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On directable nondeterministic trapped automata*

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Abstract

A finite automaton is said to be directable if it has an input word, a directing word, which takes it from every state into the same state. For nondeterministic (n.d.) automata, directability can be generalized in several ways. In [8], three such notions, D1-, D2-, and D3-directability, are introduced. In this paper, we introduce the trapped n.d. automata, and for each $i = 1, 2, 3$, present lower and upper bounds for the lengths of the shortest Di-directing words of n-state Di-directable trapped n.d. automata. It turns out that for this special class of n.d. automata, better bounds can be found than for the general case, and some of the obtained bounds are sharp.

1 Introduction

An input word w is called a *directing* (or *synchronizing) word* of an automaton $\mathcal{A}$ if it takes $\mathcal{A}$ from every state to the same state. Directable automata have been studied exstensively, we mention only some of the related works (see *e.g.* [3],[4],[5],[7],[10],[12]). Directable n.d. automata have received less attention. Directability of n.d. automata can be defined in several meaningful ways. The following three nonequivalent definitions are introduced and studied in [8]. An input word w of an n.d. automaton $\mathcal{A}$ is said to be

(1) D1-*directing* if it takes $\mathcal{A}$ from every state to the same singleton set,

(2) D2-*directing* if it takes $\mathcal{A}$ from every state to the same fixed set A', where $\emptyset \subseteq A' \subseteq A$, and

(3) D3-*directing* if there is a state c such that $c \in aw$, for every $a \in A$.

The D1-directability of complete n.d. automata was investigated by Burkhard [1]. He gave a sharp exponential bound for the lengths of minimum-length D1-directing words of complete n.d. automata. In [6] on games of composing relations over a finite set Goralčik *it et al.*, in effect, studied D1- and D3-directability and they proved that neither for D1- nor for D3-directing words, the bound can be polynomial

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for n.d. automata. Carpi [2] considered a particular class of n.d. automata, the class of unambigous n.d. automata, and presented $O(n^3)$ bounds for the lengths of their shortest D1-directing words.

Here we study trapped n.d. automata that have a trap state, *i.e.*, a state which is stable for any input symbol, and present lower and upper bounds for the lengths of their shortest directing words of the three different types.

2 Preliminaries

Throughout this paper X always denotes a finite nonempty alphabet. The set of all finite words over X is denoted by X^* and λ denotes the empty word. The length of a word $w \in X^*$ is denoted by $|w|$. For any $p, q \in X^*$, the word p is called a *prefix* of q if there exists a word $s \in X^*$ such that $ps = q$. For the sake of simplicity, we use the notation $[n]$ for the set $\{1, \ldots, n\}$.

By a *nondeterministic (n.d.) automaton* we mean a system $\mathcal{A} = (A, X)$, where A is a nonempty finite set of *states*, X is the *input alphabet*, and each input symbol $x \in X$ is realized as a binary relation $x^{\mathcal{A}} (\subseteq A \times A)$. For any $a \in A$ and $x \in X$, let

$$ax^{\mathcal{A}} = \{b : b \in A \text{ and } (a, b) \in x^{\mathcal{A}}\}.$$

Moreover, for every $B \subseteq A$, we denote by $Bx^{\mathcal{A}}$ the set $\bigcup\{ax^{\mathcal{A}} : a \in B\}$. Now, for any word $w \in X^*$ and $B \subseteq A$, $Bw^{\mathcal{A}}$ can be defined inductively as follows:

(1) $B\lambda^{\mathcal{A}} = B$,

(2) $Bw^{\mathcal{A}} = (Bp^{\mathcal{A}})x^{\mathcal{A}}$ for $w = px$, where $p \in X^*$ and $x \in X$.

If $w = x_1 \ldots x_m$ and $a \in A$, then let $aw^{\mathcal{A}} = \{a\}w^{\mathcal{A}}$. This yields that $w^{\mathcal{A}} = x_1^{\mathcal{A}} \ldots x_m^{\mathcal{A}}$. If there is no danger of confusion, then we write simply aw and Bw for $aw^{\mathcal{A}}$ and $Bw^{\mathcal{A}}$, respectively.

