On Diophantine quintuples

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1 Introduction

The Greek mathematician Diophantus of Alexandria noted that the set $\{\frac{1}{16}, \frac{33}{16}, \frac{17}{4}, \frac{105}{16}\}$ has the following property: the product of any two of its distinct elements increased by 1 is a square of a rational number (see [5]). Fermat first found a set of four positive integers with the above property, and it was $\{1, 3, 8, 120\}$.

Let n be an integer. A set of positive integers $\{x_1, x_2, \ldots, x_m\}$ is said to have *the property* $D(n)$ if for all $1 \leq i < j \leq m$ the following holds: $x_i x_j + n = y_{ij}^2$, where y_{ij} is an integer. Such a set is called *a Diophantine* m*-tuple.*

Davenport and Baker [4] showed that if d is a positive integer such that the set $\{1, 3, 8, d\}$ has the property of Diophantus, then d has to be 120. This implies that the Diophantine quadruple $\{1, 3, 8, 120\}$ cannot be extended to the Diophantine quintuple with the property $D(1)$. Analogous result was proved for the Diophantine quadruple $\{2, 4, 12, 420\}$ with the property $D(1)$ [17], for the Diophantine quadruple $\{1, 5, 12, 96\}$ with the property $D(4)$ [15] and for the Diophantine quadruples $\{k-1, k+1, 4k, 16k^3-4k\}$ with the property $D(1)$ for almost all positive integers k [9].

Euler proved that every Diophantine pair $\{x_1, x_2\}$ with the property $D(1)$ can be extended in infinitely many ways to the Diophantine quadruple with the same property (see [12]). In [6] it was proved that the same conclusion is valid for the pair with the property $D(l^2)$ if the additional condition that $x_1 x_2$ is not a perfect square is fulfilled.

Arkin, Hoggatt and Strauss [3] proved that every Diophantine triple with the property $D(1)$ can be extended to the Diophantine quadruple. More precisely, if $x_i x_j + 1 = y_{ij}^2$, then we can set $x_4 = x_1 + x_2 + x_3 + 2x_1x_2x_3 + 2y_{12}y_{13}y_{23}$. For the Diophantine quadruple obtained in this way, they proved the existence of a positive rational number x_5 with the property that $x_i x_5 + 1$ is a square of a rational number for $i = 1, 2, 3, 4$.

Using this construction, in [2, 7, 8, 11] some formulas for Diophantine quintuples in the terms of polynomials, Fibonacci, Lucas, Pell and Pell-Lucas numbers were obtained.

In the present paper we prove that for all positive rational numbers q, x_1, x_2, x_3, x_4 such that $x_i x_j + q^2 = y_{ij}^2$, $y_{ij} \in \mathbf{Q}$, for $1 \le i < j \le 4$, and $x_1x_2x_3x_4 \ne q^4$, there exists a positive rational number x_5 such that $x_i x_5 + q^2$ is a square of a rational number for $i = 1, 2, 3, 4$. As a corollary we get the result that for *all* Diophantine quadruples $\{x_1, x_2, x_3, x_4\}$ with the property $D(1)$ there exists a rational number x_5 such that $x_i x_5 + 1$ is a square of a rational number for $i = 1, 2, 3, 4$.

2 Extension of Diophantine quadruples

THEOREM 1 *Let q, x_1, x_2, x_3, x_4 be rational numbers such that $x_i x_j + q^2 = y_{ij}^2$, $y_{ij} \in \mathbf{Q}$, for all $1 \le i < j \le 4$. Assume that $x_1x_2x_3x_4 \ne q^4$. Then the rational number $x_5 = A/B$, where*

$$\begin{aligned} A &= q^3[2y_{12}y_{13}y_{14}y_{23}y_{24}y_{34} + qx_1x_2x_3x_4(x_1+x_2+x_3+x_4) \\ &+ 2q^3(x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_4 + x_2x_3x_4) + q^5(x_1+x_2+x_3+x_4)]\,, \end{aligned}$$

$$B = (x_1x_2x_3x_4 - q^4)^2\,,$$

has the property that $x_i x_5 + q^2$ is a square of a rational number for $i = 1, 2, 3, 4$. To be more precise, for $i \in \{1, 2, 3, 4\}$ it holds:

$$x_i x_5 + q^2 = \left(q\frac{x_i y_{jk} y_{jl} y_{kl} + q y_{ij} y_{ik} y_{il}}{x_1x_2x_3x_4 - q^4}\right)^2,$$

where $\{i, j, k, l\} = \{1, 2, 3, 4\}$.

