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ON DISPLAYING NEGATIVE MODALITIES

Abstract. We extend Takuro Onishi's result on displaying substructural negations by formulating display calculi for non-normal versions of impossibility and unnecessity operators, called *regular* and *co-regular negations*, respectively, by Dimiter Vakarelov. We make a number of connections between Onishi's work and Vakarelov's study of negation. We also prove a decidability result for our display calculus, which can be naturally extended to obtain decidability results for a large number of display calculi for logics with negative modal operators.

Keywords: display calculus; bi-intuitionistic logic; negative modalities; unnecessity; impossibility; decidability; distributive logic

1. Introduction

This paper was originally conceived as a study of display systems for some negative operators. After the early draft of this paper was finished I was made aware that similar results are contained in Takuro Onishi's work [24].[1] So this paper in its current form will be presented as a number of follow-up remarks and results on Onishi's work. Still, a proper introduction is in order.

Display calculi are a generalization of Gentzen's sequent calculi and were introduced by Nuel Belnap in [1]. Technically, the idea behind display calculi is to formalize the way we can combine different kinds of information in our reasoning. While in sequent calculi formulas are combined using a polyvalent comma, in display calculi one instead uses an assortment of formal structural connectives. Both the antecedent

[1] I would like to thank Prof. Heinrich Wansing for telling me about Onishi's work.

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and the succedent of a sequent are then formal terms called *structures*, which are built from formulas using structural connectives. At the core of display calculi lies the so-called *display property*, which, given a sequent and an occurrence of a structure in it, allows one to pass from it to an equivalent sequent, in which this occurrence is displayed, meaning it is the entire antecedent or the entire succedent. There are a number of benefits of using display calculi: firstly, they are well suited for studying substructural logics, secondly, they are modular, meaning it is easy to combine together several different frameworks. Moreover, there is a very general result proved in [1], which allows one to obtain cut-elimination for a display calculus quite routinely.

The operators we will be dealing with are the negative modalities of impossibility and unnecessity which are essentially negative counterparts of familiar necessity and possibility operators. Traditionally, display systems for modal logic were classical (see [36] and references therein) to the point that possibility was mostly interpreted via the well-known definition using classical negation. My intent is to study a more general framework and consider these operators over intuitionistic logic instead.

There exist in the literature a number of different approaches to formulating intuitionistic modal logics, each stemming from different properties of modal logics one might want to preserve, when passing from classical non-modal base to the intuitionistic one. Thus, for instance, Fisher Servi's logic FS (see [14, 15, 16]) aims to preserve in some sense the usual translation of modal logic into first-order logic as well as Gödel's translation, while keeping necessity and possibility dual in some weak sense; in [23] a minimal logic in which necessity and possibility are dual (albeit over positive logic) is considered; the investigation in [33] is dedicated to very abstract modalities over intuitionsitic logic satisfying only the law of replacement of equivalent formulas. A number of approaches are discussed in [32] and [39]. The approach we will be using here is the one reminiscent of Kosta Došen's and Milan Božić's take on intuitionistic modal logics [3, 5]. Došen's and Božić's goal was to separate treatments of four types of modal operators — necessity, possibility, unnecessity and impossibility — and to formulate separate systems dealing with each of these in such a way that Kripke-style semantical interpretation would remain intact. The natural reason to study four types of modal operators separately is that they are simply not interdefinable over intuitionistic logic.

As it turns out in display framework it is more natural to consider modalities not over intuitionistic logic, but instead over bi-intuitionistic logic (also called Heyting-Brouwer logic) defined by Cecylia Rauszer in [25]. Bi-intutionistic logic is obtained by adding to intuitionistic logic a new binary operator of dual implication (or subtraction), which is dual to implication in the same sense that conjunction is dual to disjunction. I was delighted to learn that Onishi in his work comes to the same conclusion and presents a display treatment of impossibility and unnecessity over bi-intuitionistic logic. Modal bi-intuitionistic logic was also considered, for instance, in [30].

Bi-intuitionistic logic itself was displayed by both Rajeev Goré in [19] and Heinrich Wansing in [37]. Notice that before Goré's work [18] display calculi for intuitionistic logic made use of the well-known Gödel-McKinsey-Tarski translation, which was much more complicated than his calculus and the intuitionistic part of display calculi for bi-intuitionistic logic mentioned above.

There is one more technical complication which comes with displaying modal operators and it has to do with the process of temporalizing these modal operators. Roughly speaking, every modal operator has to come with a backward-looking companion, where 'backward-looking' refers to the Kripke-style interpretation. The reason for this is to make sure that the above-mentioned display property is satisfied. In our case the companion to impossibility is simply a backward-looking impossibility and the companion to unnecessity is a backward-looking unnecessity. A pair of such impossibility operators is an example of a Galois connection, while a pair of unnecessity operators is an example of a dual Galois connection. These connections exemplify what Michael Dunn in [9] calls the *abstract law of residuation*, a concept which lies at the core of his theory of gaggles, which is closely connected to display calculi as outlined by Greg Restall in [28]. Notice also, that a pair of impossibilities is called split negation in [10] (see also [13]) and briefly considered in display context by Greg Restall in [28]. The idea of considering a very general impossibility as a negation has been considered, for example, by Došen [6], while unnecessity as a negation has been explored, for instance, in [34].

To summarize, in his work [24] Onishi has introduced and displayed logic BiN, which is a temporalized modal bi-intuitionistic logic with four modalities: a pair of impossibility operators and a pair of unnecessity operators. He has also studied operators obtained by merging impossibility and unnecessity together and used them to describe a *generalized kite of negations*. As far as I can tell, term *kite of negations* was coined by Dunn in [12]. Dunn's kite of negations was also generalized by what Yaroslav Shramko calls a *uniform kite of negations* [31].

Now, bi-intuitionistic temporalized modal logic might sound like it is trying a bit too much to get away from classical logic. Yet there are a number of benefits. First, this framework does cover classical modal logics — all we need to do is to add a couple of structural rules. Next, by providing a display calculus for this system we also provide a display treatment for all its fragments, including intuitionistic (modal) logics, dual intuitionistic (modal) logic, as well as *distributive logic* — a logic in a language without a conditional, which is algebraically characterized by the class of distributive lattices. This logic was used by Vakarelov in [35] to introduce two general kinds of negations called regular and co-regular negations, which could then be naturally extended to obtain normal and co-normal negations, respectively. As it turns out normal and co-normal negations are just the impossibility and unnecessity of Onishi's logic BiN. Moreover, bi-intuitionistic logic is naturally a conservative extension of distributive logic with a couple of conditionals.

This leads me to the purpose of this paper. The main goal is to build some bridges between Onishi's work and Vakarelov's framework and outline how a bunch of modal and non-modal systems can be displayed. There are two main technical results. First, as was already mentioned (co-)normal negation (under a different name) of Vakarelov was displayed by Onishi. The interesting part of his display calculus is that there are no structural rules corresponding to modal operators, which means there is nothing we can hope to subtract to obtain display the treatment of (co-)regular negation which is weaker. The question is then: is this a limitation of the display method? It turns out the answer is negative, albeit a bit boring as it says more about (co-)regular negation than it does about display calculi. We will display (co-)regular negation by treating it as a definable operator in a conservative extension of Onishi's system BiN. The second technical result involves providing a decidability result for all systems outlined in the paper.

The paper is structured as follows. In Section 2 both bi-intuitionistic logic HB and Onishi's logic BiN, which is obtained from it by adding two impossibility operators and two unnecessity operators, are introduced. The exposition here will be semantical in terms of Kripke-style frames. Display systems for both these logics are formulated in Section 3 along with some relevant results. In Section 4 Vakarelov's work on negations is outlined, starting with the distributive logic DL and then introducing regular, co-regular, normal and co-normal negations. Section 5 is dedicated to providing a display treatment for regular and co-regular negations, which boils down to a display calculus δBiRN. The paper is concluded in Section 6, where the decidability of δBiRN is proved from which decidability for a number of different display calculi can be derived.

2. Bi-intuitionistic logic and negative modalities

In this section, following Onishi's [24] we will introduce two systems: bi-intuitionistic (or Heyting-Brouwer) logic HB of Cecylia Rauszer and Onishi's system BiN, which is obtained from it by adding four negative modal operators: a pair of impossibility operators and a pair of unnecessity operators.

The non-modal logical connectives we will work with include conjunction $\wedge$, disjunction $\vee$, implication $\rightarrow$, dual implication (also called co-implication or subtraction) $\leftarrow$, as well as logical constants $\perp$ and $\top$. We will also have four modal operators: forward-looking impossibility $\rhd$, backward-looking impossibility $\lhd$ as well as forward- and backward-looking unnecessities $\blacktriangleright$ and $\blacktriangleleft$. Terms forward-looking and backward-looking simply refer to how these connectives interact on the semantical level. For a set $\mathcal{L}$ of logical connectives For $\mathcal{L}$ denotes the set of all formulas constructed from a countable set of propositional variables *Prop* using connectives in $\mathcal{L}$ the usual way. We will omit mentions of the language if it will be clear from the context.

One small difference between our approach and Onishi's is that we will be explicitly using truth and falsity constants instead of defining them through implication and dual implication, as this will be more convenient.

