On discourses addressed by infidel logicians

Walter Carnielli and Marcelo E. Coniglio
Department of Philosophy
Institute of Philosophy and Human Sciences (IFCH)
and
Centre for Logic, Epistemology and the History of Science (CLE)
State University of Campinas (UNICAMP) - Brazil
e-mail: {carniell,coniglio}@cle.unicamp.br

Abstract

We attempt to address here some alleged criticisms against the philosophical import of the so-called Brazilian approach to paraconsistency, especially by providing some demanded epistemic elucidations to the whole enterprise of the logics of formal inconsistency. In the way of elucidating, we substantiate the view that difficulties in reasoning under contradictions in both the Buddhist and the Aristotelian tradition can be accomodated within the precepts of the Brazilian school of paraconsistency.

Keywords: paraconsistent logics, paraconsistentism, dialetheism, logics of formal inconsistency[1].

1 On repugnancies and contradictions

In his well-known invective, Bishop Berkeley tried to reveal contradictions in the infinitesimal calculus, perplexed as he was by the "evanescent increments" that are neither finite nor infinitely small quantities (and "nor yet nothing", merely "ghosts of departed quantities"). In Section XLIX, B of *The Analyst; or, a Discourse Addressed to an Infidel Mathematician* of 1734 (cf. [Ber34], Berkeley asks:

> "Whether the Object of Geometry be not the Proportions of assignable Extensions? And whether, there be any need of considering Quantities either infinitely great or infinitely small? "Whether [mathematicians] do not submit to Authority, take things upon Trust, and believe Points inconceivable? Whether they have not their Mysteries, and what is more, their Repugnancies and Contradictions?"

In an indirect sense, Berkeley's *The Analyst* was very influential in the development of mathematics. The piece was a direct attack against the foundations

[1] This paper advances on some points raised on [CC08] where we attempted to refute certain criticisms to our development of the logics of formal inconsistency.

and principles of calculus and, in particular, Newton and Leibniz's notion of infinitesimal change (or *fluxions*)[2]. It is not to be denied that the resulting controversy gave some impetus to the later foundations of calculus and led it to be reworked in a much more formal and rigorous form, using limits.

However, Berkeley's criticisms did not have effect over everyone: Leonard Euler, for instance, paid little attention to the invectives against the use of infinite series, and found an astonishing new proof to the fact, originally proved by Euclid, that there are infinitely many prime numbers. Departing from Euclid's original purely combinatorial proof, Euler realized (cf. [San06]) that a distinction between divergent and convergent series could explain an infinitude; indeed, by comparing infinite sums and products

$$\frac{2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \ldots}{1 \cdot 2 \cdot 4 \cdot 6 \cdot 10 \ldots} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} \ldots$$

or, in contemporary notation:

$$\prod_p \frac{p}{p-1} = \sum_n \frac{1}{n} \quad \text{for } p \text{ primes, } n \geq 1$$

it is easy to see that the right-hand harmonic series is divergent, hence there must be infinitely many primes, otherwise there would be an equality between a divergent and a convergent series. Had Euler been afraid of the label "mathematician" and its connotation of magician and astrologist instead of a geometer, he would never dare to use the continuous into the discrete. Euler's idea turned out to be extremely fruitful: mixing the "forbidden" analysis into the investigation of prime numbers allows a much more powerful technique than mere combinatorial counting, as further results by Dirichlet and others have revealed. What was at stake was not so much how many numbers there are, but how are they distributed. Analytic number theory was thus born, which ultimately paving the way to Riemann Hypothesis and owing its existence to free-thinkers like Euler (see [Der03]).

Now, if science has some repetitive patterns, it may not be a coincidence that paraconsistent negations, that is, negations such that a contradiction does not imply everything, raise some perplexities of an analogous sort. This is certainly the case, in particular, with the kind of negation supported by the *Logics of Formal Inconsistency* (*LFI*s), introduced in [CM02] and further developed in [CCM07]. Some criticisms are supported on the idea that paraconsistent negations are not negations (in the same way as infinitesimal numbers would not be numbers). Other criticisms concerning specifically the core of *LFI*s, fail to see a central point: how is it possible that A and $\neg A$ can be simultaneously held as true (and be not explosive), and the "consistency" of A be also held as true? How can something and its negation be true and consistent? Is it not inherent in the nature of consistency to require that anything and its negation have necessarily different truth status?

[2]It is believed that the "infidel mathematician" in question was either Edmond Halley, or Isaac Newton himself.

Actually, we want to argue that this is not only plainly possible, but usual: logicians, infidel or not, perform this type of reasoning very often, and it is precisely the fact that A and $\neg A$ are true, while it is not the case that A is consistent (see Section 3) which blocks the deductive explosion. On the other hand, with respect to the first criticism, a possible answer is that paraconsistent negations can be seen as generalized negations, in the same way as infinitesimal numbers are generalized (non standard) real numbers.

