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Janusz Matkowski, Janusz Matkowski, Andrzej S. Nowak

Institutions: Silesian University, University of Zielona Góra

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Matkowski, Janusz and Nowak, Andrzej S.

University of Zielona Góra

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Janusz Matkowski [1] and Andrzej S. Nowak [2]

Abstract: In this paper, we apply the idea of k-local contraction of Rincón-Zapatero and Rodrigues-Palmero (2003, 2007) to study discounted stochastic dynamic programming models with unbounded returns. Our main results concern the existence of a unique solution to the Bellman equation and are applied to the theory of stochastic optimal growth. Also a discussion of some subtle issues concerning k-local and global contractions is included.

Key words: Stochastic dynamic programming, Bellman functional equation, contraction mapping, stochastic optimal growth

JEL Classification Numbers: C61, D90, E20

1. Introduction

The theory of stochastic dynamic programming (or Markov decision processes) with uncountable state space started with the fundamental work of Blackwell (1965). His ideas were extended in many directions, with a number of applications to economics, engineering, and operations research were presented. For a good survey the reader is referred to Bertsekas and Shreve (1978); Hernández-Lerma and Lasserre (1999); Puterman (2005) and other books and articles. A large part of the theory of stochastic optimal growth lies in the framework of dynamic programming. The classical paper of Brock and Mirman (1972) as well as the book by Stokey et al. (1989) are very much related to Blackwell's work and deal with an infinite state space models. However, many issues considered by economists (like properties of trajectories, steady states for specific models, etc.) are not covered in the aforementioned books on stochastic dynamic programming (control processes). In many applications of decision processes to operations research or economics it is natural to use unbounded return functions. The bounded case with discounted evaluation directly leads to the Banach contraction mapping theorem, see, e.g., Bertsekas and Shreve (1978) or Stokey et al. (1989). The unbounded case, however, requires different methods (techniques): "weighted norms" in the underlying function spaces, or limits of solutions for "truncated models", see Hernández-Lerma and Lasserre (1999); Stokey et al. (1989) and others. A large survey of the existing literature on various economic models with unbounded returns can be found in a recent volume edited by Dana et al. (2006). Here, we mention important works by Boyd III (1990); Boyd III and Becker (1997); Le Van and Morhaim (2002); Le Van and Vailakis (2005) representing different methods and levels of gen-

[1] Faculty of Mathematics, Computer Science and Econometrics, University of Zielona Góra, Podgórna 50, 65-246 Zielona Góra and Institute of Mathematics, Silesian University, Katowice, Bankowa 14, Poland; e-mail: j.matkowski@wmie.uz.zgora.pl

[2] Corresponding author: Faculty of Mathematics, Computer Science and Econometrics, University of Zielona Góra, Podgórna 50, 65-246 Zielona Góra, Poland; e-mail: a.nowak@wmie.uz.zgora.pl

erality. Moreover, the papers by Rincón-Zapatero and Rodrigues-Palmero (2003, 2007), which are point of our departure, contain a great deal of information on this topic, including models with recursive utility.

The aim of this paper is to apply the *valuable idea* of Rincón-Zapatero and Rodrigues-Palmero (2003) to k-local contraction to study stochastic dynamic programming models with unbounded return functions. Our main results concern the existence of a unique solution to the Bellman equation and are applied to the theory of stochastic optimal growth. We give two applications motivated by the work of Stokey et al. (1989). Before describing our model and stating the results, we discuss the basic idea of Rincón-Zapatero and Rodrigues-Palmero (2003) in detail. It turns out that Proposition 1(b) stated in Rincón-Zapatero and Rodrigues-Palmero (2003) is *false*. In Section 2 we give a counterexample to support our claim. Proposition 1(b) is fundamental for the further research demonstrated by Rincón-Zapatero and Rodrigues-Palmero (2003, 2007). For our purpose, we present in Section 3 a modification of their approach and state some fixed point results related to their Proposition 2 [3]

Sections 4-6 contain applications to dynamic programming with stochastic transition functions and economic growth, respectively. Our results on stochastic optimal growth theory are new and can be applied to multi-sector models. The weighted norm approaches of Boyd III (1990); Boyd III and Becker (1997) in economic theory or Hernández-Lerma and Lasserre (1999) in the theory of Markov decision processes are of different nature.

2. Local contractions: a counterexample

In an interesting paper by Rincón-Zapatero and Rodrigues-Palmero (2003) the existence and uniqueness of solutions of the Bellman equation in the unbounded case is a starting point of the considerations. The proposed method is based on the Banach Fixed Point Principle and on an ingenious idea of construction of a special metric space. Unfortunately, part (b) of Proposition 1 in Rincón-Zapatero and Rodrigues-Palmero (2003), the basis for this paper, is *false*. Below we give a *counterexample*. In Section 3, we present some modifications of the results in Rincón-Zapatero and Rodrigues-Palmero (2003) which are very useful to study Markov decision processes, in particular stochastic optimal growth models with unbounded returns.