2.1. Bi-intuitionistic logic

Let us define semantically bi-intuitionistic logic HB in the language $\mathcal{L}^b = \{\wedge, \vee, \rightarrow, \leftarrow, \perp, \top\}$. Two different Hilbert-style axiomatizations of HB were introduced in [25] and [26], which reflects the fact that bi-intuitionistic logic can be dually characterized by both its set of theorems and its set of counter-theorems (or anti-theorems [4]).

As usual, we can define intuitionistic negation as $\neg A := A \rightarrow \perp$ and dual intuitionistic negation as $-A := \top \leftarrow A$.

By a *frame* we call a tuple $\mathcal{W} = \langle W, \leq \rangle$ such that W is a non-empty set and $\leq$ is a partial order on W. A *model* $\mu = \langle \mathcal{W}, v \rangle$ is a frame together with a valuation $v\colon \mathit{Prop} \to 2^W$ satisfying *intuitionistic heredity* condition:

$$\forall p \in \mathit{Prop}\ \forall x, y \in W\ (x \leq y \wedge x \in v(p) \Rightarrow y \in v(p)).$$

For a model $\mu = \langle W, \leq, v \rangle$ and $x \in W$ we define inductively valuation clauses for connectives the following way:

$$\begin{array}{lcl}
\mu, x \vDash \top; & \mu, x \nvDash \bot; & \\
\mu, x \vDash p & \Longleftrightarrow & x \in v(p) \text{ for } p \in \mathit{Prop}; \\
\mu, x \vDash A \wedge B & \Longleftrightarrow & \mu, x \vDash A \text{ and } \mu, x \vDash B; \\
\mu, x \vDash A \vee B & \Longleftrightarrow & \mu, x \vDash A \text{ or } \mu, x \vDash B; \\
\mu, x \vDash A \to B & \Longleftrightarrow & \forall y \geq x\, (\mu, y \vDash A \Rightarrow \mu, y \vDash B); \\
\mu, x \vDash A \leftarrow B & \Longleftrightarrow & \exists y \leq x\, (\mu, y \vDash A \text{ and } \mu, y \nvDash B).
\end{array}$$

As usual, the intuitionistic heredity condition can be extended to all formulas; that is for any model $\mu = \langle W, \leq, v \rangle$ and formula A we have

$$\forall x, y \in W\ (\mu, x \vDash A \wedge x \leq y \Rightarrow \mu, y \vDash A).$$

To keep things uniform we will identify all logics with sets of sequents of the form $A \vdash B$, where A and B are formulas. We say that $A \vdash B$ *valid in a model* $\mu = \langle W, \leq, v \rangle$ if $\mu, x \vDash A$ imply $\mu, x \vDash B$ for all $x \in W$; otherwise, we say that $A \vdash B$ *is refuted in* μ. Similarly, we say that $A \vdash B$ is *valid in a frame* $\mathcal{W}$ if it is valid in every model $\mu = \langle \mathcal{W}, v \rangle$ over $\mathcal{W}$; otherwise, it *is refuted in* $\mathcal{W}$.

Defining logics this way allows us to sidestep one ambiguity concerning dual intuitionistic logic. Because of peculiar nature of dual implication and the deduction theorem associated with it dual intuitionistic logic does not fare well with the tradition to identify a logic with its set of theorems. Namely, the consequence relation of dual intuitionistic logic cannot be recovered from its set of theorems, but can instead be recovered from its set of counter-theorems, i.e. formulas, which are refuted in every world of every frame.

That said, let us put

$$\mathsf{HB} := \{A \vdash B \mid A, B \in \mathsf{For}\,\mathcal{L}^b \text{ and } A \vdash B \text{ is valid in every frame}\}.$$

Let us outline some logics, which we can get from HB. We can define intuitionistic logic H as the $\leftarrow$-free fragment of HB (that is with the set of all sequents in HB, which do not contain symbol $\leftarrow$) and dual intuitionistic logic B as the $\rightarrow$-free fragment of HB. From intuitionistic logic we can also obtain positive logic P by considering its $\bot$-free fragment; Johansson's minimal logic J by removing the valuation clause for $\bot$ and treating it as a distinguished propositional variable; and classical logic CL by considering all $\leftarrow$-free sequents, which are valid in every frame with equality in place of partial order. We will briefly outline later how these logics can be displayed.

2.2. Negative modalities over HB

We now introduce negative modalities to HB. We put $\mathcal{L}^n := \mathcal{L}^b \cup \{\triangleright, \triangleleft, \blacktriangleright, \blacktriangleleft\}$. Individually each of $\triangleright$ and $\triangleleft$ is just a normal impossibility operator, and each of $\blacktriangleright$ and $\blacktriangleleft$ is just a normal unnecessity operator. Having some sort of companion for every modal operator is just a technical peculiarity of display calculi.

Following [9], we can demonstrate how these pairs of operators can naturally arise. Consider a set X with a binary relation $R \subseteq X^2$ on it and for a subset $A \subseteq X$ put

$$\begin{aligned} \triangleright A &= \{x \mid \forall y\,(xRy \Rightarrow y \notin A)\}, & \triangleleft A &= \{x \mid \forall y\,(yRx \Rightarrow y \notin A)\}, \\ \blacktriangleright A &= \{x \mid \exists y\,(xRy \wedge y \notin A)\}, & \blacktriangleleft A &= \{x \mid \exists y\,(yRx \wedge y \notin A)\}. \end{aligned}$$

Then there are natural relations between them, i.e., for $A, B \subseteq X$

$$A \subseteq \triangleright B \iff B \subseteq \triangleleft A, \qquad \blacktriangleright A \subseteq B \iff \blacktriangleleft B \subseteq A.$$

A pair $(\triangleright, \triangleleft)$ thus defined is an example of a Galois connection, while $(\blacktriangleright, \blacktriangleleft)$ is an example of a dual Galois connection [9]. Notice that our pair of impossibility operators is essentially a pair of split negations for which a display treatment was suggested by Restall in [29].

Again we provide the semantical characterization of logic BiN, obtained by adding these four modalities to HB. By a *normal frame* we call $\mathcal{W} = \langle W, \leq, R_\triangleright, R_\blacktriangleright \rangle$, where $\langle W, \leq \rangle$ is a frame and $R_\triangleright, R_\blacktriangleright \subseteq W^2$ are two *accessibility relations* satisfying

$$\leq \circ R_\triangleright \subseteq R_\triangleright, \quad \leq \circ R_\triangleleft \subseteq R_\triangleleft, \quad \leq^{-1} \circ R_\blacktriangleright \subseteq R_\blacktriangleright, \quad \leq^{-1} \circ R_\blacktriangleleft \subseteq R_\blacktriangleleft,$$

where $R_\triangleleft := R_\triangleright^{-1}$ and $R_\blacktriangleleft := R_\blacktriangleright^{-1}$. A *normal model* is a normal frame together with a valuation as above.

Valuation clauses for modal operators are

$$\begin{aligned}
\mu, x \models \rhd A &\iff \forall y\,(xR_{\rhd} y \Rightarrow \mu, y \not\models A),\\
\mu, x \models \lhd A &\iff \forall y\,(xR_{\lhd} y \Rightarrow \mu, y \not\models A),\\
\mu, x \models \blacktriangleright A &\iff \exists y\,(xR_{\blacktriangleright} y \wedge \mu, y \not\models A),\\
\mu, x \models \blacktriangleleft A &\iff \exists y\,(xR_{\blacktriangleleft} y \wedge \mu, y \not\models A).
\end{aligned}$$

So, to figure out whether a forward-looking modality is true at a world we consult all worlds accessible from it, while for backward-looking modalities we do essentially the same thing, but instead go backwards, hence the terminology.

As before, intuitionistic heredity can be extended to all formulas. The definition of validity of a sequent $A \vdash B$ in a normal model (frame) is exactly the same as before. Then we define

$$\mathsf{BiN} := \{A \vdash B \mid A, B \in \mathsf{For}\,\mathcal{L}^n \ \&\ A \vdash B \text{ is valid in any normal frame}\}.$$

3. Display system for BiN

In this section following Onishi we will introduce a display system $\delta\mathsf{BiN}$ for logic BiN, as well as a display system $\delta\mathsf{HB}$ for bi-intuitionistic logic. Two slightly different display treatments for bi-intuitionistic logic were suggested by Wansing [37] and Goré [19]. The difference between those two does not matter much for our purposes and, following Onishi, we settle on Goré's variant.

As was already mentioned, the main difference of display calculi in comparison with standard sequent calculi is that we will use formal terms built from formulas using so-called structural connectives in places of both antecedents and succedents of sequents. To formulate a display system for logic BiN we will use the following structural connectives: nullary I, binary connectives $\circ$ and $\bullet$ and unary $\sharp$ and $\flat$. Thus, X is a *structure* if i) X is a formula; or ii) $X = I$; or iii) $X = (Y \times Z)$ for some structures Y and Z, where $\times \in \{\circ, \bullet\}$; or iv) $X = *Y$ for some structure Y, where $* \in \{\sharp, \flat\}$.

A *sequent* then is an expression $X \vdash Y$, where X and Y are structures.