But let us first examine an example of an argument that so much baffled Berkeley. The example is a contemporary rephrasing of a more geometric, given by Berkeley himself, on lemma 2 of Section 1 in *Philosophiae Naturalis Principia Mathematica*, by Isaac Newton (1726), and is a typical argument using infinitesimal quantities: in order to find the derivative $f'(x)$ of the function $f(x) = x^2$, let dx be an infinitesimal. Then

$$\begin{aligned} f'(x) &= \frac{f(x+dx) - f(x)}{dx} \\ &= \frac{(x+dx)^2 - x^2}{dx} \\ &= \frac{x^2 + 2x.dx + dx^2 - x^2}{dx} \\ &= \frac{2x.dx + dx^2}{dx} \\ &= 2x + dx \\ &= 2x \end{aligned}$$

since dx is infinitely small.

The fundamental problem pointed out by Berkeley is that dx is first treated as *non-zero* (because we divide by it), but later discarded as if it were zero :

> "These Expressions [dx, ddx, etc] *indeed are clear and distinct*, and the Mind finds no difficulty in conceiving them to be continued beyond any assignable Bounds. But if we remove the Veil and look underneath, if laying aside the Expressions we set ourselves attentively to consider the things themselves, which are supposed to be expressed or marked thereby, we shall discover much Emptiness, Darkness, and Confusion; nay, if I mistake not, direct Impossibilities and Contradictions. Whether this be the case or no, every thinking Reader is entreated to examine and judge for himself."

In a sense, Berkeley was right: the idea of *infinitesimal* was at that time a naive concept, namely "a number whose absolute value is less than any non-zero positive number". Thus, using the order properties of real numbers, it can be easily proven that *there are no non-zero real infinitesimals*! In Berkeley's perspective, infinitesimals are not numbers but aberrant intruders and so should be removed from the mathematical reality. But, how can Berkeley be completely right, and infinitesimal calculus be what it is today?

Karl Weierstrass and others, by using the rigorous notion of limit, have been able to give a formal mathematical foundation for calculus at the second half of the nineteenth century: the epsilon-delta interpretation. This mathematical formulation, without using "abnormalities" such as infinitesimals, removed all concerns about the illegitimacy of calculus. This was, however a sort of foundational compromise, with concessions to both sides: the notion of infinitesimal had to be replaced by a process (the limits), and in this Berkeley's criticisms to the 18th century mathematicians were in principle correct, but the thing was not impossible, and in this he was wrong. Two centuries later, anyhow, the fluxions vindicated their reputation in the hands of logicians.

2 Infidelities and perplexities: are paraconsistent negations genuine negations?

What Bishop Berkeley couldn't see is that the feasibility of infinitesimals depends on the width of the mathematical context: they are not mathematically "impossible" or " wrong", they just find a place in a wider mathematical scenario. As it is well known, the original appealing idea of Leibniz and Newton[3] of describing differential calculus by using infinitesimal quantities can be recovered in rigorous, mathematical terms by means of contemporary model theory, the non-standard models. What this achievement shows is that infinitesimals are numbers of a new kind, which are introduced conservatively by extending the previous field of numbers.

Such numbers of new species are nothing else than infinitesimals turned into possible: Abraham Robinson's *nonstandard analysis* in the 60's (cf. [Rob66]) considers the set of real numbers extended by the hyperreals which contains numbers less (in absolute value) than any positive real number. Thus, an infinitesimal is a *non-standard* number whose absolute value is less than any non-zero positive *standard* number. Other paradigms were introduced to deal with infinitesimals, such as John Conway's *surreal numbers*, cf. [Con76] and also [Knu74] (which are algebraically equivalent to hyperreals), Edward Nelson's *internal set theory* (cf. [Nel77]) and *synthetic differential geometry* (also known as *smooth infinitesimal analysis*), cf. [Law98], based on category (topos) theory and giving a new approach to Robinson's nonstandard analysis.

The latter consider *nilsquare* or nilpotent infinitesimals, that is, numbers x such that $x^2 = 0$ holds but $x = 0$ is not necessarily true. This allows rigorous algebraic proofs using infinitesimals as the one given above, which so much irritated Bishop Berkeley.

Besides infinitesimals, there are several examples of generalized mathematical structures losing "classical" features. Non-Euclidean geometry is one of them. Concerning numbers and their operations, the following are obvious examples (assuming a "classical" perspective in each case):

[3]But there are some recent historical evidences that the elements of the infinitesimal calculus, developed between the 14th and 16th centuries in Kerala, India, have been transmitted to Europe by Jesuit missionaries (cf. [AJ07]).

- Integer numbers are not numbers from the point of view of natural numbers: there cannot be any x such that $x + 2 = 1$!
- Rational numbers are not numbers from the point of view of integer arithmetic: there cannot be any x such that $2x = 1$!
- Real numbers are not numbers from the point of view of rational numbers: there cannot be any x such that $x^2 = 2$!
- Complex numbers are not numbers from the point of view of real numbers: there cannot be any x such that $x^2 = -1$!

