ON DISCRETIZATION SCHEMES FOR STOCHASTIC EVOLUTION EQUATIONS

ISTVÁN GYÖNGY AND ANNIE MILLET

Abstract. Stochastic evolutional equations with monotone operators are considered in Banach spaces. Explicit and implicit numerical schemes are presented. The convergence of the approximations to the solution of the equations is proved.

1. Introduction

Let $V \hookrightarrow H \hookrightarrow V^*$ be a *normal triple* of spaces with dense and continuous embeddings, where V is a reflexive Banach space, H is a Hilbert space, identified with its dual by means of the inner product in H, and V^* is the dual of V. Let $W = (W_t)_{t \geq 0}$ be an r-dimensional Brownian motion carried by a stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$. In this paper, we study the approximation of the solution to the evolution equation

$$u_t = u_0 + \int_0^t A_s(u_s)\, ds + \sum_{j=1}^r \int_0^t B_s^j(u_s)\, dW_s^j \,, \tag{1.1}$$

where u_0 is a H-valued $\mathcal{F}_0$-measurable random variable, A and B are (non-linear) adapted operators defined on $[0, +\infty[\times V \times \Omega$ with values in V^* and H^r respectively.

The conditions imposed on A_s are satisfied by the following classical example: $V = W_0^{1,p}(D)$, $H = L^2(D)$ $V^* = W^{-1,q}(D)$ and

$$A_s(u) = \sum_{i=1}^d \frac{\partial}{\partial x_i} \left(\left| \frac{\partial u}{\partial x_i} \right|^{p-2} \frac{\partial u}{\partial x_i} \right) ,$$

where D is a bounded domain of $\mathbb{R}^d$, $p \in]2, +\infty[$ and q are conjugate exponents. In [7] the monotonicity method is used in the deterministic case to prove that if $u_0 \in H$ and $B = 0$, equation (1.1) has a unique solution in $L^p_V([0,T])$ such that $u_t = 0$ on $]0,T] \times \partial D$. Using the monotonicity method, the existence and uniqueness of a solution u to (1.1) is proved in [9] and [6]. This result can be fruitfully applied also to linear stochastic PDEs, in particular to the equations of nonlinear filtering theory (see [8], [10] and [11]). The existence and uniqueness theorem from [6] is extended in [2] to equation (1.1) with martingales and martingale measures in place of W. Inspired by [5], the method of monotonicity is interpreted in [4] as a minimization method for some convex functionals.

In the present paper we introduce an implicit time discretization u^m, space-time explicit and implicit discretization schemes u_n^m and $u^{n,m}$ of u defined in terms of a constant time mesh $\delta_m = \frac{T}{m}$ and of a sequence of finite dimensional subspaces V_n of V. One particular

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case of such spaces is that used in the Galerkin method or in the piecewise linear finite elements methods. To define space-time discretizations of u, we denote by $\Pi_n : V^* \to V_n$ a V_n-valued projection.

For $0 \leq i \leq m$, set $t_i = \frac{iT}{m}$. The explicit V_n-valued space-time discretization of u is defined for an initial condition $u_0 \in H$ by $u^n_m(t_0) = u^n_m(t_1) = \Pi_n u_0$ and for $1 \leq i < m$,

$$u^n_m(t_{i+1}) = u^n_m(t_i) + \delta_m \, \Pi_n \tilde{A}^m_{t_i}(u^n_m(t_i)) + \sum_{j=1}^{r} \Pi_n \tilde{B}^{m,j}_{t_i}(u^n_m(t_i)) \left(W^j_{t_{i+1}} - W^j_{t_i} \right), \tag{1.2}$$

where for $x \in V$, $\tilde{A}^m_{t_i}(x) \in V^*$ and $(\tilde{B}^{m,j}_{t_i}(x) \, , \, 1 \leq j \leq r) \in H^r$ denote the averages of the processes $A_{.}(x)$ and $B_{.}(x)$ over the time interval $[t_{i-1}, t_i]$.

The V-valued implicit time discretization of u is defined for an initial condition $u_0 \in H$ by $u^m(t_0) = 0$, $u^m(t_1) = u_0 + \delta_m \, A^m_{t_1}(u^m(t_1))$, and for $1 \leq i < m$,

$$u^m(t_{i+1}) = u^m(t_i) + \delta_m \, A^m_{t_i}(u^m(t_{i+1})) + \sum_{j=1}^{r} \tilde{B}^{m,j}_{t_i}(u^m(t_i)) \left(W^j_{t_{i+1}} - W^j_{t_i} \right), \tag{1.3}$$

where for $x \in V$, $A^m_{t_i}(x) \in V^*$ denotes the average of the process $A_{.}(x)$ over the time interval $[t_i, t_{i+1}]$ and as above $(\tilde{B}^{m,j}_{t_i}(x) \, , \, 1 \leq j \leq r) \in H^r$ denotes the average of $B_{.}(x)$ over the time interval $[t_{i-1}, t_i]$.

Finally, the implicit V_n-valued space-time discretization of u is defined for $u_0 \in H$ by $u^{m,m}(t_0) = 0$, $u^{n,m}(t_1) = \Pi_n u_0 + \delta_m \, \Pi_n A^m_{t_1}(u^{n,m}(t_1))$, and for $1 \leq i < m$,

$$u^{n,m}(t_{i+1}) = u^{n,m}(t_i) + \delta_m \, \Pi_n A^m_{t_i}(u^{n,m}(t_{i+1})) + \sum_{j=1}^{r} \Pi_n \tilde{B}^{m,j}_{t_i}(u^{n,m}(t_i)) \left(W^j_{t_{i+1}} - W^j_{t_i} \right), \tag{1.4}$$

where $A^m_{t_i}$ and $\tilde{B}^{m,j}_{t_i}$ have been defined above.

The processes v equal to u^m, u^n_m or $u^{n,m}$ are defined between t_i and t_{i+1} as stepwise constant adapted stochastic processes, i.e., $v(t) := v(t_i)$ for $t \in]t_i, t_{i+1}[$. We prove that for m large enough, (1.3) (resp. (1.4)) has a unique solution u^m (resp. $u^{n,m}$), which converge weakly to u in a weighted space of p-integrable processes, and that the approximations at terminal time T converge strongly to $u(T)$ in $L^2_H(\Omega)$ as $m \to +\infty$ (resp. n and m go to infinity). As one expects, the convergence of the explicit approximation u^n_m to u in these spaces requires some condition relating the time mesh T/m and the spaces V_n. The existence of the solution to (1.3) or (1.4), as well as that of a limit for some subsequence u^{m_k}, $u^{n_k}_{m_k}$ or u^{n_k,m_k} is proved using apriori estimates, which are based on the coercivity, monotonicity and growth assumptions made on the operators A_s and B_s. The identification of u as the limit is obtained by means of the minimization property of u. Note that the conditions imposed on the operators A_s and B_s involve constants which may depend on time. This allows the operators to approach degeneracy. However, this lack of uniform non-degeneracy has to be balanced by a suitable growth condition which depends on time as well. Thus, as a by-product of the identification of the weak limit of the explicit and implicit space-time discretization schemes, we obtain the existence of a solution to (1.1) under slightly more general conditions than those used in [8], [6] or [4].

Section 2 states the conditions imposed on the operators A and B, the spaces V_n and the maps Π_n, gives examples satisfying these conditions, describes precisely the explicit and implicit schemes, and states the corresponding convergence results. The third section provides the proofs of the main theorems and an appendix gathers some technical tools.

As usual we denote by C a constant which may change from line to line. All the processes considered will be adapted with respect to the filtration $(\mathcal{F}_t\,,\,t\geq 0)$.

2. Description of the results

We first state the precise assumptions made on the operators. Let V be a separable reflexive Banach space, embedded continuously and densely into a Hilbert space H, which is identified with its dual, H^* by means of the inner product $(\cdot,\cdot)$ in H. Then the adjoint embedding $H\hookrightarrow V^*$ of $H^*\equiv H$ into V^*, the dual of V, is also dense and continuous. Let $\langle v,x\rangle=\langle x,v\rangle$ denote the duality product for $v\in V$ and $x\in V^*$. Observe that $\langle v,h\rangle=(v,h)$ for $h\in H$ and $v\in V$. Let $(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\geq 0},P)$ be a stochastic basis, satisfying the usual conditions and carrying an r-dimensional Wiener martingale $W=(W_t)_{t\geq 0}$ with respect to $(\mathcal{F}_t)_{t\geq 0}$.

Fix $T>0$, $p\in[2,+\infty[$ and let $q=\frac{p}{p-1}$ be the conjugate exponent of p. Let L^1 (resp. L^2) denote the space of integrable (resp. square integrable) real functions over $[0,T]$. Let

$$A:[0,T]\times V\times\Omega\to V^*\,,\quad B:[0,T]\times V\times\Omega\to H^r$$

be such that for every $v,w\in V$ and $1\leq j\leq r$, $\langle w,A_s(v)\rangle$ and $(B^j_s(v),w)$ are adapted processes and the following conditions hold:

(C1) The pair (A,B) satisfies the *monotonicity condition*, i.e., almost surely for all $t\in[0,T]$, x and y in V:

$$2\,\langle x-y,A_t(x)-A_t(y)\rangle+\sum_{j=1}^r|B^j_t(x)-B^j_t(y)|^2_H\leq 0\,. \tag{2.1}$$

(C2) The pair (A,B) satisfies the *coercivity condition* i.e., there exist non-negative integrable functions $K_1,\bar K_1$ and $\lambda:]0,T]\to]0,+\infty[$ such that almost surely

$$2\,\langle x,A_t(x)\rangle+\sum_{j=1}^r|B^j_t(x)|^2_H+\lambda(t)\,|x|^p_V\leq K_1(t)|x|^2_H+\bar K_1(t) \tag{2.2}$$

for all $t\in]0,T]$ and $x\in V$.

(C3) The operator A is *hemicontinuous* i.e., almost surely

$$\lim_{\varepsilon\to 0}\langle A_t(x+\varepsilon y),z\rangle=\langle A_t(x),z\rangle\,. \tag{2.3}$$

for all $t\in[0,T]$, x,y,z in V.

(C4) (*Growth condition*) There exist a non-negative function $K_2\in L^1$ and a constant $\alpha\geq 1$ such that almost surely

$$|A_t(x)|^q_{V^*}\leq\alpha\lambda^q(t)\,|x|^p_V+\lambda^{q-1}(t)K_2(t) \tag{2.4}$$

for all $t\in]0,T]$ and $x\in V$.

We also impose some integrability of the initial condition u_0:

(C5) $u_0:\Omega\to H$ is $\mathcal{F}_0$-measurable and such that $E(|u_0|^2_H)<+\infty$.

Remark 2.1. *From* **(C2)** *and* **(C4)** *it is easy to get that almost surely*

$$\sum_{j=1}^r|B^j_t(x)|^2_H\leq(2\,\alpha+1)\,\lambda(t)|x|^p_V+K_1(t)|x|^2_H+K_3(t) \tag{2.5}$$

for all $t\in]0,T]$ and $x\in V$, where $K_3(t)=\bar K_1(t)+\frac{2}{q}\,K_2(t)\in L^1$.

Proof. For every $t \in]0,T]$ and $x \in V$,

$$\begin{aligned} |\langle x\,,\, A_t(x)\rangle| &= |x|_V\,|A_t(x)|_{V^*} \le \alpha^{\frac{1}{q}}\,\lambda(t)\,|x|_V^{1+\frac{p}{q}} + \lambda(t)^{\frac{q-1}{q}}\,|x|_V\,K_2(t)^{\frac{1}{q}} \\ &\le \alpha^{\frac{1}{q}}\,\lambda(t)|x|_V^p + \frac{1}{p}\,\lambda(t)\,|x|_V^p + \frac{1}{q}\,K_2(t)\,. \end{aligned}$$

Thus, (2.2) and (2.4) yield (2.5). □

Note that, unlike in [4], [6] and [8], the coercivity constant $\lambda(t)$ can vary with t (for example, one can suppose that $\lambda(t) = \lambda\, t$ for some constant $\lambda > 0$), which means that the operators can be more and more degenerate as $t \to 0$. However, this bad behavior has to be balanced by some more and more stringent growth conditions.

We remark that the monotonicity condition (C1) can be weakened as follows:

(C1bis) There exists a non negative function $K \in L^1_+$ such that almost every $(t,\omega) \in [0,T]\times\Omega$ and every $x,y \in V$

$$2\,\langle x-y, A_t(x)-A_t(y)\rangle + \sum_{j=1}^r |B_t^j(x)-B_t^j(y)|_H^2 \le K(t)\,|x-y|_H^2\,.$$

Indeed, if u be a solution to (1.1) and $\gamma_t := \exp\Big(\frac{1}{2}\int_0^t K(s)\,ds\Big)$, then $v_t = \gamma_t^{-1}u_t$ is a solution of the equation

$$v_t = u_0 + \int_0^t \bar{A}_s(v_s)\,ds + \sum_{j=1}^r \int_0^t \bar{B}_s^j(v_s)\,dW_s^j\,,$$

where for every $t \in [0,T]$ and $x \in V$:

$$\bar{A}_t(x) := \gamma_t^{-1}\,A_t\big(\gamma_t\,x\big) - \frac{1}{2}K(t)\,x, \quad\text{and}\quad \bar{B}_t(x) := \gamma_t^{-1}\;B_t\big(\gamma_t\,x\big)\,.$$

If (A,B) satisfies (C1bis) then it is easy to see that $(\bar{A},\bar{B})$ satisfies (C1). Clearly, if A is hemicontinuous, then $\bar{A}$ is also hemicontinuous. If (A,B) satisfies the coercivity condition (C2), then $(\bar{A},\bar{B})$ also satisfies (C2). If A satisfies the growth condition (C4) then it is an easy exercise to check that $\bar{A}$ also satisfies (C4), provided $p \ge 2$ and $K(t) \le C\lambda(t)$ for all t with some constant C.