An n.d. automaton $\mathcal{A} = (A, X)$ is *complete* if $ax \neq \emptyset$ holds, for all $a \in A$ and $x \in X$. Complete n.d. automata are called c.n.d. automata for short. A state of an n.d. automaton $\mathcal{A}$ is called a *trap* if it is stable for any input symbol, *i.e.*, $ax = \{a\}$, for every input symbol x of $\mathcal{A}$. An n.d. automaton is called *trapped* if it has a trap. Let us denote the class of trapped n.d. automata by **T**. Regarding some recent results on trapped deterministic automata, we refer to the works [9],[11],[12]. Following [8] we define the directability of n.d. automata as follows. Let $\mathcal{A} = (A, X)$ be an n.d. automaton. For any word $w \in X^*$, let us consider the following conditions:

(D1) $(\exists c \in A)(\forall a \in A)(aw = \{c\})$,

(D2) $(\forall a, b \in A)(aw = bw)$,

(D3) $(\exists c \in A)(\forall a \in A)(c \in aw)$.

For any $i = 1, 2, 3$, if w satisfies Di, then w is called a Di-*directing word* of $\mathcal{A}$ and in this case $\mathcal{A}$ is said to be Di-*directable*. Let us denote by $\mathrm{D}_i(\mathcal{A})$ the set of Di-directing words of $\mathcal{A}$, moreover, let $\mathbf{Dir}(i)$ and $\mathbf{CDir}(i)$ denote the classes of

Di-directable n.d. automata and c.n.d. automata, respectively. Now, we can define the following functions. For any $i = 1, 2, 3$ and $\mathcal{A} = (A, X) \in \mathbf{Dir}(i)$, let

$$\mathrm{d}_i(\mathcal{A}) = \min\{|w| : w \in \mathrm{D}_i(\mathcal{A})\},$$

$$\mathrm{d}_i(n) = \max\{\mathrm{d}_i(\mathcal{A}) : \mathcal{A} \in \mathbf{Dir}(i) \ \& \ |A| = n\},$$

$$\mathrm{cd}_i(n) = \max\{\mathrm{d}_i(\mathcal{A}) : \mathcal{A} \in \mathbf{CDir}(i) \ \& \ |A| = n\}.$$

The functions $\mathrm{d}_i(n), \mathrm{cd}_i(n)$, $i = 1, 2, 3$, are studied in [8], where lower and upper bounds depending on n are presented for them. Similar functions can be defined for any class of n.d. automata. For a class $\mathbf{K}$ of n.d. automata, let

$$\mathrm{d}_i^{\mathbf{K}}(n) = \max\{\mathrm{d}_i(\mathcal{A}) : \mathcal{A} \in \mathbf{Dir}(i) \cap \mathbf{K} \ \& \ |A| = n\},$$

$$\mathrm{cd}_i^{\mathbf{K}}(n) = \max\{\mathrm{d}_i(\mathcal{A}) : \mathcal{A} \in \mathbf{CDir}(i) \cap \mathbf{K} \ \& \ |A| = n\}.$$

Obviously, $\mathrm{cd}_i^{\mathbf{K}}(n) \leq \mathrm{d}_i^{\mathbf{K}}(n)$, for $i = 1, 2, 3$.

In what follows, we study the case when the considered class is $\mathbf{T}$, the class of trapped n.d. automata. It is worth noting that if a trapped n.d. automaton is Di-directable, then it has only one trap.

3 Directable trapped n.d. automata

First we deal with the D3-directability. We consider D3-directable trapped c.n.d. automata, and using certain deterministic automata, introduced by Rystsov [12], we present an exact bound for this class. Then we study D3-directable trapped n.d. automata and present lower and upper bounds for the lengths of their shortest D3-directing words. For trapped c.n.d. automata the following statement is valid.

