) | L_p\|^q + 2^{k\sigma_p q} \right)^{1/q} \\
&\le c_2 2^{-k\frac{n}{p}} \cdot \|f | B^{\sigma_p}_{pq}\| \left(\sum_{j=1}^{k-1} 2^{j(\frac{n}{p}-n)q} 2^{(j-k)nq} 2^{-(j-k)n\cdot\frac{1}{p}\cdot q} \|\bar\varphi^\vee | L_p\|^q + 2^{k\sigma_p q} \right)^{1/q} \\
&\le c_3 2^{-k\frac{n}{p}} \cdot \|f | B^{\sigma_p}_{pq}\| \left(\sum_{j=1}^{k} 2^{k\sigma_p q} \right)^{1/q} \\
&= c_3 k^{1/q} 2^{-kn} \cdot \|f | B^{\sigma_p}_{pq}\|. \qquad (2.8)
\end{aligned}
$$

Now (2.3) together with (2.4), (2.6), and (2.8) give the upper estimate.

Step 2. It remains to prove that the estimate is sharp. Let $\psi \in S(\mathbb{R}^n)$ be a non-negative function with support in $\{x \in \mathbb{R}^n : |x| \le 1/8\}$ and $\int_{\mathbb{R}^n} \psi(x)\mathrm{d}x = 1$. We show that for $0 < q \le \infty$,

$$\|\psi(2^k\cdot) | B^{\sigma_p}_{p,q}\| \ge c2^{-kn}k^{1/q}, \qquad k \in \mathbb{N}.$$

Let us take a function $\kappa \in S(\mathbb{R}^n)$ with

$$(\mathrm{D}^\alpha \kappa^\vee)(0) = 0, \qquad |\alpha| \le r, \tag{2.9}$$

where $r > \sigma_p - 1$, according to [Ryc99, Th. BPT]. In particular, by [Ryc99, Rem. 3] these conditions on κ are sufficient for our purposes; we refer as well to [Vyb08, Sect. 2.2]. Furthermore, we require

$$\kappa(x) = 1, \qquad \text{if} \qquad x \in M = \{z \in \mathbb{R}^n : |z - (1/2, 0 \ldots, 0)| < 1/4\}. \tag{2.10}$$

We construct a function κ that satisfies (2.9) and (2.10). Let us first consider the one-dimensional case $n = 1$. Put

$$f(x) = \frac{\mathrm{d}^r}{\mathrm{d}x^r} f_0(x), \qquad f_0 \in S(\mathbb{R}).$$

Then we have that

$$f^\vee(\xi) = -i^r \left(x^r f_0^\vee\right)(\xi) = -i^r \xi^r f_0^\vee(\xi).$$

In particular, for $l < r$ we calculate

$$\left(\frac{\mathrm{d}^l}{\mathrm{d}x^l} f^\vee\right)(0) = -i^r \sum_{j=0}^{l} \binom{l}{j} \frac{r!}{(r-j)!} \xi^{r-j} \frac{\mathrm{d}^{l-j}}{\mathrm{d}x^{l-j}} f_0^\vee(\xi)\Big|_{\xi=0} = 0,$$

from which we see that f satisfies the moment conditions. Needing

$$f(x) = 1, \qquad \frac{1}{4} < x < \frac{3}{4},$$

we put $f_0(x) := x^r \cdot \beta(x)$, where $\beta \in S(\mathbb{R})$ is chosen such that

$$\beta(x) = \frac{1}{r!}, \qquad x \in B_1(\frac{1}{2}).$$

The previous considerations can easily be extended to higher dimensions by setting

$$g(x_1, \ldots, x_n) = f_0(x_1) f_0(x_2 - \frac{1}{2}) \cdot \cdots \cdot f_0(x_n - \frac{1}{2}),$$