Example 2.2. *A large class of linear and semi-linear stochastic partial differential equations of parabolic type satisfies the above conditions. Below we present a class of examples of nonlinear equations. Let D be a bounded domain of $\mathbb{R}^d$, $p \in [2,+\infty[$, $V = W_0^{1,p}(D)$, $H = L^2(D)$, $V^* = W^{-1,q}(D)$. Let the operators A_t, B_t^j be defined by*

$$A_t(u,\omega) := \sum_{i=1}^d \frac{\partial}{\partial x_i} f_i(t,x,\nabla u(x),\omega),$$

$$B_t^k(u,\omega) := g^k(t,x,\nabla u(x),\omega) + h^k(t,x,u(x),\omega), \quad k = 1,2,...,r$$

for $u \in V$, $t \in [0,T]$ and $\omega \in \Omega$, where ∇u denotes the gradient of u, i.e., $\nabla u = (\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, ..., \frac{\partial u}{\partial x_d})$, and $f_i = f_i(t,x,z,\omega)$, $g^j = g^j(t,x,z,\omega)$, $h^j = h^j(t,x,s,\omega)$ are some real valued functions of $t \in [0,\infty[$, $x,z \in \mathbb{R}^d$ and $s \in \mathbb{R}$, such that the following conditions are satisfied:

(i) The functions f_i, g^j and h^j are Borel measurable in t,x,z,s for each fixed ω, and are $\mathcal{F}_t$-adapted stochastic processes for each fixed t,x,z,s.

(ii) The functions f_i *and* g^j *are differentiable in* $z = (z_1, z_2, ..., z_d)$*, and there exists a constant* $\varepsilon > 0$*, such that for almost every* $\omega \in \Omega$ *and all* t, x, z *the matrix*

$$\big(S_{ij}\big) := \big(2f_{iz_j} - (1+\varepsilon)\sum_{k=1}^{r} g^k_{z_i} g^k_{z_j}\big)$$

is positive semidefinite, where $f_{iz_j} := \frac{\partial}{\partial z_j} f_i$*,* $g^k_{z_j} := \frac{\partial}{\partial z_j} g^k$*.*

(iii) There exists a function $K : [0,T] \to [0,\infty[$*,* $K \in L^1$*, such that*

$$\sum_{k=1}^{r} |h^k(t,x,u) - h^k(t,x,v)|^2 \le K(t)|u-v|^2,$$

$$\sum_{k=1}^{r} \int_{\mathbb{R}^d} |h^k(t,x,0)|^2\, dx \le K(t)$$

for almost every $\omega \in \Omega$ *and all* $t \in [0,T], x \in \mathbb{R}^d$*,* $u, v \in \mathbb{R}$*.*

(iv) There exist a constant $\varepsilon > 0$ *and a function* $\lambda : [0,T] \to]0,\infty[$*,* $\lambda \in L^1$*, such that almost surely*

$$2\sum_{i=1}^{d} z_i f_i(t,x,z) - (1+\varepsilon)\sum_{k=1}^{r} |g^k(t,x,z)|^2 \ge \lambda(t)|z|^p,$$

$$\sum_{i=1}^{d} |f_i(t,x,z)| \le \alpha\lambda(t)|z|^{p-1} + \lambda^{\frac{1}{p}}(t) K_1^{\frac{1}{q}}(t,x)$$

for all $t \in [0,T]$*,* $x, z \in \mathbb{R}^d$*, where* $\alpha > 0$ *is a constant and* $K_1 : [0,T] \times \mathbb{R}^d \to [0,\infty[$ *is a function such that for every* $t \in [0,T]$*,* $\int_0^T K_1(t,x)\,dx < \infty$ *and* $\int_0^T \int_{\mathbb{R}^d} K_1(t,x)\,dx\,dt < \infty$*.*

It is an easy exercise to verify that under these conditions A *and* (B^j) *satisfy conditions (C2)-(C4) and (C1bis). A simple example of nonlinear functions* f_i*,* g^k *and* h^k*, satisfying the above conditions (i)-(iv), is for* $p \in]2,+\infty[$

$$\begin{aligned} f_i(t,x,z,\omega) &:= a_i(t,x,\omega)|z_i|^{p-2} z_i, \\ g^k(t,x,z,\omega) &:= 2p^{-1}\sum_{i=1}^{d} b^k_i(t,x,\omega)|z_i|^{\frac{p}{2}}, \\ h^k(t,x,u,\omega) &:= c^k(t,x,\omega)|u| + d^k(t,x,\omega) \end{aligned}$$

for $t \in [0,T]$*,* $x, z = (z_1, ..., z_d) \in \mathbb{R}^d$*,* $u \in \mathbb{R}$*,* $\omega \in \Omega$*, where* a_i*,* b^k_i*,* c^k *and* d^k *are real valued functions such that the following conditions hold:*

(1) The functions a_i*,* b^k_i*,* c^k *and* d^k *are Borel functions of* t, x *for each fixed* ω*, and are* $\mathcal{F}_t$*-adapted stochastic processes for each fixed* x*.*

(2) There exist constants $\varepsilon > 0$*,* $\alpha > 0$ *and a function* $\lambda : [0,T] \to]0,\infty[$*,* $\lambda \in L^1$*, such that almost surely*

$$\Big(2(p-1)\, a_i(t,x)\,\delta_{ij} \, - (1+\varepsilon)\sum_{k=1}^{r} b^k_i b^k_j(t,x), 1 \le i,j \le d\Big) \ge \lambda(t) I$$

$$\sum_{i=1}^{d} a_i(t,x) \le \alpha\lambda(t)$$

for all $t \in [0,T]$ *and* $x \in \mathbb{R}^d$*, where* I *is the identity matrix, and* $\delta_{ij} = 1$ *for* $i = j$ *and* $\delta_{ij} = 0$ *otherwise.*

(3) There exist functions $K : [0,T] \to [0,\infty[$ and $L : [0,T] \times \mathbb{R}^d \to [0,\infty[$ such that almost surely

$$\sum_{k=1}^{r} |c^k(t,x,\omega)|^2 \le K(t), \quad \sum_{k=1}^{r} |d^k(t,x,\omega)|^2 \le L(t,x)$$

for all t, x, and

$$\int_0^T K(t)\,dt < \infty\,, \quad \int_0^T \int_{\mathbb{R}^d} L(t,x)\,dx\,dt < \infty.$$

We remark that though for $p=2$ the function $g^k(t,x,z,\omega) := \sum_{i=1}^d b_i^k(t,x,\omega)|z_i|$ is not differentiable at points z such that $z_i = 0$ for some i, it is easy to see that the corresponding operators A, B^k still satisfy conditions (C2)-(C4) and (C1bis) also in this case.

Note that the conditions (C2)-(C4) slightly extend those used in [8], [6] or [4], where the function λ is supposed to be constant.

Definition 2.3. *An adapted continuous H-valued process u is a solution to (1.1) if*
(i) $E\int_0^T |u_t|_V^p\,\lambda(t)\,dt < \infty$.
(ii) For every $t \in [0,T]$ and $z \in V$

$$\langle u_t, z\rangle = \langle u_0, z\rangle + \int_0^t \langle A_s(u_s), z\rangle\,ds + \sum_{j=1}^{r} \int_0^t (B_s^j(u_s), z)\,dW_s^j \quad a.s. \tag{2.6}$$

Notice that under condition (C4) and (2.5), i.e., for example under conditions (C2) and (C4), it is easy to see that an adapted continuous H-valued u is a solution to (1.1) as soon as (2.6) is satisfied for all z in a dense subset of V. The following theorem extends the existence and uniqueness theorem proved in [8] and [6].

Theorem 2.4. *Let conditions (C1)-(C5) hold. Then equation (1.1) has a unique solution u.*

Remark 2.5. *The uniqueness of the solution to equation (1.1) follows easily from conditions (C1) and (C4). Moreover, if u is a solution of equation (1.1), then conditions (C2) and (C5) imply*

$$\sup_{t\in[0,T]} E|u_t|_H^2 < \infty. \tag{2.7}$$

Proof of Remark 2.5. Let $u^{(1)}$ and $u^{(2)}$ be solutions to (1.1). Then for $\delta_t := u_t^{(1)} - u_t^{(2)}$ we have

$$\delta_t = \int_0^t z_s^*\,dY_s + h_t, \quad dY_t \times dP - a.e., \tag{2.8}$$

where

$$\begin{aligned} z_t^* : &= \lambda^{-1}(t)\big[A_t(u_t^{(1)}) - A_t(u_t^{(2)})\big],\ dY_t = \lambda(t)dt,\\ h_t : &= \sum_{j=1}^{r} \int_0^t \big[B_s^j(u_s^{(1)}) - B_s^j(u_s^{(2)})\big]\,dW_s^j. \end{aligned}$$

Notice that almost surely

$$\Big|\int_0^T |\delta_t|_V^p\,dY_t\Big| \ \le\ 2^{p-1}\sum_{i=1}^{2}\int_0^T |u_t^{(i)}|_V^p\,\lambda(t)\,dt < \infty\,,$$

$$\Big|\int_0^T |z_t^*|_{V^*}^q\,dY_t\Big| \;\le\; 2^{q-1}\sum_{i=1}^2\int_0^T |A_t(u_t^{(i)})|_{V^*}^q\,\lambda^{1-q}(t)\,dt$$
$$\le\; 2^{q-1}\sum_{i=1}^2\int_0^T \alpha\,|u_t^{(i)}|_V^p\,\lambda(t)\,dt + 2^q\int_0^T K_2(t)\,dt<\infty\,,$$

and hence almost surely

$$\int_0^T |\delta_t|_V\,|z_t^*|_{V^*}\,dY_t \;\le\; \frac{2^{p-1}}{p}\sum_{i=1}^2\int_0^T |u_t^{(i)}|_V^p\,\lambda(t)\,dt$$
$$+\frac{2^{q-1}}{q}\,\alpha\sum_{i=1}^2\int_0^T |u_t^{(i)}|_V^p\,\lambda(t)\,dt + 2^q\int_0^T K_2(t)\,dt<\infty\,.$$

Thus the conditions of Theorem 1 from [1] on Itô's formula holds for the semi-martingale y defined by the right-hand side of (2.8). Hence the monotonicity condition (C1) yields

$$0 \;\le\; |\delta_t|_H^2 = \int_0^t 2\langle\delta_s\,,\,z_s^*\rangle\,dY_s + [h]_t + m_t$$
$$=\; 2\int_0^t\Big[\langle u_s^{(1)}-u_s^{(2)}, A_s(u_s^{(1)})-A_s(u_s^{(1)})\rangle + \sum_{j=1}^r |B_s^j(u_s^{(1)})-B_s^j(u_s^{(2)})|_H^2\Big]\,ds + m_t \le m_t,$$

where $[h]$ is the quadratic variation of h, and m is a continuous local martingale starting from 0. By the above inequality m is non-negative; hence almost surely $m_t=0$ for all $t\in[0,T]$, which proves that almost surely $u_t^{(1)}=u_t^{(2)}$ for all $t\in[0,T]$.

In order to prove the second statement of the remark we set $\gamma(t):=\exp(-\int_0^t K_1(s)\,ds)$, where K_1 is from condition (C2). Let u be a solution of equation (1.1). Then by using Itô's formula for $\gamma(t)|u(t)|_H^2$ and condition (C2) we get

$$\gamma(t)|u(t)|_H^2 \le |u_0|_H^2 + \int_0^t \gamma(s)\bar K_1(s)\,ds + M(t),$$

where M is a continuous local martingale starting from 0. Hence

$$E|u(t)|_H^2 \le \gamma^{-1}(T)\Big[E|u_0|_H^2 + \int_0^T \gamma(s)\bar K_1(s)\,ds\Big]$$

for all $t\in[0,T]$, which proves (2.7). □

We note that if u is a solution of equation (1.1) then under conditions (C2), (C4) and (C5) one can also show by standard arguments from [8], [6] (or see [2]) that $E\big(\sup_{t\in[0,T]}|u_t|_H^2\big)<\infty$. In the present paper we do not need this estimate, therefore we do not prove it.

Our aim is to show that the explicit and implicit numerical schemes presented below converge to a stochastic process u, which is a solution of equation (1.1). Thus, as a byproduct we prove also the existence part of Theorem 2.4.

First we characterize the solution of equation (1.1) as a minimiser of certain convex functionals. This characterization, which is a translation of the method of monotonicity used for example in [8], [6] and [2], gives a way of proving our approximation theorems.

Fix $T>0$. If X is a separable Banach space, φ is a positive adapted stochastic process and $p\in[1,\infty[$, then $\mathcal{L}_X^p(\varphi)$ denotes the Banach space of the X-valued adapted stochastic processes $\{z_t : t\in[0,T]\}$ with the norm

$$|z|_{\mathcal{L}_X^p(\varphi)} := \Big(E\int_0^T |z_t|_X^p\,\varphi(t)\,dt\Big)^{1/p}<\infty\,,$$

where $|x|_X$ denotes the norm of x in X. If $\varphi = 1$, then we use also the notation $\mathcal{L}^p_X$ for $\mathcal{L}^p_X(1)$. Let L^p_X denote the Banach space of X-valued random variables ξ with the norm

$$|\xi|_{L^p_X} := \big(E|\xi|^p_X\big)^{1/p}.$$

Let X be embedded in the Banach space Y, and let $x = \{x_t : t \in [0,T]\}$ and $y = \{y_t : t \in [0,T]\}$ be stochastic processes, such that $x_t(\omega) = y_t(\omega)$ for $dt \times P$-almost every (t,ω). Then we say that x is an X-valued modification of y, or that y is a Y-valued modification of x.

Definition 2.6. *Let $\mathcal{A}$ denote the space of triplets (ξ, a, b) satisfying the following conditions:*
- $\xi : \Omega \to H$ *is $\mathcal{F}_0$-measurable and such that* $E|\xi|^2_H < +\infty$*;*
- $a : [0,T] \times \Omega \to V^*$ *is a predictable process such that* $E\int_0^T |a_s|^q_{V^*}\, \lambda^{1-q}(s)\, ds < +\infty$*;*
- $b : [0,T] \times \Omega \to H^r$ *is a predictable process such that* $\sum_{j=1}^r E\int_0^T |b^j_s|^2_H\, ds < +\infty$*;*
- *There exists a V-valued adapted process $x \in \mathcal{L}^p_V(\lambda)$ such that*

$$x_t = \xi + \int_0^t a_s\, ds + \sum_{j=1}^r \int_0^t b^j_s\, dW^j_s \tag{2.9}$$

for $dt \times P$-almost all $(t,\omega) \in [0,T] \times \Omega$.

Let $(\xi, a, b) \in \mathcal{A}$, x defined by (2.9), and $y \in \mathcal{L}^p_V(\lambda) \cap \mathcal{L}^2_H(K_1)$. Set

$$F_y(\xi,a,b) := E|u_0 - \xi|^2_H + E\int_0^T \Big[2\,\langle x_s - y_s\,,\, a_s - A_s(y_s)\rangle + \sum_{j=1}^r |b^i_s - B^j_s(y_s)|^2_H\Big]\, ds\,, \tag{2.10}$$

and

$$G(\xi,a,b) := \sup\{F_y(\xi,a,b) \,:\, y \in \mathcal{L}^p_V(\lambda) \cap \mathcal{L}^2_H(K_1)\}\,.$$

Due to the growth condition (C4), for $y \in \mathcal{L}^p(\lambda)$, $A_.(y_.) \in \mathcal{L}^q_{V^*}(\lambda^{1-q})$. Clearly, $\langle x\,,\, z\rangle \in \mathcal{L}^1$ for $x \in \mathcal{L}^p(\lambda)$ and $z \in \mathcal{L}^q_{V^*}(\lambda^{1-q})$, by Hölder's inequality. Hence (2.5), (C4) and (C5) imply that the functionals F_y and G are well-defined. Notice also that G can take the value $+\infty$.