Theorem 1. *For any* $n \geq 1$, $\mathrm{cd}_3^{\mathbf{T}}(n) = (n-1)n/2$.

Proof. First we prove that $(n-1)n/2 \leq \mathrm{cd}_3^{\mathbf{T}}(n)$. This inequality follows from Theorem 6.1 in [12]. Since the proof is short, we recall it for the sake of completeness.

For every $n > 1$, let us define the c.n.d. automaton $\mathcal{B}_n = (\{0, 1, \ldots, n-1\}, \{x_1, \ldots, x_{n-1}\})$ as follows. Let $0x_1 = 1x_1 = \{0\}$, and $jx_1 = \{j\}$, $j = 2, \ldots, n-1$. Moreover, for all $2 \leq k \leq n-1$ and $j \in \{0, 1, \ldots, n-1\}$, let

$$jx_k = \begin{cases} \{j-1\} & \text{if } j = k, \\ \{j+1\} & \text{if } j = k-1, \\ \{j\} & \text{otherwise.} \end{cases}$$

Obviously, $\mathcal{B}_n$ is a D3-directable trapped c.n.d. automaton with the trap 0. Let us observe that for any $j \in \{0, 1, \ldots, n-1\}$, jp is a singleton set whenever $p \in X^*$, moreover, $jw = \{0\}$ for any D3-directing word of $\mathcal{B}_n$, because 0 is a trap state. Therefore, $\{0, 1, \ldots, n-1\}w = \{0\}$, for any $w \in \mathrm{D}_3(\mathcal{B}_n)$. Now, let us assign to every nonempty subset J of states a weight, denoted by $g(J)$, which is the sum of the numbers contained in J, *i.e.*,

$$g(J) = \sum_{j \in J} j.$$

Then $g(\{0,1,\ldots,n-1\}) = (n-1)n/2$ and for any nonempty subset J of $\{0,1,\ldots,n-1\}$ and input sign x_k, $k \in [n-1]$,

$$|g(J) - g(Jx_k)| \leq 1.$$

From these facts it follows that the length of any D3-directing word of $\mathcal{B}_n$ is not less than $(n-1)n/2$, because this word brings the state set of weight $(n-1)n/2$ into the set $\{0\}$ with weight 0. Hence, $(n-1)n/2 \leq \mathrm{d}_3(\mathcal{B}_n)$. On the other hand, it is easy to check that the word

$$w = x_1x_2x_1x_3x_2x_1 \ldots x_{n-1}x_{n-2} \ldots x_2x_1$$

is a D3-directing word of $\mathcal{B}_n$ and $|w| = (n-1)n/2$. Consequently,

$$\mathrm{d}_3(\mathcal{B}_n) = (n-1)n/2.$$

Since $\mathcal{B}_n$ is a D3-directable trapped c.n.d. automaton of n states, the equality above implies $(n-1)n/2 \leq \mathrm{cd}_3^{\mathrm{T}}(n)$.

In order to prove that this bound is sharp, we prove that for any D3-directable trapped c.n.d. automaton $\mathcal{A} = (A, X)$ of $n(>1)$ states, there exists a D3-directing word whose length is not greater than $(n-1)n/2$. To simplify the notation, we assume that $A = \{0,1,\ldots,n-1\}$ and 0 is the trap of $\mathcal{A}$. Since $\mathcal{A}$ is a D3-directable c.n.d. automaton and $0x = \{0\}$, for all $x \in X$, there exists for any state $j \in A$ a word $x_1 \ldots x_m$ of minimum-length such that $0 \in jx_1 \ldots x_m$. Moreover, there are states $j_1, \ldots, j_{m-1} \in A$ such that $j_t \in jx_1 \ldots x_t$ and $0 \in j_tx_{t+1} \ldots x_m$, for all $t = 1, \ldots, m-1$. Since $x_1 \ldots x_m$ is a minimum-length word satisfying $0 \in jx_1 \ldots x_m$, the states $j, j_1, \ldots, j_{m-1}, 0$ must be pairwise different. Therefore, by $|A| = n$, we obtain $m \leq n-1$. Observe that for any $2 \leq t \leq m$, $x_t \ldots x_m$ is a minimum-length word satisfying $0 \in j_{t-1}x_t \ldots x_m$. Based on these observations, by renaming the states, we may suppose that for any state $j \in A$, there exists a word p_j such that $0 \in jp_j$ and $|p_j| \leq j$. By using the pairs j, p_j, $j = 0, \ldots, n-1$, we present a procedure for finding a D3-directing word with length, not greater than $(n-1)n/2$.