Proof. Let $i \in \{1, 2, 3, 4\}$ and $\{i, j, k, l\} = \{1, 2, 3, 4\}$. Then we have:

$$\begin{aligned} &(x_1x_2x_3x_4 - q^4)^2(x_i x_5 + q^2) \\ &\quad = 2q^3 x_i y_{12}y_{13}y_{14}y_{23}y_{24}y_{34} + q^4 x_1x_2x_3x_4 x_i(x_1+x_2+x_3+x_4) \\ &\qquad + 2x_i q^6(x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_4 + x_2x_3x_4) \\ &\qquad + x_i q^8(x_1+x_2+x_3+x_4) + q^2x_1^2x_2^2x_3^2x_4^2 - 2q^6x_1x_2x_3x_4 + q^{10} \\ &\quad = q^2[2qx_i y_{12}y_{13}y_{14}y_{23}y_{24}y_{34} + q^2x_i^2x_jx_kx_l(x_i+x_j+x_k+x_l) \\ &\qquad + 2q^4x_i^2(x_jx_k + x_jx_l + x_kx_l) + 2q^4x_ix_jx_kx_l + q^6x_i^2 \\ &\qquad + q^6(x_ix_j + x_ix_k + x_ix_l) + x_i^2x_j^2x_k^2x_l^2 - 2q^4x_ix_jx_kx_l + q^8] \\ &\quad = q^2[2qx_i y_{12}y_{13}y_{14}y_{23}y_{24}y_{34} + x_i^2(x_jx_k + q^2)(x_jx_l + q^2)(x_kx_l + q^2) \\ &\qquad + q^2(x_ix_j + q^2)(x_ix_k + q^2)(x_ix_l + q^2)] \\ &\quad = q^2(2qx_i y_{12}y_{13}y_{14}y_{23}y_{24}y_{34} + x_i^2y_{jk}^2y_{jl}^2y_{kl}^2 + q^2y_{ij}^2y_{ik}^2y_{il}^2) \\ &\quad = [q(x_i y_{jk}y_{jl}y_{kl} + q y_{ij}y_{ik}y_{il})]^2\,, \end{aligned}$$

which proves the theorem. ■

Since the signs of y_{ij} are arbitrary, we have two choices for x_5. Let x_5^+ and x_5^- denote these two numbers, and let x_5^+ be the number which corresponds to the case where all y_{ij} are nonnegative.

COROLLARY 1 *Let* $\{x_1, x_2, x_3, x_4\} \subset \mathbf{N}$ *be the set with the property* $D(1)$. *Then there exists a rational number* x_5, $0 < x_5 < 1$, *such that* $x_i x_5 + 1$ *is a square of a rational number for* $i = 1, 2, 3, 4$.

Proof. We claim that the number x_5^+, obtained by applying the construction from Theorem 1 to the set $\{x_1, x_2, x_3, x_4\}$, has the desired property. Indeed, it is sufficient to prove that $x_5^+ < 1$. Let us introduce the following notation:

$$\begin{aligned}
\sigma_1 &= x_1 + x_2 + x_3 + x_4 \\
\sigma_2 &= x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4 \\
\sigma_3 &= x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_4 + x_2x_3x_4 \\
\sigma_4 &= x_1x_2x_3x_4 \\
X &= \sigma_1\sigma_4 + 2\sigma_3 + \sigma_1 \\
Y &= y_{12}y_{13}y_{14}y_{23}y_{24}y_{34} \,.
\end{aligned}$$

The proof that $x_5^+ = \dfrac{2Y + X}{(\sigma_4 - 1)^2} < 1$ is completed by showing that

$$2X < (\sigma_4 - 1)^2 \quad \text{and} \quad 4Y < (\sigma_4 - 1)^2 \,. \tag{1}$$