Theorem 2.7. *(i) Suppose that conditions (C1)-(C5) hold and let u be a solution to (1.1). Then*

$$\inf\{G(\xi,a,b) \,:\, (\xi,a,b) \in \mathcal{A}\} = G(u_0, A_.(u_.), B_.(u_.)) = 0\,.$$

(ii) Assume conditions (C2)-(C5). Suppose that there exist $(\hat\xi, \hat a, \hat b) \in \mathcal{A}$ and some subset $\mathcal{V}$ of $\mathcal{L}^p_V(\lambda) \cap \mathcal{L}^2_H(K_1)$ dense in $\mathcal{L}^p_V(\lambda)$, such that

$$F_y(\hat\xi, \hat a, \hat b) \le 0\,, \quad \forall y \in \mathcal{V}. \tag{2.11}$$

Then $\hat\xi = u_0$,

$$u_t = u_0 + \int_0^t \hat a_s\, ds + \sum_{j=1}^r \int_0^t \hat b^j_s\, dW^j_s, \quad t \in [0,T]$$

is a solution to (1.1), and $G(u_0, \hat a, \hat b) = 0$.

This theorem, which is formulated under stronger assumptions in [4], is proved in the Appendix for the sake of completeness.

Let $V_n \subset V$ be a finite dimensional subset of V and let $\Pi_n : V^* \to V_n$ be a bounded linear operator for every integer $n \geq 1$. Suppose that the following conditions hold:

(H1) The sequence $(V_n\,,\, n\geq 1)$ is increasing, i.e., $V_n \subset V_{n+1}$, and $\cup_n V_n$ is dense in V.

(H2) For $x \in V_n$, $\Pi_n x = x$ and for every $h,k\in H$, $x\in V$ and $y \in V^*$

$$(\Pi_n h\,,\, k) = (h\,,\, \Pi_n k) \quad \text{and} \quad \langle \Pi_n x\,,\, y\rangle = \langle x\,,\, \Pi_n y\rangle\,.$$

(H3) For every $h\in H$, $|\Pi_n h|_H \leq |h|_H$ and $\lim_n |h-\Pi_n h|_H = 0$.

For $v \in V_n$, let $|v|_{V_n} = |v|_V$ denote the restriction of the V-norm to V_n, and let $|v|_{H_n} = |v|_H$ denote the restriction of the H-norm to V_n. We denote by H_n the Hilbert space V_n endowed with the norm $|\,.\,|_{H_n}$. We have $V_n = H_n \equiv H_n^* = V_n^*$ as topological spaces, where V_n^* is the dual of V_n, and H_n is identified with its dual H_n^* with the help of the inner product in H_n. The conditions (H2) and (H3) clearly imply that $\Pi_n\circ\Pi_n = \Pi_n$. In particular, if $\{e_i \in V : i = 1.2....\}$ is a complete orthonormal basis in H, then the spaces $V_n := \text{span}(e_i\,,\, 1\leq i\leq n)$, and the projections Π_n defined by $\Pi_n y := \sum_{i=1}^n \langle e_i, y\rangle\, e_i$ for $y\in V^*$ satisfy (H1)-(H3).

We now describe several discretization schemes. Let $m\geq 1$, and set $\delta_m := T\,m^{-1}$, $t_i := i\delta_m$ for $0\leq i\leq m$.

2.1. **Explicit space-time discretization.** For $0\leq i\leq m$, $t\in [t_i, t_{i+1}[$ and $1\leq j\leq r$, define the operators $\tilde{A}^m_t$ and $\tilde{B}^{m,j}_t$ on V by:

$$\begin{aligned}
\tilde{A}^m_t(x) &:= \tilde{A}^m_{t_0}(x) = \tilde{B}^{m,j}_t(x) = \tilde{B}^{m,j}_{t_0}(x) = 0 \;\text{ for } i=0\,,\\
\tilde{A}^m_t(x) &:= \tilde{A}^m_{t_i}(x) = \frac{1}{\delta_m}\int_{t_{i-1}}^{t_i} A_s(x)\,ds \in V^* \text{ for } \;1\leq i\leq m\,, \qquad (2.12)\\
\tilde{B}^{m,j}_t(x) &:= \tilde{B}^{m,j}_{t_i}(x) = \frac{1}{\delta_m}\int_{t_{i-1}}^{t_i} B^j_s(x)\,ds \in H \text{ for } 1\leq i\leq m\,. \qquad (2.13)
\end{aligned}$$

We define an approximation u^n_m of u by explicit space-time discretization of equation (1.1) as follows:

$$\begin{aligned}
u^n_m(t) &:= u^n_m(t_i) \text{ for } \; t\in]t_i, t_{i+1}[\,, \quad 0\leq i\leq m-1\,,\\
u^n_m(t_0) &:= u^n_m(t_1) = \Pi_n u_0\,,\\
u^n_m(t_{i+1}) &:= u^n_m(t_i) + \delta_m\,\Pi_n \tilde{A}^m_{t_i}\big(u^n_m(t_i)\big) \qquad (2.14)\\
&\quad + \sum_{j=1}^r \Pi_n \tilde{B}^{m,j}_{t_i}\big(u^n_m(t_i)\big)\,\big(W^j_{t_{i+1}} - W^j_{t_i}\big),\; 1\leq i\leq m-1.
\end{aligned}$$

Notice that the random variables $u^n_m(t_i)$ are $\mathcal{F}_{t_i}$-measurable and $\Pi_n\tilde{B}^{m,j}_{t_i}\big(u^n_m(t_i)\big)$ is independent of $\big(W^j_{t_{i+1}} - W^j_{t_i}\big)$. For every $n\geq 1$ let $\mathcal{B}_n = (e_k\,,\, k\in I(n)\,)$ denote a basis of V_n, such that $\mathcal{B}_n\subset\mathcal{B}_{n+1}$, and such that $\mathcal{B} = \cup_n\mathcal{B}_n$ is a complete orthonormal basis of H. For every $n\geq 1$ set

$$C_{\mathcal{B}}(n) := \sum_{k\in I(n)} |e_k|_V^2\,. \qquad (2.15)$$

The following theorem establishes the convergence of u^n_m to a solution u of (1.1), and hence proves the existence of a solution to the equation (1.1).

Theorem 2.8. *Suppose conditions (C1)-(C5) with $0 < \lambda \le 1$, $p = 2$, and conditions (H1)-(H3). Assume that n and m converge to ∞ such that*

$$\frac{C_{\mathcal{B}}(n)}{m} \to 0 \,. \tag{2.16}$$

Then the sequence of processes u_m^n converges weakly in $\mathcal{L}_V^2(\lambda)$ to the solution u of equation (1.1), and $u_m^n(T)$ converges to u_T strongly in L_H^2.

When $D =]0,1[$, $V = W_0^{1,2}(D)$, $H = L^2(D)$, $Au = \dfrac{\partial^2 u}{\partial x^2}$, and V_n corresponds to the piecewise linear finite elements methods then condition (2.16) reads $\frac{n^3}{m} \to 0$. In this case condition (2.16) can be weakened substantially. (See, e.g., [3]).

2.2. **Implicit discretization schemes.** For every $j = 1, \cdots, r$ and $i = 0, \cdots, m-1$ let A^m denote the following average:

$$A_t^m(x) := A_{t_i}^m(x) = \frac{1}{\delta_m} \int_{t_i}^{t_{i+1}} A_s(x)\, ds \text{ for } t_i \le t < t_{i+1} \,. \tag{2.17}$$

We define an approximation u^m for u by an implicit time discretization of equation (1.1) as follows:

$$\begin{aligned} u^m(t_0) &:= 0 \,, \\ u^m(t_1) &:= u_0 + \delta_m A_{t_0}^m\big(u^m(t_1)\big) \,, \\ u^m(t_{i+1}) &:= u^m(t_i) + \delta_m A_{t_i}^m\big(u^m(t_{i+1})\big) \\ &\quad + \sum_{j=1}^r \tilde{B}_{t_i}^{m,j}\big(u^m(t_i)\big)\big(W_{t_{i+1}}^j - W_{t_i}^j\big) \,, \quad 1 \le i < m \,, \\ u^m(t) &:= u^m(t_i) \text{ for } \ t \in]t_i, t_{i+1}[\,, \quad 0 \le i < m \,, \end{aligned} \tag{2.18}$$

where the operators A_s^m and $\tilde{B}_s^{m,j}$ have been defined in (2.17) and (2.13).

From the above scheme we get another approximation $u^{n,m}$ for u by space discretization:

$$\begin{aligned} u^{n,m}(t_0) &:= 0 \,, \\ u^{n,m}(t_1) &:= \Pi_n u_0 + \delta_m \Pi_n A_{t_0}^m\big(u^{n,m}(t_1)\big) \,, \\ u^{n,m}(t_{i+1}) &:= u^{n,m}(t_i) + \delta_m \Pi_n A_{t_i}^m\big(u^{n,m}(t_{i+1})\big) \\ &\quad + \sum_{j=1}^r \Pi_n \tilde{B}_{t_i}^{m,j}\big(u^{n,m}(t_i)\big)\big(W_{t_{i+1}}^j - W_{t_i}^j\big) \,, \quad 1 \le i < m \,, \\ u^{n,m}(t) &:= u^{n,m}(t_i) \text{ for } \ t \in]t_i, t_{i+1}[\,, \quad 0 \le i < m \,. \end{aligned} \tag{2.19}$$

The following theorem establishes the existence and uniqueness of u^m and of $u^{n,m}$ for m large enough.

Theorem 2.9. *Let $p \in [2, +\infty[$ and assume (C1)-(C5). Then for any sufficiently large integer m equation (2.18) has a unique solution $\{u^m(t_i) : i = 0, 1, ..., m\}$ such that $E\big(|u^m(t_i)|_V^p\big) < +\infty$ for each $i = 0, \cdots, m$. If in addition to (C1)-(C5) conditions (H2) and (H3) also hold, then there is an integer $m_0 \ge 1$ such that for every $m \ge m_0$ and $n \ge 1$ equation (2.19) has a unique solution $\{u^{n,m}(t_i) : i = 0, 1, ..., m\}$ satisfying $E\big(|u^{n,m}(t_i)|_V^p\big) < +\infty$ for each $i = 0, 1, 2, ..., m$ and $n \ge 1$.*

Once the existence of the solutions to (2.18) and to (2.19) is established, it is easy to see that $u^m = \{u^m(t) : t \in [0,T]\}$ and $u^{n,m} = \{u^{n,m}(t) : t \in [0,T]\}$ are V-valued adapted processes. Now we formulate our convergence result for the above implicit schemes.

Theorem 2.10. *Let $p \in [2,+\infty[$ and assume conditions (C1)-(C5). Then for $m \to \infty$ the sequence of processes u^m converges weakly in $\mathcal{L}^p_V(\lambda)$ to the solution u of equation (1.1), and the sequence of random variables $u^m(T)$ converges strongly to u_T in L^2_H. If in addition to (C1)-(C5) conditions (H1)-(H3) also hold, then as m,n converge to infinity, $u^{n,m}$ converge weakly to the solution u of equation (1.1) in $\mathcal{L}^p_V(\lambda)$, and the random variables $u^{n,m}(T)$ converge to u_T strongly in L^2_H.*

3. Proof of the results

3.1. **Convergence of the explicit scheme.** We reformulate the equation (2.14) in an integral form. For fixed integer $m \geq 1$ set $t_i := i\delta_m$,

$$\kappa_1(t) := t_i \quad \text{for} \quad t \in [t_i, t_{i+1}[, \text{ and } \quad \kappa_2(t) := t_{i+1} \quad \text{for} \quad t \in]t_i, t_{i+1}] \tag{3.1}$$

for integers $i \geq 0$ and let $\kappa_2(t_0) = t_0$. Then (2.14) can be reformulated as follows:

$$\begin{aligned} u^n_m(t) &= \Pi_n u_0 + \int_0^{(\kappa_1(t)-\delta_m))^+} \Pi_n A_s\big(u^n_m(\kappa_2(s))\big)\, ds \\ &\quad + \sum_{j=1}^r \int_0^{\kappa_1(t)} \Pi_n \tilde{B}^{m,j}_s\big(u^n_m(\kappa_1(s))\big)\, dW^j_s\,. \end{aligned} \tag{3.2}$$

The following lemma provides important bounds for the approximations. Set

$$\rho := \rho(n,m) := \alpha\, C_{\mathcal{B}}(n)\delta_m\,,$$

and for every $\gamma \in]0,1[$, let

$$I_\gamma = \{(n,m) \,:\, n,m \geq 1\,,\ \rho(n,m) \leq \gamma\}\,,$$

where α is the constant from condition (C4), and $C_{\mathcal{B}}(n)$ is defined by (2.15).