Throughout this paper N and R denote, respectively, the set of positive integers and the set of real numbers. As in in Rincón-Zapatero and Rodrigues-Palmero (2003) assume that X is a topological space such that $X = \bigcup_{j=1}^{\infty} K_j$ where $\{K_j\}$ is an increasing sequence of compact subsets of X. We assume that

$$X = \bigcup_{j=1}^{\infty} \mathrm{Int}(K_j).$$

[3] After finishing the first draft of this paper, we obtained a communication from Filipe Martins-da-Rocha and Vailakis (2008) where a different counterexample is shown and different corrections to Rincón-Zapatero and Rodrigues-Palmero (2003) are given. We would like to thank Filipe Martins-da-Rocha and Yiannis Vailakis for some useful comments on our work.

Let $C(X)$ denote the set of all continuous real-valued functions on X. Define

$$d_j(\phi, \psi) := \max_{x \in K_j} |\phi(x) - \psi(x)|, \quad j \in N.$$

Clearly, $\{d_j\}$ is a countable family of semimetrics and d defined by

$$d(\phi, \psi) := \sum_{j=1}^{\infty} 2^{-j} \frac{d_j(\phi, \psi)}{1 + d_j(\phi, \psi)} \quad \text{for all} \quad \phi, \psi \in C(X) \tag{1}$$

is a complete metric on $C(X)$.

Following Rincón-Zapatero and Rodrigues-Palmero (2003, 2007), we say that an operator $T : C(X) \mapsto C(X)$ is a 0-*local contraction* relative to a set $G \subset C(X)$ if

$$d_j(T\phi, T\psi) \leq \beta_j d_j(\phi, \psi) \quad \text{for each} \quad j \in N \quad \text{and for all} \quad \phi, \psi \in G, \tag{2}$$

where $0 \leq \beta_j < 1$ for every $j \in N$.

Here and in the sequel $\mathbf{0}$ denotes the function ψ such that $\psi(x) = 0$ for all $x \in X$.

In Rincón-Zapatero and Rodrigues-Palmero (2003, 2007), a set $G \subset C(X)$ is called "bounded", if there is a sequence of positive real numbers $\{m_j\}$ such that $d_j(\phi, \mathbf{0}) \leq m_j$ for each $\phi \in G$ and $j \in N$. Thus, if the set G contains an unbounded function ϕ, then the sequence $\{m_j\}$ must be unbounded as well.

A key role in Rincón-Zapatero and Rodrigues-Palmero (2003) is plays the following statement (Proposition 1): If an operator $T : C(X) \mapsto C(X)$ is a 0-local contraction relative to a bounded set $G \subset C(X)$, then there exists a constant $\alpha \in [0, 1)$ such that

$$d(T\phi, T\psi) \leq \alpha d(\phi, \psi) \quad \text{for all} \quad \phi, \psi \in G. \tag{3}$$

It turns out that this proposition is false. An "a contrario" argument used in the proof is erroneous just before the Lebesgue dominated convergence theorem is applied.

Example 1: Assume that $X = (0, 1]$ and $K_j = [\frac{1}{j}, 1]$ for each $j \in N$. Let $\{m_j\}$ be an increasing sequence of positive numbers. Consider the "bounded set" $G \subset C(X)$ (in the sense of Rincón-Zapatero and Rodrigues-Palmero (2003, 2007)) containing functions f_i ($i \in N$) such that $d_j(f_i, \mathbf{0}) = m_i$ for all $j \geq i$, and $d_j(f_i, \mathbf{0}) = 0$ for all $1 < j < i$. For instance take

$$f_i(x) = \begin{cases} m_i & \text{if} \quad 0 < x \leq \frac{1}{i} \\ i(i-1)m_i \left(\frac{1}{i-1} - x\right) & \text{if} \quad \frac{1}{i} < x \leq \frac{1}{i-1} \\ 0 & \text{if} \quad \frac{1}{i-1} < x \leq 1 \end{cases}$$

for $i \in N$, $i > 1$, and $f_1 = m_1$. Assume that $\phi \in G$ if and only if there is some i such that $0 \leq \phi(x) \leq f_i(x)$ for all $x \in X$. Let $T\psi(x) := \beta\psi(x)$ for some $\beta \in (0, 1)$. Then $T : G \mapsto G$. Clearly, T is a 0-local contraction relative to the set G with $\beta_j = \beta$ for all $j \in N$. Take $i > 1$. Since $T\mathbf{0} = \mathbf{0}$ and $d_j(Tf_i, \mathbf{0}) = \beta m_i$ for all $j \geq i$, and $d_j(Tf_i, \mathbf{0}) = 0$ for all $j \in N$, $j < i$, we have

$$d(Tf_i, T\mathbf{0}) = \sum_{j=1}^{\infty} 2^{-j} \frac{d_j(Tf_i, \mathbf{0})}{1 + d_j(Tf_i, \mathbf{0})} = \sum_{j=i}^{\infty} 2^{-j} \frac{d_i(Tf_i, \mathbf{0})}{1 + d_i(Tf_i, \mathbf{0})}$$
$$= \frac{d_i(Tf_i, \mathbf{0})}{1 + d_i(Tf_i, \mathbf{0})} \sum_{j=i}^{\infty} 2^{-j} = 2^{-i+1} \frac{\beta m_i}{1 + \beta m_i}.$$