Initialization. Let $t = 0$, $B_0 = \{0\}$, $p_{i_0} = \lambda$, and $R_0 = \{1, 2, \ldots n-1\}$.

Iteration.

- *Step 1.* Terminate if $A = B_t$. Otherwise proceed to Step 2.
- *Step 2.* For each $j \in R_t$, let $k_j^{(t)}$ denote the smallest number in the set jp_{i_t}. Select the least element in $\{k_j^{(t)} : j \in R_t\}$ and denote it by i_{t+1}. Let

$$B_{t+1} = \{j : j \in A \;\&\; 0 \in jp_{i_0} \ldots p_{i_{t+1}}\}.$$

and

$$R_{t+1} = \{k_j^{(t)} : j \in A \setminus B_{t+1}\}.$$

Increase the value of t by 1 and proceed to the next iteration.

To verify the correctness of the above procedure, we note the following facts.

(i) For any i_{t+1}, there exists a $j \in A \setminus B_t$ such that i_{t+1} is an element of the set $jp_{i_0} \dots p_{i_t}$. Then $B_t \cup \{j\} \subseteq B_{t+1}$, and hence, $B_t \subset B_{t+1}$.

(ii) If $j \in A \setminus B_t$, then $0 \notin jp_{i_0} \dots p_{i_t}$ yielding $k_j^{(t)} > 0$. Therefore, R_t is a set of positive integers.

(iii) If $A \neq B_t$, then there is a $j \in A \setminus B_t$ with $jp_{i_0} \dots p_{i_t} \neq \emptyset$ since $\mathcal{A}$ is a c.n.d. automaton, and thus, $R_t \neq \emptyset$. Consequently, $A \neq B_t$ implies $R_t \neq \emptyset$.

From these facts it follows that there exists a positive integer $s \leq n-1$ such that $A = B_s$. Now, by the definition of B_s, we obtain that

$$w = p_{i_0} \dots p_{i_s}$$

is a D3-directing word of $\mathcal{A}$. Let $r_t = |R_t|$, $t = 0, \dots, s-1$. From the definition of R_t it follows that

$$n - 1 \geq r_0 > r_1 > \dots > r_{s-1} > 0.$$

On the other hand, since $|R_t| = r_t$, the least number i_{t+1} of $\{k_j^{(t)} : j \in R_t\}$ is not greater than $n - r_t$. This yields that $|p_{i_{t+1}}| \leq n - r_t$, $t = 0, \dots, s-1$. Since $|p_{i_0}| = 0$, we obtain that

$$|w| \leq \sum_{t=0}^{s-1} (n - r_t).$$

Let us observe that the numbers $n - r_t$, $t = 0, \dots, s-1$ are pairwise different and each of them is contained in the set $[n-1]$. Therefore, the upper bound of $|w|$ is the sum of some distinct elements of $[n-1]$. But this sum is not greater than the sum $(n-1)n/2$ of all the elements of $[n-1]$. Consequently, $|w| \leq (n-1)n/2$. If $n = 1$, then the statement is obviously also valid. This completes the proof of Theorem 1. □

For D3-directable trapped n.d. automata, we have the following bounds.