Lemma 3.1. *Let $p = 2$ and conditions (C1)-(C5) with $0 < \lambda \leq 1$ and (H1)-(H3) hold. Then for every $\gamma \in (0,1)$*

$$\sup_{(n,m)\in I_\gamma} \sup_{s\in[0,T]} E\big|u^n_m(s)\big|^2_H < \infty\,, \tag{3.3}$$

$$\sup_{(n,m)\in I_\gamma} E\int_0^T \big|u^n_m(\kappa_2(s))\big|^2_V\, \lambda(s)\, ds < \infty\,, \tag{3.4}$$

$$\sup_{(n,m)\in I_\gamma} E\int_0^T \big|A_s\big(u^n_m(\kappa_2(s))\big|^2_{V^*} \lambda^{-1}(s)\ ds < \infty\,, \tag{3.5}$$

$$\sup_{(n,m)\in I_\gamma} \sum_{j=1}^r E\int_0^T \big|\Pi_n\, \tilde{B}^{m,j}_s\big(u^n_m(\kappa_1(s))\big|^2_H\, ds < \infty\,. \tag{3.6}$$

Proof. For any $i = 1, \cdots, m-1$,

$$\begin{aligned} E|\, u^n_m(t_{i+1})\,|^2_H &= E|\, u^n_m(t_i)\,|^2_H + \delta^2_m\, E|\Pi_n \tilde{A}^m_{t_i}\big(u^n_m(t_i)\big)|^2_H \\ &\quad + \delta_m\, E\Big[2\,\langle u^n_m(t_i), \Pi_n\, \tilde{A}^m_{t_i}\big(u^n_m(t_i)\big)\rangle + \sum_{j=1}^r \big|\Pi_n\, \tilde{B}^{m,j}_{t_i}\big(u^n_m(t_i)\big)\big|^2_H\Big]\,. \end{aligned}$$

Adding these equalities, using (H2) and (2.12) we deduce

$$E|\,u_m^n(t_{i+1})\,|_H^2 = E|\,\Pi_n u_0\,|_H^2 + \delta_m \sum_{k=1}^{i} E \int_{t_k}^{t_{k+1}} \big|\Pi_n\,\tilde{A}_{t_k}^m\big(u_m^n(t_k)\big)\big|_H^2\,dt$$
$$+ \sum_{k=1}^{i} E \int_{t_{k-1}}^{t_k} 2\,\langle\,u_m^n(t_k)\,,\;A_s\big(u_m^n(t_k)\big)\rangle\,ds + \sum_{k=1}^{i}\sum_{j=1}^{r} \int_{t_k}^{t_{k+1}} E\big|\Pi_n\,\tilde{B}_s^{m,j}\big(u_m^n(t_k)\big)\big|_H^2\,ds\,.$$

Property (H3), the coercivity condition (C2) and the growth condition (C4) with $0<\lambda\le 1$ and the Bunjakovskii-Schwarz inequality yield for every $i=1,\cdots,m-1$

$$\begin{aligned} E|u_m^n(t_{i+1})|_H^2 &\le E|u_0|_H^2 + \delta_m \sum_{k=1}^{i} \int_{t_{k-1}}^{t_k} \sum_{l\in I(n)} E\big[\langle A_s\big(u_m^n(t_k)\big)\,,\,e_l\rangle^2\big]\,ds \\ &+ \int_0^{t_i} E\Big[2\,\big\langle u_m^n(\kappa_2(s)), A_s\big(u_m^n(\kappa_2(s))\big)\big\rangle + \sum_{j=1}^{r} |\Pi_n\,B_s^j\big(u_m^n(u(\kappa_2(s))\big)|_H^2\Big]\,ds \qquad (3.7)\\ &\le E|u_0|_H^2 + \delta_m\,C_{\mathcal{B}}(n)\,E\int_0^{t_i} |A_s\big(u_m^n(\kappa_2(s))\big)|_{V^*}^2\,ds \\ &\quad - E\int_0^{t_i} \lambda(s)\,|u_m^n(\kappa_2(s))|_V^2\,ds + \int_0^{t_i} \bar{K}_1(s)\,ds + E\int_0^{t_i} K_1(s)\,|u_m^n(\kappa_2(s))|_H^2\,ds \\ &\le E|u_0|_H^2 - E\int_0^{t_i} \lambda(s)\big(1-\alpha\,\delta_m\,C_{\mathcal{B}}(n)\big)\,|u_m^n(\kappa_2(s))|_V^2\,ds \\ &\quad + \int_0^{t_i} \big[\bar{K}_1(s) + \delta_m\,C_{\mathcal{B}}(n)K_2(s)\big]\,ds + E\int_0^{t_i} K_1(s)\,|u_m^n(\kappa_2(s))|_H^2\,ds\,. \end{aligned}$$

Hence

$$\begin{aligned} E|u_m^n(t_{i+1})|_H^2 &+ \varepsilon \int_0^{t_i} E|u_m^n(\kappa_2(s))|_V^2\,\lambda(s)\,ds \le E|u_0|_H^2 \\ &+ \int_0^{t_i} K_1(s)\,E|u_m^n(\kappa_2(s))|_H^2\,ds + \int_0^{t_i} \big[\bar{K}_1(s) + \alpha^{-1}\gamma K_2(s)\big]\,ds \qquad (3.8) \end{aligned}$$

for $i=1,\cdots,m-1$ and $(n,m)\in I_\gamma$, where $\varepsilon := 1-\gamma>0$. Therefore, the integrability of K_1, $\bar{K}_1$ and K_2 yields the existence of some positive constant C, which is independent of n and m, and the existence of some positive constants α_i^m, $1\le i\le m$ with $\sup_m \sum_{i=0}^{m-1} \alpha_i^m < +\infty$, such that

$$E\big[\big|u_m^n\big(k\,\delta_m\big)\big|_H^2\big] \le C + C\sum_{i=0}^{k-1} \alpha_i^m\,E\big[\big|u_m^n\big(i\,\delta_m\big)\big|_H^2\big]$$

for all $k\in\{1,\cdots,m\}$ and $(n,m)\in I_\gamma$. Hence by a discrete version of Gronwall's lemma

$$\sup_{(n,m)\in I_\gamma}\ \sup_{0\le i\le m} E\big[\big|u_m^n\big(i\,\delta_m\big)\big|_H^2\big] =: C_{\gamma,\varepsilon} < +\infty\,, \qquad (3.9)$$

which gives (3.3). The inequalities (3.8) with $i=m$ and (3.9) yield (3.4). Finally, by (C4), (2.5), (2.13) and (H3) we have:

$$E\int_0^T \big|A_s\big(u_m^n(\kappa_2(s))\big)\big|_{V^*}^2\,\lambda^{-1}(s)ds \le \alpha E\int_0^T |u_m^n(\kappa_2(s))|_V^2\,\lambda(s)\,ds + \int_0^T K_2(s)\,ds,$$

and for $j=1,\cdots,r$:

$$E\int_0^T |\Pi_n\,\tilde B_t^{m,j}\big(u_m^n(\kappa_1(t))\big)|_H^2\,dt \le \int_0^T \frac{1}{\delta_m}\int_{(\kappa_1(t)-\delta_m)^+}^{\kappa_1(t)} E\big|B_s^j\big(u_m^n(\kappa_2(s))\big)\big|_H^2\,ds\,dt$$
$$\le E\int_0^T \big|B_s^j\big(u_m^n(\kappa_2(s))\big)|_H^2\,ds \le (2\alpha+1)\,E\int_0^T \lambda(s)\,|u^{n,m}(\kappa_2(s))|_V^p\,ds$$
$$+\int_0^T K_1(s)\,E|u_m^n(\kappa_2(s))|_H^2\,ds + \int_0^T K_3(s)\,ds\,.$$

Hence (3.3) and (3.4) imply (3.5) and (3.6). □

Proposition 3.2. *Let $p=2$ and conditions (C1)-(C5) with $0<\lambda\le 1$ and (H1)-(H3) hold. Let (n,m) be a sequence from I_γ for some $\gamma\in(0,1)$, such that m and n converge to infinity. Then it contains a subsequence, denoted also by (n,m), such that:*

(i) $u_m^n(T)$ converges weakly in L^2_H to some random variable $u_{T\infty}$,

(ii) $u_m^n(\kappa_2(\cdot))$ converges weakly in $\mathcal{L}^2_V(\lambda)$ to some process v_∞,

(iii) $A_\cdot(u_m^n(\kappa_2(\cdot)))$ converges weakly in $\mathcal{L}^2_{V^*}(\lambda^{-1})$ to some process a_∞,

(iv) for any $j=1,\cdots,r$, $\Pi_n\tilde B_\cdot^{m,j}(u_m^n(\kappa_1(\cdot))$ converges weakly in $\mathcal{L}^2_H$ to some process b^j_∞,

(v) $(u_0,a_\infty,b_\infty)\in\mathcal{A}$, and for $dt\times P$-almost every $(t,\omega)\in[0,T]\times\Omega$

$$v_\infty(t) = u_0+\int_0^t a_\infty(s)\,ds+\sum_{j=1}^r\int_0^t b^j_\infty(s)\,dW^j(s), \tag{3.10}$$
$$u_{T\infty} = u_0+\int_0^T a_\infty(s)\,ds+\sum_{j=1}^r\int_0^T b^j_\infty(s)\,dW^j(s)\quad\text{(a.s.)}. \tag{3.11}$$

Proof. Assertions (i)-(iv) follow immediately from Lemma 3.1. It remains to prove (3.10) and (3.11). Fix $N\ge1$ and let $\varphi=\{\varphi(t):t\in[0,T]\}$ be an adapted V_N-valued process such that $|\varphi(t)|_V\le N$ for all (t,ω). From (3.2) and (H2), for $n\ge N$ we have

$$E\int_0^T\langle u_m^n(t)\,,\,\varphi(t)\rangle\,\lambda(t)\,dt = E\int_0^T (u_0,\varphi(t))\,\lambda(t)\,dt + J_1+J_2-R-\sum_{j=1}^r R_j, \tag{3.12}$$

with

$$J_1 := E\int_0^T\Big\langle\int_0^t A_s\big(u_m^n(\kappa_2(s))\big)\,ds,\varphi(t)\Big\rangle\,\lambda(t)\,dt\,,$$
$$J_2 := \sum_{j=1}^r E\int_0^T\Big(\int_0^t \Pi_n\,\tilde B_s^{m,j}\big(u_m^n(\kappa_1(s))\big)\,dW_s^j,\,\varphi(t)\Big)\,\lambda(t)\,dt\,,$$
$$R := E\int_0^T\Big\langle\int_{(\kappa_1(t)-\delta_m)^+}^t A_s\big(u_m^n(\kappa_2(s))\big)\,ds\,,\,\varphi(t)\Big\rangle\,\lambda(t)\,dt\,,$$
$$R_j := E\int_0^T\Big(\int_{\kappa_1(t)}^t \Pi_n\,\tilde B_s^{m,j}\big(u_m^n(\kappa_1(s))\big)\,dW_s^j\,,\,\varphi(t)\Big)\,\lambda(t)\,dt\,.$$

For $(n,m)\in I_\gamma$ and $(n,m)\to\infty$, using (3.5) we obtain

$$|R|\le N\,E\int_0^T\int_{(\kappa_1(t)-\delta_m)^+}^t |A_s\big(u_m^n(\kappa_2(s))\big)|_{V^*}\,ds\,dt$$

$$\leq 2\,N\,\delta_m \Big(E\int_0^T |A_s(u_m^n\big(\kappa_2(s)\big))|_{V^*}^2\,\lambda(s)^{-1}\,ds\Big)^{\frac12}\,T^{\frac12} \to 0\,. \tag{3.13}$$

For $j=1,\cdots,r$ Schwarz's inequality with respect to $dt\times P$, the isometry of stochastic integrals, (3.6) and $|\varphi(t)|_H\leq C\,|\varphi(t)|_V\leq C\,N$ yield:

$$\begin{aligned}|R_j| &\leq C\,\Big(E\int_0^T |\varphi(t)|_H^2\,dt\Big)^{\frac12}\,\Big(E\int_0^T \Big|\int_{\kappa_1(t)}^t \Pi_n\,\tilde B_s^{m,j}\big(u_m^n(\kappa_1(s))\big)\,dW_s^j\Big|_H^2\,dt\Big)^{\frac12}\\ &\leq C\,N\,\sqrt{T}\,\Big(E\int_0^T\int_{\kappa_1(t)}^t \Big|\Pi_n\,\tilde B_s^{m,j}\big(u_m^n(\kappa_1(s))\big)\Big|_H^2\,ds\,dt\Big)^{\frac12}\\ &\leq CN\,\sqrt{T\,\delta_m}\,\Big(E\int_0^T\Big|\Pi_n\,\tilde B_s^{m,j}\big(u_m^n(\kappa_1(s))\big)\Big|_H^2\,ds\Big)^{\frac12}\to 0.\end{aligned} \tag{3.14}$$

For $j=1,\cdots,r$ and $g\in\mathcal{L}_H^2$ let

$$F_j(g)(t):=\int_0^t g_s\,dW_s^j,\quad t\in[0,T] \tag{3.15}$$

Then by the isometry of stochastic integrals

$$\|F_j(g)\|_{\mathcal{L}_H^2(\lambda)}^2=\int_0^T E\Big(\int_0^t |g_s|_H^2\,ds\Big)\,\lambda(t)\,dt\leq\int_0^T\lambda(t)\,dt\,\|g\|_{\mathcal{L}_H^2}^2\,,$$

which means that the operator F_j defined by (3.15) is a continuous linear operator from $\mathcal{L}_H^2$ into $\mathcal{L}_H^2(\lambda)$, and hence it is continuous also in the weak topologies. Thus (iv) implies

$$J_2\to\sum_{j=1}^r E\int_0^T\Big(\int_0^t b_\infty^j(s)\,dW_s^j,\,\varphi(t)\Big)\,\lambda(t)\,dt\,. \tag{3.16}$$

Similarly, the linear operator $G:\mathcal{L}_{V^*}^2(\lambda^{-1})\to\mathcal{L}_{V^*}^2(\lambda)$ defined by $G(g)_t=\int_0^t g(s)\,ds$ is continuous and hence continuous with respect to the weak topologies. Indeed,

$$\begin{aligned}\|G(g)\|_{\mathcal{L}_{V^*}^2(\lambda)}^2 &\leq E\int_0^T\lambda(t)\Big(\int_0^T\lambda(s)^{-1}\,|g(s)|_{V^*}^2\,ds\Big)\,\Big(\int_0^t\lambda(s)\,ds\Big)\,dt\\ &\leq \Big(\int_0^T\lambda(t)\,dt\Big)^2\,\|g\|_{\mathcal{L}_{V^*}^2(\lambda^{-1})}^2\,.\end{aligned}$$

Since $\varphi\in\mathcal{L}_V^2(\lambda)$, (iii) implies

$$J_1\to E\int_0^T\Big\langle\int_0^t a_\infty(s)\,ds\,,\,\varphi(t)\Big\rangle\,\lambda(t)\,dt. \tag{3.17}$$

Clearly (ii) implies

$$E\int_0^T\big\langle u_m^n(t),\varphi(t)\big\rangle\,\lambda(t)\,dt\to E\int_0^T(v_\infty(t),\varphi(t))\,\lambda(t)\,dt. \tag{3.18}$$

Thus from (3.12) we get (3.10) by (3.13), (3.14), (3.16) - (3.18), and by taking into account that $\cup_N V_N$ is dense in V. A similar, simpler argument yields that for every random variable $\psi\in L_{V_N}^2$ such that $E|\psi|_V^2\leq N$:

$$E\,\langle u_m^n(T)\,,\,\psi\rangle=E(u_0\,,\,\psi)+\tilde J_1+\tilde J_2-\tilde R\,, \tag{3.19}$$

where as $n,m\to+\infty$ with $(n,m)\in I_\gamma$,

$$\begin{aligned}\tilde J_1 &= E\Big\langle \int_0^T A_s\big(u^n_m(\kappa_2(s))\big)\,ds\,,\,\psi\Big\rangle \to E\Big\langle \int_0^T a_\infty(s)\,ds\Big\rangle\,,\\ \tilde J_2 &= \sum_{j=1}^r E\Big(\int_0^T \Pi_n\tilde B^{m,j}_s\big(u^n_m(\kappa_1(s))\big)\,dW^j_s\,,\,\psi\Big) \to \sum_{j=1}^r E\Big(\int_0^T b^j_\infty(s)\,dW^j_s\,,\,\psi\Big)\,,\\ |\tilde R| &= E\Big(\int_{T-\delta_m}^T A_s\big(u^n_m(\kappa_1(s))\big)\,,\,\psi\Big)\le CN\sqrt{\delta_m}\,.\end{aligned}$$

Thus, as $n,m\to\infty$ with $(n,m)\in I_\gamma$, $E\,(u^n_m(T)\,,\,\psi)\to E(u_{T\infty}\,,\,\psi)$. Since $\cup_N V_N$ is dense in H, this concludes the proof. □

Proposition 3.3. *Let $p=2$, (C1)-(C5) with $0<\lambda\le1$ and (H1)-(H3) hold. Let (n,m) be a sequence of pair of positive integers such that m and n converge to infinity, and $C_{\mathcal B}(n)/m\to0$. Then the assertions of Proposition 3.2 hold, and for every $y\in\mathcal L^p(\lambda)$:*

$$\int_0^T E\Big[2\langle v_\infty(t)-y(t)\,,\,a_\infty(t)-A_t(y(t))\rangle+\sum_{j=1}^r|b^j_\infty(t)-B^j_t(y_t)|^2_H\Big]\,dt\le0\,. \tag{3.20}$$

The process v_∞ has an H-valued continuous modification, u_∞, which is the solution of equation (1.1), and $E|u^n_m(T)-u_\infty(T)|^2_H\to0$.