There are certain subtle similarities between Berkeley's criticism to infinitesimals and some criticism to paraconsistent negations found in the literature. For instance, the well-known Slater's criticism in [Sla95] is based on the contention that negations of paraconsistent logics would not be proper negation operators given that they are not any 'contradictory-forming functor', but just a 'subcontrary-forming one'.

But arguing in this direction requires strong presuppositions on what a negation should be. Since, supposedly, natural language is our basic source of inspiration for understanding negation, it is advisable to pay close attention to it before sermonizing. Is there really just one negation, and has it necessarily the role of suppressing whatever comes after it? R. Giora in [Gio06] (pp. 1009-1010) argues against the view that negation is unique in natural language, and against the purported functional asymmetry of affirmation and negation, a view that supports the "suppression hypothesis"−which assumes that negation necessarily suppresses what is inside its scope:

> "Indeed, many discourse functions assumed to uniquely distinguish negatives from affirmatives, such as denying, rejecting, disagreeing, repairing (both linguistically and metalinguistically), eliminating from memory, communicating the opposite, attenuating or reducing the accessibility of concepts and replacing them with alternative opposites, are equally enabled by affirmatives. Similarly, discourse roles assumed to uniquely distinguish affirmatives from negatives, such as representing events, conveying agreement, confirmation, or affective support, highlighting and intensifying information, introducing new topics, conveying an unmarked interpretation, establishing comparisons, effecting discourse coherence and discourse resonance, are equally enabled by negatives. Such evidence attesting to some functional affinity between negative and affirmative interpretations can only be explained by processing mechanisms that do not operate obligatorily but are instead sensitive to global discourse considerations."

In analogy with the case of infinitesimals, paraconsistent negations could be seen as extending the classical ones by generalizing some of its features. And this makes sense with the above landscape of many negations: the more properties a negation operator has, the more restricted and specific the operator is. Thus, paraconsistent negations are neither logically "wrong" nor "impossible", but they are part of an enhanced and more general logical scenario, in the same way that infinitesimals are legitimate numbers in a wider sense.

It is really interesting to pay attention for some moments to the model theory of propositional quantified logic and again to compare this model theory with the underlying model theory of algebraic extension of fields (from the real to the complex numbers).

Recall that, if $\mathfrak{A}$ and $\mathfrak{B}$ are two first-order structures for the same language such that $\mathfrak{A}$ is a substructure of $\mathfrak{B}$ and $\varphi(x)$ is a formula of that language with just x as free variable, and where just the logical operators $\exists$, $\wedge$ and $\vee$ occurs in $\varphi(x)$, then

$$\mathfrak{A} \models \exists x \varphi \quad \text{implies} \quad \mathfrak{B} \models \exists x \varphi.$$

In particular, taking the language of fields consisting of symbols $+, \cdot, 0, 1$ and $\mathfrak{A}$ and $\mathfrak{B}$ as being the structures of real numbers and complex numbers over that language, respectively, then

$$\mathbb{R} \models \exists x \varphi \quad \text{implies} \quad \mathbb{C} \models \exists x \varphi$$

for any $\varphi(x)$ as above. As is well-known, the converse implication is not valid because $\mathbb{C}$ is an algebraic extension of $\mathbb{R}$ and so it satisfies an existential sentence that the latter does not satisfies. For instance,

$$\mathbb{C} \models \exists x(x.x + 1 = 0) \quad \text{but} \quad \mathbb{R} \not\models \exists x(x.x + 1 = 0).$$

This means that $\mathbb{R}$ is not an elementary substructure of $\mathbb{C}$.

Consider now $\mathbf{PCL}$ and $\mathbf{C}_1$, propositional classical logic and da Costa's paraconsistent logic over the signature $\wedge, \vee, \rightarrow, \neg$, respectively. Consider a fixed set $\mathcal{V}$ of propositional letters and let For be the algebra of formulas generated over that signature from $\mathcal{V}$. Let $V_{\mathbf{PCL}}$ and $V_{\mathbf{C}_1}$ be the sets of bivaluations characterizing $\mathbf{PCL}$ and $\mathbf{PCL}_1$, respectively. Then $PCL = \langle For, V_{\mathbf{PCL}} \rangle$ can be conceived as a semantical structure interpreting quantified propositional logic in an obvious way, representing $\mathbf{PCL}$. Similarly, $C_1 = \langle For, V_{\mathbf{C}_1} \rangle$ can be seen as a structure for that language representing $\mathbf{C}_1$. Since $V_{\mathbf{PCL}} \subseteq V_{\mathbf{C}_1}$ then

$$PCL \models \exists p_1 \ldots \exists p_n \, \varphi \quad \text{implies} \quad C_1 \models \exists p_1 \ldots \exists p_n \, \varphi$$

for any $\varphi(p_1, \ldots, p_n)$ depending on propositional letters $p_1, \ldots, p_n$ without quantifiers. In a sense, PCL is a "substructure" of C_1. But the converse implication does not hold, and so PCL is not an "elementary substructure" of C_1. In fact,

$$C_1 \models \exists p(p \wedge \neg p) \quad \text{but} \quad PCL \not\models \exists p(p \wedge \neg p).$$