Suppose that there exists an $\alpha \in [0, 1)$ such that (3) holds. Taking $\phi = f_i$ and $\psi = \mathbf{0}$ in (3) we get

$$d(Tf_i, T\mathbf{0}) \le \alpha d(f_i, \mathbf{0}) = \alpha \sum_{j=1}^{\infty} 2^{-j} \frac{d_j(f_i, \mathbf{0})}{1 + d_j(f_i, \mathbf{0})} = \sum_{j=i}^{\infty} 2^{-j} \frac{d_i(f_i, \mathbf{0})}{1 + d_i(f_i, \mathbf{0})} = 2^{-i+1} \frac{\alpha m_i}{1 + m_i}.$$

It follows that

$$\frac{\beta m_i}{1 + \beta m_i} \le \frac{\alpha m_i}{1 + m_i}$$

whence

$$m_i \le \frac{\alpha - \beta}{\beta(1 - \alpha)}.$$

Since $i \in N$ is arbitrary fixed, we have shown that the sequence $\{m_j\}$ is bounded and, consequently, the set G must be bounded in the usual sense. Note that, by the last inequality, the sequence $\{m_j\}$ can be unbounded only if $\alpha \ge 1$. Thus the unboudedness of the sequence $\{m_j\}$ excludes the contractivity of T.

The above example shows that the metric d given by (1) does not have the properties expected by Rincón-Zapatero and Rodrigues-Palmero (2003, 2007). This metric "kills" the contractivity of mappings on "bounded sets"!

3. Fixed points of local contractions

In this section, we apply the basic idea of Proposition 2 in Rincón-Zapatero and Rodrigues-Palmero (2003). Let X is be a nonempty set. By $\{K_j\}$ we shall denote a strictly increasing (in the sense of inclusion) sequence of subsets of X and assume that

$$X = \bigcup_{j=1}^{\infty} K_j. \tag{4}$$

Let $F(X)$ be a vector space of functions $\phi : X \mapsto R$. For any $j \in N$, define the seminorm on $F(X)$ by

$$\|\phi\|_j := \sup_{x \in K_j} |\phi(x)|, \quad \phi \in F(X).$$

We assume that the set off all functions $\phi \in F(X)$ with domain restricted to any K_j endowed with the norm $\| \cdot \|_j$ is a Banach space. Let $c > 1$ and $m = \{m_j\}$ be an increasing unbounded sequence of positive real numbers. Denote by $F_m(X)$ the set of all $\phi \in F(X)$ such that

$$\sum_{j=1}^{\infty} \frac{\|\phi\|_j}{m_j c^j} < \infty.$$

The function $\|\cdot\| : F_m(X) \mapsto R$ defined by

$$\|\phi\| := \sum_{j=1}^{\infty} \frac{\|\phi\|_j}{m_j c^j} \tag{5}$$

is a complete norm on $F_m(X)$, so $(F_m(X), \|\cdot\|)$ is a Banach space. Define

$$F_{mb}(X) := \{\phi \in F(X) : \|\phi\|_j \leq m_j \quad \textit{for all} \quad j \in N\}.$$

Clearly, $F_{mb}(X)$ is a closed subset of $F_m(X)$.

Let $G \subset F_m(X)$ and $k \in \{0, 1\}$. Inspired by Rincón-Zapatero and Rodrigues-Palmero (2003), we say that a mapping $T : F_m(X) \mapsto F(X)$ is a *k-local contraction* (relative to the set G) if there is a $\beta \in [0, 1)$ such that

$$\|T\phi - T\psi\|_j \leq \beta \|\phi - \psi\|_{j+k} \qquad \text{for all} \quad \phi, \psi \in G \text{ and } \ j \in N.$$

Note that this definition is in some sense stronger than that of Rincón-Zapatero and Rodrigues-Palmero (2003).

Proposition 1: *Let $T : F_m(X) \mapsto F(X)$ be a 0-local contraction relative to $G = F_m(X)$. Then*

$$\|T\phi - T\psi\| \leq \beta \|\phi - \psi\|, \tag{6}$$

for any ϕ, $\psi \in F_m(X)$. If $T\mathbf{0} \in F_m(X)$, then T maps $F_m(X)$ into itself and has a unique fixed point $\phi^* \in F_m(X)$. If, in addition,

$$\|T\mathbf{0}\|_j \leq (1 - \beta) m_j \quad \textit{for all} \quad j \in N,$$

then $T : F_{mb}(X) \mapsto F_{mb}(X)$ and has a unique fixed point $\phi^* \in F_{mb}(X)$.