Proof. Since $C_{\mathcal B}(n)/m\to0$, with finitely many exceptions all pairs (n,m) from the given sequence belong to I_γ. Thus we can apply Proposition 3.2 and get a subsequence, denoted again by (n,m), such that assertions (i)–(v) of Proposition 3.2 hold. Notice that $v_\infty\in\mathcal L^2_V(\lambda)$ and $a_\infty\in\mathcal L^2_{V^*}(\lambda^{-1})$. Thus from (3.10) by Theorem 1 from [1] on Itô's formula we get that v_∞ has an H-valued continuous modification u_∞, and a.s.

$$E|u_\infty(T)|^2_H=E|u_0|^2_H+E\int_0^T\Big[2\,\langle v_\infty(s),a_\infty(s)\rangle+\sum_{j=1}^r|b^j_\infty(s)|^2_H\Big]\,ds\,. \tag{3.21}$$

Moreover, by (3.10) and (3.11) we get $u_\infty(T)=u_{T\infty}$. For $y\in\mathcal L^2_V(\lambda)$ such that $\sup_{0\le t\le T}E|y_t|^2_H<+\infty$, let

$$\begin{aligned}F^n_m(y) &:= E\int_0^T\Big[2\,\langle u^n_m(\kappa_2(t))-y(t)\,,\,A_t(u^n_m\big(\kappa_2(t))\big)-A_t(y_t)\rangle\\ &\quad+\sum_{j=1}^r\big|\Pi_nB^j_t(u^n_m\big(\kappa_2(t))\big)-\Pi_nB^j_t(y(t))\big|^2_H\Big]\,dt\,.\end{aligned}$$

Notice that the growth condition (C4) and (2.5) imply that for $x,z\in\mathcal L^2_V(\lambda)$, $\langle x_.\,,\,A_.(z_.)\rangle\in\mathcal L^1$ and $B^j(y)\in\mathcal L^2_H(K_1)$ for $j=1,\cdots,r$; since the estimates (3.4), (3.5) and (2.5) hold, $F^n_m(y)$ is well-defined and is finite. By the monotonicity condition (C1), (H3) and by inequality (3.7)

$$0\ge F^n_m(y)\ge E|u^n_m(T)|^2_H-E|u_0|^2_H+2\,E\int_0^T\langle y_t\,,\,A_t(y_t)\rangle\,dt-R-2J_1-2J_2-2J_3+J_4, \tag{3.22}$$

with

$$R:=\delta_mE\int_0^{T-\delta_m}\sum_{l\in I(n)}\langle A_s\big(u^n_m(\kappa_2(s))\big)\,,\,e_l\rangle^2\,ds\,,$$

$$J_1 := E\int_0^T \langle u_m^n(\kappa_2(t))\,,\,A_t(y_t)\rangle\,dt\,,$$
$$J_2 := E\int_0^T \big\langle y_t\,,\,A_t\big(u_m^n(\kappa_2(t))\big)\big\rangle\,dt\,,$$
$$J_3 := \sum_{j=1}^r E\int_0^T \big(\Pi_n B_t^j\big(u_m^n(\kappa_2(t))\big)\,,\,B_t^j(y_t)\big)\,dt\,,$$
$$J_4 := \sum_{j=1}^r E\int_0^T |\Pi_n B_t^j(y_t)|_H^2\,dt.$$

Since $\lambda^{-1}\geq 1$, (3.5) implies that for $C_{\mathcal{B}}(n)/m\to 0$

$$|R|\leq T\,\frac{C_{\mathcal{B}}(n)}{m}\,E\int_0^T \big|A_s(u_m^n(\kappa_2(s))\big|_{V^*}^2\,ds\to 0. \tag{3.23}$$

By Proposition 3.2, as $C_{\mathcal{B}}(n)/m\to 0$,

$$J_1 \;=\; E\int_0^T \big\langle u_m^n(\kappa_2(t))\,,\,A_t(y_t)\,\lambda(t)^{-1}\big\rangle\,\lambda(t)\,dt\to E\int_0^T \big\langle u_\infty(t)\,,\,A_t(y_t)\,\big\rangle\,dt\,, \tag{3.24}$$
$$J_2 \;=\; E\int_0^T \big\langle \lambda(t)y_t\,,\,A_t\big(u_m^n(\kappa_2(t))\big)\big\rangle\,\lambda^{-1}(t)dt\to E\int_0^T \big\langle y_t\,,\,a_\infty(t)\,\big\rangle\,dt. \tag{3.25}$$

Notice that

$$E\int_0^T \Big(\Pi_n\tilde{B}_t^{m,j}\big(u_m^n(\kappa_1(t))\big)\,,\,B_t^j(y_t)\Big)\,dt = E\int_0^T \Big(\Pi_n B_t^j\big(u_m^n(\kappa_2(t))\big)\,,\,S_m B_t^j(y_t)\Big)\,dt\,,$$

where S_m is the averaging operator, defined by

$$(S_m Z)(t) := \begin{cases} \delta_m^{-1}\int_{\kappa_1(t)+\delta_m}^{\kappa_1(t)+2\delta_m} Z_s\,ds & \text{if}\quad 0\leq t\leq T-\delta_m\,,\\ 0 & \text{if}\quad T-\delta_m<t\leq T\end{cases} \tag{3.26}$$

for $Z\in\mathcal{L}_H^2$. Hence, taking into account Proposition 3.2 (iv) and

$$\lim_{m\to\infty} E\int_0^T |(S_m Z)_t - Z_t|_H^2\,dt = 0\,,\quad \forall Z\in\mathcal{L}_H^2\,,$$

as $C_{\mathcal{B}}(n)/m\to 0$ we get

$$J_3\to\sum_{j=1}^r E\int_0^T \Big(b_\infty^j(t), B_t^j(y_t)\Big)\,dt\,. \tag{3.27}$$

Using (H3) and the dominated convergence theorem, since $B_\cdot(y_\cdot)\in\mathcal{L}_H^2$, we obtain

$$J_4\to\sum_{j=1}^r E\int_0^T |B_t(y_t)|_H^2\,dt\,. \tag{3.28}$$

Since $u_m^n(T)$ converges weakly in L_H^2 to $u_{T\infty}=u_\infty(T)$,

$$d := \liminf_{n,m\to\infty} E|u_m^n(T)|_H^2 - E|u_\infty(T)|_H^2\geq 0\,. \tag{3.29}$$

Letting $n,m\to\infty$ with $C_{\mathcal{B}}(n)/m\to 0$ in (3.22) and using (3.21), (3.23) - (3.25) and (3.27) - (3.29), we deduce that for $y\in\mathcal{L}_V^2(\lambda)$ with $\sup_t E|y_t|_H^2<+\infty$ and F_y defined by

(2.10):

$$
\begin{aligned}
0 \quad \geq \quad & d + E|u_\infty(T)|_H^2 - E|u_0|_H^2 + 2\,E\int_0^T \langle y_t\,,\, A_t(y_t)\rangle\,dt - 2\,E\int_0^T \langle u_\infty(t)\,,\, A_t(y_t)\rangle\,dt \\
& -2E\int_0^T \langle y_t\,,\, a_\infty(t)\rangle\,dt + \sum_{j=1}^r E\int_0^T \Big[|B_t^j(y_t)|_H^2 - 2\,(b_\infty^j(t), B_t^j(y_t))\Big]\,dt \\
= \quad & d + F_y(u_0, a_\infty, b_\infty),
\end{aligned}
\tag{3.30}
$$

by (3.23) - (3.25), (3.27) - (3.29), and taking into account (3.21). Hence by Theorem 2.7 (ii), u is a solution to equation (1.1). Taking $y := u$ in the above inequality we get $d \leq 0$, and hence $d = 0$. Thus the approximations $u_m^n(T)$ converge weakly in L_H^2 and their L_H^2-norms converge to that of $u_\infty(T)$, which imply the strong convergence of $u_m^n(T)$ in L_H^2 to $u(T)$. □

Now we conclude the proof of Theorem 2.8. Let (n, m) be a sequence of pairs of positive integers such that m and n converge to infinity and $C_{\mathcal{B}}(n)/m \to 0$; the previous proposition proves the existence of a subsequence of the explicit approximations u_m^n that converges weakly in $\mathcal{L}_V^2(\lambda)$ to a solution u_∞ of the equation (1.1), and such that $u_m^n(T)$ converges strongly to $u_\infty(T)$ along the same subsequence. Since by Remark 2.5 the solution to (1.1) is unique, the whole sequence u_m^n converges weakly in $\mathcal{L}_V^2(\lambda)$ to the solution of the equation (1.1), and the whole sequence $u_m^n(T)$ converges strongly in L_H^2 to $u_\infty(T)$. □

3.2. **Existence and uniqueness of solutions to the implicit schemes.** The following proposition establishes existence and uniqueness of the solution to the equation $Dx = y$ and provides an estimate of the norm of x in terms of that of y.

Proposition 3.4. *Let $D : V \to V^*$ be such that:*

(i) D is monotone, i.e., for every $x, y \in V$, $\langle D(x) - D(y), x - y\rangle \geq 0$.

(ii) D is hemicontinuous, i.e., $\lim_{\varepsilon\to 0}\langle D(x + \varepsilon y), z\rangle = \langle D(x), z\rangle$ for every $x, y, z \in V$.

(iii) D satisfies the growth condition, i.e., there exists $K > 0$ such that for every $x \in V$,

$$
|D(x)|_{V^*} \leq K\,(1 + |x|_V^{p-1}).
\tag{3.31}
$$

(iv) D is coercive, i.e., there exist constants $C_1 > 0$ and $C_2 \geq 0$ such that

$$
\langle D(x), x\rangle \geq C_1\,|x|_V^p - C_2\,, \quad \forall x \in V\,.
$$

Then for every $y \in V^*$, there exists $x \in V$ such that $D(x) = y$ and

$$
|x|_V^p \leq \frac{C_1 + 2\,C_2}{C_1} + \frac{1}{C_1^2}\,|y|_{V^*}^2\,.
\tag{3.32}
$$

If there exists a positive constant C_3 such that

$$
\langle D(x_1) - D(x_2), x_1 - x_2\rangle \geq C_3\,|x_1 - x_2|_{V^*}^2\,, \quad \forall x_1, x_2 \in V\,,
\tag{3.33}
$$

then for any $y \in V^*$, the equation $D(x) = y$ has a unique solution $x \in V$.

This result is known, or can easily be obtained from well-known results. We include its proof in the Appendix for the convenience of the reader.

Proof of Theorem 2.9. To prove this theorem, we need to check the conditions of the previous proposition for the operators $D : V \to V^*$ and $D_n : V_n \to V_n$, defined by

$$
D := I - \int_{t_i}^{t_{i+1}} A_s\,ds \quad \text{and} \quad D_n := I_n - \int_{t_i}^{t_{i+1}} \Pi_n A_s\,ds
$$

for each $i=0,1,2,...,m-1$, where $I: V\to V^*$ denotes the canonical embedding and I_n denotes the identity operators on V_n. Hence $u^m(t_i)$ and $u^{n,m}(t_i)$ can be uniquely defined recursively for $0\le i\le m$ by the equations (2.18) and (2.19), respectively.

We first check that D satisfies the strong monotonicity condition. Let $x,y\in V$. Then (C1) implies

$$\langle D(x)-D(y),x-y\rangle = |x-y|_H^2 - \int_{t_i}^{t_{i+1}} \langle A_s(x)-A_s(y),x-y\rangle\,ds \ge |x-y|_H^2\,.$$

Let us check that D is hemicontinuous. Let $x,y,z\in V$ and $\varepsilon\in\mathbb{R}$:

$$\langle D(x+\varepsilon y),z\rangle = \langle x+\varepsilon\, y,z\rangle - \int_{t_i}^{t_{i+1}} \langle A_s(x+\varepsilon y)\,,\,z\rangle\,ds\,.$$

As $\varepsilon\to 0$, condition (C3) implies that for every $s\in[t_i,t_{i+1}]$, $\langle A_s(x+\varepsilon y)\,,\,z\rangle$ converges to $\langle A_s(x),z\rangle$. Hence, using condition (C4) we have that for every $\varepsilon\in]0,1]$:

$$|\langle A_s(x+\varepsilon y\,,\,z\rangle| \le C\,\alpha^{\frac{1}{q}}\,\lambda(s)\,\Big[|x|_V^{p-1}+|y|_V^{p-1}\Big]\,|z|_{V^*} + \frac{1}{p}\,\lambda(s)+\frac{1}{q}\,K_2(s)\in\mathcal{L}^1\,.$$

Thus, we get the hemicontinuity of D by the Lebesgue theorem on dominated convergence.