In analogy with the extension $\mathbb{C}$ of $\mathbb{R}$, which adds new objects outside the classical (real) scope satisfying unexpected (or "absurd") properties, the paraconsistent model-theoretic extension of classical logic by C_1 adds new valuations v (or new formulas p) satisfying "exotic" or "absurd" properties such as $v(p) = v(\neg p) = 1$. Recalling that the inconsistency operator $\bullet$ of LFI's allows or guarantees such "exotic" properties (see next section) then, in some sense, the inconsistency operator $\bullet$ or, more precisely, inconsistent formulas of the form $\bullet\varphi$, play a very similar role of non-real complex numbers, that is, complex numbers $a + b\mathbf{i}$ such that $b \neq 0$.

C. Dutilh Novaes, in [DN08], p. 470, convincingly argues that "there is no real negation" and that "paraconsistent negation is as "real" as any other. So, how much denying are you in "real negation"? The comparison is analogous to the joke about naive tourists who keep asking how much is the price of this and that in "real money": the problem only arises if you insist that your money, or your negation, is unique, and *is* the real one.

3 Can you sustain a consistent contradiction?

But more pointy criticisms to paraconsistency concern explicitly the *LFI*s of [CM02] and [CCM07]. The main feature of this approach is that sets $\bigcirc(p)$ of formulas (just depending on p) are considered, to convey the idea that $\bigcirc(A)$ expresses the fact (or the information, or even the hypothesis) that "A is consistent". Thus $\Gamma \vdash A$ and $\Gamma \vdash \neg A$ do not ensure that Γ is trivial. Instead, Γ is logically explosive iff

$$\Gamma \vdash A \quad \text{and} \quad \Gamma \vdash \neg A \quad \text{and} \quad \Gamma \vdash \bigcirc(A)$$

In most *LFI*s the set set $\bigcirc(p)$ is a singleton, defining a consistency connective (primitive or not) denoted by $\circ$; thus, $\circ A$ means "A is consistent", and the usual "Classical Explosion Principle"

$$\text{(exp)} \;\; A \Rightarrow (\neg A \Rightarrow B)$$

is replaced by a weaker version, the "Gentle Explosion Principle":

$$\text{(bc)} \;\; \circ A \Rightarrow (A \Rightarrow (\neg A \Rightarrow B))$$

So, a contradiction (involving A) *plus* the information that A is consistent produces a trivial set. The *inconsistency* of a sentence A can be expressed by a sentence of the form $\bullet A$ where $\bullet$ is an *inconsistency operator*. In most LFI's both operators are related as follows: $\bullet A \equiv \neg\circ A$ and $\circ A \equiv \neg\bullet A$. Let us go a bit further in order to appreciate the importance of this approach to logic and reasoning.

A well-known (by now) 12th century example of a derivation by Petrus Abelardus in his *Dialectica* is recalled at page 217 of W. Kneale and M. Kneale in [KK85]: the conclusion *si Socrates est lapis, est asinus* ("if Socrates is a stone, he is an ass") as a consequence of the validity of the Disjunctive Syllogism $\alpha \vee \beta, \neg\alpha \vdash \beta$. Indeed, from the hypothesis *Socrates est lapis* one derives *Socrates est lapis* or *Socrates est asinus.* But surely *Socrates non est lapis*, *ergo Socrates est asinus.*

W. Kneale and M. Kneale recognize this "very interesting contention" of Abelard as the beginning of the long Medieval debate on paradoxes of implication:

"On the other hand, he thinks that we have departed from the highest standard of rigour as soon as we put forward a consequentia which involves the assumption of two distinct substances $\cdots$ For he says that in such a case the sense of the consequent is not contained in the sense of the antecedent and that the truth of the whole can be established only by special knowledge of nature ('posterius ex naturae discretione et proprietatis naturae cognitione')."

They recognize that "it is difficult to find any satisfactory interpretation for this passage". But the reason, as Abelard himself explains, cf. page 284 of [Aba70], is that the nature of man and stone are incomparable:

"*quod novimus natura ita hominem et lapidem esse disparata*"

Now, it is just an immediate theorem of *LFI*s that Disjunctive Syllogism is not to be held unrestrictedly, but just for situations with some aside assumptions (or proofs) of certainty or consistency are present, i.e.:

$$\alpha \vee \beta, \neg\alpha \nvdash \beta$$

Disjunctive Syllogism does not hold in general, but does hold in a controlled form:

$$\circ\alpha, \alpha \vee \beta, \neg\alpha \vdash \beta$$

is a valid rule, where $\circ$ is the consistency connective (in the *LFI*s).

This property can be easily proven in almost all systems developed in [CCM07], and actually, besides a conceptual interest, it is essential for the development of an effective and natural "paraconsistent logic programming", since it is just the appropriate generalization of the famous resolution rule.