We will distinguish every occurrence of a structure in a sequent as either *a-part* (antecedent part) or *s-part* (succedent part) the following way:

1. X is an a-part and Y is an s-part of $X \vdash Y$;
2. if $(Z \circ V)$ is an a-part (s-part) of $X \vdash Y$, then so are Z and V;
3. if $(Z \bullet V)$ is an a-part or an s-part of $X \vdash Y$ then Z is an a-part and V is an s-part of $X \vdash Y$;
4. if $\sharp Z$ is an a-part (s-part) of $X \vdash Y$ then Z is an s-part (a-part);
5. if $\flat Z$ is an a-part (s-part) of $X \vdash Y$ then Z is an s-part (a-part).

To provide some intuition on structural connectives we define the *canonical translation* τ of sequents with structures into sequents without them.

DEFINITION 1. For structures X and Y put $\tau(X \vdash Y) = \tau_1(X) \vdash \tau_2(Y)$, where

$$\begin{aligned} \tau_1(A) &:= A; & \tau_2(A) &:= A; \\ \tau_1(I) &:= \top; & \tau_2(I) &:= \bot; \\ \tau_1(X \circ Y) &:= \tau_1(X) \wedge \tau_1(Y); & \tau_2(X \circ Y) &:= \tau_2(X) \vee \tau_2(Y); \\ \tau_1(X \bullet Y) &:= \tau_1(X) \leftarrow \tau_2(Y); & \tau_2(X \bullet Y) &:= \tau_1(X) \rightarrow \tau_2(Y); \\ \tau_1(\sharp X) &:= \blacktriangleright \tau_2(X); & \tau_2(\sharp X) &:= \triangleright \tau_1(X); \\ \tau_1(\flat X) &:= \blacktriangleleft \tau_2(X); & \tau_2(\flat X) &:= \triangleleft \tau_1(X). \end{aligned}$$

Thus, for example, you can think of $X \circ Y$ as corresponding to conjunction if it is an a-part and to disjunction if it is an s-part of a sequent in much the same way that comma is interpreted in standard sequent calculi.

The first ingredient of any display calculus is so-called *display equivalences*. Display equivalences for $\delta\mathsf{BiN}$ are:

$$(de)\,\frac{X \vdash (Y \bullet Z)}{\overline{(X \circ Y) \vdash Z}} \qquad (de)\,\frac{(X \bullet Y) \vdash Z}{\overline{X \vdash (Y \circ Z)}}$$

$$(de)\,\frac{(X \circ Y) \vdash Z}{\overline{Y \vdash (X \bullet Z)}} \qquad (de)\,\frac{X \vdash (Y \circ Z)}{\overline{(X \bullet Z) \vdash Y}}$$

$$(de)\,\frac{X \vdash \sharp Y}{\overline{Y \vdash \flat X}} \qquad (de)\,\frac{\sharp X \vdash Y}{\overline{\flat Y \vdash X}}$$

The double line means that the rule is invertible — we can pass from its conclusion to the premiss. We say that two sequents are *display equivalent* if we can derive one from the other and vice versa using display equivalences alone.

Notice that our display equivalences (as well as the set of structural rules) are very different than the ones originally defined by Belnap in [1]. Relations between different display equivalences were investigated by Belnap himself in [2] and later by Goré in [17].

Display equivalences allow us to prove the following crucial theorem

THEOREM 1 (Display). *If a structure Z is an s-part (a-part) in $X \vdash Y$ then there is a display equivalent to $X \vdash Y$ sequent of the form $W \vdash Z$ ($Z \vdash W$).*

What display theorem tells us is that we can display any structure occurring in a sequent by making it either the entire antecedent or the entire succedent of a display equivalent sequent. On the technical level this theorem is the reason why we had to have backward-looking modalities on top of forward-looking ones — they allow us to formulate display equivalences, which in turn allow us to prove this theorem.

The axioms of $\delta\mathsf{BiN}$ are (p is an arbitrary propositional variable):

$$(ap)\ \ p \vdash p, \qquad (a\bot)\ \ \bot \vdash I, \qquad (a\top)\ \ I \vdash \top.$$

Introduction rules are

$$(\top\vdash)\ \frac{I \vdash X}{\top \vdash X} \qquad\qquad (\vdash\bot)\ \frac{X \vdash I}{X \vdash \bot}$$

$$(\vdash\wedge)\ \frac{X \vdash A \qquad Y \vdash B}{X \circ Y \vdash A \wedge B} \qquad\qquad (\wedge\vdash)\ \frac{A \circ B \vdash X}{A \wedge B \vdash X}$$

$$(\vee\vdash)\ \frac{A \vdash X \qquad B \vdash Y}{A \vee B \vdash X \circ Y} \qquad\qquad (\vdash\vee)\ \frac{X \vdash A \circ B}{X \vdash A \vee B}$$

$$(\to\vdash)\ \frac{X \vdash A \qquad B \vdash Y}{A \to B \vdash X \bullet Y} \qquad\qquad (\vdash\to)\ \frac{X \vdash A \bullet B}{X \vdash A \to B}$$

$$(\leftarrow\vdash)\ \frac{X \vdash A \qquad B \vdash Y}{X \bullet Y \vdash A \leftarrow B} \qquad\qquad (\vdash\leftarrow)\ \frac{A \bullet B \vdash X}{A \leftarrow B \vdash X}$$

$$(\vdash\rhd)\ \frac{X \vdash \sharp A}{X \vdash \rhd A} \qquad\qquad (\rhd\vdash)\ \frac{X \vdash A}{\rhd A \vdash \sharp X}$$

$$(\vdash\lhd)\ \frac{X \vdash \flat A}{X \vdash \lhd A} \qquad\qquad (\lhd\vdash)\ \frac{X \vdash A}{\lhd A \vdash \flat X}$$

$$(\blacktriangleright\vdash)\ \frac{\sharp A \vdash X}{\blacktriangleright A \vdash X} \qquad\qquad (\vdash\blacktriangleright)\ \frac{A \vdash X}{\sharp X \vdash \blacktriangleright A}$$

$$(\blacktriangleleft\vdash)\ \frac{\flat A \vdash X}{\blacktriangleleft A \vdash X} \qquad\qquad (\vdash\blacktriangleright)\ \frac{A \vdash X}{\flat X \vdash \blacktriangleleft A}$$

Finally, the structural rules are

$$(m\vdash)\ \frac{X \vdash Y}{X \circ Z \vdash Y} \qquad\qquad (\vdash m)\ \frac{X \vdash Y}{X \vdash Y \circ Z}$$

$$(a\vdash)\ \begin{array}{c} X \circ (Y \circ Z) \vdash W \\ \hline\hline (X \circ Y) \circ Z \vdash W \end{array} \qquad\qquad (\vdash a)\ \begin{array}{c} W \vdash X \circ (Y \circ Z) \\ \hline\hline W \vdash (X \circ Y) \circ Z \end{array}$$

$$(i\vdash)\ \frac{X \circ I \vdash Y}{X \vdash Y} \qquad (\vdash i)\ \frac{X \vdash Y \circ I}{X \vdash Y}$$

$$(w\vdash)\ \frac{X \circ X \vdash Y}{X \vdash Y} \qquad (\vdash w)\ \frac{X \vdash Y \circ Y}{X \vdash Y}$$

We also have the cut rule stated in the following form:

$$(cut)\ \frac{X \vdash A \qquad A \vdash Y}{X \vdash Y}$$

We denote by $\delta\mathsf{BiN}$ the system which consists of all rules and axioms above. From it we can obtain system $\delta\mathsf{HB}$ by considering the non-modal fragment of $\delta\mathsf{HB}$ (that is by removing all modalities from the logical language, $\sharp$ and $\flat$ from the structural language and removing all rules, which explicitly mention these symbols).

In both systems we can derive the following introduction rules for intuitionistic and dual intuitionistic negations:

$$(\neg\vdash)\ \frac{X \vdash A}{\neg A \vdash X \bullet Y} \qquad (\vdash\neg)\ \frac{X \vdash A \bullet I}{X \vdash \neg A}$$

$$(-\vdash)\ \frac{I \bullet A \vdash X}{-A \vdash X} \qquad (\vdash -)\ \frac{A \vdash Y}{X \bullet Y \vdash -A}$$

Our system differs slightly from Onishi's in that we explicitly have logical constants in our language and thus have introduction rules for them. Onishi treated $\top$ as a shorthand for $(p \to p)$ and $\bot$ as a shorthand for $(p \leftarrow p)$.

We can state the following completeness result for HB and BiN

THEOREM 2 (weak completeness). [24] *For any formulas* $A, B \in \mathsf{For}\,\mathcal{L}^n$ ($\mathsf{For}\,\mathcal{L}^b$) *a sequent* $A \vdash B$ *is derivable in* $\delta\mathsf{BiN}$ ($\delta\mathsf{HB}$) *iff* $A \vdash B \in \mathsf{BiN}$ ($A \vdash B \in \mathsf{HB}$).

The stronger version of this result utilizes the canonical translation defined above.

THEOREM 3 (strong completeness). [24] *For any structures* X *and* Y *(without occurrences of* $\flat$, $\sharp$ *and modal operators) a sequent* $X \vdash Y$ *is derivable in* $\delta\mathsf{BiN}$ ($\delta\mathsf{HB}$) *iff* $\tau_1(X) \vdash \tau_2(Y) \in \mathsf{BiN}$ ($\tau_1(X) \vdash \tau_2(Y) \in \mathsf{HB}$).