Theorem 2. *For any* $n \geq 2$, $\max\{\lfloor n^{\frac{1}{3}} - 1 \rfloor !, (n-2)^2 + 1\} \leq \mathrm{d}_3^{\mathrm{T}}(n) \leq 2^{n-1} - 1$.

Proof. The first member in the lower bound comes from the general case (*cf.* [8]), where the automata, providing this bound, are trapped automata. The second member in the lower bound can be derived from Černý's well-known examples (*cf.* [3]) as follows. One can equip Černý's automaton of $n-1$ states with a trap state and a new input symbol, denoted by $\diamond$ and z, respectively. Let $\diamond z = \{\diamond\}$, $0z = \{\diamond\}$, and $jz = \emptyset$, for all $j = 1, \ldots, n-1$. Now, redefine the remaining transitions as follows. If $ax = b$, then let $ax = \{b\}$ be the new transition. Then we obtain an n.d. automaton of n states whose shortest D3-directing words are of length $(n-2)^2 + 1$.

To obtain the upper bound, let us consider an arbitrary D3-directable trapped n.d. automaton $\mathcal{A} = (A, X)$ of $n(> 1)$ states. Let $A = \{a_1, \ldots, a_n\}$ and a_n be the trap of $\mathcal{A}$. Let $w = x_1 \ldots x_m$ be a minimum-length D3-directing word of $\mathcal{A}$. Then $a_n w = \{a_n\}$ and by the D3-directability of $\mathcal{A}$, $a_n \in a_j x_1 \ldots x_m$, $j = 1, \ldots, n$. For all $j \in [n-1]$ and $k \in [m]$, let us select an element a_{jk} from $a_j x_1 \ldots x_k$ such that $a_n \in a_{jk} x_{k+1} \ldots x_m$. Such elements exist, because for every $j \in [n-1]$, $a_n \in a_j x_1 \ldots x_m$. Now, let $S_k = \{a_n\} \cup \{a_{1k}, \ldots, a_{n-1,k}\}$, for all $k \in [m]$, and $S_0 = \{a_1, \ldots, a_n\}$. Let us observe that $a_j x_1 \ldots x_k \cap S_k \neq \emptyset$, for every $k \in [m]$, and if $a_t \in S_k$ for some $t \in [n]$ and $k \in [m]$, then $a_n \in a_t x_{k+1} \ldots x_m$. By using these observations, it is easy to see that if $S_j = S_l$ for some $0 \leq j < l \leq m$, then $x_1 \ldots x_j x_{l+1} \ldots x_m$ is a D3-directing word of $\mathcal{A}$ which is a contradiction. Consequently, the sets $S_0, S_1, \ldots, S_m$ must be pairwise different. Since $a_n \in S_k$, $k = 0, \ldots, m$, the number of these sets can not exceed 2^{n-1}. Therefore, $|w| \leq 2^{n-1} - 1$. This ends the proof of Theorem 2. □

Remark 1. It is worth noting that the proof above with a small changing can be applied for the general case, and one obtains the upper bound $2^n - 1$ for $d_3(n)$ which is a significant improvement of the upper bound, given in [8].

Now, we study D1-directable trapped c.n.d. automata. By a slight modification of the automata, introduced by Burkhard [1], we prove the following sharp bound.

Theorem 3. *For any* $n \geq 1$, $\mathrm{cd}_1^{\mathrm{T}}(n) = 2^{n-1} - 1$.

Proof. First we prove that $2^{n-1} - 1$ is a lower bound for $\mathrm{cd}_1^{\mathrm{T}}$. To do so, for every integer $n > 1$, we present a D1-directable trapped c.n.d. automaton, having a minimum-length D1-directing word w with $|w| = 2^{n-1} - 1$.