and finally

$$\kappa(x) = \mathrm{D}^{(r,\ldots,r)} g(x), \qquad x \in \mathbb{R}^n,$$

gives the desired function, if we choose $r > \sigma_p - 1$.
Simple calculation shows that if $j = 1, 2, \ldots, k$ and $|x - (\frac{1}{2} \cdot \frac{1}{2^j}, 0 \ldots, 0)| < \frac{1}{2^j}\frac{1}{8}$, then

$$\operatorname{supp}_y \psi(2^k x + 2^{k-j} y) \subset M.$$

For these x we get

$$\mathcal{K}(2^{-j}, \psi(2^k \cdot))(x) = \int_{\mathbb{R}^n} \kappa(y) \psi(2^k x + 2^{k-j} y) \mathrm{d}y = \int_{\mathbb{R}^n} \psi(2^k x + 2^{k-j} y) \mathrm{d}y = 2^{(j-k)n}.$$

Hence,

$$\|\mathcal{K}(2^{-j}, \psi(2^k \cdot))|L_p\| \ge 2^{-\frac{jn}{p}} 2^{(j-k)n} = 2^{-jn(\frac{1}{p}-1)} 2^{-kn} = 2^{-j\sigma_p} 2^{-kn}.$$

This yields

$$\begin{aligned}\|\psi(2^k \cdot)|B^{\sigma_p}_{p,q}\| &\ge c \left(\sum_{j=1}^{k} 2^{j\sigma_p q} \|\mathcal{K}(2^{-j}, \psi(2^k \cdot))|L_p\|^q\right)^{1/q} \\ &= c 2^{-kn} \left(\sum_{j=1}^{k} 1\right)^{1/q} = c 2^{-kn} k^{1/q},\end{aligned}$$

which is the desired result. $\square$

3 Applications

3.1 Besov spaces with positive smoothness on $\mathbb{R}^n$

With the help of the previous results on dilation operators, we want to discuss the connection and diversity of three different approaches to Besov spaces with positive smoothness in this section.
In addition to the Fourier-analytical approach, cf. Definition 1.1, we now present two further characterizations – associated to definitions by differences and subatomic decompositions – before we come to some comparison.

The classical approach: Besov spaces $\mathrm{B}^s_{p,q}(\mathbb{R}^n)$

If f is an arbitrary function on $\mathbb{R}^n$, $h \in \mathbb{R}^n$ and $k \in \mathbb{N}$, then

$$(\Delta_h^1 f)(x) = f(x+h) - f(x) \quad \text{and} \quad (\Delta_h^{k+1} f)(x) = \Delta_h^1(\Delta_h^k f)(x), \quad k \in \mathbb{N}.$$

For convenience we may write Δ_h instead of Δ_h^1. Furthermore, the k*-th modulus of smoothness* of a function $f \in L_p(\mathbb{R}^n)$, $0 < p \leq \infty$, $k \in \mathbb{N}$, is defined by

$$\omega_k(f,t)_p = \sup_{|h|\leq t} \|\Delta_h^k f \mid L_p(\mathbb{R}^n)\|, \quad t > 0. \tag{3.1}$$

We shall simply write $\omega(f,t)_p$ instead of $\omega_1(f,t)_p$ and $\omega(f,t)$ instead of $\omega(f,t)_\infty$.

Definition 3.1 *Let $0 < p,q \leq \infty$, $s > 0$, and $r \in \mathbb{N}$ such that $r > s$. Then the Besov space $\mathrm{B}^s_{p,q}(\mathbb{R}^n)$ contains all $f \in L_p(\mathbb{R}^n)$ such that*

$$\|f|\mathbf{B}^s_{p,q}(\mathbb{R}^n)\|_r = \|f|L_p(\mathbb{R}^n)\| + \left(\int_0^1 t^{-sq}\omega_r(f,t)_p^q \,\frac{\mathrm{d}t}{t}\right)^{1/q} \tag{3.2}$$

(with the usual modification if $q = \infty$) is finite.