Without loss of generality we can assume that $x_1 < x_2 < x_3 < x_4$. If $x_1 = 1$, then $x_2 \neq 2$. Therefore, $x_2 \geq 3$, $x_3 \geq 4$ and $x_4 \geq 5$. Hence $\sigma_4 \geq 60$. Furthermore, from

$$\frac{1}{x_1x_2x_3} + \frac{1}{x_1x_2x_4} + \frac{1}{x_1x_3x_4} + \frac{1}{x_2x_3x_4} \leq \frac{13}{60} < \frac{1}{4}$$

it follows that $52 \leq 4\sigma_1 < \sigma_4$. In the same manner we can see that $59 \leq \sigma_2 < \sigma_4$ and $107 \leq \sigma_3 < 2\sigma_4$ (see also [12]). Hence

$$(\sigma_4 - 1)^2 - 2X > \sigma_4^2 - 2\sigma_4 + 1 - \frac{\sigma_4^2}{2} - 8\sigma_4 - \frac{\sigma_4}{2} = \frac{1}{2}(\sigma_4^2 - 21\sigma_4 + 2) > 0$$

(since $\sigma_4 \geq 60$). To get the second inequality from (1), we note that

$$Y^2 = (x_1x_2 + 1)(x_1x_3 + 1)(x_1x_4 + 1)(x_2x_3 + 1)(x_2x_4 + 1)(x_3x_4 + 1)$$

$$
\begin{aligned}
&= \sigma_4^3 + \sigma_2\sigma_4^2 - \sigma_4^2 + \sigma_1\sigma_3\sigma_4 + \sigma_1^2\sigma_4 - 2\sigma_2\sigma_4 + \sigma_3^2 - \sigma_4 + \sigma_1\sigma_3 + \sigma_2 + 1 \\
&< \sigma_4^3 + \sigma_4^3 - \sigma_4^2 + \frac{\sigma_4^2}{2} + \frac{\sigma_4^2}{16} - 118\sigma_4 + 4\sigma_4^2 - \sigma_4 + \frac{\sigma_4^2}{2} + \sigma_4 + 1 \\
&= \frac{41}{16}\sigma_4^3 + \frac{7}{2}\sigma_4^2 - 118\sigma_4 + 1\,.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&(\sigma_4 - 1)^4 - 16Y^2 \\
&\qquad > \sigma_4^4 - 4\sigma_4^3 + 6\sigma_4^2 - 4\sigma_4 + 1 - 41\sigma_4^3 - 56\sigma_4^2 + 1888\sigma_4 - 16 \\
&\qquad = \sigma_4^4 - 45\sigma_4^3 - 50\sigma_4^2 + 1884\sigma_4 - 15 > 0
\end{aligned}
$$

(since $\sigma_4 \geq 60$), which completes the proof. ■

COROLLARY 2 *Let q, x_1, x_2, x_3 be rational numbers such that $x_ix_j + q^2 = y_{ij}^2$, $y_{ij} \in \mathbf{Q}$ for all $1 \leq i < j \leq 3$. Let*

$$x_4 = [2y_{12}y_{13}y_{23} + 2x_1x_2x_3 + q^2(x_1 + x_2 + x_3)]/q^2\,,$$

$$x_5 = \frac{4y_{12}y_{13}y_{23}(x_1y_{23} + y_{12}y_{13})(x_2y_{13} + y_{12}y_{23})(x_3y_{12} + y_{13}y_{23})}{(x_1x_2x_3x_4 - q^4)^2}\,.$$

Then the set $\{x_1, x_2, x_3, x_4, x_5\}$ has the property that the product of its any two distinct elements increased by q^2 is equal to the square of a rational number. In the notation of Theorem 1, we have

$$x_5 = \frac{4q^3y_{12}y_{13}y_{14}y_{23}y_{24}y_{34}}{(x_1x_2x_3x_4 - q^4)^2}\,.$$