We check that the operator D satisfies the growth condition (3.31). Let $x\in V$. Then by condition (C4) for $p\in[2,+\infty[$ we have

$$|D(x)|_{V^*}\le |x|_{V^*} + C_1\int_{t_i}^{t_{i+1}} \Big[\lambda(s)\,|x|_V^{p-1}+\frac{1}{p}\,\lambda(s)+\frac{1}{q}\,K_2(s)\Big]\,ds \le C_2\,[1+|x|_V^{p-1}]$$

with some constants C_1,C_2.

We check that for m large enough, D satisfies the coercivity condition. Let $x\in V$; then using (C2) we have:

$$\begin{aligned}\langle D(x),x\rangle \ge\; & |x|_H^2 + \int_{t_i}^{t_{i+1}} \frac{1}{2}\,\big[\lambda(s)\,|x|_V^p - K_1(s)\,|x|_H^2 - \bar{K}_1(s)\big]\,ds\\ \ge\; & \frac{1}{2}\,\Big(\int_{t_i}^{t_{i+1}}\lambda(s)\,ds\Big)\,|x|_V^p - \frac{1}{2}\int_{t_i}^{t_{i+1}}\bar{K}_1(s)\,ds + |x|_H^2\,\Big[1-\frac{1}{2}\int_{t_i}^{t_{i+1}} K_1(s)\,ds\Big]\,.\end{aligned}$$

Since K_1 is integrable, for large enough m, $\delta_m=t_{i+1}-t_i=T\,m^{-1}$ is small enough to imply that $\int_{t_i}^{t_{i+1}}K_1(s)\,ds<2$; thus D is coercive. Using (H2), (H3) and the equivalence of the norms $|\,.\,|_{V_n}$, $|\,.\,|_{H_n}$ and $|\,.\,|_{V_n^*}$ on V_n, we similar arguments show that D_n satisfies conditions (i)-(iv) too.

We finally prove by induction that the random variables $u^{n,m}(t_i)$ and $u^m(t_i)$ belong to L_V^p. This is obvious for $t_0=0$, and for t_1 it follows immediately from the estimate (3.32). Let i be an integer in $\{1,\,\cdots\,,\,m-1\}$, assume that $E(|u^{n,m}(t_k)|_V^p)<+\infty$ for every $k\in\{1,\,\cdots\,,\,i\}$ and set

$$y=u^{n,m}(t_i)+\sum_{j=1}^r\int_{t_i}^{t_{i+1}}\Pi_n\tilde{B}_s^{m,j}(u^{n,m}(t_i))\,dW_s^j\in V_n\,.$$

Then by the isometry of stochastic integrals, and by Remark 2.1 for $p\in[2,+\infty[$ we have

$$\begin{aligned}E|y|_{V^*}^2 \;&\le\; C_1\,E|u^{n,m}(t_i)|_V^2 + C_1\,E\int_{t_i}^{t_{i+1}}|\Pi_n\tilde{B}_s^{m,j}(u^{n,m}(t_i))|_H^2\,dt\\ &\le\; C\Big[1+\int_{t_{i-1}}^{t_i}\lambda(s)\,ds\Big]\,E|u^{n,m}(t_i)|_V^p\end{aligned}$$

$$+C\int_{t_{i-1}}^{t_i} K_1(s)\, E|u^{n,m}(t_i)|_H^2\, ds + C\Big[1+\int_{t_{i-1}}^{t_i} K_3(s)\, ds\Big] < \infty\,.$$

Hence (3.32) shows that $E(|u^{n,m}(t_{i+1})|_V^p) < +\infty$. In the same way we get the finiteness of the p-th moments of the V-norm of $u^m(t_i)$ for $i=0,1,2,..m$. □

3.3. **Convergence of the implicit schemes.** We first prove some a priori estimates on the processes u^m and $u^{n,m}$ and give an evolution formulation to the equations satisfied by these processes.

Recall that for $0\le i<m$ and $t\in]t_i,t_{i+1}[$, we set $\kappa_1(t)=t_i$ and $\kappa_2(t)=t_{i+1}$, while for $i=0,\cdots,m$, we set $\kappa_1(t_i)=\kappa_2(t_i)=t_i$. Let A^m and $\tilde{B}^{m,j}$ be defined in (2.17) and (2.13). Then equations (2.18) and (2.19) can be cast in the integral form:

$$u^m(t) = u_0\, 1_{\{t\ge t_1\}} + \int_0^{\kappa_1(t)} A_s\big(u^m(\kappa_2(s))\big)\, ds + \sum_{j=1}^r \int_0^{\kappa_1(t)} \tilde{B}_s^{m,j}\big(u^m(\kappa_1(s))\big)\, dW_s^j\,, \quad (3.34)$$

and

$$u^{n,m}(t) = \Pi_n u_0 1_{\{t\ge t_1\}} + \int_0^{\kappa_1(t)} \Pi_n A_s\big(u^{n,m}(\kappa_2(s))\big)\, ds + \sum_{j=1}^r \int_0^{\kappa_1(t)} \Pi_n \tilde{B}_s^{m,j}\big(u^{n,m}\big(\kappa_1(s)\big)\big)\, dW_s^j, \quad (3.35)$$

respectively.

Lemma 3.5. *Let conditions (C1)-(C5) and (H1)-(H3) hold. Then there exist an integer m_1 and some constants $L_i, 1\le i\le 4$, such that:*

$$\sup_{s\in[0,T]} E\big|u^{n,m}(s)\big|_H^2 \le L_1\,, \quad (3.36)$$

$$E\int_0^T \big|u^{n,m}(\kappa_2(s))\big|_V^p\, \lambda(s)\, ds \le L_2\,, \quad (3.37)$$

$$E\int_0^T \big|A_s\big(u^{n,m}(\kappa_2(s))\big)\big|_{V^*}^q\, \lambda(s)^{1-q}\, ds \le L_3\,, \quad (3.38)$$

$$\sum_{j=1}^r E\int_0^T \big|\tilde{B}_s^{m,j}\big(u^{n,m}(\kappa_1(s))\big)\big|_H^2\, ds \le L_4 \quad (3.39)$$

for all $m\ge m_1$ and $n\ge 1$. Under conditions (C1)-(C5) the above estimates hold with the implicit approximations u^m in place of $u^{n,m}$ for all sufficiently large m.

Proof. We only prove the estimates for $u^{n,m}$. The proof of the estimates for u^m is essentially the same, and we omit it. We set $\Delta W_{t_i}^j = W_{t_{i+1}}^j - W_{t_i}^j$ for $i=0,\cdots,m-1$, $j=1\cdots,r$. Then from the definition of the approximations $u^{n,m}$ we get

$$|u^{n,m}(t_1)|_H^2 - |\Pi_n u_0|_H^2 = 2\langle u^{n,m}(t_1), A_0^m(u^{n,m}(t_1))\rangle \delta_m - |\Pi_n A_0^m(u(t_1))|_H^2 \delta_m^2$$

and for $i=1,\cdots,m-1$:

$$u^{n,m}(t_{i+1})|_H^2 - |u^{n,m}(t_i)|_H^2 = 2\big\langle u^{n,m}(t_{i+1}), A_{t_i}^m(u^{n,m}(t_{i+1}))\big\rangle\, \delta_m - \big|\Pi_n A_{t_i}^m(u^{n,m}(t_{i+1}))\big|_H^2 \delta_m^2$$
$$+ 2\sum_{j=1}^r \big(u^{n,m}(t_i), \Pi_n \tilde{B}_{t_i}^{m,j}(u^{n,m}(t_i))\big)\, \Delta W_{t_i}^j + \Big|\sum_{j=1}^r \Pi_n \tilde{B}_{t_i}^{m,j}(u^{n,m}(t_i)\, \Delta W_{t_i}^j\Big|_H^2.$$

Hence adding these equations and taking expectation we obtain

$$E|u^{n,m}(t_k)|_H^2 = E|\Pi_n(u_0)|_H^2 + 2\, E\int_0^{t_k} \big\langle\, A_s\big(u^{n,m}(\kappa_2(s))\big), u^{n,m}(\kappa_2(s))\big\rangle\, ds$$

$$+\sum_{j=1}^r E\int_{t_1}^{t_k} \big|\Pi_n \tilde B^{m,j}_s\big(u^{n,m}(\kappa_1(s))\big)\big|_H^2\,ds - \delta_m E\int_0^{t_k} \big|\Pi_n A_t\big(u^{n,m}(\kappa_2(s))\big)\big|_H^2\,ds$$

for $k=1,2,...,m$, which implies

$$\begin{aligned}E|u^{n,m}(t_k)|_H^2 &\le E|u_0|_H^2 - \delta_m\, E\int_0^{t_k} \big|\Pi_n A_s\big(u^{n,m}(\kappa_2(s))\big)|_H^2\,ds \\ &+ E\int_0^{t_k}\Big\{2\,\big\langle A_s\big(u^{n,m}(\kappa_2(s))\big)\,,\,u^{n,m}(\kappa_2(s))\big\rangle + \sum_{j=1}^r \big|B^j_s(u^{n,m}(\kappa_2(s)))\big|_H^2\Big\}\,ds \quad (3.40)\\ &\le\; E|u_0|_H^2 - E\int_0^{t_k}\lambda(s)\,|u^{n,m}(\kappa_2(s))|_V^p\,ds + E\int_0^{t_k} K_1(s)\,|u^{n,m}(\kappa_2(s))|_H^2\,ds \\ &+\int_0^{t_k}\bar K_1(s)\,ds\,,\end{aligned}$$

by the definition of $\tilde B^{m,j}$, the coercivity condition (C2), and by (H2). For m large enough, $\gamma_m=\sup\{\int_{t_{k-1}}^{t_k} K_1(s)\,ds\;:\;1\le k\le m\}\le\frac12$. Consequently, there exists an integer m_1 such that for all $n\ge 1$, $m\ge m_1$ and $k=1,2,...,m$:

$$\frac12\,E|u^{n,m}(t_k)|_H^2 + E\int_0^{t_k}|u^{n,m}(\kappa_2(s))|_V^p\,\lambda(s)\,ds \le C + \int_0^{t_{k-1}} K_1(s)\,E|u^{n,m}(\kappa_2(s))|_H^2\,ds\,. \tag{3.41}$$

Hence a discrete version of Gronwall's lemma implies the existence of a constant $C>0$ such that

$$\sup_{n\ge1}\ \sup_{m\ge m_1}\ \sup_{0\le k\le m} E\big|u^{n,m}\big(k\,\delta_m\big)\big|_H^2 = C<\infty\,, \tag{3.42}$$

which implies (3.36). The inequalities (3.41) and (3.42) yield (3.37). Notice that by the growth condition (C4)

$$E\int_0^T \big|A_s(u^{n,m}\big(\kappa_2(s))\big)\big|_{V^*}^q\,\lambda(s)^{1-q}\,ds \le \alpha\,E\int_0^T |u^{n,m}(\kappa_2(s))|_V^p\,\lambda(s)\,ds + \int_0^T K_2(s)\,ds$$

and by the definition of $\tilde B^{m,j}$ and by Remark 2.1,

$$\begin{aligned}E\int_0^T |\tilde B^{m,j}_s(u^{n,m}(\kappa_1(s)))|_H^2\,ds &\le E\int_0^{T-\delta_m} \big|B^j_s(u^{n,m}(\kappa_2(s))\big|_H^2\,ds\\ &\le (2\alpha+1)E\int_0^T|u^{n,m}(\kappa_2(s))|_V^p\,\lambda(s)\,ds + E\int_0^T K_1(s)|u^{n,m}(\kappa_2(s))|_H^2\,ds + \int_0^T K_3(s)ds.\end{aligned}$$

Thus estimates (3.36) and (3.37) imply estimates (3.38) and (3.39). □

Proposition 3.6. *Let conditions (C1)–(C5) and (H1)-(H3) hold. Then for any sequence $(n,m)\to\infty$ of pairs of positive integers there exists a subsequence, denoted also by (n,m), such that:*

(i) $u^{n,m}(T)$ converges weakly to $u_{\infty T}$ in L^2_H,
(ii) $u^{n,m}(\kappa_2(.))$ converges weakly in $\mathcal{L}^p_V(\lambda)$ to v_∞,
(iii) $A_.(u^{n,m}(\kappa_2(.)))$ converges weakly in $\mathcal{L}^q_{V^*}(\lambda^{-1})$ to a_∞,
(iv) $\Pi_n\tilde B^{m,j}_.(u^{n,m}(\kappa_1(.)))$ converges weakly in $\mathcal{L}^2_H$ to b^j_∞ for each $j=1,2,...,r$.
(v) $(u_0,a_\infty,b_\infty)\in\mathcal{A}$, and for $dt\times P$-almost every $(t,\omega)\in[0,T]\times\Omega$

$$v_\infty(t) \;=\; u_0+\int_0^t a_\infty(s)\,ds + \sum_{j=1}^r\int_0^t b^j_\infty(s)\,dW^j(s), \tag{3.43}$$

$$u_{T\infty} = u_0 + \int_0^T a_\infty(s)\,ds + \sum_{j=1}^r \int_0^T b^j_\infty(s)\,dW^j(s) \quad \text{(a.s.)}, \tag{3.44}$$

$$F_y(u_0, a_\infty, b_\infty) \le 0\,, \quad \forall y \in \mathcal{L}^p_V(\lambda)\,. \tag{3.45}$$

(vi) The process v_∞ has an H-valued continuous modification, u_∞, which is the solution of equation (1.1). Moreover, the sequence $u^{n,m}(T)$ converges strongly in L^2_H to $u_{\infty T} = v_\infty(T)$.

Under conditions (C1)-(C5) the above assertions hold with u^m and $\tilde{B}^{m,j}$, in place of $u^{n,m}$ and $\Pi_n \tilde{B}^{m,j}$, respectively.