Remark 3.2 These are the *classical Besov spaces*, in particular, when $1 \leq p,q \leq \infty$, $s > 0$. The study for all admitted s, p and q goes back to [SO78], we also refer to [BS88, Ch. 5, Def. 4.3] and [DL93, Ch. 2, §10]. There are as well many older references in the literature devoted to the cases $p,q \geq 1$. A recent approach including atomic characterisations is given in [HN07] based on [Net89].
Definition 3.1 is independent of r, meaning that different values of $r > s$ result in (quasi-)norms which are equivalent. Furthermore the spaces are (quasi-)Banach spaces (Banach spaces if $p,q \geq 1$). Note that we deal with subspaces of $L_p(\mathbb{R}^n)$, in particular we have the embedding

$$\mathbf{B}^s_{p,q}(\mathbb{R}^n) \hookrightarrow L_p(\mathbb{R}^n), \qquad s > 0, \quad 0 < q \leq \infty, \quad 0 < p \leq \infty.$$

The classical scale of Besov spaces contains many well-known function spaces. For example, if $p = q = \infty$, one recovers the Hölder-Zygmund spaces $\mathcal{C}^s(\mathbb{R}^n)$,

$$\mathbf{B}^s_{\infty,\infty}(\mathbb{R}^n) = \mathcal{C}^s(\mathbb{R}^n), \quad s > 0. \tag{3.3}$$

We add the following homogeneity estimate, which will serve us later on. Let $R > 0$, $s > 0$, and $0 < p,q \leq \infty$. Then

$$\left\|f(R\cdot)|\mathbf{B}^s_{p,q}(\mathbb{R}^n)\right\| \leq \; c \; \max\left(R^{-\frac{n}{p}}, R^{s-\frac{n}{p}}\right) \left\|f|\mathbf{B}^s_{p,q}(\mathbb{R}^n)\right\|. \tag{3.4}$$

To prove this we simply observe that

$$\begin{aligned}
\left\|f(R\cdot)|\mathbf{B}^s_{p,q}(\mathbb{R}^n)\right\| &= \|f(R\cdot)|L_p(\mathbb{R}^n)\| + \left(\int_0^1 t^{-sq}\omega_r(f(R\cdot),t)_p^q \frac{\mathrm{d}t}{t}\right)^{1/q} \\
&= R^{-\frac{n}{p}} \|f|L_p(\mathbb{R}^n)\| + R^{-\frac{n}{p}} \left(\int_0^1 t^{-sq}\omega_r(f,Rt)_p^q \,\frac{\mathrm{d}t}{t}\right)^{1/q} \\
&= R^{-\frac{n}{p}} \|f|L_p(\mathbb{R}^n)\| + R^{s-\frac{n}{p}} \left(\int_0^R \tau^{-sq}\omega_r(f,\tau)_p^q \,\frac{\mathrm{d}\tau}{\tau}\right)^{1/q} \\
&\leq c \max\left(R^{-\frac{n}{p}}, R^{s-\frac{n}{p}}\right) \left\|f|\mathbf{B}^s_{p,q}(\mathbb{R}^n)\right\|.
\end{aligned}$$

The subatomic approach: Besov spaces $\mathfrak{B}^s_{p,q}(\mathbb{R}^n)$

The subatomic approach provides a constructive definition for Besov spaces, expanding functions f via building blocks and suitable coefficients, where the latter belong to certain sequence spaces. For further details on the subject we refer to [Tri06, HS08]. Let

$$\mathbb{R}^n_{++} := \{y \in \mathbb{R}^n : y = (y_1, \dots, y_n),\ y_j > 0\}.$$