Proof: It is easy to see that (6) holds. Assume that $T\mathbf{0} \in F_m(X)$. Note that, for all $\phi \in F_m(X)$,

$$\|T\phi\| = \|T\phi - T\mathbf{0} + T\mathbf{0}\| \leq \|T\phi - T\mathbf{0}\| + \|T\mathbf{0}\| \leq \beta \|\phi\| + \|T\mathbf{0}\| < \infty.$$

Then T maps $F_m(X)$ into itself and is a contraction. Suppose now that for each $\phi \in F_{mb}(X)$ and $j \in N$, we have $\|T\phi\|_j \leq (1 - \beta) m_j$. Then

$$\|T\phi\|_j \leq \|T\phi - T\mathbf{0}\|_j + \|T\mathbf{0}\|_j \leq \beta \|\phi\| + (1 - \beta) m_j \leq \beta m_j + (1 - \beta) m_j = m_j.$$

The existence of a unique fixed point for T in $F_m(X)$ or $F_{mb}(X)$ follows from the Banach contraction principle. □

Proposition 2: *Let $T : F_m(X) \mapsto F_m(X)$ be a 1-contraction relative to $G = F_m(X)$. If*

$$\gamma := \beta c \sup \left\{ \frac{m_{j+1}}{m_j} : j \in N \right\} < 1,$$

then T is a contraction mapping from $F_m(X)$ into itself with the contractivity coefficient γ and has a unique fixed point $\phi^* \in F_m(X)$.

Proof: For $\phi, \psi \in F_m(X)$ we have

$$\|Tf - Tg\| = \sum_{j=1}^{\infty} c^{-j} \frac{\|Tf - Tg\|_j}{m_j} \le \sum_{j=1}^{\infty} \beta c^{-j} \frac{\|f - g\|_{j+1}}{m_j} = \sum_{j=1}^{\infty} \left(\beta c \frac{m_{j+1}}{m_j} \right) c^{-j-1} \frac{\|f - g\|_{j+1}}{m_{j+1}}$$
$$\le \gamma \sum_{j=1}^{\infty} \frac{\|f - g\|_{j+1}}{m_{j+1}} \le \gamma \sum_{j=1}^{\infty} \frac{\|f - g\|_j}{m_j} = \gamma \|f - g\| .$$

Thus T is contractive and by Banach's theorem has a unique fixed point $\phi^* \in F_m(X)$. □

Remark 1: We have shown that having a k-local contraction mapping T on a space of functions $F(X)$, one can construct a Banach space using some subset, say S, of $F(X)$ on which T is contractive. Then the unique fixed point of T in S can be obtained by taking the limit (in the norm on S) of the iterations $T^n \phi_0$ with an arbitrary fixed function $\phi_0 \in S$.

Remark 2: In this paper, we are mainly interested in two special cases:
(a) X is a metric space, the sets K_j are compact and

$$X = \bigcup_{j=1}^{\infty} \mathrm{Int}(K_j),$$

$F(X)$ is the space $C(X)$ of all continuous functions on X. Then $F_m(X)$ and $F_{mb}(X)$ will be denoted by $C_m(X)$ and $C_{mb}(X)$, respectively.
(b) (X, Σ) is a measurable space, $\{K_j\}$ is an increasing sequence of measurable sets satisfying (4), $F(X)$ is the space $M(X)$ of all measurable functions on X, bounded on every set K_j. Then $F_m(X)$ and $F_{mb}(X)$ will be denoted by $M_m(X)$ and $M_{mb}(X)$, respectively.

4. The model and main results

We start with some preliminaries. Let (X, Σ) be a measurable space, Y a separable metric space. A set-valued mapping A from X into the family of nonempty subsets of Y is called (weakly) *measurable* if $A^{-1}(D) := \{x \in X : A(x) \cap D \neq \emptyset\} \in \Sigma$ for every open set $D \subset Y$. Assume now that X is a metric space. Then a set-valued mapping A is called *continuous* if $A^{-1}(D)$ is closed for each closed set $D \subset Y$ and open for every open set $D \subset Y$. Clearly, a continuous set-valued mapping A is measurable if Σ is the Borel σ-algebra on X. It is well-known that any measurable mapping A having nonempty compact values $A(x)$ for all $x \in X$ admits a measurable selector, see Kuratowski and Ryll-Nardzewski (1965).

Fix a measurable *compact* set-valued mapping A and define

$$C := \{(x, a) : x \in X, a \in A(x)\}. \tag{7}$$

Then C is a measurable subset of $X \times Y$ endowed with the product σ-algebra, see Himmelberg (1975).

Lemma 1: *Let $g : C \mapsto R$ be a measurable function such that $a \mapsto g(x, a)$ is continuous on*

$A(x)$ for each $x \in X$. Then

$$g^*(x) := \max_{a \in A(x)} g(x,a)$$

is measurable and there exists a measurable mapping $f^* : X \mapsto Y$ such that

$$f^*(x) \in \arg\max_{a \in A(x)} g(x,a)$$

for all $x \in X$.

This fact follows from the measurable selection theorem of Kuratowski and Ryll-Nardzewski (1965) and Lemma 1.10 in Nowak (1984).

If in addition we assume that X is a metric space and A is continuous, then g^* is a continuous function by Berge's maximum theorem, see pages 115-116 in Berge (1963).

A discrete-time *Markov decision process* considered in this paper is defined by the objects: X, Y, $\{A(x)\}_{x \in X}$, u, q, and β satisfying the following assumptions:

A1: X is the *state space* endowed with a σ-algebra Σ.

A2: Y is a separable metric *space of actions* of the decision maker. For any $x \in X$, $A(x)$ is a compact subset of Y representing the set of all actions available in state $x \in X$. It is assumed that the set-valued mapping $x \mapsto A(x)$ is measurable. Define C as in (7).