The difference in two flavors of completeness is that weak completeness tells you that display calculus can express everything the logic can, while strong completeness tells you that logic can express everything the display calculus can.

Finally, the cut elimination result for display calculi is proved using eight conditions listed by Belnap in [1]. Out of these only one has to be confirmed rigorously rather than by a simple observation. We state it for future use:

C8 Suppose there are derivable sequents $X \vdash A$ and $A \vdash Y$, derived in such a way that the last instance of a rule in both is an introduction rule for A, then either $X \vdash Y$ is identical to one of $X \vdash A$ or $A \vdash Y$, or we can infer $X \vdash Y$ from premisses of these introduction rules, while only applying (*cut*) to proper subformulas of A.

Thus, we have

Theorem 4 (Cut elimination [24]). *System* $\delta\mathsf{BiN}$ ($\delta\mathsf{HB}$) *admits cut elimination, that is any sequent derivable in* $\delta\mathsf{BiN}$ ($\delta\mathsf{HB}$) *has a proof without applications of cut.*

Let us also state another of these conditions, which is essentially the subformula property:

C1 Suppose (r) is an instance of some rule in $\delta\mathsf{BiN}$ then every formula in premisses of (r) is a subformula of some formula in the conclusion of (r).

4. Vakarelov's study of negations

In this section we connect Onishi's system to Vakarelov's study of negations [35].

Vakarelov has introduced two very general kinds of negation called *regular* and *co-regular negations*, respectively. His motivation was largely a semantical one. To make his exposition as general as possible he defined those over the so-called distributive logic — a logic which can be algebraically characterized by distributive lattices.

The first part of this section is dedicated to distributive logics as well as some other natural candidates for the non-modal base we might consider, while the second part is dedicated to regular and co-regular negations themselves.

4.1. Distributive logic

Thus, let us start with distributive logic DL. It is defined as a sequent calculus $s\mathsf{DL}$ in the language $\mathcal{L}^l = \{\wedge, \vee, \top, \bot\}$ with axioms

$$A \vdash A, \qquad \bot \vdash A, \qquad A \vdash \top, \qquad (A \wedge B) \vdash A, \qquad (A \wedge B) \vdash B,$$
$$A \vdash (A \vee B), \quad ; B \vdash (A \vee B), \quad ; A \wedge (B \vee C) \vdash (A \wedge B) \vee (A \wedge C)$$

and rules

$$\frac{A \vdash B \qquad B \vdash C}{A \vdash C} \qquad \frac{C \vdash A \qquad C \vdash B}{C \vdash A \wedge B} \qquad \frac{A \vdash C \qquad B \vdash C}{A \vee B \vdash C}$$

Vakarelov further remarks that this system can be extended to a sequent calculus for intuitionistic logic $s\mathsf{H}$ formulated in the language $\mathcal{L}^i = \mathcal{L}^l \cup \{\rightarrow\}$ by adding to it the following axiom and rule:

$$A \wedge (A \rightarrow B) \vdash B \qquad \frac{A \wedge B \vdash C}{A \vdash B \rightarrow C}.$$

While this was not outlined in [35], we can also extend it with dual implication by adding the following:

$$B \vdash A \vee (B \leftarrow A) \qquad \frac{A \vdash B \vee C}{A \leftarrow B \vdash C}.$$

By adding these to $s\mathsf{DL}$ we obtain a system $s\mathsf{B}$ for dual intuitionistic logic in the language $\mathcal{L}^d = \mathcal{L}^l \cup \{\leftarrow\}$ and by adding those to $s\mathsf{H}$ we obtain a system $s\mathsf{HB}$ for bi-intuitionistic logic.

The semantical interpretation of all connectives above is exactly the same as above, so we can define:

$$\begin{aligned} \mathsf{DL} &:= \{A \vdash B \mid A, B \in \mathsf{For}\,\mathcal{L}^l \text{ and } A \vdash B \text{ is valid in every frame}\},\\ \mathsf{H} &:= \{A \vdash B \mid A, B \in \mathsf{For}\,\mathcal{L}^i \text{ and } A \vdash B \text{ is valid in every frame}\},\\ \mathsf{B} &:= \{A \vdash B \mid A, B \in \mathsf{For}\,\mathcal{L}^d \text{ and } A \vdash B \text{ is valid in every frame}\}. \end{aligned}$$

Then from the results in [24] and [35], as well as the subformula property of display calculi we can obtain the following four-way completeness result:

Theorem 5. *For any formulas $A, B \in \mathsf{For}\,\mathcal{L}^l$ ($\mathsf{For}\,\mathcal{L}^i$, $\mathsf{For}\,\mathcal{L}^d$, $\mathsf{For}\,\mathcal{L}^b$) the following are equivalent:*

1. *$A \vdash B$ is derivable in $s\mathsf{DL}$ ($s\mathsf{H}$, $s\mathsf{B}$, $s\mathsf{HB}$);*
2. *$A \vdash B$ lies in DL (H, B, HB);*
3. *$A \vdash B$ is valid in every frame;*
4. *$A \vdash B$ is derivable in $\delta\mathsf{HB}$.*

The only new part of this theorem involves proving that our axiom and rule adequately capture the dual implication of HB. This is done dually to how implication is considered in [35] by making use of what Vakarelov calls the co-extension lemma.

The last theorem suggests a natural way to define display calculi for DL, H and B by considering corresponding fragments of HB ($\leftarrow$-free in case of H and etc.). Notice, though, that defining them this way only gives us the weaker version of the completeness result (as stated in Theorem 3). To get the stronger version of completeness we would have to modify our systems in some way. For instance, a display calculus for H with a stronger version of completeness (albeit with different structural connectives and hence different display equivalences) was suggested by Goré in [18].

As a side note let us mention a couple other ways we could modify δHB to get some familiar logics over which modal operators were studied in the literature. A number of different ways to extend δHB to obtain a display calculus for classical logic are outlined in [19]. In [11] modal operators (albeit positive) were studied over the positive logic P, which is simply a $\bot$-free fragment of intuitionistic logic. Accordingly we can obtain a display calculus δP for it by considering the $\bot$-free fragment of δH (that is by removing introduction rules for $\bot$ and considering formulas not containing the symbol). Again, this only gives us the weaker version of the completeness result. In [8] it was shown that KC is the smallest superintuitionistic logic over which a number of natural modal operators obtained by composition of intuitionistic negation and basic operators of necessity, possibility, unnecessity and impossibility behave naturally in some sense. It is well known that KC can be axiomatized modulo H by either the weak law of excluded middle $\neg A \vee \neg\neg A$ or by one of De Morgan's laws $\neg(A \wedge B) \to \neg A \vee \neg B$. A display calculus for KC can be obtained by first extending δHB with the following structural rule:

$$\frac{X \vdash Y \bullet I}{I \bullet (X \bullet I) \vdash Y \bullet I}$$

and then taking the $\leftarrow$-free fragment of the resulting system.

Here is how the weak law of excluded middle can be derived using it:

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{A \vdash A}{\neg A \vdash A \bullet I}}{I \bullet (\neg A \bullet I) \vdash A \bullet I}}{I \bullet (\neg A \bullet I) \vdash \neg A}}{I \bullet \neg A \vdash \neg A \bullet I}}{I \vdash \neg A \circ \neg\neg A}}{I \vdash \neg A \vee \neg\neg A}$$

The full completeness result can be obtained quite routinely given the semantical interpretation of KC; we leave the details to a interested reader.

4.2. Regular and co-regular negations

Finally we turn to the regular and co-regular negations of Vakarelov.

Regular negation $\rhd_r$ is axiomatized modulo sDL by the following axiom and rule

$$\rhd_r A \wedge \rhd_r B \vdash \rhd_r(A \vee B) \qquad\qquad \frac{A \vdash B}{\rhd_r B \vdash \rhd_r A}.$$

While co-regular negation $\blacktriangleright_r$ is axiomatized by

$$\blacktriangleright_r(A \wedge B) \vdash \blacktriangleright_r A \vee \blacktriangleright_r B \qquad\qquad \frac{A \vdash B}{\blacktriangleright_r B \vdash \blacktriangleright_r A}.$$

Let us denote by sBiR a sequent calculus in the language $\mathcal{L}^r = \mathcal{L}^b \cup \{\rhd_r, \blacktriangleright_r\}$ obtained by adding to sHB axioms and rules for regular and co-regular negation.

From the results of [35] we can infer the following semantical interpretation. By a *regular frame* we call a tuple $\mathcal{W} = \langle W, \leq, R_\rhd, R_\blacktriangleright, N, N'\rangle$, where $\langle W, \leq, R_\rhd, R_\blacktriangleright\rangle$ is a normal frame and $N, N' \subseteq W$ are cones in $\langle W, \leq\rangle$, that is $x \in N$ and $x \leq y$ implies $y \in N$ and similarly for N'. Elements of N are called *normal worlds* and elements of N' are called *co-normal worlds*. A *regular model* is a regular frame together with a valuation as before. Valuations clauses for regular and co-regular negations are:

$$\begin{aligned} \mu, x \models \rhd_r A &\iff x \in N \text{ and } \forall y\,(xR_\rhd y \Rightarrow \mu, y \not\models A); \\ \mu, x \models \blacktriangleright_r A &\iff x \in N' \text{ or } \exists y\,(xR_\blacktriangleright y \wedge \mu, y \not\models A). \end{aligned}$$

The notions of the validity of formulas and sequents are defined as above.