Proof. Let $z_1 = x_1$, $z_2 = x_2$, $z_3 = x_3$, $z_4 = 0$. Then the rational numbers z_1, z_2, z_3, z_4 satisfy the conditions of Theorem 1, and its application gives us the number

$$z_5 = [2y_{12}y_{13}y_{23} + 2x_1x_2x_3 + q^2(x_1 + x_2 + x_3)]/q^2\,.$$

Set $x_4 = z_5$. We can now apply Theorem 1 on the numbers x_1, x_2, x_3, x_4. Let x_5 be the number which is obtained by this construction. Observe that, by Theorem 1, for all $i \in \{1, 2, 3\}$

$$qy_{i4} = x_iy_{jk} + y_{ij}y_{ik}\,, \tag{2}$$

where $\{i,j,k\}=\{1,2,3\}$. Let us introduce the following notation:

$$\begin{aligned}
\Sigma_1 &= x_1+x_2+x_3 \\
\Sigma_2 &= x_1x_2+x_1x_3+x_2x_3 \\
\Sigma_3 &= x_1x_2x_3 \\
V &= y_{12}y_{13}y_{23} \\
W &= y_{14}y_{24}y_{34}\,.
\end{aligned}$$

We have

$$V^2=(x_1x_2+q^2)(x_1x_3+q^2)(x_2x_3+q^2)=\Sigma_3^2+q^2\Sigma_1\Sigma_3+q^4\Sigma_2+q^6\,.$$

From (2) it follows that

$$\begin{aligned}
q^3W &= (x_1y_{23}+y_{12}y_{13})(x_2y_{13}+y_{12}y_{23})(x_3y_{12}+y_{13}y_{23}) \\
&= 4\Sigma_3^2+3q^2\Sigma_1\Sigma_3+2q^4\Sigma_2+q^6+V(4\Sigma_3+q^2\Sigma_1)\,.
\end{aligned}$$

Now it is easy to check that, in notation of Corollary 1,

$$q^4\sigma_1\sigma_4+2q^6\sigma_3+q^8\sigma_1=2q^3VW\,. \tag{3}$$

Consequently,

$$\begin{aligned}
x_5 &= \frac{4q^3VW}{(x_1x_2x_3x_4-q^4)^2} \\
&= \frac{4q^3y_{12}y_{13}y_{14}y_{23}y_{24}y_{34}}{(x_1x_2x_3x_4-q^4)^2} \\
&= \frac{4y_{12}y_{13}y_{23}(x_1y_{23}+y_{12}y_{13})(x_2y_{13}+y_{12}y_{23})(x_3y_{12}+y_{13}y_{23})}{(x_1x_2x_3x_4-q^4)^2}\,.
\end{aligned}$$

■

Let us now consider the question when one or both (since x_5^+ and x_5^- can be equal) of the numbers x_5^+ and x_5^- will be equal to zero. For the obvious reason, such extension of a Diophantine quadruple we will call trivial. We will see that the answer to this question is closely connected to the construction of Corollary 2. From now on, we assume that $q\neq 0$.

PROPOSITION 1 *In the notation of Theorem 1, we have* $x_5^+=x_5^-=0$ *if and only if there exist* $1\le i<j\le 4$ *such that* $x_ix_j=-q^2$ *and* $x_i+x_j=x_k+x_l$*, where* $\{i,j,k,l\}=\{1,2,3,4\}$.

Proof. From $x_5^+ = x_5^-$ we conclude that there exist $1 \leq i < j \leq 4$ such that $y_{ij} = 0$, i.e. $x_i x_j = -q^2$. Substituting this into expression for x_5 we obtain

$$x_5 = \frac{q^2(x_i + x_j - x_k - x_l)}{x_k x_l + q^2}\,. \tag{4}$$

Consequently, the condition $x_5 = 0$ implies that $x_i + x_j = x_k + x_l$.

Conversely, suppose that x_1, x_2, x_3, x_4 satisfy the condition of the proposition. Then $y_{ij} = 0$, and (4) implies that $x_5^+ = x_5^- = 0$. ∎

PROPOSITION 2 *In the notation of Theorem 1, we have* $0 \in \{x_5^+, x_5^-\}$ *if and only if there exists* $i \in \{1,2,3,4\}$ *such that*

$$x_i = [2y_{jk}y_{jl}y_{kl} + 2x_j x_k x_l + q^2(x_j + x_k + x_l)]/q^2\,,$$

where $\{i,j,k,l\} = \{1,2,3,4\}$.