Proof. We prove the lemma for subsequences of $u^{n,m}$. The proof for the sequence u^m is essentially the same, and we omit it. The assertions (i)-(iv) are immediate consequences of Lemma 3.5. We need only prove assertions (v) and (vi). For fixed $N \ge 1$ let $\varphi = \{\varphi(t) : t \in [0,T]\}$ be a V_N-valued adapted stochastic process such that $|\varphi(t)|_V \le N$ for all $t \in [0,T]$ and $\omega \in \Omega$. Then from equation (3.35) for $n \ge N$ we have

$$\begin{aligned}
E\int_0^T (u^{n,m}(t), \varphi(t))\,\lambda(t)\,dt &= E\int_0^T 1_{\{t \ge t_1\}} \big(\Pi_n u_0\,,\, \varphi(t)\big)\,\lambda(t)\,dt \\
&\quad + E\int_0^T \Big\langle \int_0^{\kappa_1(t)} \Pi_n A_s(u^{n,m}(\kappa_2(s)))\,ds\,,\, \varphi(t)\Big\rangle\,\lambda(t)\,dt \\
&\quad + \sum_{j=1}^r E\int_0^T \Big(\int_0^{\kappa_1(t)} \Pi_n \tilde{B}^{m,j}_s(u^{n,m}(\kappa_1(s)))\,dW^j_s\,,\, \varphi(t)\Big)\,\lambda(t)\,dt \\
&= E\int_0^T (u_0\,,\, \varphi(t))\,\lambda(t)\,dt + J_1 + J_2 - R_1 - R_2 - R_3\,,
\end{aligned} \tag{3.46}$$

with

$$\begin{aligned}
J_1 &:= E\int_0^T \Big\langle \int_0^t A_s\big(u^{n,m}(\kappa_2(s))\big)\,ds\,,\, \varphi(t)\Big\rangle\,\lambda(t)\,dt\,, \\
J_2 &:= \sum_{j=1}^r E\int_0^T \Big(\int_0^t \Pi_n \tilde{B}^{m,j}_s\big(u^{n,m}(\kappa_1(s))\big)\,dW^j_s\,,\, \varphi(t)\Big)\,\lambda(t)\,dt\,, \\
R_1 &:= E\int_0^{t_1} (u_0\,,\, \varphi(t))\,\lambda(t)\,dt\,, \\
R_2 &:= E\int_0^T \Big\langle \int_{\kappa_1(t)}^t A_s\big(u^{n,m}(\kappa_2(s))\big)\,ds\,,\, \varphi(t)\Big\rangle\,\lambda(t)\,dt\,, \\
R_3 &:= \sum_{j=1}^r E\int_0^T \Big(\int_{\kappa_1(t)}^t \Pi_n \tilde{B}^{m,j}_s\big(u^{n,m}(\kappa_1(s))\big)\,dW^j_s\,,\, \varphi(t)\Big)\,\lambda(t)\,dt\,.
\end{aligned}$$

Clearly, for $n, m \to \infty$:

$$\begin{aligned}
|R_1| &\to 0, \\
|R_2| &\le N\,E\int_0^T \lambda(t) \int_{\kappa_1(t)}^t |A_s(u^{n,m}(\kappa_2(s))|_{V^*}\,ds\,dt \\
&\le N\Big\{E\int_0^T\int_0^T \lambda(t)\lambda(s)^{-\frac{q}{p}}\,|A_s(u^{n,m}(\kappa_2(s))|^q_{V^*}\,ds\,dt\Big\}^{\frac{1}{q}}
\end{aligned} \tag{3.47}$$

$$\times\Big\{\int_0^T \lambda(t)\int_{\kappa_1(t)}^t \lambda(s)\,ds\,dt\Big\}^{\frac{1}{p}} \to 0 \tag{3.48}$$

by Hölder's inequality, Lebesgue's theorem on dominated convergence, and by virtue of estimate (3.38). By the isometry of H-valued stochastic integrals

$$\begin{aligned}|R_3| &\le N\int_0^T E\Big(\int_{\kappa_1(t)}^t \sum_{j=1}^r \big|\tilde B_s^{m,j}\big(u^{n,m}(\kappa_1(s))\big)\big|_H^2\,ds\Big)^{1/2}dt\\ &\le N\sqrt{\delta_m}\,E\int_0^T\sum_{j=1}^r \big|\tilde B_s^{m,j}\big(u^{n,m}(\kappa_1(s))\big)\big|_H^2\,ds \to 0\end{aligned} \tag{3.49}$$

as $n,m\to\infty$, by virtue of estimate (3.39). The arguments used to prove (3.16) in Proposition 3.2 yield as $n,m\to\infty$

$$J_2 \to \sum_{j=1}^r E\int_0^T \Big\langle \int_0^t b_\infty^j(s)\,dW_s^j, \varphi(t)\Big\rangle\,\lambda(t)\,dt\,. \tag{3.50}$$

Similarly, for $g\in\mathcal{L}^q_{V^*}(\lambda^{1-q})$, let $G(g)_t=\int_0^t g(s)\,ds$. Then Hölder's inequality implies that

$$\begin{aligned}\|G(g)\|^q_{\mathcal{L}^q_{V^*}(\lambda)} &\le E\int_0^T\lambda(t)\Big(\int_0^t |g(s)|^q_{V^*}\,\lambda T(s)^{-\frac{q}{p}}\,ds\Big)\Big(\int_0^t\lambda(s)\,ds\Big)^{\frac{q}{p}}dt\\ &\le \Big(\int_0^T\lambda(t)\,dt\Big)^q\,\|g\|^q_{\mathcal{L}^q_{V^*}(\lambda^{1-q})}\,.\end{aligned}$$

Hence, the operator G is bounded from $\mathcal{L}^q_{V^*}(\lambda^{1-q})$ to $\mathcal{L}^q_{V^*}(\lambda)$. Thus this operator is weakly continuous. Therefore as $m,n\to\infty$

$$J_1 \to E\int_0^T\Big\langle\int_0^t a_\infty(s)\,ds\,,\,\varphi(t)\Big\rangle\,\lambda(t)\,dt\,. \tag{3.51}$$

Letting now $n,m\to\infty$ in equation (3.46), we obtain

$$\begin{aligned}E\int_0^T (v_\infty(t)\,,\,\varphi(t))\,\lambda(t)\,dt = {}& E\int_0^T\big(u_0,\varphi(t)\big)\,\lambda(t)\,dt + E\int_0^T\Big\langle\int_0^t a_\infty(s)\,ds\,,\,\varphi(t)\Big\rangle\,\lambda(t)\,dt\\ &+E\int_0^T\Big(\sum_{j=1}^r\int_0^t b_\infty(s)^j\,dW_s^j\,,\,\varphi(t)\Big)\,\lambda(t)\,dt\end{aligned}$$

by (3.47)-(3.51) for any V_N-valued adapted stochastic process φ with $\sup_{t,\omega}|\varphi(t,\omega)|_H\le N$. Since N can be arbitrary large, equation (3.43) follows immediately. As in the proof of (3.11), a similar argument based on an analog of (3.21) for a $L^2_{V_N}$ random variable ψ with $E|\psi|_V^2\le N$ yields equation (3.44). An argument similar to that proving (3.49) yields

$$E|u_\infty(T)|_H^2 = E|u_0|_H^2 + E\int_0^T\Big[2\langle v_\infty(s),a_\infty(s)\rangle + \sum_{j=1}^r |b_\infty(s)|_H^2\Big]\,ds\,. \tag{3.52}$$

Moreover, by (3.43) and (3.44) we get $u_\infty(T)=u_{\infty T}$ (a.s.). To prove inequality (3.45) set

$$\begin{aligned}F_y^{n,m} := E\int_0^T 2\Big\{&\big\langle u^{n,m}(\kappa_2(t))-y(t)\,,\,A_t(u^{n,m}(\kappa_2(t)))-A_t(y_t)\big\rangle\\ &+\sum_{j=1}^r\big|B_t^j(u^{n,m}(\kappa_2(t)))-\Pi_n B_t^j(y(t))\big|_H^2\Big\}\,dt\end{aligned}$$

for $y \in \mathcal{L}^p_V(\lambda) \cap \mathcal{L}^2_H(K_1)$. By (C4), (2.5) and Lemma 3.5, $F^{n,m}_y$ is well-defined and it is finite. By the monotonicity condition and by inequality (3.40) with $k := m$ we obtain:

$$0 \geq F^{n,m}_y \geq \; E|u^{n,m}(T)|^2_H - E|u_0|^2_H + 2\, E \int_0^T \langle y_t \; A_t(y_t)\rangle \, dt - 2\, L^{n,m}_1$$
$$- 2\, L^{n,m}_2 + L^n_3 - 2\, L^{n,m}_4 + \delta_m \, E \int_{\delta_m}^T \big| \Pi_n A_s(u^{n,m}(\kappa_2(s))) \big|^2_H \, ds\,, \qquad (3.53)$$

with

$$\begin{aligned} L^{n,m}_1 &:= E \int_0^T \big\langle u^{n,m}(\kappa_2(t))\,, A_t(y_t)\big\rangle \, dt\,, \\ L^{n,m}_2 &:= E \int_0^T \big\langle y_t\,,\, A_t\big(u^{n,m}(\kappa_2(t))\big)\big\rangle \, dt\,, \\ L^n_3 &:= \sum_{j=1}^r E \int_0^T \big|\Pi_n B^j_t(y_t)\big|^2_H \, dt\,, \\ L^{n,m}_4 &:= \sum_{j=1}^r E \int_0^T \Big(\Pi_n B^j_t\big(u^{n,m}(\kappa_2(t))\big)\,,\, B^j_t(y_t)\Big) \, dt\,. \end{aligned}$$

Using (i)-(iv) and the arguments used to prove (3.24), (3.25),(3.27) and (3.29) we deduce:

$$\lim_{n,m\to\infty} L^{n,m}_1 = E\int_0^T \langle v_\infty(t)\,,\, A_t(y_t)\rangle \, dt, \qquad (3.54)$$

$$\lim_{n,m\to\infty} L^{n,m}_2 = E\int_0^T \langle y_t\,,\, a_\infty(t)\rangle \, dt, \qquad (3.55)$$

$$\lim_{n\to\infty} L^n_3 = \sum_{j=1}^r \int_0^T |B^j_t(y_t)|^2_H \, dt\,, \qquad (3.56)$$

$$\lim_{n,m\to\infty} L^{n,m}_4 = \sum_{j=1}^r E\int_0^T \big(b^j_\infty(t)\,,\, B^j_t(y_t)\big) \, dt. \qquad (3.57)$$

Furthermore, for some constant $d \geq 0$:

$$\liminf_{n,m\to\infty} E|u^{n,m}(T)|^2_H = d + E|u_\infty(T)|^2_H. \qquad (3.58)$$

Thus, letting $n, m \to \infty$ in (3.53), by (3.54)-(3.58) we deduce:

$$\begin{aligned} 0 \;\geq\; & d + E|u_\infty(T)|^2_H - E|u_0|^2_H - 2E\int_0^T \langle v_\infty(t)\,,\, A_t(y_t)\rangle \, dt - 2E\int_0^T \langle y_t\,,\, a_\infty(t)\rangle \, dt \\ & +2\, E\int_0^T \langle y_t\,,\, A_t(y_t)\rangle \, dt + \sum_{j=1}^r E\int_0^T \big[|B^j_t(y_t)|^2_H - 2\,(b^j_\infty(t), B^j_t(y_t))\big] \, dt \\ = \;& d + F_y(u_0, a_\infty, b_\infty)\,. \end{aligned}$$

Then we proceed as after (3.30) at the end of the proof of Proposition 3.3 and finish the proof of the proposition. □

Now we conclude the proof of Theorem 2.10. By the previous proposition, from any sequence (n, m) of pairs of positive integers such that $m, n \to \infty$, there exists a subsequence, (n_k, m_k), such that the approximations u^{n_k,m_k} converge weakly in $\mathcal{L}^p_V(\lambda)$ to the solution u of equation (1.1), and the approximations $u^{n_k,m_k}(T)$ converge strongly in L^2_H to $u(T)$. Hence, taking into account that the solution of equation (1.1) is unique, we get that these convergence statements hold for any sequences of approximations $u^{n,m}$ and $u^{n,m}(T)$ as $n,m\to\infty$. The proof of Theorem 2.10 is complete. □

4. Appendix

We start with a technical lemma ensuring that a map from V to V^* is continuous.

Lemma 4.1. *Let V be a Banach space and V^* its topological dual, $D:V\to V^*$ satisfy the conditions (i)-(iii) of Proposition 3.4. Then D is continuous from $(V,|\,.\,|_V)$ into V^* endowed with the weak star topology $\sigma(V^*,V)$. In particular, if V is a finite dimension vector space, then D is continuous.*

Proof. Let $x\in V$ and $(x_n,\, n\geq 1)$ be a sequence of elements of V such that $\lim_n |x-x_n|_V = 0$. The monotonicity property (i) implies that for every $y\in V$ and $n\geq 1$,

$$\langle D(x_n)-D(y), x_n-x\rangle + \langle D(x_n)-D(y), x-y\rangle \geq 0\,.$$

Furthermore, since $(|x_n|_V\,,\, n\geq 1)$ is bounded, the growth condition (iii) implies that

$$\begin{aligned}|\langle D(x_n)-D(y), x_n-x\rangle| &\leq \big[|D(x_n)|_{V^*}+|D(y)|_{V^*}\big]\,|x_n-x|_V\\ &\leq C\,(1+|x_n|_V^{p-1}+|y|_V^{p-1})\,|x_n-x|_V\to 0\end{aligned}$$

as $n\to+\infty$; hence, $\liminf_n\langle D(x_n)-D(y),x-y\rangle\geq 0$. Since $(|x_n|_V, n\geq 1)$ is bounded, the growth condition implies the existence of a subsequence $(n_k, k\geq 1)$ such that $D(x_{n_k})\to D_\infty\in V^*$ is the weak star topology as $k\to+\infty$; clearly,

$$\langle D_\infty - D(y), x-y\rangle \geq 0\ ,\forall y\in V\,. \tag{4.1}$$

To conclude the proof, we check that $D_\infty = D(x)$; indeed, this yields that the whole sequence $(D(x_n), n\geq 1)$ converges weakly to $D(x)$. For any $z\in V$ and $\varepsilon>0$, apply (4.1) with $y=x-\varepsilon z$; then dividing by $\varepsilon>0$ and using the hemicontinuity property (ii), we deduce that for any $z\in V$,

$$\lim_{\varepsilon\to 0}\langle D_\infty - D(x-\varepsilon z), z\rangle = \langle D_\infty - D(x), z\rangle \geq 0\,.$$