Denote

$$\begin{aligned} \mathsf{BiR} := \{\, & A \vdash B \mid A, B \in \mathsf{For}\,\mathcal{L}^r \text{ and} \\ & A \vdash B \text{ is valid in every regular frame}\,\}. \end{aligned}$$

Then, by modifying slightly the result of [35] we can obtain

Theorem 6. *For any sequent S in $\mathcal{L}^r$: $S \in$ BiR iff S is derivable in sBiR.*

Vakarelov has further distinguished two other classes of negations: *normal* and *co-normal* negations are, respectively, regular and co-regular negations satisfying:

$$\top \vdash \rhd_r \bot \qquad\qquad \blacktriangleright_r \top \vdash \bot.$$

We call these *normality axioms*. Notice that normal and co-normal negations are just the impossibility and unnecessity operators of system BiN.

This in turn reveals one interesting feature of δBiN: recall that we did not have any structural rules governing our structural connectives $\sharp$ and $\flat$, yet as a result we obtained a system in which normality axioms can be derived for free. Does it mean then it is impossible to display regular and co-regular negations, since there is nothing we can hope to remove from the display calculus for normal and co-normal negations to get there? It turns out it is possible, but we need to add things instead of removing them.

We close this section by outlining how one can get a Hilbert-style axiomatization of BiN:

1. a Hilbert style axiomatization of HB from [25];
2. rules and axioms for $\rhd_r$ and $\blacktriangleright_r$ from the beginning of this section plus similar rules and axioms for backward-looking modalities (obviously, we need to rename modalities to $\rhd$ and $\blacktriangleright$, respectively);
3. normality axioms for both forward-looking and backward-looking modalities;
4. axioms $A \to \rhd\lhd A$, $A \to \lhd\rhd A$, $\blacktriangleright\blacktriangleleft A \to A$ and $\blacktriangleleft\blacktriangleright A \to A$ — these formulas express connections between backward- and forward-looking modalities.

5. Displaying regular and co-regular negations

In this section we will formulate a display calculus for regular and co-regular negations.

Our solution is slightly roundabout in that we will display regular versions of these negations through theirs normal versions. This is reminiscent of early display systems for intuitionistic logic which were obtained by displaying classical modal logic $S4$ and introducing intuitionistic implication by means of the well-known Gödel-McKinsey-Tarksi translation.

Thus, we will first introduce and display an extension of BiR denoted by BiRN and then derive a display calculus for BiR from it. Logic BiRN is formulated in a language containing excessive amount of connectives, that is we put $\mathcal{L}^f = \mathcal{L}^n \cup \{\rhd_r, \lhd_r, \blacktriangleright_r, \blacktriangleleft_r, n, n'\}$. Connectives $\lhd_r$ and $\blacktriangleleft_r$ are just backward-looking variants of $\rhd_r$ and $\blacktriangleright_r$, respectively, while n and n' are new constants, which we will use to model normal and co-normal worlds, respectively.

5.1. Logic BiRN

We define BiRN semantically by introducing new valuation clauses on regular models. Thus, for a regular model $\mu = \langle W, \leq, R_{\triangleright}, R_{\blacktriangleright}, N, N' \rangle$ put:

$$\begin{aligned}
\mu, x \vDash n &\iff x \in N, \\
\mu, x \vDash n' &\iff x \in N', \\
\mu, x \vDash \triangleleft_r A &\iff x \in N \text{ and } \forall y\,(xR_{\triangleleft}y \Rightarrow \mu, y \nvDash A), \\
\mu, x \vDash \blacktriangleleft_r A &\iff x \in N' \text{ or } \exists y\,(xR_{\blacktriangleleft}y \wedge \mu, y \nvDash A).
\end{aligned}$$

Then put

$$\begin{aligned}
\mathsf{BiRN} := \{\, A \vdash B \mid{} & A, B \in \mathsf{For}\,\mathcal{L}^f \text{ and} \\
& A \vdash B \text{ is valid in every regular frame}\,\}.
\end{aligned}$$

Notice that in this system formulas $\triangleright_r A$ and $\triangleright A \wedge n$ are semantically equivalent and similarly for $\blacktriangleright_r A$ and $\blacktriangleright A \vee n'$ and backward-looking modalities.

5.2. Displaying BiRN

We will now define a display calculus δBiRN for BiRN. First we add two new nullary structural connectives η and η' to our structural language. These will correspond to n and n', respectively. Since they are nullary we do not need to add any further display equivalences for them.

Then δBiRN is obtained by adding the following axioms and rules to δBiN:

$$\eta \vdash n \qquad \frac{\eta \vdash X}{n \vdash X} \qquad \frac{X \vdash \eta'}{X \vdash n'} \qquad n' \vdash \eta'$$

Introduction rules for $\triangleright_r$ can be obtained by keeping in mind the semantical equivalence between $\triangleright_r A$ and $\triangleright A \wedge n$; thus we have

$$(\triangleright_r \vdash 1)\ \frac{\eta \vdash X}{\triangleright_r A \vdash X} \qquad (\triangleright_r \vdash 2)\ \frac{X \vdash A}{\triangleright_r A \vdash \sharp X} \qquad (\vdash \triangleright_r)\ \frac{X \vdash \sharp A}{\eta \circ X \vdash \triangleright_r A}$$

Rules for $\triangleleft_r$ are obtained from these by replacing $\triangleright_r$ with $\triangleleft_r$ and $\sharp$ with $\flat$.

For $\blacktriangleright_r$ we think of $\blacktriangleright_r A$ as $\blacktriangleright A \vee n'$ and derive

$$(\blacktriangleright_r \vdash)\ \frac{\sharp A \vdash X}{\blacktriangleright_r A \vdash X \circ \eta'} \qquad (\vdash \blacktriangleright_r 1)\ \frac{X \vdash \eta'}{X \vdash \blacktriangleright_r A} \qquad (\vdash \blacktriangleright_r 2)\ \frac{A \vdash X}{\sharp X \vdash \blacktriangleright_r A}$$

Again, rules for $\blacktriangleleft_r$ are obtained similarly by replacing $\blacktriangleright_r$ with $\blacktriangleleft_r$ and $\sharp$ with $\flat$.

We do not add any additional structural rules.

Notice that even though n and n' look virtually the same from the semantical perspective, we have chosen different rules for them in our display calculus: n has $\top$-like rules, while n' has $\bot$-like rules. To demonstrate why, we can show that $\rhd_r A$ and $\rhd A \wedge n$ are equivalent on the syntactic level and similarly for $\blacktriangleright_r A$ and $\blacktriangleright A \vee n'$:

$$
\dfrac{\dfrac{\dfrac{\dfrac{A \vdash A}{\rhd_r A \vdash \sharp A}}{\rhd_r A \vdash \rhd A} \qquad \dfrac{\eta \vdash n}{\rhd_r A \vdash n}}{\rhd_r A \circ \rhd_r A \vdash \rhd A \wedge n}}{\rhd_r A \vdash \rhd A \wedge n}
\qquad\qquad
\dfrac{\dfrac{\dfrac{\dfrac{A \vdash A}{\rhd A \vdash \sharp A}}{\eta \circ \rhd A \vdash \rhd_r A}}{n \circ \rhd A \vdash \rhd_r A}}{n \wedge \rhd A \vdash \rhd_r A}
$$

$$
\dfrac{\dfrac{\dfrac{\dfrac{A \vdash A}{\sharp A \vdash \blacktriangleright A}}{\blacktriangleright_r A \vdash \blacktriangleright A \circ \eta'}}{\blacktriangleright_r A \vdash \blacktriangleright A \circ n'}}{\blacktriangleright_r A \vdash \blacktriangleright A \vee n'}
\qquad\qquad
\dfrac{\dfrac{\dfrac{\dfrac{A \vdash A}{\sharp A \vdash \blacktriangleright_r A}}{\blacktriangleright A \vdash \blacktriangleright_r A} \qquad \dfrac{n' \vdash \eta'}{n' \vdash \blacktriangleright_n A}}{\blacktriangleright A \vee n' \vdash \blacktriangleright_r A \circ \blacktriangleright_r A}}{\blacktriangleright A \vee n' \vdash \blacktriangleright_r A}
$$

Similar equivalences can be proved for $\lhd_r$ and $\blacktriangleleft_r$.

Let us show that δBiRN admits cut-elimination. We only need to prove that **C8** (cf. Section 4) is satisfied for it and we only consider the case of $\rhd_r A$. There are two distinct cases depending on which left introduction rule for $\rhd_r A$ was used in the derivation

$$
\dfrac{\dfrac{X \vdash \sharp A}{\eta \circ X \vdash \rhd_r A} \qquad \dfrac{\eta \vdash Y}{\rhd_r A \vdash Y}}{\eta \circ X \vdash Y}
\quad\leadsto\quad
\dfrac{\eta \vdash Y}{\eta \circ X \vdash Y}
$$

$$
\dfrac{\dfrac{X \vdash \sharp A}{\eta \circ X \vdash \rhd_r A} \qquad \dfrac{Y \vdash A}{\rhd_r A \vdash \sharp Y}}{\eta \circ X \vdash \sharp Y}
\quad\leadsto\quad
\dfrac{\dfrac{\dfrac{Y \vdash A \qquad \dfrac{X \vdash \sharp A}{A \vdash \flat X}}{Y \vdash \flat X}}{X \vdash \sharp Y}}{\eta \circ X \vdash \sharp Y}
$$

Thus we have

Theorem 7. *Calculus* δBiRN *admits cut elimination.*

Next, we want to show that BiRN is sound and complete with respect to δBiRN. We follow Onishi's proof of completeness for BiN.