Proof. We can assume that $y_{ij} \neq 0$ since otherwise the assertion of the proposition follows from Proposition 1. If $x_5 = 0$, then $x_i x_5 + q^2 = q^2$ for $i = 1,2,3,4$. Hence, if $0 \in \{x_5^+, x_5^-\}$, then Theorem 1 implies that for appropriate choice of the sign of y_{ij} we have

$$x_i y_{jk} y_{jl} y_{kl} + q y_{ij} y_{ik} y_{il} = \pm(x_1 x_2 x_3 x_4 - q^4)\,,$$

where $\{i,j,k,l\} = \{1,2,3,4\}$. Hence, there is no loss of generality is assuming that

$$x_1 y_{23} y_{24} y_{34} + q y_{12} y_{13} y_{14} = x_2 y_{13} y_{14} y_{34} + q y_{12} y_{23} y_{24}\,.$$

This gives $(x_1 y_{34} - q y_{12}) y_{23} y_{24} = (x_2 y_{34} - q y_{12}) y_{13} y_{14}$. Set $x_1 y_{34} - q y_{12} = \alpha y_{13} y_{14}$. Then $x_2 y_{34} - q y_{12} = \alpha y_{23} y_{24}$, and so

$$\begin{aligned} &\alpha(x_1 y_{23} y_{24} y_{34} + q y_{12} y_{13} y_{14}) \\ &\quad = x_1 y_{34}(x_2 y_{34} - q y_{12}) + q y_{12}(x_1 y_{34} - q y_{12}) \\ &\quad = x_1 x_2 y_{34}^2 - q^2 y_{12}^2 = x_1 x_2 x_3 x_4 - q^4\,. \end{aligned}$$

We thus get $\alpha = \pm 1$ and $x_1 y_{34} - q y_{12} = \pm y_{13} y_{14}$. Squaring this relation we obtain

$$x_1^2 x_3 x_4 + q^2 x_1^2 + q^2 x_1 x_2 + q^4 - 2q x_1 y_{12} y_{34} = x_1^2 x_3 x_4 + q^2 x_1 x_3 + q^2 x_1 x_4 + q^4\,,$$

and (if $x_1 \neq 0$) $2y_{12}y_{34} = q(x_1 + x_2 - x_3 - x_4)$. Squaring again we obtain the quadratic equation in x_4:

$$q^2x_4^2 - 2x_4[q^2(x_1 + x_2 + x_3) + 2x_1x_2x_3] \\ + q^2(x_1^2 + x_2^2 + x_3^2 - 2x_1x_2 - 2x_1x_3 - 2x_2x_3 - 4q^2) = 0\,,$$

with the solutions

$$x_4 = [q^2(x_1 + x_2 + x_3) + 2x_1x_2x_3 \pm 2y_{12}y_{13}y_{23}]/q^2\,. \quad (5)$$

We have been working under assumption that $x_1 \neq 0$. Now suppose that $x_1 = 0$. In the same manner, using Corollary 2, it can be proved that $x_1 = 0$ implies

$$x_4 = x_2 + x_3 \pm y_{23}\,,$$

which is exactly the relation (5) for $x_1 = 0$.

This proves one implication of the proposition. The opposite implication is direct consequence of the relation (3). ■

3 Examples

Example 1 Let us first show that the condition $x_1x_2x_3x_4 \neq q^4$ from Theorem 1 is not superfluous. Indeed, the set $\{25600, 50625, 82944, 518400\}$ has the property $D(86400^2)$ and

$$25600 \cdot 50625 \cdot 82944 \cdot 518400 = 86400^4.$$

As an illustration of the situation from Proposition 1 let us adduce the set $\{-25, 25, -24, 24\}$ with the property $D(625)$ and the set $\{-1, 64, 48, 15\}$ with the property $D(64)$. In both cases the construction from Theorem 1 gives $x_5^+ = x_5^- = 0$.

