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ON DISTANCE SETS OF LARGE SETS OF INTEGER POINTS

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Abstract. Distance sets of large sets of integer points are studied in dimensions at least 5. To any $\varepsilon > 0$ a positive integer Q_ε is constructed with the following property; If A is any set of integer points of upper density at least ε, then all large multiples of Q_ε^2 occur as squares of distances between the points of the set A.

1. Introduction.

A result of Fürstenberg, Katznelson and Weiss [FKW] states that if A is a measurable subset of $\mathbb{R}^2$ of positive upper density, then its distance set: $d(A) = \{|x-y| :\ x \in A,\ y \in A\}$ contains all large numbers.

Our aim is to prove a similar result for subsets of $\mathbb{Z}^n$ $(n > 4)$ of positive density ε, namely that: $d^2(A) = \{|m-l|^2 :\ m \in A,\ l \in A\}$ contains all large multiples of a fixed number Q_ε^2, which depends only on the density ε and the dimension n.

Note that one cannot take $Q_\varepsilon = 1$ as the set A may fall into a fixed congruence class of some integer q, and if $q \leq \varepsilon^{-1/n}$ then such a set A would have density $q^{-n} \geq \varepsilon$, and all elements of $d^2(A)$ would be divisible by q^2. Moreover this implies that Q'_ε divides Q_ε, where Q'_ε is the least common multiple of all $q \leq \varepsilon^{-1/n}$. In particular by the prime number theorem $Q_\varepsilon \geq Q'_\varepsilon \geq exp\,(c\,\varepsilon^{-1/n})$ with some $c > 0$. The number Q_ε we construct will be similar and will satisfy the upper bound: $Q_\varepsilon \leq exp\,(C_n\,\varepsilon^{-6/n-4})$, where C_n is a constant depending only on the dimension n.

Such results are impossible in dimensions $n \leq 3$. Indeed, even if one takes $A = \mathbb{Z}^n$, the equation: $d = |m-l|^2$ has no solution if $d = 4^a(8k+1)$ by Gauss' characterization, however every number has multiples of this form. We prove our result in dimensions $n \geq 5$ leaving the case $n = 4$ open.

A corollary is that the gaps between consecutive distances $d < d'$, $d,\ d' \in d(A)$ satisfy: $d' - d \leq C_\varepsilon d^{-1/2}$ where ε denotes the upper density of the set A. Distance sets of discrete subsets of $\mathbb{R}^n$ have been studied before, in [IŁ] it was shown that the gaps between consecutive distances from discrete subsets A of $\mathbb{R}^2$ tend to 0, if A has a point in every square of size $\sqrt{5}$. In fact this was proved more generally when the distances are associated to convex sets, see also [K] for similar higher dimensional results. Our proof may be generalized when the distances are defined by certain positive homogeneous polynomials, however we do not pursue such generalizations here.

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2. Main results.

We say that a set $A \subseteq \mathbb{Z}^n$ has upper density at least ε, and write $\delta(A) \geq \varepsilon$, if there exists a sequence of cubes B_{R_j} of sizes $R_j \to \infty$, not necessarily centered at the origin, such that

$$|A \cap B_{R_j}| \geq \varepsilon R_j^n \quad \forall j \tag{2.1}$$

where $|A|$ denotes the number of elements of the set A. As usual $\mathbb{Z}^n$ denotes the integer lattice and $\mathbb{N}$ stands for the natural numbers.

Theorem 1. *Let $n \geq 5$, $\varepsilon > 0$ and let $A \subseteq \mathbb{Z}^n$ such that $\delta(A) \geq \varepsilon$.*

Then there exists $Q_\varepsilon \in \mathbb{N}$ depending only on ε, and $\Lambda_A \in \mathbb{N}$ depending on the set A, such that

$$\lambda Q_\varepsilon^2 \in d^2(A) = \{|m - l|^2 : \; m \in A, \; l \in A\} \tag{2.2}$$

for every $\lambda \geq \Lambda_A$.