Let us expand the canonical translation (see Definition 1) to accommodate new structural connectives by putting $\tau_1(\eta) = \tau_2(\eta) = n$ and $\tau_1(\eta') = \tau_2(\eta') = n'$.

For two sets of formulas $\Gamma, \Delta \subseteq \mathsf{For}\,\mathcal{L}^f$ put $\Gamma \vdash \Delta$ if a sequent

$$(\ldots(A_1 \circ \cdots) \circ A_n) \vdash (\ldots(B_1 \circ \cdots) \circ B_m)$$

is derivable in $\delta\mathsf{BiRN}$ for some $A_1, \ldots, A_n \in \Gamma$, $B_1, \ldots, B_m \in \Delta$. By a *maximal consistent pair* we will call a pair (Γ, Δ) of sets of formulas such that $\Gamma \nvdash \Delta$ and $\Gamma \cup \Delta = \mathsf{For}\,\mathcal{L}^f$.

We have:

Lemma 1 (extension [24]). *If $\Gamma \nvdash \Delta$ for some $\Gamma, \Delta \subseteq \mathsf{For}\,\mathcal{L}^f$ then there is a maximal consistent pair (Γ', Δ') such that $\Gamma \subseteq \Gamma'$ and $\Delta \subseteq \Delta'$.*

We can define the *canonical model* $\mu^c = \langle W^c, \leq^c, R^c_{\triangleright}, R^c_{\blacktriangleright}, N^c, N'^c, v^c\rangle$, where

1. $W^c = \{\Gamma \mid (\Gamma, \mathsf{For}\,\mathcal{L}^f \setminus \Gamma)$ is a maximal consistent pair$\}$;
2. $\Gamma \leq^c \Delta$ iff $\Gamma \subseteq \Delta$;
3. $\Gamma R^c_{\triangleright} \Delta$ iff $\forall A\,(\triangleright A \in \Gamma \Rightarrow A \notin \Delta)$;
4. $\Gamma R^c_{\blacktriangleright} \Delta$ iff $\exists A\,(\blacktriangleright A \in \Gamma$ and $A \notin \Delta)$;
5. $\Gamma \in N^c$ iff $n \in \Gamma$;
6. $\Gamma \in N'^c$ iff $n' \in \Gamma$;
7. $\Gamma \in v^c(p)$ iff $p \in \Gamma$ for propositional variable p.

It is easy to see that the canonical model is a regular model.

Lemma 2 (canonical). *For any $\Gamma \in W^c$ and formula A we have*

$$\mu^c, \Gamma \vDash A \iff A \in \Gamma.$$

Proof. By induction on complexity of A. For connectives in $\mathcal{L}^n$ see [24]. For $A = n$ or $A = n'$ the result is obtained immediately by definition.

Let us consider the case of $\triangleright_r$. We have the following equivalences

$$\mu^c, \Gamma \vDash \triangleright_r A \iff \mu^c, \Gamma \vdash \triangleright A \wedge n \iff \triangleright A \wedge n \in \Gamma \iff \triangleright_r A \in \Gamma.$$

The first one is obvious, the second one follows from Onishi's proof and the last one is easily obtained using the definition of a maximal consistent pair and the fact that sequents $\triangleright_r A \vdash \triangleright A \wedge n$ and $\triangleright A \wedge n \vdash \triangleright_r A$ are derivable in $\delta\mathsf{BiRN}$ (see above).

The remaining cases are considered similarly. $\square$

Lemma 3. 1. *For any structure X sequents $X \vdash \tau_1(X)$ and $\tau_2(X) \vdash X$ are derivable in* δBiRN.

2. *If $\tau_1(X) \vdash \tau_2(Y)$ is derivable in* δBiRN *then so is $X \vdash Y$.*

Proof. The first item is proved by induction on the complexity of X, while the second one follows from the first by a couple application of cuts. $\square$

Theorem 8 (completeness). *A sequent $X \vdash Y$ is derivable in* δBiRN *iff* $\tau_1(X) \vdash \tau_2(Y) \in$ BiRN.

Proof. $\Longrightarrow$. By induction on the complexity of derivation of $X \vdash Y$.

$\Longleftarrow$. Assume that $X \vdash Y$ is not derivable in δBiRN. Then neither is $\tau_1(X) \vdash \tau_2(Y)$ by the previous lemma. By the extension lemma there is a maximal consistent pair (Γ, Δ) such that $\tau_1(X) \in \Gamma$ and $\tau_2(Y) \notin \Gamma$. Then by the completeness lemma $\mu^c, \Gamma \models \tau_1(X)$ and $\mu^c, \Gamma \not\models \tau_2(Y)$, hence the sequent $\tau_1(X) \vdash \tau_2(Y)$ is refuted on the canonical model. Then $\tau_1(X) \vdash \tau_2(Y) \notin$ BiRN by definition. $\square$

Notice that this is the stronger version of the completeness result, from which the weaker version automatically follows. Now we know exactly how to display regular and co-regular negations by considering an appropriate fragment of δBiRN, as can be inferred from the following:

Theorem 9. *For $A, B \in$ For $\mathcal{L}^r$ the following are equivalent*:

1. *$A \vdash B$ is valid in every regular frame*;
2. *$A \vdash B$ is derivable in* sBiR;
3. *$A \vdash B$ is derivable in* δBiRN.

To summarize, we can display regular and co-regular negations by taking an appropriate fragment of δBiRN. As before, we will only get the weaker version of completeness result by doing so.

One last remark here. Looking at δBiRN it is easy to see, how a display calculus for Johansson's minimal logic J could be obtained. It would be enough to take a display calculus for positive logic (as outlined above) and add to it a nullary connective that has the same introduction rules as n' and no additional structural rules.

5.3. Regaining normality

Here we show how normal and co-normal negations can be regained from δBiRN in a natural way. Consider a system obtained from δBiRN by adding the following structural rules.

$$(norm \vdash)\ \frac{\eta \circ X \vdash Y}{X \vdash Y} \qquad\qquad (\vdash norm)\ \frac{X \vdash Y \circ \eta'}{X \vdash Y}$$

In the resulting system we can then prove both normality and co-normality axioms:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\bot \vdash I}{\bot \vdash \flat I}}{I \vdash \sharp \bot}}{\eta \circ I \vdash \rhd_r \bot}}{I \vdash \rhd_r \bot}}{\top \vdash \rhd_r \bot} \qquad\qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{I \vdash \top}{\flat I \vdash \top}}{\sharp \top \vdash I}}{\blacktriangleright_r \top \vdash I \circ \eta'}}{\blacktriangleright_r \top \vdash I}}{\blacktriangleright_r \top \vdash \bot}$$

Moreover we can prove:

$$\dfrac{\dfrac{A \vdash A}{\rhd_r A \vdash \sharp A}}{\rhd_r A \vdash \rhd A} \qquad\qquad \dfrac{\dfrac{\dfrac{A \vdash A}{\rhd A \vdash \sharp A}}{\eta \circ \rhd A \vdash \rhd_r A}}{\rhd A \vdash \rhd_r A}$$

$$\dfrac{\dfrac{\dfrac{A \vdash A}{\sharp A \vdash \blacktriangleright A}}{\blacktriangleright_r A \vdash \blacktriangleright A \circ \eta'}}{\blacktriangleright_r A \vdash \blacktriangleright A} \qquad\qquad \dfrac{\dfrac{A \vdash A}{\sharp A \vdash \blacktriangleright_r A}}{\blacktriangleright A \vdash \blacktriangleright_r A}$$

We omit the details of proving that rules $(norm \vdash)$ and $(\vdash norm)$ correspond exactly to normality and co-normality axioms, respectively.

6. Decidability

In this section we adapt Restall's method [29] to develop a general decidability recipe for δBiRN and all other systems we have mentioned.

A careful observation shows that there are only four problematic rules in our system: $(i \vdash)$, $(\vdash i)$, $(w \vdash)$ and $(\vdash w)$. They are problematic insofar as their premisses contain more information than conclusions. Basically, the idea behind the proof is to make sure that these rules are not applied unless absolutely necessary. But to get there we would need a bunch of preliminary results.