But on what concerns Abelard and the allegation that the truth involved in a reasoning such that which proves that *Socrates est asinus* can only be established by special knowledge of nature, our extra hypothesis $\circ\alpha$ in the controlled form of Disjunctive Syllogism above is perfectly able to express the proviso that the nature of man and of stone cannot be disparate in order to perform a reasoning of this sort. Indeed, just take $\circ\alpha$ to be true, in this case, if you are prepared to defend any reasoned connection between the nature of Socrates and the nature of a stone. If so, you may derive *Socrates est asinus*; if not, you know where your mistake is. Thus the *LFIS*s, basic and simple as they are, not only are coherent to Abelard's advice but also may help to *express* that restriction.

Let us face the criticisms against this approach. There seem to be, to start with, several cases of misunderstandings, misapprehensions or pure disregards[4] on what paraconsistent logic is, or rather *are*: for instance, R. Sorensen at p. 114 in [Sor03] manifests his firm (and wrong) belief that paraconsistent logics (in the plural!) must reject weakening (the inference rule from p to $p \vee q$):

[4]For some strange reasons, this seems to be more acute among writers in certain groups: although the phenomenon of "relative own-language preference" on citations is well-known, in this case it seems to be"relative own-group preference". As an attitude, it is similar to a famous whimsy by the Brazilian writer Oswald de Andrade applied to his impression on a book by a rival: "I didn't read, and didn't like it".

> "Paraconsistent logics are designed to safely confine the explosion. For instance, they reject the inference rule "p, therefore, p or q" on the grounds that a valid argument must have premises that are relevant to the conclusion. They extend this relevance requirement to conditionals in an effort to head off the paradoxes of implication."

He blames dialetheistic logics for this unability, and in a hasty generalization continues, at the same page:

> "Dialetheists portray themselves as friends of contradiction. They remind me of ranchers who present themselves as friends of the horses they castrate. A gelding is not just a tamer sort of stallion; it is not a stallion at all. The dialetheist's "contradiction" may look like contradictions and sound like contradictions, but they cannot perform a role essential to being a contradiction; they cannot serve as the decisive endpoint of a reductio ad absurdum. At best they can be the q in a modus tollens argument: If p then q; not q, therefore, not p. So in the end, I think Priest falls into Antisthenes' skepticism about contradictions."

The quotation refers to Antisthenes of Athens (445-360 B.C.), a student of Socrates previously trained under the Sophists. Antisthenes thinks there are no contradictions, and Sorensen puts Antisthenes and the dialetheists (and by his reduction, all paraconsistentists) in the same side: their contradictions would not be as such. We cannot respond for dialetheists, but perhaps Sorensen would be happy to learn that *LFI*s neither reject the weakening rule, nor pose themselves as friends of geldings or stallions: they just separate geldings from stallions, as we argued above. Consistent contradictions *do* serve as the decisive endpoints of *reductio ad absurdum* reasoning: it is proven in [CCM07] that the following *reductio* rule holds in most *LFI*s:

$$\text{If } \ \Gamma \vdash \circ\alpha, \quad \Delta, \beta \vdash \alpha \text{ and } \quad \Lambda, \beta \vdash \neg\alpha \quad \text{then} \quad \Gamma, \Delta, \Lambda \vdash \neg\beta)$$

This illustrates an instance of a more general phenomenon: Any classical rule can be recovered within a class of *LFI*s known as **C**-systems, if a sufficient number of consistency assumptions are assumed (see the "Derivability Adjustment Theorem" in [CCM07], Remark 21).

Some authors consider, however, that this perspective has inherent problems. Specifically, the following passage of [CM02] p. 27 has been criticized:

> "So one may conjecture that consistency is exactly what a contradiction might be lacking to become explosive if it was not explosive from the start. Roughly speaking, we are going to suppose that a consistent contradiction is likely to explode, even if a regular contradiction is not."

One of the main criticisms comes from F. Berto in [Ber07], who akes at p. 162:

> "How could we have α, $\neg\alpha$ *and* keep claiming that α is consistent?"

Well, you *cannot* and that is precisely what is meant! In the realm of classical logic, the corresponding objection is: How could we have α *and* $\neg\alpha$ in our theory? Of course, a classical logician simply *cannot*, and this is exactly what the "Classical Explosion Principle" says:

$$\alpha, \neg\alpha \vdash \beta$$

for any β, that is: if you have *any* contradiction, you are in trouble.

This principle, as explained, is generalized in the *LFI*s by the above "Gentle Explosion Principle":

$$\alpha, \neg\alpha, \circ\alpha \vdash \beta$$

for any β, that is: if you have any *consistent* contradiction, then you have troubles!

The situation is rather similar to chess: no piece can capture the King but the King can be under the threat of being captured, that is, under check. In this contradictory situation (the King cannot be captured and is being threatened with capture) the King has to be protected, and there may be several moves do protect him. However, if there is no move which can put the King out of check (that is, if the threat is fatal), this is a checkmate and the game is over. But this point seems to have been overlooked by Berto ([Ber07], p. 162) who abandons the game:

> "These difficulties seem to speak against the philosophical import of the Brazilian approach to paraconsistency (which is why it has been dealt quickly in this Chapter)."