From [10, (13)], for $a = 2$ and $k = 3$, we obtain the Diophantine quadruple $\{2, 20, 44, 72\}$ with the property $D(81)$. It is easy to check that this quadruple does not satisfy the conditions of Proposition 2. Therefore the numbers x_5^+ and x_5^- are different from 0. Indeed, $x_5^+ = \frac{4860}{169}$ and $x_5^- = -\frac{1156680}{1054729}$. Using x_5^+, we obtain the Diophantine quintuple $\{338, 3380, 4860, 7436, 12168\}$ with the property $D(39^4)$.

If we apply the construction from Theorem 1 to the original set of Diophantus $\{\frac{1}{16}, \frac{33}{16}, \frac{17}{4}, \frac{105}{16}\}$, we obtain $x_5^+ = \frac{549120}{101^2}$ and $x_5^- = \frac{-26880}{421^2}$.

The definition of a Diophantine m-tuple can be extended to the subsets of $\mathbf{Q}$. Let q be a rational number. We call a set $A = \{x_1, x_2, \ldots, x_m\} \subset$

$\mathbf{Q}\setminus\{0\}$ *a (rational) Diophantine m-tuple with the property $D(q)$* if the product of any two distinct elements of A increased by q is equal to the square of a rational number. The construction of the rational Diophantine quintuple with the property $D(1)$ which extends the given Diophantine triple was described in [3]. That construction is equivalent to the construction from Corollary 2. But Theorem 1 makes possible the extension of the Diophantine quadruples which are not of the form $\{x_1, x_2, x_3, x_4\}$ from Corollary 2. One such quadruple is the set $\{2, 20, 44, 72\}$ from Example 1. Let us now examine two ways for generation of such Diophantine quadruples.

EXAMPLE 2 Let $\{x_1, x_2, x_3, x_4\} \subset \mathbf{Q}$ be an arbitrary set with the property $D(q^2)$ and let $x_5 \in \mathbf{Q}$ be the number which is obtained by applying Theorem 1 to this set. Then the set $\{x_2, x_3, x_4, x_5\}$ also has the property $D(q^2)$, and we can apply Theorem 1 again. In this way we obtain $x_6 \in \mathbf{Q}$ such that the set $\{x_2, x_3, x_4, x_5, x_6\}$ has the property $D(q^2)$.

For example, if $x_1 = k-1$, $x_2 = k+1$, $x_3 = 4k$ and $x_4 = 16k^3-4k$, then the set $\{x_1, x_2, x_3, x_4\}$ has the property $D(1)$ ([6, p. 22]) and we obtain

$$x_5 = \frac{4k(2k-1)(2k+1)(4k^2-2k-1)(4k^2+2k-1)(8k^2-1)}{(64k^6-80k^4+16k^2-1)^2},$$

and $x_6 = P(k)/Q(k)$, where

$$\begin{aligned} P(k) &= (8k^3-4k^2+1)(8k^3+4k^2-4k-1)(8k^3-12k^2+1) \\ &\quad \times (8k^4+4k^3-8k^2-k+1)(32k^4-8k^3+28k^2+3) \\ &\quad \times (32k^4+8k^3-12k^2+1)(32k^4+24k^3-12k^2-4k+1) \\ &\quad \times (32k^4+40k^3+4k^2-4k+1)\,, \end{aligned}$$

$$\begin{aligned} Q(k) &= (131072k^{14}+131072k^{13}-184320k^{12}-180224k^{11}+96256k^{10} \\ &\quad +86016k^9-26880k^8-18432k^7+4480k^6+1792k^5-480k^4 \\ &\quad -64k^3+32k^2-1)^2\,. \end{aligned}$$