The first thing to do is to modify the system slightly. Thus, let us replace rules $(w \vdash)$, $(\vdash w)$, $(\vdash \wedge)$, $(\vee \vdash)$, $(\rightarrow\vdash)$ and $(\vdash\leftarrow)$ with the following new versions:

$$(w \vdash)' \ \frac{X \circ X \vdash Y}{X \vdash Y}, \ X \neq U \circ V$$

$$(\vdash w)' \ \frac{X \vdash Y \circ Y}{X \vdash Y}, \ Y \neq U \circ V$$

$$(\vdash \wedge)' \ \frac{X \vdash A \qquad X \vdash B}{X \vdash A \wedge B} \qquad (\vee \vdash)' \ \frac{A \vdash X \qquad B \vdash X}{A \vee B \vdash X}$$

$$(\rightarrow\vdash)' \ \frac{X \vdash A \qquad X \circ B \vdash Y}{X \circ (A \rightarrow B) \vdash Y} \qquad (\vdash\leftarrow)' \ \frac{X \vdash A \circ Y \qquad B \vdash Y}{X \vdash (A \leftarrow B) \circ Y}$$

We denote by $\delta\mathsf{BiRN}'$ the system obtained by replacing the old rules with these new ones.

Lemma 4. *A sequent S is derivable in $\delta\mathsf{BiRN}$ iff it is derivable in $\delta\mathsf{BiRN}'$.*

Proof. The only non-trivial part is to show that one can derive $(\rightarrow\vdash)$ using $(\rightarrow\vdash)'$ (and similarly for $(\vdash\leftarrow)$). Notice that every structure can be presented in the form $Y = Y_1 \bullet (Y_2 \bullet \cdots \bullet (Y_n \bullet Z) \dots)$ with $n \geq 0$, where Z is not of the form $Z_1 \bullet Z_2$. Then we can get the following derivation scheme:

$$\dfrac{\dfrac{X \vdash (Y_2 \bullet (\cdots \bullet (Y_n \bullet Z) \dots)}{}}{}$$

$$(\rightarrow\vdash')\ \cfrac{\cfrac{\cfrac{X \vdash A}{X \circ (Y_1 \circ \cdots \circ Y_n) \vdash A} \qquad \cfrac{\cfrac{\cfrac{B \vdash Y}{B \vdash Y_1 \bullet (Y_2 \bullet (\cdots \bullet (Y_n \bullet Z) \dots)}}{(Y_1 \circ \cdots \circ Y_n) \circ B \vdash Z}}{(X \circ (Y_1 \circ \cdots \circ Y_n)) \circ B \vdash Z}}{(X \circ (Y_1 \circ \cdots \circ Y_n)) \circ (A \rightarrow B) \vdash Z}}{\cfrac{X \circ (A \rightarrow B) \vdash (Y_2 \bullet (\cdots \bullet (Y_n \bullet Z) \dots)}{X \circ (A \rightarrow B) \vdash Y}}$$

□

So technically we will be proving the decidability of $\delta\mathsf{BiRN}'$. In what follows by *derivability/derivable sequents* we will mean derivability/derivable sequents in $\delta\mathsf{BiRN}'$.

Now onto the proof itself. We say that sequents S_1 and S_2 are *equivalent* and write $S \sim S'$, if one can be derived from the other using only display equivalences, $(a \vdash)$ and $(\vdash a)$. We will refer to these rules as $\sim$-rules. Notice that since all $\sim$-rules are reversible, we can infer that equivalent sequents are interderivable.

We say that structures X and X' are *a-similar* (*s-similar*) if for any structure V sequents $X \vdash V$ and $X' \vdash V$ ($V \vdash X$ and $V \vdash X'$) are equivalent. For example, structures $A \bullet (B \bullet C)$ and $(A \circ B) \bullet C$ are s-similar, but not a-similar.

Definition 2. We say that a structure occurrence X in sequent S is *congruent* to a structure occurrence X' in sequent S' if there is a sequence of pairs $(S_1, X_1), \ldots, (S_n, X_n)$ such that $X_1 = X$, $S_1 = S$, $S_n = S'$, $X_n = X'$ and for each $i \leq n$ X_i is a structure occurrence in S_i such that one of the following conditions holds:

(a) S_{i+1} is obtained from S_i by an application of a $\sim$-rule and $X = X'$ occupy the same position in structure occurrences assigned to some structure-variable in the definition of this rule;
(b) $S_i = X_i \vdash V$, $S_{i+1} = X_{i+1} \vdash V$ and X_i, X_{i+1} are a-similar;
(c) $S_i = V \vdash X_i$, $S_{i+1} = V \vdash X_{i+1}$ and X_i, X_{i+1} are s-similar.

Let us illustrate (a). Consider an instance of a $\sim$-rule:

$$\frac{X \circ Y \vdash Z}{X \vdash Y \bullet Z}$$

For any substructure V of X in the premiss of this rule there is a corresponding V in the succedent, and these two V's will be congruent by item (a) above. Notice that in most cases there is no structure occurrence in $X \vdash Y \bullet Z$, which is congruent to $(X \circ Y)$ in $X \circ Y \vdash Z$ (in theory some substructure of Z could be congruent to $(X \circ Y)$).

In what follows we will often use the term structure to denote both structures and structure occurrences as those will hopefully be easy to tell apart.

Remark 1. If X in S is congruent to X' in S' then S is equivalent to S'.

We say that an occurrence of I in S is *a-superfluous* (*s-superfluous*) if it is congruent to the indicated I in some sequent $X \circ I \vdash Y$ ($X \vdash Y \circ I$). We say that an occurrence of structure Z, such that $Z \neq I$ and $Z \neq V \circ W$, is *a-superfluous* (*s-superfluous*) in S if it is congruent to one of the indicated Z in $Z \circ Z \vdash X$ ($X \vdash Z \circ Z$). By a *superfluous* structure we will sometimes call one which is either a-superfluous or s-superfluous.

Superfluous structures correspond to structures that can be eliminated by an application of one of the 'problematic' rules. Observe that restrictions for $(w \vdash)'$ and $(\vdash w)'$ justify our restriction on the form of superfluous structures.

Remark 2. An a-(s-)superfluous structure is always an a-(s-)part of a sequent.

Definition 3. Let us define the *result of deleting a structure occurrence* X *from a sequent* S, which we denote by $S \setminus X$ (in all cases X is the indicated occurrence): if S is one of $X \vdash Y \circ Z$, $X \vdash Y \bullet Z$, $Y \circ Z \vdash X$ or $Y \bullet Z \vdash X$ then $S \setminus X$ is $Y \vdash Z$; if X is a substructure of $(Y * Z)$ in S, where $* \in \{\circ, \bullet\}$ and $X = Y$ then $S \setminus X$ is obtained from S by replacing the indicated $(X \circ Y)$ or $(X \bullet Y)$ with Y.

Notice that the cases outlined above are not exhaustive, meaning that the operation of deleting a structure from a sequent only makes sense in some situations. Where it does make sense we say that X *can be deleted* from S.

Lemma 5. *Suppose X is superfluous in S. Then X can be deleted from S. Moreover,*

($*$) *if S is $X \vdash Y$ ($Y \vdash X$) and $X \neq I$ then there is a proper substructure X' of Y, which is congruent to X (in S).*

Proof. We only consider the case of a-superfluous structure $X \neq I$. By definition there are pairs $F_1, \ldots, F_n$ with $F_i = (X_i, S_i)$, such that $F_1 = (X, S)$, $F_n = (X, X \circ X \vdash V)$ for some V and for each j one of the conditions (a)–(c) above is satisfied.

For the main part we just have to consider all situations not present in Definition 3 to show that X cannot be a-superfluous in these situations. We only outline one such case. Suppose X is a-superfluous in $X \vdash \sharp Y$. It is easy to see then that for each i the following holds: either X_i is the entire antecedent of S_i or X_i is a substructure of $\flat X_i$ in S_i. Yet this property does not hold for F_n, thus X could not be a-superfluous in $X \vdash \sharp Y$.

To prove ($*$) simply observe that it is trivially satisfied for F_n and that if it is satisfied for some F_{i+1} then it is also satisfied for F_i. □

Our main goal now is to show that if a structure is superfluous in some sequent then the result of deleting it is interderivable with the original sequent.

Lemma 6. *Suppose X is a-(s-)superfluous in S and it is congruent to X' in S'. Then X' is a-(s-)superfluous in S' and sequents $S \setminus X$ and $S' \setminus X'$ are equivalent.*

Proof. We consider the case when X is a-superfluous in S. We use simple induction argument following the definition of congruent structures. By definition we have pairs $F_1, \ldots, F_n$ such that $F_i = (X_i, S_i)$,

$F_1 = (X, S)$, $F_n = (X', S')$ and for each i we have one of conditions (a)–(c) above. Since every X_i in S_i is congruent to X_j in S_j and X in S is congruent to X in $X \circ X \vdash V$ then all X_j are obviously a-superfluous. Hence every X_i can be deleted from S_i by Lemma 5. Now let us show that $S_i \setminus X_i$ is equivalent to $S_{i+1} \setminus X_{i+1}$ for every i. Notice that F_{i+1} could not be obtained from F_i by (c). If F_{i+1} is obtained from F_i by (b) then $S_i \setminus X_i = S_{i+1} \setminus X_{i+1}$. If F_{i+1} is obtained from F_i by (a) then $S_{i+1} \setminus X_{i+1}$ can be obtained from $S_i \setminus X_i$ by an application of the same rule which leads from S_i to S_{i+1}. □

Lemma 7. *Suppose X is superfluous in S then $S \setminus X$ and S are inter-derivable. Moreover, there is a derivation of $S \setminus X$ from S, which contains only one instance of one of the rules $(i \vdash)$, $(\vdash i)$, $(w \vdash)$, $(\vdash w)$; and a derivation of S from $S \setminus X$ which does not contain any instances of these rules.*

Proof. Assume that X is a-superfluous in S. Then we have the following derivation schemes:

$$(1)\,\dfrac{S}{\dfrac{X \circ X \vdash Y}{(2)\;\dfrac{X \vdash Y}{(3)\;S \setminus X}}} \qquad\qquad (1)\,\dfrac{S}{\dfrac{Z \circ I \vdash Y}{(2)\;\dfrac{Z \vdash Y}{(3)\;S \setminus I}}}$$

Here, (1) is a number of instances of $\sim$-rules (see Remark 1); (2) is an instance of $(w \vdash)$ or $(i \vdash)$ going from top to bottom and an instance of $(m \vdash)$ the other way around; (3) again consists of $\sim$-rules and is obtained by Lemma 6. □

We need a number of new notions. For a sequent S by $r(S)$ we denote the result of deleting every superfluous structure in S starting with leftmost one and proceeding to the right. We say that a sequent S is *reduced* if it does not have superfluous structures. We say that S is *semi-reduced* if it either reduced or it can be made reduced by deleting up to two superfluous structures. Finally, we say that a proof Π is *reduced* if i) it does no contain loops (i.e., there no sequent S which appears twice on some branch) and ii) every sequent in Π is semi-reduced.