Definition 3.3 *Let k be a non-negative C^∞ function in $\mathbb{R}^n$ with*

$$\operatorname{supp} k \subset \left\{y \in \mathbb{R}^n : |y| < 2^{J-\varepsilon}\right\} \cap \mathbb{R}^n_{++} \tag{3.5}$$

for some fixed $\varepsilon > 0$ and some fixed $J \in \mathbb{N}$, satisfying

$$\sum_{m\in\mathbb{Z}^n} k(x-m) = 1,\ \ x \in \mathbb{R}^n. \tag{3.6}$$

Let $\beta \in \mathbb{N}^n_0$, $j \in \mathbb{N}_0$, $m \in \mathbb{Z}^n$, and let $k^\beta(x) = (2^{-J}x)^\beta k(x)$. Then

$$k^\beta_{j,m}(x) = k^\beta(2^j x - m) \tag{3.7}$$

denote the building blocks related to $Q_{j,m}$.

Definition 3.4 *Let $\varrho \geq 0$, $s \in \mathbb{R}$, $0 < p, q \leq \infty$, and*

$$\lambda = \left\{\lambda^\beta_{j,m} \in \mathbb{C} : \beta \in \mathbb{N}^n_0,\ m \in \mathbb{Z}^n,\ j \in \mathbb{N}_0\right\}.$$

Then the sequence space $b^{s,\varrho}_{p,q}$ is defined as

$$b^{s,\varrho}_{p,q} := \left\{\lambda : \|\lambda | b^{s,\varrho}_{p,q}\| < \infty\right\} \tag{3.8}$$

where

$$\|\lambda | b^{s,\varrho}_{p,q}\| = \sup_{\beta\in\mathbb{N}^n_0} 2^{\varrho|\beta|} \left(\sum_{j=0}^\infty 2^{j(s-n/p)q} \left(\sum_{m\in\mathbb{Z}^n} |\lambda^\beta_{j,m}|^p \right)^{q/p} \right)^{1/q} \tag{3.9}$$

(with the usual modification if $p = \infty$ and/or $q = \infty$).

Remark 3.5 It might not be obvious immediately, but the building blocks $k^\beta_{j,m}$ in our subatomic approach differ from the atoms a – used to characterize the spaces $B^s_{p,q}(\mathbb{R}^n)$ in Theorem 1.7 – mainly by the imposed moment conditions on the latter and some unimportant technicalities. In particular, the normalizing factors $2^{j(s-\frac{n}{p})}$ are incorporated in the sequence spaces $b^{s,\varrho}_{p,q}$ in the subatomic approach; recall Definition 1.5. We refer as well to [Tri01, Th. 3.6].

Definition 3.6 *Let $s > 0$, $0 < p \leq \infty$, $0 < q \leq \infty$, $\varrho \geq 0$. Then $\mathfrak{B}^s_{p,q}(\mathbb{R}^n)$ contains all $f \in L_p(\mathbb{R}^n)$ which can be represented as*

$$f(x) = \sum_{\beta\in\mathbb{N}^n_0} \sum_{j=0}^\infty \sum_{m\in\mathbb{Z}^n} \lambda^\beta_{j,m} k^\beta_{j,m}(x), \quad x \in \mathbb{R}^n, \tag{3.10}$$

with coefficients $\lambda = \left\{\lambda^\beta_{j,m}\right\}_{\beta\in\mathbb{N}^n_0, j\in\mathbb{N}_0, m\in\mathbb{Z}^n} \in b^{s,\varrho}_{p,q}$, and equipped with the norm

$$\left\|f | \mathfrak{B}^s_{p,q}(\mathbb{R}^n)\right\| = \inf \left\|\lambda | b^{s,\varrho}_{p,q}\right\| \tag{3.11}$$

where the infimum is taken over all possible representations (3.10).