Let us define the c.n.d. automaton $\mathcal{A}_n = ([n], X^{(n)})$ as follows. For every integer $2 \leq k \leq n-2$, let us consider all of the k-element subsets of the set $A' = \{2, \ldots, n\}$. Let us order these sets in a chain such that the first set is $\{n-k, \ldots, n-1\}$ and the last one is $\{n-k+1, \ldots, n\}$. We denote this sequence by $A_1^{(k)}, \ldots, A_{\binom{n-1}{k}}^{(k)}$. Now, let $X_k = \{x_r^{(k)} : r = 1, \ldots, \binom{n-1}{k} - 1\}$, $V = \{v_1, \ldots, v_{n-1}\}$, $Y = \{y_1, \ldots, y_{n-2}\}$, and

$$X^{(n)} = V \cup Y \cup \left(\bigcup \{X_k : 2 \leq k \leq n-2\}\right).$$

The transitions of $\mathcal{A}_n$ are defined as follows. For any $x \in X^{(n)}$, let $1x = \{1\}$. Moreover, for any $x_r^{(k)} \in X_k$, $v_t \in V$, $y_s \in Y$, and state $j \in A'$, let

$$jv_t = \begin{cases} \{j-1\} & \text{if } t = j-1, \\ A' & \text{otherwise,} \end{cases}$$

$$jx_r^{(k)} = \begin{cases} A_{r+1}^{(k)} & \text{if } j \in A_r^{(k)}, \\ A' & \text{otherwise,} \end{cases}$$

$$jy_s = \begin{cases} A_1^{(s)} & \text{if } 2 \le s \le n-2 \;\&\; n-s \le j \le n\}, \\ \{n\} & \text{if } s = 1 \;\&\; j \in \{n-1, n\}, \\ A' & \text{otherwise.} \end{cases}$$

Obviously, $\mathcal{A}_n$ is a trapped c.n.d. automaton, its trap is the state 1. Let us consider the word $w \in X^{(n)*}$, given by

$$w = y_{n-2}x_1^{(n-2)} \dots x_{\binom{n-1}{n-2}-1}^{(n-2)} y_{n-3} \dots y_2 x_{1,2}^{(2)} \dots x_{\binom{n-1}{2}-1}^{(2)} y_1 v_{n-1} \dots v_1 .$$

It is easy to check that w is a D1-directing word of $\mathcal{A}_n$, namely $[n]w = \{1\}$. Moreover, w is the unique minimum-length D1-directing word of $\mathcal{A}_n$. This fact is based on the following observation.

If px is a prefix of w, then for any $x' \in X^{(n)}$, different from x, there exists a prefix q of p such that $[n]px' = [n]q$.

Since w is a minimum-length D1-directing word of $\mathcal{A}_n$ and its length is equal to $2^{n-1} - 1$, we obtain $2^{n-1} - 1 \le \mathrm{cd}_1^{\mathrm{T}}(n)$.

Regarding the upper bound, let us observe that if $w = x_1 \dots x_m$ is a minimum-length D1-directing word of a trapped c.n.d. automaton $\mathcal{A} = (A, X)$ of $n(> 1)$ states with a trap $a_n \in A$, then $Aw = \{a_n\}$. Moreover, the sequence $A, Ax_1, \dots, Ax_1 \dots x_m$ consists of pairwise different nonempty subsets of A and each of them contains a_n. The number of these subsets is at most 2^{n-1}, and so, the length of w is not greater than $2^{n-1} - 1$. Hence, we obtain that $\mathrm{cd}_1^{\mathrm{T}}(n) \le 2^{n-1} - 1$. The statement is obviously also valid for $n = 1$. This ends the proof of Theorem 3. □

In what follows, we shall use the following observation.