Notice that as a corollary of Lemma 7 we immediately obtain

Remark 3. A sequent S is derivable iff $r(S)$ is derivable.

Finally, we get to the main lemma of our proof.

Lemma 8. *Every derivable reduced sequent has a reduced proof.*

Proof. The fact that we can eliminate loops from the proof is obvious, so assume that we have a reduced sequent S with a proof Π, which does not contain loops. Next, we replace every sequent S' in Π with $r(S')$. Since all axioms of $\delta\mathsf{BiRN}'$ are reduced and so is S, it is enough to show that given any instance of a rule $S_1, \dots, S_n/S$ in Π we can derive $r(S)$ from $r(S_1), \dots, r(S_n)$ using only semi-reduced sequents.

Let us first describe the general scheme we will follow in all cases. Take an instance of a rule (r)

$$\frac{S_1 \qquad \dots \qquad S_n}{S_0}$$

Step 1. Start simultaneously deleting superfluous structures from $S_0, \dots, S_n$ until we obtain sequents $S'_1, \dots, S'_n$ such that $r(S_i) = r(S'_i)$ for all i and either $S'_i = S'_0$ for some $1 \le i \le n$ or deleting any further superfluous structure Z results in an expression, which is no longer an instance of (r).

Step 2. Fill the gaps in the following derivation using only semi-reduced sequents.

$$\dfrac{\dfrac{\dfrac{\dfrac{r(S_1)}{\vdots}}{S'_1} \qquad \dots \qquad \dfrac{\dfrac{r(S_n)}{\vdots}}{S'_n}}{\dfrac{\dfrac{S'_0}{\vdots}}{r(S_0)}}}{}$$

Lemma 7 tells us exactly how to fill in these gaps. More specifically it tells us that superfluous structures can only appear on branches from $r(S_i)$ to S'_i for $1 \le i \le n$.

Step 3. Count how many superfluous structures are in S'_i for $1 \le i \le n$. If the answer for each i is "no more than two" then we are done.

Let us demonstrate, how the scheme works for some rules.

Rule $(a\vdash)$. Consider $X \circ (Y \circ Z) \vdash W$ and assume U is superfluous in it. Because of the restriction on the form of superfluous structures U has to be a substructure of X, Y, Z or W. If it is a proper substructure of one of those, then deleting it gives us an instance of the same rule. If U is one of X, Y or Z then deleting it from both premiss and conclusion gives us the same sequent. If $U = W$ and $W \neq I$ then by $(*)$ there is a congruent U', which is again a substructure of X, Y or Z and we can delete U' instead. On the other hand, reasoning as in the proof of Lemma 5 it is easy to show that W cannot be superfluous if $W = I$. Thus, we can complete Step 1 by deleting all superfluous structures in $X \circ (Y \circ Z) \vdash W$.

Rule $(\vdash \wedge)'$. Consider $X \vdash A$. By $(*)$ we know that if Y is superfluous in $X \vdash A$ then there is a structure Z congruent to Y, which is contained in X. Now, if Z does not contain a part, which is congruent to the indicated A in $X \vdash A$, then it is superfluous in both $X \vdash B$ and $X \vdash A \wedge B$ and deleting it from all three sequent yields an instance of the same rule. After deleting all such superfluous structures (denote by X' the result of deleting them from X) we complete Step 1 of our scheme. Naturally, what we are left with is at most one superfluous structure in each of $X' \vdash A$, $X' \vdash B$, which is what we needed.

Rule $(\wedge \vdash)$. Again, if some structure Z is superfluous in $A \circ B \vdash X$ but does not involve the indicated A and B, then it is superfluous in $A \wedge B \vdash X$ too and deleting it from both gives as an application of $(\wedge \vdash)$ again. We finish Step 1 by deleting all such superfluous structures. But now the resulting sequent $A \circ B \vdash X'$ might contain at most two superfluous structures — one containing structure congruent to the indicated A and one containing structure congruent to the indicated B. Yet this is still within the constraints of semi-reduced sequents. Notice that this case is the reason for allowing at most two superfluous structures in semi-reduced sequents.

Rule $(\rightarrow\vdash)'$. Reasoning as above we assume that $X \vdash A$ contains at most one superfluous structure, which has a part congruent to the indicated A. Now, consider $X \circ B \vdash A$ and suppose Z is superfluous in $X \circ B \vdash Y$. If Z is a substructure of Y and does not involve B then it is superfluous in $X \circ (A \rightarrow B) \vdash Y$ too and deleting it from both of these sequents results in an instance of the same rule. If Z is $X \circ B$ then by $(*)$ there is a substructure Z' of Y, which is congruent to it and we proceed as above. Now, if Z is a substructure of X, then, given that Y is not of the form $Y_1 \bullet Y_2$, we can reason as in Lemma 5 to show that no substructure of X can be congruent to substructure of Y. From which is easily follows that Z has to involve the indicated B, which means that there is at most one superfluous structure in the resulting sequent $X' \circ B \vdash Y'$.

All other rules are considered similarly. □

By an *atom* of some sequent S we will call every structure occurrence Z in S which is either a formula or a nullary structural connective. We can define the multiset $Atom(S)$ of atoms of S in a natural way.

THEOREM 10. *It is decidable whether a sequent S is derivable in $\delta\mathsf{BiRN}'$.*

Proof. Consider a sequent S. It follows from Lemma 7 that S is interderivable with the reduced sequent $r(S)$. Thus we can assume that S is reduced. By Lemma 8 S is derivable iff it has a reduced proof. Then we want to show that there is only a finite number of candidates for a reduced proof of S.

Let us define a multiset $\mathbb{X}(S)$ the following way: for each formula occurrence A (including those which are subformulas of other formulas) in S put into $\mathbb{X}(S)$ four instances of A; also put four instances of I (η, η') for each occurrence of I (η, η') in S. Naturally, $\mathbb{X}(S)$ is a finite multiset.

Then let $Tree(S)$ be the set of all finite trees T such that i) every node of T is a sequent S' such that $Atom(S') \subseteq \mathbb{X}(S)$; ii) T does not contain loops; iii) branching in T is limited to two. Obviously, $Tree(S)$ is a finite set.

Finally it is routine to check that if Π is a reduced proof of S then Π is an element of $Tree(S)$, which concludes our proof. □

Why did we multiply everything by four? A sequent $S = B \vdash ((A \circ A) \bullet C) \circ ((A \circ A) \bullet C)$ is an example of a semi-reduced sequent which contains four instances of A whereas $r(S) = B \vdash A \bullet C$ contains only one.

To summarize, our decision procedure for a sequent S in δBiRN is the following.

I. Compute $r(S)$. By Lemmas 4 and 7 S is derivable in δBiRN iff $r(S)$ is derivable in δBiRN$'$. See also Remark 3.

II. By Lemma 8 $r(S)$ is derivable in δBiRN$'$ iff it has a reduced proof. List all candidates for a reduced proof of $r(S)$ as in Theorem 10. By the same theorem there is only a finite number of those.

III. If one of the candidates is a reduced proof of $r(S)$ then S is derivable in δBiRN; it is not derivable otherwise.

Naturally, this procedure is not very effective at all, but our goal here was the decidability result itself so we leave it at that.

There are some simple consequences of this decision procedure. First, we have naturally obtained decidability for every fragment of δBiRN, including Onishi's δBiN, bi-intuitionistic logic and so on. Second, we can automatically extend this result for some extensions of δBiRN. More specifically, extending it with any structural rule $S_1, \ldots, S_n/S$ such that $Atom(S_i) \subseteq Atom(S)$ for all $i \leq n$ preserves this result. The combination of these two covers most of logics outlined in the paper including something as obscure as a system obtained by adding a normal modal unnecessity operator to KC.

In [20] Marcus Kracht proved that it is undecidable whether a modal display calculus over classical base is decidable, hence showing that the cut-elimination and subformula properties are not enough for decidability. The possibility of a similar result for our modal logics over a bi-intuitionistic base is an interesting open question.

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