It is perhaps M. Bremer ([Bre05] , p. 117), however, who makes Berto to resign so quickly:

> "introducing 'consistent contradictions' [...] awaits epistemic elucidation: If we have A and $\neg A$, then we should take $\circ A$ as false, shouldn't we? And how can we take A to be consistent and have A and $\neg A$ at the same time?"

Indeed, Bremer had previously announced, in [Bre98] p. 53, his intention to abandon what has called "da Costa systems" from a certain point on due to what seemed to him unsurmountable philosophical difficulties:

> "Da Costa-Systeme werden deshalb hier nicht weiter betrachtet"[5].

Notwithstanding, as observed above this criticism can be similarly posed to classical logicians as well: "If we have A, then we should take $\neg A$, as false, shouldn't we? And how can we take A and have A and have $\neg A$ at the same time?"

Obviously you *cannot*, of course, and that is precisely what the "Classical Explosion Principle" tells you!

Bremer, in [Bre98], p. 50, also claims:

[5] "da Costa' systems are, consequently, not treated here from this point on".

"Die da Costa-Negation ist überhaupt nich rekursiv!"[6]

The reason, he says, is that the truth-value of $\neg A$ cannot be computed from the truth-value of A. But this is just non-functionality, and if this were any philosophical criticism, it could be posed to several other logics, including intuitionistic and almost all modal logics, and to all *LFI*'s as well. It is hard to see the point behind his criticism. The fact is that da Costa (and *LFI*) negations are *bounded non-deterministic* functions, and there is no purpose in disqualifying them for such reasons. It happens that all *LFI*'s (including da Costa calculi in the hierarchy C_n) are decidable (thus, recursive); this is accomplished by the original valuation semantics in the case of C_n, and by the possible-translations semantics (cf. [CCM07] in the general case of *LFI*'s.

This is amply confirmed by the non-deterministic semantics of A. Avron and his collaborators (cf. e.g. [AL05]). But specifically for da Costa calculi C_n decidability results by means of the procedure of quasi-matrices are known since three decades, cf. [Alv76] and [dCA77] (some mistakes being fixed in [Mar99], Section 2.2.2). Disqualifying non determinism in logic matters thus not seem to be so easy. Non-deterministic Turing machines are equivalent with deterministic Turing machines: issues on complexity apart, they are the same. Even modal logics of non-deterministic partial recursive functions, which are extensions of the classical propositional logic, are studied in [Nau05], and moreover, proved to be decidable. Confusions of this sort just adds to the difficult on appraising the real ideas behind *LFI*'s. There is no better epistemic elucidation, and no better way of assessing the "philosophical import" of something than appreciating it from the fair perspective.

4 From contradictoriness to Buddhism, and back

It was not, apparently, any idea of a paraconsistent logician, as noted by J. Garfield and G. Priest in [GP02], and in [GP03], to propose that some contradictions in Buddhist reasoning would make them endorse paraconsistent logic: they agree, but credit T. Tillemans in [Til99] for the observation. Garfield and Priest do a decisive step by studying the possibilities that those contradictions could be seen as structurally analogous to those arising in the Western tradition. By a penetrating analysis of *catuskoti* or tetralemma, the four-cornered negations of Nāgārjuna, a Buddhist thinker of the II century, they aptly conclude that Nāgārjuna is not

> "$\cdots$ an irrationalist, a simple mystic, or crazy; on the contrary: he is prepared to go exactly where reason takes him: to the transconsistent."

It is not doubt that Nāgārjuna reasoning involves contradictions, and J. Garfield and G. Priest offer what they call "dialetheist's comfort" (admitting the possibility of true contradictions) to consign the view that Nāgārjuna is indeed

[6] "da Costa's negation is absolutely non-recursive!"

a highly rational thinker. A similar defense that contradictions in Buddhism are esentially dialetheist is found in [YDP08].

As an example, the reasoning involved in a tetralemma is the following. When an interlocutor poses four questions to the Buddha about the re-birth of an *arhat* (illuminated person), as:

- "Is the arhat reborn?
- Is the arhat not reborn?
- Is the arhat both reborn and not reborn?
- Is the arhat neither reborn nor not reborn?"

the Buddha replies to each question saying that one cannot say so.

Now, looking at the the three first questions, we might conclude that the Buddah would be forced to accept true contradictions–but is that really so? According to M. Siderits in [Sid08]:

> "[...] the third possibility involves equivocation on 'existent': that the arhat does exist when 'existent' is taken in one sense, but does not exist when it is taken in some other sense. For when the Buddha rejects both of the first two lemmas, this generates an apparent contradiction. And one way of seeking to resolve this contradiction is to suppose that there is equivocation at work."

Siderits then concludes:

> "To consider this possibility is not to envision that there might be true contradictions. It is a way of trying to avoid attributing to the speaker the view that a contradiction holds."