Remark 3.7 The definitions given above follow closely [Tri06, Sect. 9.2]. The spaces $\mathfrak{B}^s_{p,q}(\mathbb{R}^n)$ are (quasi-)Banach spaces (Banach spaces for $p, q \geq 1$) and independent of k and ϱ (in terms of equivalent (quasi-)norms). Furthermore, we have the embedding

$$\mathfrak{B}^s_{p,q}(\mathbb{R}^n) \hookrightarrow L_p(\mathbb{R}^n), \qquad 0 < p \leq \infty,$$

see [Tri06, Th. 9.8].
Concerning the convergence of (3.10) one obtains as a consequence of $\lambda \in b^{s,\varrho}_{p,q}$ that the series on the right-hand side converges absolutely in $L_p(\mathbb{R}^n)$ if $p < \infty$, and in $L_\infty(\mathbb{R}^n, w_\sigma)$ if $p = \infty$, where $w_\sigma(x) = (1 + |x|^2)^{\sigma/2}$ with $\sigma < 0$. Since this implies unconditional convergence we may simplify (3.10) and write in the sequel

$$f = \sum_{\beta,j,m} \lambda^\beta_{j,m} k^\beta_{j,m}.$$

Connections and diversity

We now discuss the coincidence and diversity of the above presented concepts of Besov spaces and may restrict ourselves to positive smoothness $s > 0$. In view of our Remarks 1.2, 3.2, and 3.7 concerning the different nature of these spaces, it is obvious that there cannot be established a complete coincidence of all approaches when $s < \sigma_p$, since $\mathbf{B}^s_{p,q}(\mathbb{R}^n)$ and $\mathfrak{B}^s_{p,q}(\mathbb{R}^n)$ are always subspaces of $L_p(\mathbb{R}^n)$ and thus contain *functions*, whereas the elements of $B^s_{p,q}(\mathbb{R}^n)$ are *distributions* which can be *interpreted* as regular distributions ('functions') if, and only if, (1.4) is satisfied. However, when $s > \sigma_p$, the outcome is optimal in the sense that all approaches result in the same Besov space.

Theorem 3.8 *Let* $s > 0$, $0 < p \leq \infty$, $0 < q \leq \infty$.

(i) *Then*

$$\mathbf{B}^s_{p,q}(\mathbb{R}^n) = \mathfrak{B}^s_{p,q}(\mathbb{R}^n) \tag{3.12}$$

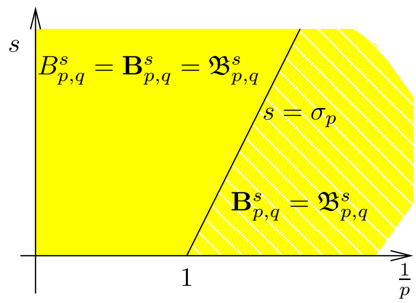
(*in the sense of equivalent norms*).

(ii) *Let* $s > \sigma_p$, *then*

$$B^s_{p,q}(\mathbb{R}^n) = \mathbf{B}^s_{p,q}(\mathbb{R}^n) = \mathfrak{B}^s_{p,q}(\mathbb{R}^n) \tag{3.13}$$

(*in the sense of equivalent norms*).

Remark 3.9 The first equality in (3.13) is longer known, see [Tri83, Thm. 2.5.12], [Tri92, Thm. 2.6.1] with forerunners in case of $p, q \geq 1$, see [Tri78, 2.5.1, 2.7.2], whereas the second equality in (3.13) is a consequence of the recently proved coincidence (3.12), see [Tri06, Prop. 9.14] (with forerunners in [Tri97, Sect. 14.15], [Tri01, Thm. 2.9]). It essentially relies on the atomic decomposition, cf. [Net89], [HN07, Thm. 1.1.14]. In the figure aside we have indicated the situation in the usual $(\frac{1}{p}, s)$-diagram.



Our results on the norms of the dilation operators T_k established in Theorem 2.1, now lead to new insights when dealing with the limiting case $s = \sigma_p$.

Corollary 3.10 *Let* $0 < p < 1$, *and* $0 < q \leq \infty$. *Then*

$$B^{\sigma_p}_{p,q}(\mathbb{R}^n) \neq \mathbf{B}^{\sigma_p}_{p,q}(\mathbb{R}^n)$$

(in terms of equivalent norms).