Lemma. *For every $n \ge 1$, $\mathrm{cd}_1^{\mathrm{T}}(n) = \mathrm{cd}_2^{\mathrm{T}}(n)$ and $\mathrm{d}_1^{\mathrm{T}}(n) = \mathrm{d}_2^{\mathrm{T}}(n)$.*

Proof. Let us observe that for any trapped n.d. automaton $\mathcal{A} = (A, X)$ of n states, $\mathrm{D}_1(\mathcal{A}) = \mathrm{D}_2(\mathcal{A})$. Indeed, $\mathrm{D}_1(\mathcal{A}) \subseteq \mathrm{D}_2(\mathcal{A})$ follows from the definition. Now, let $w \in \mathrm{D}_2(\mathcal{A})$. Then $aw = bw$ for every pair of states. This yields that $\{a_n\} = a_n w = aw$ is valid for any state $a \in A$, where a_n denotes the trap state of $\mathcal{A}$. This means that $w \in \mathrm{D}_1(\mathcal{A})$, implying $\mathrm{D}_2(\mathcal{A}) \subseteq \mathrm{D}_1(\mathcal{A})$. Therefore, $\mathrm{D}_1(\mathcal{A}) = \mathrm{D}_2(\mathcal{A})$. From this equality it follows that $\mathrm{cd}_1^{\mathrm{T}}(n) = \mathrm{cd}_2^{\mathrm{T}}(n)$ and $\mathrm{d}_1^{\mathrm{T}}(n) = \mathrm{d}_2^{\mathrm{T}}(n)$. □

Now, we can conclude the following statement from Theorem 3 by our Lemma.

Theorem 4. *For any $n \ge 1$, $\mathrm{cd}_2^{\mathrm{T}}(n) = 2^{n-1} - 1$.*

For D1- and D2-directable trapped n.d. automata, we have the following bounds.

Theorem 5. *For any $n \ge 1$, $2^{n-1} - 1 \le \mathrm{d}_1^{\mathrm{T}}(n) = \mathrm{d}_2^{\mathrm{T}}(n) \le 2(2^{n-1} - 1)$.*

Proof. $d_1^T(n) = d_2^T(n)$ is provided by our Lemma. By Theorem 3, we have that $2^{n-1} - 1 \leq cd_1^T(n)$. On the other hand, $cd_1^T(n) \leq d_1^T(n)$, and therefore, $2^{n-1} - 1 \leq d_1^T(n)$.

Regarding the upper bound, let us consider an arbitrary D1-directable trapped n.d. automaton $\mathcal{A} = (\{a_1, \ldots, a_n\}, X)$ with $n \geq 2$. Without loss of generality, we may suppose that a_n is the trap of $\mathcal{A}$. First, let us observe that $\mathcal{A}$ is a D3-directable n.d. automaton, as well. Let w_1 be a minimum-length D3-directing word of $\mathcal{A}$. By Theorem 2, $|w_1| \leq 2^{n-1} - 1$. Since $\mathcal{A}$ is a trapped n.d. automaton, $a_n \in a_j w_1$, for all $j \in [n]$. Then for every $j \in [n]$ and $p \in X^*$, $a_j w_1 p \neq \emptyset$. Now, let $w_2 = x_1 \ldots x_m$ be a minimum-length word such that $Aw_2 = \{a_n\}$. Such a word there exists since $\mathcal{A}$ is D1-directable. Let us consider the sequence $A, Ax_1, \ldots, Ax_1 \ldots x_m$. We show that these sets are pairwise different. If it is not so, then there are integers $0 \leq r < s \leq m$ such that $Ax_1 \ldots x_r = Ax_1 \ldots x_s$. Then $Ax_1 \ldots x_r x_{s+1} \ldots x_m = \{a_n\}$ which is a contradiction. Since $a_n \in Ap$ for every prefix p of w_2, we obtain that $m \leq 2^{n-1} - 1$. Now, we prove that $w_1 w_2$ is a D1-directing word of $\mathcal{A}$. Let $j \in [n]$ be arbitrary. Then $a_n \in a_j w_1$ and $a_j w_1 \subseteq A$. Moreover, $a_j w_1 w_2 \neq \emptyset$ and $a_j w_1 w_2 \subseteq Aw_2 = \{a_n\}$, and hence, $a_j w_1 w_2 = \{a_n\}$. On the other hand, $|w_1 w_2| = 2^{n-1} - 1 + 2^{n-1} - 1 = 2(2^{n-1} - 1)$. Consequently, $d_1^T(n) \leq 2(2^{n-1} - 1)$ if $n \geq 2$. On the other hand, $d_1^T(n) \leq 0$ is obvious. This completes the proof of Theorem 5. □