A3: $u : C \to R$ is a (product) measurable *instantaneous return function.*

A4: q is a transition probability from C to X, called the *law of motion* among states. If x_t is a state at the beginning of period t of the process and an action $a_t \in A(x_t)$ is selected, then $q(\cdot|x_t, a_t)$ is the probability distribution of the next state x_{t+1}.

A5: $\beta \in (0,1)$ and is called the *discount factor.*

A *policy* is a sequence $\pi = \{\pi_t\}$ where π_t is a measurable mapping which associates an action $a_t \in A(s_t)$ for any admissible history of the process up to state s_t. Let Π denote the set of all policies. Note that we restrict our attention to non-randomized policies which are enough to study the discounted models. For a more formal definition of a general policy the reader is referred to Bertsekas and Shreve (1978) or Hernández-Lerma and Lasserre (1999). As usual, a *stationary policy* can be identified with a measurable mapping $\varphi : X \mapsto Y$ such that $\varphi(x) \in A(x)$ for each $x \in X$. More formally, a stationary policy is a constant sequence π with $\pi_t = \varphi$. We denote by Φ the set of all stationary policies and identify Φ with the nonempty set of measurable selectors of the mapping $x \mapsto A(x)$. Clearly, if a policy $\phi \in \Phi$ is used, then the action selected at state x_t of the process is $a_t = \varphi(x_t)$.

For each initial state $x_1 = x$ and any policy $\pi \in \Pi$, the *expected discounted return* over an infinite future is defined as:

$$J(x,\pi) := E_x^\pi \left(\sum_{t=1}^{\infty} \beta^{t-1} u(x_t, a_t) \right), \tag{8}$$

where E_x^π denotes the expectation operator with respect to the unique conditional probability measure P_x^π defined (on the space of histories, endowed with the product σ-algebra, starting at the state x) by π and the transition probability q according to the Ionescu Tulcea Theorem, see Proposition V.1.1 in Neveu (1965). We shall accept conditions under which the expected returns (8) are well-defined.

We now describe some regularity assumptions on the return and transition probability functions.

C1: Let X be a metric space and $\{K_j\}$ a strictly increasing family of compact sets such that

$$X = \bigcup_{j=1}^{\infty} \mathrm{Int}(K_j). \tag{9}$$

Let $C_c(X)$ be the space of all continuous functions on X with compact supports. Suppose that the set-valued mapping $x \mapsto A(x)$ is continuous. In addition, assume that the return function u is continuous and, for any $v \in C_c(X)$,

$$(x,a) \mapsto \int_X v(y) q(dy|x,a)$$

is also continuous on the set C.

If X is not necessarily a topological space, we accept the following regularity condition.

C2: For every $x \in X$, any measurable set $D \subset X$, the functions $a \mapsto u(x,a)$ and $a \mapsto q(D|x,a)$ are continuous on $A(x)$.

Remark 3: The continuity assumptions of the above type are typical in the theory of Markov decision processes, see Schäl (1975) and Hernández-Lerma and Lasserre (1999). From **C1**, it follows that q is continuous on C if the space of probability measures on the σ-compact state space X is endowed with the vague topology. From **C2** we can conclude easily that $a \mapsto \int_X v(y)q(dy|x,a)$ is continuous on $A(x)$ for any $x \in X$ and every bounded measurable function v on X.

Under **C1** or **C2** we can define

$$u_j(x) := \max_{a \in A(x)} |u(x,a)|, \quad x \in K_j \quad \text{and} \quad r_j := \sup_{x \in K_j} u_j(x). \tag{10}$$

Consider the sequences $\{m_j\}$ and $\{K_j\}$ as in Section 2. Assume that (4) holds. We can now describe our basic assumptions.

D1: For every $j \in N$ and $x \in K_j$, $a \in A(x)$, we have $q(K_j|x,a) = 1$.

D2: Assume that there exists $c > 1$ such that

$$\gamma := c\beta \sup_{j \in N} \frac{m_{j+1}}{m_j} < 1. \tag{11}$$

Moreover, there exists a function $h \in M_m(X)$ ($h \in C_m(X)$ when X is a metric space) such that for every $j \in N$ and $x \in K_j$, $|u_j(x)| \le h(x)$. In addition, for $x \in K_j$, $a \in A(x)$, $j \in N$, we have $q(K_{j+1}|x,a) = 1$.

Note that (11) implies that $\sum_{t=1}^{\infty}(c\beta)^t m_t < \infty$.

Lemma 2: *Assume (4) and either* **D1** *or* **D2***. Then the expected returns (8) are finite.*

Proof: Suppose that **D1** holds. Choose any $j \in N$ and $x \in K_j$. For any $t \ge 2$, we have $E_x^\pi\left(|u(x_t,a_t)|\right) \le m_j$. Hence $|J(x,\pi)| \le \frac{m_j}{1-\beta}$. Let **D2** be satisfied. Using the norm (5), define $r := \|h\|$. Observe that $\|h\|_i \le r m_i c^i$ for all $i \in N$. Let $x \in K_j$. Then for any $t \ge 2$ we have

$$|E_x^\pi\left(u(x_t,a_t)\right)| \le E_x^\pi\left(h(x_t)\right) \le r m_{j+t-1} c^{j+t-1}.$$