It turns out that this factorization of the numerator of x_6 is not accidental. Namely, it can be checked that, in notation of Theorem 1, $x_6 = P/Q$, where

$$\begin{aligned} P = q^3&(y_{12}y_{13}y_{14}+qy_{12}y_{13}+qy_{12}y_{23}+qy_{13}y_{23}) \\ &\times (y_{12}y_{13}y_{14}+qy_{12}y_{13}-qy_{12}y_{23}-qy_{13}y_{23}) \\ &\times (y_{12}y_{13}y_{14}-qy_{12}y_{13}+qy_{12}y_{23}-qy_{13}y_{23}) \\ &\times (y_{12}y_{13}y_{14}-qy_{12}y_{13}-qy_{12}y_{23}+qy_{13}y_{23})(y_{23}y_{24}+y_{23}y_{34}+y_{24}y_{34}) \\ &\times (y_{23}y_{24}+y_{23}y_{34}-y_{24}y_{34})(y_{23}y_{24}-y_{23}y_{34}+y_{24}y_{34}) \\ &\times (-y_{23}y_{24}+y_{23}y_{34}+y_{24}y_{34})\,, \end{aligned}$$

$$Q = x_1^4(4x_2x_3x_4y_{12}y_{13}y_{14}y_{23}y_{24}y_{34} - qx_1^2x_2^2x_3^2x_4^2 + 2q^5x_1x_2x_3x_4 - q^9)^2 .$$

Proposition 3 *Let x_1, x_2, and x_3 be rational numbers such that the denominator of*

$$x_4 = \frac{8(x_3 - x_1 - x_2)(x_1 + x_3 - x_2)(x_2 + x_3 - x_1)}{(x_1^2 + x_2^2 + x_3^2 - 2x_1x_2 - 2x_1x_3 - 2x_2x_3)^2}$$

is different from 0. *Then $x_1x_4 + 1$, $x_2x_4 + 1$ and $x_3x_4 + 1$ are squares of rational numbers.*

Proof. It follows immediately that

$$x_1x_4 + 1 = \left(\frac{x_2^2 - 2x_2x_3 + x_3^2 - 3x_1^2 + 2x_1x_2 + 2x_1x_3}{x_1^2 + x_2^2 + x_3^2 - 2x_1x_2 - 2x_1x_3 - 2x_2x_3}\right)^2 ,$$

and analogous relations hold for $x_2x_4 + 1$ and $x_3x_4 + 1$. ■

Example 3 Let us observe that the set $\{x_1, x_2, x_3\}$ in Proposition 3 does not need to have the property $D(1)$. Let us take for example $x_1 = F_{2n+1}$, $x_2 = F_{2n+3}$ and $x_3 = F_{2n+5}$. Then the set $\{x_1, x_2, x_3\}$ has the property $D(-1)$ for every positive integer n (see [13, 14]). Proposition 3 implies that there exists a rational number x_4 with the property that $x_ix_4 + 1$, $i = 1, 2, 3$, are squares of rational numbers. We will show that in this case the number x_4 is an integer. Indeed,

$$\begin{aligned}
&x_1^2 + x_2^2 + x_3^2 - 2x_1x_2 - 2x_1x_3 - 2x_2x_3 \\
&\quad = (x_1 - x_2 + x_3)^2 - 4x_1x_3 \\
&\quad = [F_{2n+1} - F_{2n+3} + (3F_{2n+3} - F_{2n+1})]^2 - 4F_{2n+1}F_{2n+5} \\
&\quad = 4(F_{2n+3}^2 - F_{2n+1}F_{2n+5}) = -4 .
\end{aligned}$$

Hence,

$$x_4 = \frac{8}{16} \cdot 2F_{2n+2} \cdot 2F_{2n+3} \cdot 2F_{2n+4} = 4F_{2n+2}F_{2n+3}F_{2n+4} .$$

Example 4 If $x_1x_2 + 1 = y_{12}^2$ and $x_3 = x_1 + x_2 + 2y_{12}$, then the set $\{x_1, x_2, x_3\}$ has the property $D(1)$. If we apply the construction from Proposition 3 to this set we obtain

$$x_4 = 4y_{12}(x_1 + y_{12})(x_2 + y_{12}) .$$

If we apply the construction from Corollary 2 to the set $\{x_1, x_2, x_3\}$ we obtain exactly the same result.