So, the point would be: if indeed contradictions in Buddhist reasoning are structurally analogous to those we refer to in the Western tradition, and can be put under the scope of a paraconsistent formalism, teh task would be to specify which kind of paraconsistent logic could better reflect the logic of Nāgārjuna reasoning.

A conducive strategy would be to get some inspiration on how they see negation. And, once more, coinciding to what has been defended for natural language, negation for them is not unique: B. Galloway argues in [Gal89] that the *prasajya* negation of the Madhyamaka school of Buddhist philosophy (members of a Buddhist school founded by Nāgārjuna) is not the same as that of the other schools. Would it appease the Madhyamikas to conceive them as endorsing dialetheism and swallowing contradictions as real? It does not seem to be so: quoting again Siderits, at page 132:

> "Madhyamikas say that only mad people accept contradictions"

This seems to exclude any eventuality that fellows of Nāgārjuna school would embrace dialetheism, the view that there are dialetheias, *qua* sentences which are both true and false. This difficulty is in line with another, as pointed by P. Gottlieb in [Got07]:

> "While Aristotle is clearly not a dialetheist, it is not clear where he stands on the issue of paraconsistency. Although Aristotle does argue that if his opponent rejects PNC across the board, she is committed to a world in which anything goes, he never argues that if (per impossibile) his opponent is committed to one contradiction, she is committed to anything, and he even considers that the opponent's view might apply to some statements but not to others (Metaph IV 4 1008a1012)."

The Shōbōgenzō, a masterpiece with the teachings of the Japanese Soto Zen Master Eihei Dogen of the thirteenth century, contains several Zen Koan stories that are sometimes disconcertingly contradictory. The text is full of contradictions, of several dimensions: contradiction between chapters, contradictions between paragraphs, contradictions between sentences and even contradictions *within* a sentence (see [Nea07] for a careful translation from Japanese). For instance, a discourse by Master Dogen "On Buddha Nature" ([Nea07], p. 244) contains two ostensibly contradictory statements, namely, that all sentient beings have a Buddha Nature and that all sentient beings lack a Buddha Nature.

Such a contradiction, however, is not anything like a dialetheia: Buddha Nature is not the existence of something; on the one hand, Buddha Nature is encountered everywhere, and on the other hand sentient beings do not readily find an easy or pleasant way to encounter Buddha Nature (see [Nea07], chapter 21, for a long discussion).

Gudo Wafu Nishijima, a Japanese Zen Buddhist priest and teacher, in trying to explain why the Shōbōgenzō is difficult to understand because of this intricate weave of contradictions, make a point clear in [Nis92]:

> At this point I want to make a very fundamental point about the nature of contradiction itself. We feel in the intellectual area that something called contradiction exists; that something can be illogical. But in reality, there is no such thing as a contradiction. It is just a characteristic of the real state of things. It is only with our intellect that we can detect the existence of something called contradiction.

It seems, thus, that both Buddhist and Aristotelian tradition will see perfect coherence in the distinction between *reasoning* in the presence of contradictions and *accepting* them. The former position expands the latter, and any appeal to the principle of rational accommodation would support the following conclusion: If by 'accepting a contradiction' we mean 'considering it as consistent', then both traditions would agree with our vision of paraconsistentism. Infidelity, but only at a barely minimum.

Acknowledgements. Both authors acknowledge support from FAPESP - Fundação de Amparo à Pesquisa do Estado de São Paulo, Brazil, Thematic Research Project grant LogCons 2010/51038-0, and from CNPq Brazil Research Grants. The first author has been additionally supported by the Fonds National de la Recherche, Luxembourg.

References

[Aba70] Petrus Abaelardus. Dialectica. L. M. De Rijk (editor), Assen: Van Gorcum, 1970. http://individual.utoronto.ca/pking/resources/abelard/Dialectica.txt.

[AJ07] D. F. Almeida and G. G. Joseph. Kerala mathematics and its possible transmission to europe. *Philosophy of Mathematics Education Journal*, 20, 2007. http://people.exeter.ac.uk/PErnest/pome20/.

[AL05] A. Avron and I. Lev. Non-deterministic multi-valued structures. *Journal of Logic and Computation*, 15:241–261, 2005.

[Alv76] E. Alves. *Logic and Inconsistency* (in Portuguese). PhD thesis, University of São Paulo, Brazil, 1976.

[Ber34] G. Berkeley. The Analyst; or, a Discourse Addressed to an Infidel Mathematician. D. R. Wilkins (editor), School of Mathematics, Trinity College, Dublin, 1734. http://www.maths.tcd.ie/pub/HistMath/People/Berkeley/Analyst/Analyst.html.

[Ber07] F. Berto. *How to Sell a Contradiction: The Logic and Metaphysics of Inconsistency*. Studies in Logic. College Publications, 2007.

[Bre98] M. Bremer. *Wahre Widersprche. Einfhrung in die parakonsistente Logik*. Academia, 1998.

[Bre05] M. Bremer. *An introduction to paraconsistent logics*. Peter Lang, Frankfurt, 2005.