Proof :
We use the homogeneity estimate (3.4), which for $R = 2^k$, $s > 0$, and $0 < p, q \leq \infty$ shows

$$\left\| f(2^k \cdot) | \mathbf{B}^s_{p,q} \right\| \leq \; c \; 2^{k(s-\frac{n}{p})} \left\| f | \mathbf{B}^s_{p,q} \right\| . \tag{3.14}$$

We proceed indirectly, assuming that $B^{\sigma_p}_{p,q}(\mathbb{R}^n) = \mathbf{B}^{\sigma_p}_{p,q}(\mathbb{R}^n)$ for $0 < q \leq \infty$. But then using Theorem 2.1 and (3.14), we could find a function $\psi \in B^{\sigma_p}_{p,q}$ with

$$2^{k(\sigma_p - \frac{n}{p})} k^{1/q} \|\psi | B^{\sigma_p}_{p,q}\| \leq c \|\psi(2^k \cdot) | B^{\sigma_p}_{p,q}\| \sim \|\psi(2^k \cdot) | \mathbf{B}^{\sigma_p}_{p,q}\| \leq c 2^{k(\sigma_p - \frac{n}{p})} \|\psi | \mathbf{B}^{\sigma_p}_{p,q}\| \sim c 2^{k(\sigma_p - \frac{n}{p})} \|\psi | B^{\sigma_p}_{p,q}\|,$$

which leads to

$$k^{1/q} \leq c, \qquad k \in \mathbb{N}.$$

This gives the desired contradiction. □

Remark 3.11 Alternatively, we could use the idea from [Vyb08, Rem. 3.7], and show that the moment conditions on the line $s = \sigma_p$ are absolutely necessary. This immediately leads to

$$B^{\sigma_p}_{p,q}(\mathbb{R}^n) \neq \mathfrak{B}^{\sigma_p}_{p,q}(\mathbb{R}^n)$$

in view of Remark 3.5. We sketch the proof.
Every $f \in B^{\sigma_p}_{p,q}(\mathbb{R}^n)$ may be rewritten into its optimal atomic decomposition

$$f(x) = \sum_{\nu,m} \lambda_{\nu,m} a_{\nu,m}(x), \qquad x \in \mathbb{R}^n,$$

with

$$\|\lambda | b^{\sigma_p}_{p,q}\| \leq c \|f | B^{\sigma_p}_{p,q}\|, \qquad f \in B^{\sigma_p}_{p,q}(\mathbb{R}^n),$$

see [Tri06, Ch. 1.5] for details. If no moment conditions were required here, then

$$g_k(x) = f(2^k x) = \sum_{\nu,m} \lambda_{\nu,m} a_{\nu,m}(2^k x), \qquad x \in \mathbb{R}^n$$

would represent an atomic decomposition of $f(2^k x)$. This can be seen by setting

$$g_k(x) = \sum_{\nu,m} \lambda_{\nu,m} 2^{k(\sigma_p - \frac{n}{p})} 2^{-k(\sigma_p - \frac{n}{p})} a_{\nu,m}(2^k x) = \sum_{\nu,m} \lambda^k_{\nu,m} a^k_{\nu,m}(x),$$

where $a^k_{\nu,m}(x) = 2^{-k(\sigma_p - \frac{n}{p})} a_{\nu,m}(2^k x) \sim \tilde{a}_{\nu+k,m}(x)$, since

$$\operatorname{supp} a^k_{\nu,m} \subset Q_{\nu+k,m},$$

$$|\mathrm{D}^\alpha a^k_{\nu,m}(x)| = 2^{-k(\sigma_p - \frac{n}{p}) + k|\alpha|} |\mathrm{D}^\alpha a_{\nu,m}(x)| \leq 2^{-(\nu+k)(\sigma_p - \frac{n}{p}) + (\nu+k)|\alpha|}.$$