EXAMPLE 5 Let $x_1 = 1$, $x_2 = 3$ and $x_3 = 120$. Then proposition 3 gives $x_4 = \frac{834968}{3361^2}$. The set $\{x_1, x_2, x_3, x_4\}$ has the property $D(1)$ and we can apply the construction from Theorem 1. We obtain:

$$x_5^+ = \frac{3985166705520 \cdot 481^2}{601439^2 \cdot 481^2}\,, \qquad x_5^- = \frac{426360 \cdot 601439^2}{481^2 \cdot 601439^2}\,.$$

It turns out that this cancelation is not accidental. Namely, let $\{x_1, x_2, x_3\}$ be the arbitrary set with the property $D(1)$, let x_4 be the number which is obtained by applying Proposition 3 to this set, and let x_5^+ and x_5^- be the numbers which are obtained by applying Theorem 1 to the set $\{x_1, x_2, x_3, x_4\}$. Then

$$\sqrt{x_1 x_5^+ + 1} \cdot \sqrt{x_1 x_5^- + 1} = |\frac{(a+b)(a-b)cd}{c^2 d^2}|\,,$$

where

$$\begin{aligned} a &= x_1 y_{23}[x_1^2(4x_2x_3+1) - 2x_1(x_2+x_3)(2x_2x_3-1) - (3x_2^2+2x_2x_3+3x_2^2)]\,, \\ b &= y_{12}y_{13}[x_1^2(-4x_2x_3-3) + 2x_1(x_2+x_3)(2x_2x_3+1) + (x_2-x_3)^2]\,, \\ c &= (x_1+x_2+x_3)^2 - 4(x_1x_2x_3 - y_{12}y_{13}y_{23})^2 + 4\,, \\ d &= 4(x_1x_2x_3 + y_{12}y_{13}y_{23})^2 - (x_1+x_2+x_3)^2 - 4\,. \end{aligned}$$

For $x_1 = 1$, $x_2 = 3$ and $x_3 = 120$, we get $c = 4 \cdot 481$ and $d = 4 \cdot 601439$.

4 Some open problems

One question still unanswered is whether there exists a (positive integer) Diophantine quintuple with the property $D(1)$. Corollary 1 shows that if such a quintuple exists it cannot be obtained by the construction from Theorem 1. Let us mention that the analogous result for the sets with the property $D(l^2)$, where $l > 1$, does not hold. For example, if we apply the construction from Theorem 1 to the quadruples $\{4, 21, 69, 125\}$ and $\{7, 12, 63, 128\}$ with the property $D(400)$, we obtain $x_5^+ = 384$, $x_5^- = -\frac{4032000}{1129^2}$ and $x_5^+ = 375$, $x_5^- = -\frac{11856000}{2021^2}$, respectively. Hence, the sets $\{4, 21, 69, 125, 384\}$ and $\{7, 12, 69, 125, 375\}$ are Diophantine quintuples with the property $D(400)$.

One may ask which is the least positive integer n_1, and which is the greatest negative integer n_2, for which there exists a Diophantine quintuple with the property $D(n_i)$, $i = 1, 2$. Certainly $n_1 \leq 256$ and $n_2 \geq -255$, since the sets $\{1, 33, 105, 320, 18240\}$ and $\{5, 21, 64, 285, 6720\}$ have the property $D(256)$, and the set $\{8, 32, 77, 203, 528\}$ has the property $D(-255)$.

In present paper we have considered the quintuples with the property $D(q)$, where q was a square of a rational number. However, the last set with the property $D(-255)$ indicates that there exist quintuples with the property $D(q)$, where q is not a perfect square (see also [9, 16]). Thus we came to the following open problem: For which rational numbers q there exists a rational Diophantine quintuple with the property $D(q)$? It follows easily from [6, Theorem 5] that for every rational number q there exists a rational Diophantine quadruple with the property $D(q)$.

At present it is not known whether there exists a rational number $q \neq 0$ such that there exists a rational Diophantine sextuple with the property $D(q)$. In [1], some rational "sextuples" with the property $D(1)$ were obtained, but all of them have two equal elements. Thus, they are actually quintuples with the additional property that $x_1^2 + 1$ is a perfect square. There exists also a rational Diophantine quintuple $\{x_1, \ldots, x_5\}$ with the property $D(1)$ such that x_1^2+1, x_2^2+1 and x_3^2+1 are perfect squares. However, the question of the existence of Diophantine sextuples is still open.

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