On Dirichlet series for sums of squares

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Abstract. In their famous book on the theory of numbers, Hardy and Wright recorded elegant closed forms for the generating functions of the divisor functions $\sigma_k(n)$ and $\sigma_k^2(n)$ in the terms of Riemann Zeta function $\zeta(s)$ only. In this talk, we explore other arithmetical functions enjoying this remarkable property. We are able to generalize this result and prove that if f_i and g_i are completely multiplicative, then we have

$$\sum_{n=1}^{\infty} \frac{(f_1 * g_1)(n) \cdot (f_2 * g_2)(n)}{n^s} = \frac{L_{f_1 f_2}(s) L_{g_1 g_2}(s) L_{f_1 g_2}(s) L_{g_1 f_2}(s)}{L_{f_1 f_2 g_1 g_2}(2s)}$$

where $L_f(s) := \sum_{n=1}^{\infty} f(n) n^{-s}$ is the Dirichlet series corresponding to f.

Let $r_N(n)$ be the number of solutions of $x_1^2 + \cdots + x_N^2 = n$ and $r_{2,P}(n)$ be the number of solutions of $x^2 + Py^2 = n$. One of the applications of our theorem above is to obtain closed forms, in terms of $\zeta(s)$ and Dirichlet L-functions, for the generating functions of $r_N(n), r_N^2(n), r_{2,P}(n)$ and $r_{2,P}(n)^2$ for certain N and P. We also use these generating functions to obtain asymptotic estimates of the average values for each function for which we obtain a Dirichlet series. This is a joint work with Jonathan Borwein.