Therefore we obtain

$$\|g_k | B^{\sigma_p}_{p,q}\| \leq \|\lambda^k | b^{\sigma_p}_{p,q}\| = 2^{k(\sigma_p - \frac{n}{p})} \|\lambda | b^{\sigma_p}_{p,q}\| = 2^{-nk} \|\lambda | b^{\sigma_p}_{p,q}\|,$$

and thus,

$$\|f(2^k \cdot) | B^{\sigma_p}_{p,q}\| \leq c 2^{-nk} \|f | B^{\sigma_p}_{p,q}\|.$$

But we know by Theorem 2.1 that this is *not* true in general.

References

[BPT96] H.-Q. Bui, M. Paluszyński, M.H. Taibleson. A maximal function characterization of weighted Besov-Lipschitz and Triebel-Lizorkin spaces. *Studia Math.* 119:219-246, 1996.

[BPT97] H.-Q. Bui, M. Paluszyński, M.H. Taibleson. Characterization of weighted Besov-Lipschitz and Triebel-Lizorkin spaces. The case $q < 1$. *Journ. Fourier Anal. Appl. 3*, special issue, 837-846, 1997.

[BS88] C. Bennett and R. Sharpley. *Interpolation of operators*. Academic Press, Boston, 1988.

[DL93] R. DeVore and G.G. Lorentz. *Constructive approximation*, volume 303 of *Grundlehren der Mathematischen Wissenschaften.* Springer, Berlin, 1993.

[ET96] D.E. Edmunds and H. Triebel. *Function spaces, entropy numbers, differential operators.* Cambridge Univ. Press, Cambridge, 1996.

[HN07] L.I. Hedberg and Y. Netrusov. An axiomatic approach to function spaces, spectral synthesis, and Luzin approximation. *Mem. Amer. Math. Soc.*, 188(882):97 p., 2007.

[HS08] D.D. Haroske and C. Schneider. Besov spaces with positive smoothness on $\mathbb{R}^n$, embeddings and growth envelopes. Jenaer Schriften zur Mathematik und Informatik, Math/Inf/02/08, 2008.

[Net89] Yu. Netrusov. Sets of singularities of functions in spaces of Besov and Lizorkin-Triebel type. *Trudy Mat. Inst. Steklov*, 187:162–177, 1989. Russian, Engl. Transl.: Proc. Steklov Inst. Math. 187, 185-203, 1990.

[Ryc99] V.S. Rychkov. On a theorem of Bui, Paluszyński, and Taibleson. *Proc. Steklov Inst. Math.*, 227:280-292, 1999.

[SO78] È.A. Storoženko and P. Oswald. Jackson's theorem in the spaces $L^p(\mathbf{R}^k)$, $0 < p < 1$. *Sibirsk. Mat. Ž.*, 19(4):888–901, 956, 1978. Russian; Engl. transl.: Siberian Math. J. 19 (1978), no. 4, 630–640 (1979).

[ST95] W. Sickel and H. Triebel. Hölder inequalities and sharp embeddings in function spaces of $B^s_{p,q}$ and $F^s_{p,q}$ type. *Z. Anal. Anwendungen*, 14:105–140, 1995.

[Tri78] H. Triebel. *Interpolation theory, function spaces, differential operators.* North-Holland, Amsterdam, 1978.

[Tri83] H. Triebel. *Theory of function spaces.* Birkhäuser, Basel, 1983.

[Tri92] H. Triebel. *Theory of function spaces II.* Birkhäuser, Basel, 1992.

[Tri97] H. Triebel. *Fractals and spectra.* Birkhäuser, Basel, 1997.

[Tri01] H. Triebel. *The structure of functions.* Birkhäuser, Basel, 2001.

[Tri06] H. Triebel. *Theory of function spaces III.* Birkhäuser, Basel, 2006.

[Vyb08] J. Vybíral. Dilation operators and sampling numbers. *to appear in J. of Function spaces and applications.*

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