Consequently,

$$|J(x,\pi)| \le \sum_{t=1}^{\infty} \beta^{t-1} E_x^\pi\left(|u(x_t,a_t)|\right) \le \sum_{t=1}^{\infty} r\beta^{t-1}c^{j+t-1}m_{j+t-1} = \frac{r}{\beta^j}\sum_{t=1}^{\infty}(c\beta)^{j+t-1}m_{j+t-1} < \infty,$$

because $c\beta < 1$. $\square$

The Bellman functional equation (BE) plays a crucial role in the theory of discounted Markov decision processes. We now describe its form. For any integrable function $v : X \mapsto R$, put

$$Lv(x,a) := u(x,a) + \beta\int_X v(y)q(dy|x,a), \quad (x,a) \in C.$$

Using this notation we can write BE in the form

$$v^*(x) = \max_{a\in A(x)} Lv^*(x,a), \quad x \in X. \tag{12}$$

In this paper we are interested in the existence of a unique solution to (12) in the space $C_m(X)$ (when X is a metric space) or in $M_m(X)$ in the more general state space case.

Proposition 3: *Assume* **D1***. If* **C1** *(***C2** *and $r_j < \infty$ for each $j \in N$) is satisfied, then there exist an increasing unbounded sequence $m = \{m_j\}$ and a unique function $v^* \in C_m(X)$ ($v^* \in M_m(X)$) which satisfies the Bellman equation.*

Proof: First assume **C1**. By the maximum theorem of Berge (1963), every function u_i is continuous on the compact set K_j. Therefore $r_j < \infty$ for each j. We can choose any increasing unbounded sequence $m = \{m_j\}$ such that $m_j \ge r_j$. Consider the closed subset $C_{mb}(X)$ of the Banach space $C_m(X)$. Define an operator T on $C_{mb}(X)$ by

$$Tv(x) := \max_{a\in A(x)}\left((1-\beta)u(x,a) + \beta\int_X v(y)q(dy|x,a)\right) \tag{13}$$

where $v \in C_{mb}(X)$, $x \in X$. By the maximum theorem of Berge (1963), Tv is continuous on every set K_j. From (9), it follows that Tv is continuous on X. Under our assumption on q it is now easy to see that T maps $C_{mb}(X)$ into itself. Moreover, for any v, $w \in C_{mb}(X)$, we have

$$\|Tv - Tw\|_j \le \beta\|v - w\|_j$$

for every $j \in N$. Thus, T is a 0-local contraction. By Proposition 1 and Remark 2(a), there exists a unique $w^* \in C_{mb}(X)$ such that $Tw^* = w^*$. Put $v^* = \frac{w^*}{1-\beta}$. Clearly, $v^* \in C_m(X)$ and is a solution to the Bellman equation. The proof under condition **C2** proceeds along similar lines if we apply Lemma 1, Proposition 1 and Remark 2(b). Clearly, in that case $v^* \in M_m(X)$. $\square$

Remark 4: The operator (13) can be considered for $v \in M_m(X)$. Such situations we shall meet in the sequel.

Proposition 4: *Assume* **D2**. *If* **C1** *(***C2***) is satisfied, then there exists a unique function $v^* \in C_m(X)$ ($v^* \in M_m(X)$) which satisfies the Bellman equation.*

Proof: We first assume **D2** and **C1**. The operator given by (13) can be defined for any $v \in C_m(X)$. Let $r := \|h\|$ and $u^*(x) := \max_{a\in A(x)} |u(x,a)|$. Consider the closed ball $B_r := \{v \in C_m(X) : \|v\| \leq r\}$ in $C_m(X)$. Then $u^* \in B_r$. Choose any $v \in B_r$. By the maximum theorem of Berge (1963), Tv is continuous. We shall show that $Tv \in B_r$. Define

$$\eta(x) = \max_{a\in A(x)} \left| \int_X v(y)q(dy|x,a) \right|, \quad x \in X.$$

Clearly, η is continuous. If $x \in K_j$, then under **D2**, we have $\|\eta\|_j \leq \|v\|_{j+1}$ for all $j \in N$. Consequently,

$$\|\eta\| \leq \frac{1}{\beta} \sum_{j=1}^{\infty} \frac{\|v\|_{j+1}}{m_{j+1}c^{j+1}} (\frac{c\beta m_{j+1}}{m_j}) \leq \frac{\gamma\|v\|_c}{\beta} \leq \frac{r}{\beta}.$$

Thus, $\|Tv\| \leq r$. We have shown that T maps B_r into itself. If v, $w \in C_m(X)$, then for any j, we have

$$\|Tv - Tw\|_j \leq \beta\|v - w\|_{j+1},$$

so T is a 1-contraction. By Proposition 2 and Remark 2(a), there exists a unique $w^* \in C_m(X)$ (actually, $w^* \in B_r$) such that $Tw^* = w^*$. Clearly, $v^* = \frac{w^*}{1-\beta}$ is a solution to the Bellman equation. The proof under condition **C2** makes use of Lemma 1, Proposition 2, Remark 2(b) and proceeds along similar lines. $\square$