[CC08] W. A. Carnielli and M. E. Coniglio. Aristóteles, Paraconsistentismo e a Tradição Budista. *O que nos faz pensar*, (23):163–175, 2008.

[CCM07] W. A. Carnielli, M. E. Coniglio, and J. Marcos. Logics of formal inconsistency. In D. Gabbay and F. Guenthner, editors, *Handbook of Philosophical Logic*, volume 14, pages 1–93, Amsterdam, 2007. Springer-Verlag.

[CM02] W. A. Carnielli and J. Marcos. A taxonomy of **C**-systems. In W. A. Carnielli, M. E. Coniglio, and I. M. L. D'Ottaviano, editors, *Paraconsistency - the Logical Way to the Inconsistent*, volume 228 of *Lecture Notes in Pure and Applied Mathematics*, pages 1–94, New York, 2002. Marcel Dekker.

[Con76] J. H. Conway. *On Numbers and Games*. Academic Press, New York, 1976.

[dCA77] N.C.A. da Costa and E. Alves. A semantical analysis of the calculi C_n. *Notre Dame Journal of Formal Logic*, 18(4):621–630, 1977.

[Der03] J. Derbyshire. *Prime Obsession: Bernhard Riemann and the Greatest Unsolved Problem in Mathematics*. Plume/Penguin Group, New York, 2nd. edition, 2003.

[DN08] C. Dutilh-Novaes. Contradiction: the real philosophical challenge for paraconsistent logic. In J.-Y. Béziau, W.A. Carnielli, and D. Gabbay, editors, *Handbook of Paraconsistency*, pages 465–480, Amsterdam, 2008. Elsevier.

[Gal89] B. Galloway. Some logical issues in Madhyamaka thought. *Journal of Indian Philosophy*, 17(1):1–35, 1989.

[Gio06] R. Giora. Anything negatives can do affirmatives can do just as well, except for some metaphors. *Journal of Pragmatics*, 38:981–1014, 2006.

[Got07] P. Gottlieb. Aristotle on non-contradiction. Stanford Encyclopedia of Philosophy, February 2007. `http://plato.stanford.edu/entries/aristotle-noncontradiction/#1`.

[GP02] J. Garfield and G. Priest. Nāgārjuna and the limits of though. In G. Priest, editor, *Beyond the Limits of Thought*, pages 249–270, Oxford, 2002. Oxford Univ. Press.

[GP03] J. L. Garfield and G. Priest. Nagarjuna and the limits of thought. *Philosophy East and West*, 53(1):1–21, 2003.

[KK85] W. Kneale and M. Kneale. *The Development of Logic*. Oxford University Press, 1985.

[Knu74] D. Knuth. *Surreal Numbers: How Two Ex-Students Turned on to Pure Mathematics and Found Total Happiness*. Addison-Wesley, Reading, Mass., 1974.

[Law98] F. W. Lawvere. Outline of synthetic differential geometry. Available at the Internet, February 1998. `http://www.acsu.buffalo.edu/~wlawvere/SDG_Outline.pdf`.

[Mar99] J. Marcos. Semânticas de Traduções Possíveis (Possible Translations Semantics, in Portuguese). Master's thesis, IFCH-UNICAMP, Campinas, Brazil, 1999. `http://www.cle.unicamp.br/pub/thesis/J.Marcos/`.

[Nau05] P. Naumov. On modal logics of partial recursive functions. *Studia Logica*, 81(3):295–309, 2005.

[Nea07] H. Nearman. *Shōbōgenzō- The Treasure House of the Eye of the True Teaching. A Trainee's Translation of Great Master Dogen's Spiritual Masterpiece*. Shasta Abbey Press, Mount Shasta, California, 2007.

[Nel77] E. Nelson. Internal set theory; a new approach to nonstandard analysis. *Bull. Amer. Math. Soc*, 83(6):1165–1198, 1977.

[Nis92] G. W. Nishijima. *Understanding the Shōbōgenzō*. Windbell Publications, London, 1992.

[Rob66] A. Robinson. *Non-standard Analysis*. Princeton University Press, Princeton, 2nd. revised edition edition, 1966.

[San06] E. Sandifer. Infinitely many primes. MAA Online–How Euler Did It, March 2006. `http://www.maa.org/news/columns.html`.

[Sid08] M. Siderits. Contradiction in Buddhist argumentation. *Argumentation*, 22(1):125–133, 2008.

[Sla95] B. H. Slater. Paraconsistent logics? *Journal of Philosophical Logic*, 24:233–254, 1995.

[Sor03] R. Sorensen. *A Brief History of the Paradox*. Oxford University Press, 2003.

[Til99] T. Tillemans. Is Buddhist logic non-classical or deviant? In T. Tillemans, editor, *Language, Logic and Scripture*, pages 187–205, Boston, 1999. Wisdom Publications.

[YDP08] J. L. Garfield Y. Deguchi and G. Priest. The way of the dialetheist: contradictions in Buddhism. *Philosophy East and Wes*, 58(3):395–402, 2008.