Remark 2. Since $cd_2^T(n) \leq cd_2(n) \leq d_2(n)$, we obtain that $2^{n-1} - 1$ is a lower bound for both $cd_2(n)$ and $d_2(n)$. On the other hand, the known lower bound, given for $cd_2(n)$ and $d_2(n)$ in [8], is $\lfloor n^{\frac{1}{3}} - 1 \rfloor!$ which is less than $2^{n-1} - 1$. Therefore, $2^{n-1} - 1$ is an improvement of the lower bounds of both $cd_2(n)$ and $d_2(n)$.

Remark 3. The upper bound, presented in Theorem 5, is worse than the upper bound $2^n - n - 1$, given for $d_1(n)$ in [8]. The verification of the inequality $d_1(n) \leq 2^n - n - 1$ is based on the observation that if $w = x_1 \ldots x_m$ is a minimum-length D1-directing word of an n.d. automaton $\mathcal{A} = (A, X)$, then the sets $A, Ax_1, \ldots, Ax_1 \ldots x_m$ must be pairwise different. The following example shows that this observation is not valid, moreover, $2^n - n - 1$ is not necessarily upper bound for $d_1(n)$ in general. Let $\mathcal{A} = (\{0, 1)\}, \{x, y\})$, where $0x = \{0, 1\}$, $1x = \{1\}$, $0y = \emptyset$, and $1y = 1$. Then xy is a minimum-length D1-directing word of $\mathcal{A}$, but $\{0, 1\}x = \{0, 1\}$. Moreover, $2 = |xy| \not\leq 2^2 - 2 - 1$.

References

[1] H. D. Burkhard, Zum Längenproblem homogener Experimente an determinierten und nicht-deterministischen Automaten, *Elektronische Informationsverarbeiterung und Kybernetik, EIK* **12** (1976), 301-306.

[2] A. Carpi, On synchronizing unambigous automata, *Theoretical Computer Science* **60** (1988), 285-296.

[3] J. Černý, Poznámka *k* homogénnym experimentom s konečnými automatmi, *Matematicko-fysikalny Časopis SAV* **14** (1964), 208–215.

[4] J. Černý, A. Piricka, B. Rosenauerova, On directable automata, *Kybernetika (Praha)*, **7** (1971), 289-297.

[5] D. Eppstein, Reset sequences for monotonic automata, *SIAM Journal of Computing* **19** (1990), 500-510.

[6] P. Goralčik, Z. Hedrlin, V. Koubek, J. Ryšlinková, A game of composing binary relations, *R.A.I.O. Informatique théorique/Theoretical Informatics* **16** (1982), 365-369.

[7] B. Imreh, M. Steinby, Some remarks on directable automata, *Acta Cybernetica* **12** (1995), 23–35.

[8] B. Imreh, M. Steinby, Directable nondeterministic automata, *Acta Cybernetica* **14** (1999), 105-115.

[9] T. Petković, M. Ćirić, S. Bogdanović, Decompositions of automata and transition semigroups, *Acta Cybernetica* **13** (1998), 385–403.

[10] J.-E. Pin, Sur un cas particulier de la conjecture de Cerny, *Automata, languages and programming*, ICALP'79 (Proc. Coll., Udine 1979), LNCS 62, Springer-Verlag, Berlin, 1979, 345–352.

[11] Z. Popović, S. Bogdanović, T. Petković, M. Ćirić, Trapped automata, *Publicationes Mathematicae*, **60** (2002), 661–677.

[12] I. Rystsov, Reset words for commutative and solvable automata, *Theoretical Computer Science* **172** (1997), 273-279.

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