Remark 5: If v^* is a solution to the Bellman equation, then by Lemma 1 one can find a measurable mapping $\varphi^* \in \Phi$ such that $\varphi^*(x) \in \arg\max_{a\in A(x)} Lv^*(x,a)$ for each $x \in X$. Using standard iteration arguments and Lemma 2, one can prove that

$$v^*(x) = J(x,\varphi^*) = \sup_{\pi\in\Phi} J(x,\pi), \quad x \in X,$$

i.e., φ^* is a stationary optimal policy. For more details about this iteration method the reader is referred to Schäl (1975); Bertsekas and Shreve (1978) or Puterman (2005). Also one can show that $(1-\beta)v^*$ is the limit (in the norm $\|\cdot\|$) of the sequence $T^n\mathbf{0}$, i.e., *value iteration* holds. $T^n\mathbf{0}$ is the optimal expected return in the n-period model with return function $(1-\beta)u$, see Bertsekas and Shreve (1978).

5. Extensions to the models with discontinuous return functions or non-compact action spaces

In some applications of Markov decision processes in operations research or economics it is desirable to allow for non-compact action spaces and discontinuous return functions. We describe two possibilities for extending the results of last section.

C3: Assume in **C1** that u is upper semicontinuous.

Proposition 5: *Let us replace assumption* **C1** *by* **C3** *in Propositions 3 or 4. Then the Bellman equation has a unique upper semicontinuous solution.*

Proof: Denote by $S(X)$ the set of all *upper semicontinuous functions* in $M(X)$. Put $S_m(X) := S(X) \cap M_m(X)$ and $S_{mb}(X) := S(X) \cap M_{mb}(X)$. Propositions 1 and 2 can be formulated for operators $T : S_{mb}(X) \mapsto S_{mb}(X)$ or $T : S_m(X) \mapsto S_m(X)$, because the indicated subsets are closed in the Banach space $F_m(X)$. By Proposition 7.31 in Bertsekas and Shreve (1978), for any $v \in S_m(X)$, the function $\nu(x,a) := \int_X v(y)q(dy|x,a)$ is upper semicontinuous on every set $\{(x,a) : x \in K_j, a \in A(x)\}$, $j \in N$. From the maximum theorem of Berge (1963), it follows that Tv is upper semicontinuous on K_j. Using our assumption (9), we infer that $Tv \in S(X)$. The remaining part of the proof is an adaptation of the arguments used in proving Propositions 3 and 4. □

C4: Let X, Y be Borel (subsets of complete separable metric) spaces. Assume that $C \subset X \times Y$ is a Borel set and $A(x)$ is *σ-compact* for each $x \in X$. Suppose that the sets K_j satisfying (4) are Borel and the assumption on q in **C2** holds, $u : C \mapsto R$ is Borel measurable, and for each $x \in X$, $a \mapsto u(x,a)$ is upper semicontinuous.

In this context, $M(X)$ and $M_m(X)$ consist of Borel measurable functions.

Proposition 6: *Assume* **C4**. *If* **D1** *and* $\sup_{x \in K_j} \sup_{a \in A(x)} |u(x,a)| < \infty$ *for all* $j \in N$ *or* **D2** *with* $h \in M_m(X)$ *is satisfied, then the Bellman equation*

$$v(x) = \sup_{a \in A(x)} Lv(x,a), \quad x \in X,$$

has a unique solution $v^* \in M_m(X)$.

Proof: It is sufficient to show that T defined by (13) maps $M_m(X)$ into $M(X)$. Then the assertion follows by simple adaptations of the proofs of Propositions 3 and 4. Let $v \in M_m(X)$. Then the function $\nu(x,a) := \int_X v(y)q(dy|x,a)$ is Borel measurable on C and $a \mapsto \nu(x,a)$ is continuous on $A(x)$ for each $x \in X$. Therefore Lv is Borel on C and $a \mapsto Lv(x,a)$ is upper semicontinuous on $A(x)$ for each $x \in X$. The fact that $Tv \in M(X)$ now follows from Corollary 1 in Brown and Purves (1973). □

This result, Corollary 1 in Brown and Purves (1973), and standard iteration arguments in dynamic programming, see Blackwell (1965), lead to the following conclusion.

Corollary 1: *Under assumptions of Proposition 5, for any* $\epsilon > 0$ *there exists some* $\varphi^* \in \Phi$ *such that*

$$Lv^*(x, \varphi^*(x)) + \epsilon(1-\beta) \geq \sup_{a \in A(x)} Lv^*(x,a), \quad x \in X,$$

which implies that

$$\epsilon + J(x, \varphi^*) \geq \sup_{\pi \in \Pi} J(x, \pi), \quad x \in X.$$

Remark 6: The regularity assumptions **C1, C2** or **C3** can be considerably weakened if the state and action spaces are Borel. One can assume that u is a Borel measurable function. Using universally measurable policies, it is possible to obtain (under similar assumptions to **D1** or **D2**) that there is an upper semi-analytic solution to the Bellman equation and (for any $\epsilon > 0$) there exists an ϵ-optimal universally measurable policy. For a background material for this modification consult Bertsekas and Shreve (1978). Finally, we would like to point out that our results can also be applied to zero-sum discounted stochastic games with unbounded payoffs studied in Nowak (1984, 1985) and related articles under a boundedness assumption.