Changing z into $-z$, this yields $D_\infty = D(x)$. □

Proof of Proposition 3.4. Let $(e_i\,,\, i\geq 1)$ be a sequence of elements of V which is a complete orthonormal basis of H and for every $n\geq 1$, let $\tilde V_n = \text{span}\,(e_i\,,\, 1\leq i\leq n)$, $\tilde\Pi_n : V^*\to\tilde V_n$ be defined by $\tilde\Pi_n(y)=\sum_{i=1}^n\langle e_i, y\rangle\, e_i$ for $y\in V^*$ and let $\tilde D_n = \tilde\Pi_n\circ D:\tilde V_n\to\tilde V_n$. Then $\tilde D_n$ is coercive and satisfies the assumptions of Lemma 4.1; hence it is continuous. Fix $y\in V^*$; the existence of $x_n\in\tilde V_n$ such that $\tilde D_n(x_n)=\tilde\Pi_n(y)$ is classical (see e.g. [12]). The coercivity condition implies that for every $n\geq 1$:

$$|y|_{V^*}\,|x_n|_V \geq \langle x_n, y\rangle = \langle x_n, \tilde D_n(x_n)\rangle = \langle x_n, D(x_n)\rangle \geq C_1\,|x_n|_V^p - C_2\,,$$

which implies that the sequence $(|x_n|_V, n\geq 1)$ is bounded, and the growth property implies that the sequence $(|D(x_n)|_{V^*}\,,\, n\geq 1)$ is bounded. Since V is reflexive, there exists a subsequence $(n_k\,,\, k\geq 1)$ such that the sequence $(x_{n_k}\,,\, k\geq 1)$ converges to $x_\infty\in V$ in the weak $\sigma(V,V^*)$ topology, and such that the sequence $(D(x_{n_k})\,,\, k\geq 1)$ converges to D_∞ in the weak-star topology $\sigma(V^*,V)$. We at first check that $D_\infty = y$; indeed, for every $i\geq 1$:

$$\langle D_\infty, e_i\rangle \;=\; \lim_k\langle D(x_{n_k}), e_i\rangle = \lim_k\langle \tilde D_{n_k}(x_{n_k}), e_i\rangle$$

$$= \lim_k \langle \tilde\Pi_{n_k}(y), e_i\rangle = \langle y, e_i\rangle\,.$$

We then prove that $y = D(x_\infty)$; the monotonicity property of D implies that for every $z \in \cup_n V_n$, for k large enough:

$$\begin{aligned} 0 &\le \langle D(x_{n_k}) - D(z), x_{n_k} - z\rangle \\ &\le \langle \tilde D_{n_k}(x_{n_k}), x_{n_k}\rangle - \langle D(z), x_{n_k}\rangle - \langle \tilde D_{n_k}(x_{n_k}), z\rangle + \langle D(z), z\rangle \\ &\le \langle y, x_{n_k}\rangle - \langle D(z), x_{n_k}\rangle - \langle \tilde D_{n_k}(x_{n_k}), z\rangle + \langle D(z), z\rangle\,. \end{aligned}$$

As $k \to +\infty$, we deduce that for every $z \in \cup_n V_n$, $\langle D_\infty - D(z), x_\infty - z\rangle \ge 0$. Since $\cup_n V_n$ is dense in V, we deduce that $\langle D_\infty - D(z), x_\infty - z\rangle \ge 0$ for every $z \in V$. Let $\xi \in V$; apply the previous inequality to $z = x_\infty + \varepsilon\,\xi$ for any $\varepsilon > 0$ and divide by ε. This yields that for any $\xi \in V$, $\langle D_\infty - D(x_\infty + \varepsilon\,\xi)\,,\, \xi\rangle \ge 0$; as $\varepsilon \to 0$, the hemicontinuity implies that for any $\xi \in V$, $\langle D_\infty - D(x_\infty)\,,\, \xi\rangle \ge 0$, and hence that $y = D_\infty = D(x_\infty)$. This concludes the proof of the existence of a solution x to the equation $D(x) = y$. Furthermore, the coercivity of D implies that

$$C_1\,|x|_V^p - C_2 \le \langle D(x)\,,\, x\rangle = \langle y\,,\, x\rangle \le \frac{C_1}{2}\,|x|_V^2 + \frac{1}{2C_1}\,|y|_{V^*}^2\,.$$

Hence for $p \in [2, +\infty[$, $C_1\,|x|_V^p - C_2 \le \frac{C_1}{2}\,|x|_V^p + \frac{C_1}{2} + \frac{1}{2C_1}\,|y|_{V^*}^2$, which implies (3.32). Finally, if D satisfies the strong monotonicity condition (3.33) and if $x_1, x_2 \in V$ are such that $D(x_1) = D(x_2) = y$, then

$$0 = \langle D(x_1) - D(x_2), x_1 - x_2\rangle \ge C_3\,|x_1 - x_2|_{V^*}^2\,;$$

this yields $|x_1 - x_2|_{V^*} = 0$. □

We finally sketch the proof of Theorem 2.7

Proof of Theorem 2.7. (i) The monotonicity condition (C1) implies that for every $y \in \mathcal{L}_V^p(\lambda)$ such that $\sup_{0\le t\le T} E|y_t|_H^2 < +\infty$ one has:

$$2\langle u_s - y_s\,,\, A_s(u_s) - A_s(y_s)\rangle + \sum_{j=1}^r |B_s^j(u_s) - B_s^j(y_s)|_H^2 \le 0\,.$$

This implies that $F_y(u_0, A_.(u_.), B_.(u_.)) \le 0$ for every $y \in \mathcal{L}_V^p(\lambda)$ with $\sup_{0\le t\le T} E|y_t|_H^2 < +\infty$, which yields (i).

(ii) Let $(\xi, a, b) \in \mathcal{A}$, $u_t = \xi + \int_0^t a_s\,ds + \sum_{j=1}^r \int_0^t b_s^j\,dW_s^j$ and let $\mathcal{V}$ be a subset of $\mathcal{L}_V^p(\lambda)$ of processes y such that $\sup_{0\le t\le T} E|y_t|_H^2 < +\infty$, which is dense in $\mathcal{L}_V^p(\lambda)$ and such that

$$F_y(\xi, a, b) \le 0 \quad \text{for } y \in \mathcal{V}\,. \tag{4.2}$$

We first check that (4.2) holds for $y = u + z$ where $\sup_{0\le t_l eqT} E|z_t|_V^p < +\infty$. To this end let $\{y_n\,,\, n \ge 1\}$ be a sequence of elements of $\mathcal{V}$, such that $\lim_n \|y - y_n\|_{\mathcal{L}_V^p(\lambda)} = 0$. For any $U \in \mathcal{L}_V^p(\lambda)$ such that $\sup_{0\le t\le T} E|U_t|_H^2 < +\infty$, set

$$\Phi(U) = E\int_0^T \Big[2\big\langle u_s - U(s)\,,\, a_s - A_s(U(s))\big\rangle + \sum_{j=1}^r \big|b_s^j - B_s^j(U(s))\big|_H^2\Big]\,ds\,.$$

Then $\Big|\,\Phi(y_n) - \Phi(y)\Big| \le \sum_{i=1}^3 T_i(n)$, where:

$$T_1(n) \;=\; \Big|E\int_0^T 2\,\big\langle y(s) - y_n(s)\,,\, a_s - A_s(y_n(s))\big\rangle\,ds\Big|\,,$$

$$
\begin{aligned}
T_2(n) &= \Big| E\int_0^T 2\left\langle u_s - y(s)\,,\, A_s(y(s)) - A_s(y_n(s))\right\rangle ds \Big|\,,\\
T_3(n) &= \sum_{j=1}^r \Big| E\int_0^T \Big[\big|B_s^j(y_n(s))\big|_H^2 - \big|B_s^j(y(s))\big|_H^2 + 2\left(b_s^j\,,\, B_s^j(y(s)) - B_s^j(y_n(s))\right)\Big]\,ds \Big|\,.
\end{aligned}
$$

Since $\sup_n E\int_0^T |y_n(s)|_V^p\,\lambda(s)\,ds<\infty$, the growth condition (C4) yields

$$
\begin{aligned}
T_1(n) &\le \|y-y_n\|_{\mathcal{L}_V^p(\lambda)} \Big\{ E\int_0^T \left(|a_s|_{V^*}^q + |A_s(y_n(s))|_{V^*}^q\right)\lambda^{1-q}(s)\,ds\Big\}^{\frac{1}{q}}\\
&\le C_1\,\|y-y_n\|_{\mathcal{L}_V^p(\lambda)} \Big\{ E\int_0^T \Big[|a_s|_{V^*}^q\,\lambda^{1-q}(s) + \alpha\,|y_n(s)|_V^p\,\lambda(s) + K_2(s)\Big]\,ds\Big\}^{\frac{1}{q}}\\
&\le C_2\,\|y-y_n\|_{\mathcal{L}_V^p(\lambda)}\,, \qquad (4.3)
\end{aligned}
$$

where C_1, C_2 are constants which do not depend on n. For $dt\times P$-almost every (t,ω) the operator $A_t(\omega)\,:\,V\to V^*$ is monotone and hemicontinuous, hence it is demi-continuous, i.e., the sequence $A_t(\omega,x_n)$ converges weakly in V^* to $A_t(\omega,x)$ whenever x_n converges strongly in V to x (see, e.g., Proposition 26.4 in [12]). Hence for $dt\times P-$almost every $(t,\omega)\in[0,T]\times\Omega$,

$$
\lim_n \langle z(s)\,,\, A_s(y(s)) - A_s(y_n(s))\rangle = 0\,.
$$

Furthermore, since z is bounded, condition (C4) implies

$$
\begin{aligned}
&\sup_n E\int_0^T \left|\left\langle z(s)\,,\, A_s(y(s)) - A_s(y_n(s))\right\rangle\right|^q ds\\
&\qquad\le C \sup_n E\int_0^T |A_s(y(s)) - A_s(y_n(s))|_{V^*}^q\,ds\\
&\qquad\le C_1 \sup_n E\int_0^T \left(|y(s)|_V^p + |y_n(s)|_V^p\right)\lambda(s)\,ds + C_1\int_0^T K_2(s)\,ds<\infty\,.
\end{aligned}
$$

Therefore, the sequence $\left\{\left\langle z\,,\, A(y)-A(y_n)\right\rangle,\, n\ge 1\right\}$ is uniformly integrable with respect to the measure $dt\times P$. Hence

$$
\lim_n T_2(n) = 0\,. \qquad (4.4)
$$

By Remark 2.1

$$
\begin{aligned}
&\sup_n \sum_{j=1}^r E\int_0^T \left[|B_s^j(y(s))|_H^2 + |B_s^j(y_n(s))|_H^2\right] ds\\
&\qquad\le \sup_n C E\int_0^T \left[|y(s)|_V^p + |y_n(s)|_V^p\right]\lambda(s)\,ds\\
&\qquad\quad + C\,E\int_0^T \left\{K_1(s)\left[|y(s)|_H^2 + |y_n(s)|_H^2\right] + K_3(s)\right\} ds<\infty\,.
\end{aligned}
$$

By Schwarz's inequality we deduce

$$
\begin{aligned}
T_3(n) &\le C\sum_{j=1}^r \Big\{ E\int_0^T \left[|B_s^j(y(s))|_H^2 + |B_s^j(y_n(s))|_H^2 + |b_\infty^j(s)|_H^2\right] ds\Big\}^{\frac{1}{2}}\\
&\qquad \times \Big\{ E\int_0^T |B_s^j(y_n(s)) - B_s^j(y(s))|_H^2\,ds\Big\}^{\frac{1}{2}}
\end{aligned}
$$

$$\leq \quad C \sum_{j=1}^{r} \Big\{ E \int_0^T |B_s^j(y_n(s)) - B_s^j(y(s))|_H^2 \, ds \Big\}^{\frac{1}{2}} .$$

The monotonicity assumption (C1) and the growth condition (C4) imply:

$$\sum_{j=1}^{r} E \int_0^T \big|B_s^j(y_n(s)) - B_s^j(y(s))\big|_H^2 \, ds \leq -2\, E \int_0^T \big\langle y_n(s) - y(s)\, ,\, A_s(y_n(s)) - A_s(y(s)) \big\rangle \, ds$$

$$\leq C \Big\{ E \int_0^T |y_n(s) - y(s)|_V^p \, \lambda(s) \, ds \Big\}^{\frac{1}{p}} \Big\{ E \int_0^T \big[\big(|y_n(s)|_V^p + |y(s)|_V^p \big) \lambda(s) + K_2(s) \big] ds \Big\}^{\frac{1}{q}}$$

$$\leq C \, \|y_n - y\|_{\mathcal{L}_V^p(\lambda)} \, . \tag{4.5}$$

The inequalities (4.3)-(4.5) imply $\lim_n \Phi(y_n) = \Phi(u+z)$. Consequently, (4.2) holds for $y = u + z$ with any $z \in \mathcal{L}_V^\infty$.

Fix $z \in \mathcal{L}_V^\infty$ and $\varepsilon > 0$, apply (4.2) to $y = u - \varepsilon z$ and divide by ε; this yields:

$$E \int_0^T \big\langle z_t, a_t - A_t(u_t - \varepsilon z_t) \big\rangle \, dt \geq 0 \, . \tag{4.6}$$

By the hemicontinuity condition (C3) for almost all $\omega \in \Omega$:

$$\lim_{\varepsilon \to 0} \big\langle z_t \, , \, a_t - A_t(u_t - \varepsilon z_t) \big\rangle = \big\langle z_t \, , \, a_t - A_t(u_t) \big\rangle \, , \quad \forall t \in [0,T].$$

Furthermore, since z is bounded, by (C4)

$$\sup_{0<\varepsilon\leq 1} E \int_0^T |\langle z_t \, , \, a_t - A_t(u_t - \varepsilon \, z_t) \rangle|^q \, dt < \infty \, ,$$

which implies that $\{ \langle z \, , \, a - A(u - \varepsilon z) \rangle \, , \, 0 < \varepsilon \leq 1 \}$ is uniformly integrable over $[0,T] \times \Omega$, with respect to the measure $dt \times P$. Hence letting $\varepsilon \to 0$ in (4.6) we get

$$E \int_0^T \langle z_t \, , \, a_t - A_t(u_t)) \rangle \, dt \leq 0 \quad \text{for any} \quad z \in \mathcal{L}_V^\infty \, .$$

Changing z into $-z$ and using that $\mathcal{L}_V^\infty$ is dense in $\mathcal{L}_V^p(\lambda)$ we deduce that

$$a_t(\omega) = A_t(u_t(\omega), \omega) \quad \text{for } dt \times P \text{almost every } (t, \omega) \in [0,T] \times \Omega.$$

Using again (4.2) with $y = u$ (i.e., $z = 0$), we deduce that $B_t(u_t(\omega), \omega) = b_t(\omega)$ for $dt \times P$ almost every (t, ω), and that $\xi = u_0$ (a.s.). Consequently, u is a solution to (1.1). □

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School of Mathematics, University of Edinburgh, King's Buildings, Edinburgh, EH9 3JZ, United Kingdom

E-mail address: gyongy@maths.ed.ac.uk

Laboratoire de Probabilités et Modèles Aléatoires (CNRS UMR 7599), Universités Paris 6-Paris 7, Boite Courrier 188, 4 place Jussieu, 75252 Paris Cedex 05, *and* SAMOS-MATISSE, Université Paris 1, 90 Rue de Tolbiac, 75634 Paris Cedex 13

E-mail address: amil@ccr.jussieu.fr