6. Applications to one-sector models of stochastic optimal growth

The results of Section 3 may have many applications to various models in operations research as studied in Hernández-Lerma and Lasserre (1999) or Puterman (2005) and in economics. . We now show two applications of Propositions 3 and 4 to the theory of stochastic optimal growth. We have in mind classical models studied in Brock and Mirman (1972) and Stokey et al. (1989). However, within our framework we allow for *unbounded utility* (return) functions. Let $X = [0, \infty)$ be the set of all *capital stocks*. If x_t is a capital stock at the beginning of period t, then consumption a_t in this period belongs to $A(x_t) := [0, x_t]$. The utility of consumption a_t is $U(a_t)$ where $U : X \mapsto R$ is a fixed function. The evolution of the state process is described by some function f of the investment for the next period $y_t := x_t - a_t$ and some random variable ξ_t. In the literature, f is called *production technology*, see Stokey et al. (1989). We shall view this model as a Markov decision process with $X = [0, \infty)$, $A(x) = [0, x]$, and $u(x, a) = U(a)$, $x \in X$, $a \in A(x)$. The transition probability will be specified in two different cases.

Assume that $\{\xi_t\}$ are independent and have a common probability distribution μ with support included in $[0, z]$ for some $z > 1$.

Example 2: (*A model with multiplicative shocks*) Assume that

$$x_{t+1} = f(x_t - a_t)\xi_t, \quad t \in N. \tag{14}$$

As in Stokey et al. (1989) (see pages 104 and 288), we assume that $f : X \mapsto R$ is a strictly concave continuously differentiable function such that $f(0) = 0$ and there exists some $y_0 > 0$ such that

$$f(y) > y \quad \text{for all} \quad y \in (0, y_0) \quad \text{and} \quad f(y) < y \quad \text{for all} \quad y > y_0. \tag{15}$$

Moreover, we assume that $f'(y) \to 0$ as $y \to \infty$. We consider the more interesting case when f is unbounded. Observe that the transition probability q is of the form: for any Borel set $B \subset X$, $x \in X$, $a \in A(x)$, we have

$$q(B|x, a) = \int_0^z 1_B(f(x - a)\xi)\mu(d\xi),$$

where 1_B is the indicator function of the set B. If $v \in C_c(X)$, then the integral

$$\int_X v(y)q(dy|x, a) = \int_0^z v(f(x - a)\xi)\mu(d\xi)$$

depends continuously on (x, a). From (15) and our additional assumptions on f, it follows that for any $j \in N$, there exists $y_j > y_0$ such that $f(y_j)z^j = y_j$. The sequence $\{y_j\}$ is increasing. Define $K_j := [0, y_j]$ for each $j \in N$. Note that if $y = x - a \in K_j$, then for any $\xi \in [0, z]$, we have $\xi f(y) \leq z f(y_j) < f(y_j)z^j = y_j$. From (14) we conclude that $q(K_j|x, a) = 1$ for every $x \in K_j$, $a \in A(x)$. We have shown that assumptions of Proposition 3 are satisfied. Therefore, for *arbitrary unbounded* continuous utility function U the Bellman equation has a unique continuous solution.

Example 3: (*A model with additive shocks*) Assume that

$$x_{t+1} = (1 + \rho)(x_t - a_t) + \xi_t, \quad t \in N. \tag{16}$$

Here $\rho > 0$ is a constant *rate of growth* and ξ_t an additional *random income* received in period t. The transition probability q is of the form

$$q(B|x, a) = \int_0^z 1_B((1 + \rho)(x - a) + \xi)\mu(d\xi),$$

where $B \subset X$ is a Borel set. If $v \in C_c(X)$, then the integral

$$\int_X v(y)q(dy|x, a) = \int_0^z v((1 + \rho)(x - a) + \xi)\mu(d\xi)$$

is continuous in (x, a). Fix a number $d > 0$. Define $K_1 := [0, d]$ and then recursively $K_{j+1} := [0, k_{j+1}]$ with $k_{j+1} := (1 + \rho)k_j + z$, $k_1 := d$. Assume that $U(a) := a^\sigma$, $\sigma \in (0, 1)$ is fixed. Define $m_j := \max_{a \in K_j} U(a)$. The sequence $\{m_j\}$ is increasing and unbounded and it is easy to prove that

$$\sup_{j \in N} \frac{m_{j+1}}{m_j} = \left(1 + \rho + \frac{z}{d}\right)^\sigma.$$

Therefore γ defined in (11) satisfies

$$\gamma < c\beta \left(1 + \rho + \frac{z}{d}\right)^\sigma < 1$$

only for some $c > 1$ and $\beta < 1$. Note that d can be arbitrarily large. For example, we can take d such that $z/d < \rho$. Then $\gamma < 1$ if $c\beta(1 + 2\rho)^\sigma < 1$. If ρ is small, then we can consider discount factors very close to one. From (16), it is easy to see that $q(K_{j+1}|x, a) = 1$ for each $x \in K_j$, $a \in A(x)$. Assumptions of Proposition 4 are thus satisfied. Therefore for this model the Bellman equation has a unique continuous solution.

We believe that our results can also be applied to multi-sector models of stochastic optimal growth.

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