In fact a more quantitative version will be proved

Theorem 2. *Let $n \geq 5$, $\varepsilon > 0$. Then there exist a pair $J_\varepsilon, Q_\varepsilon \in \mathbb{N}$ depending only on ε, such that the following holds.*

If $0 < \lambda_1 < \lambda_2 < \ldots < \lambda_{J_\varepsilon}$ is any sequence of natural numbers with $\lambda_{j+1} \geq 10\lambda_j$ and if $A \subseteq \mathbb{Z}^n \cap B_R$ such that $|A| \geq \varepsilon R^n$ and $R \geq 10\lambda_{J_\varepsilon}^{1/2}$, then

$$\exists \; j \leq J_\varepsilon \qquad \textit{such that} \qquad \lambda_j Q_\varepsilon^2 \in d^2(A) \tag{2.3}$$

It is clear that Theorem 2 implies Theorem 1. Indeed, let $\varepsilon > 0$, J_ε, Q_ε as in Theorem 2. If Theorem 1 does not hold for Q_ε, then there is a set $A \subseteq \mathbb{Z}^n$ with upper density $\delta(A) \geq \varepsilon$ and infinite sequence λ_j such that $10\lambda_j < \lambda_{j+1}$ and $\lambda_j Q_\varepsilon^2 \notin d^2(A)$ for all j. Choosing a cube B_R with size $R > 10\lambda_{J_\varepsilon}^{1/2}$ such that $|A \cap B_R| \geq \varepsilon R^n$ contradicts (2.3).

It will be convenient to introduce the following terminology; a triple (ε, Q, J) is called *regular* if the conclusion of Theorem 2 holds for that triple. It is clear that the triple (1,1,1) is regular as every positive integer is the sum of 5 squares, and also the regularity of (ε, Q, J) implies that of (ε', Q, J) for $\varepsilon \leq \varepsilon'$.

Thus it is enough to show that for each $\varepsilon_k = (9/10)^k$ there exists a pair of natural numbers Q_k, J_k such that $(\varepsilon_k, Q_k, J_k)$ is regular. This will be shown by induction on k, constructing $Q_k = Q_{k-1}q_k$ and J_k recursively to ε_k. The point is that induction will enable one to assume that A is well-distributed in the congruence classes of a fixed modulus q_k, to be chosen later.

Indeed for $s \in \mathbb{Z}^n$, let $A_{q_k,s} = \{m \in \mathbb{Z}^n : \; q_k m + s \in A\}$. If there is an s such that the density $\delta(A_{q_k,s}) \geq \varepsilon_{k-1}$, then by induction it follows that $\lambda Q_{k-1}^2 \in d^2(A_{q_k,s})$ for all large λ, and hence $\lambda Q_k^2 \in d^2(A)$. Thus one can assume that $\delta(A) \geq \varepsilon_k$, but $\delta(A_{q_k,s}) \leq \varepsilon_{k-1} = \frac{10}{9}\varepsilon_k$ for each $s \in \mathbb{Z}^n$.

Our proof was motivated by the short Fourier analytic proof of the Fürstenberg-Katznelson-Weiss theorem given in [B]. The starting point is to express the number of pairs $m \in A,\ l \in A$ such that $|m-l|^2 = \lambda$ in the form

$$N(A,\lambda) = \sum_{m,l} 1_A(m) 1_A(l)\, \sigma_\lambda(m-l) = \langle 1_A, 1_A * \sigma_\lambda \rangle$$

where 1_A denotes the indicator function of the set A and σ_λ stands for that of the of the set of integer points on the sphere of radius $\lambda^{1/2}$. Thus by Plancherel

$$N(A,\lambda) = \langle \hat{1}_A, \hat{1}_A\, \hat{\sigma}_\lambda \rangle = \int_{\Pi^n} |\hat{1}_A(\xi)|^2 \hat{\sigma}_\lambda(\xi)\, d\xi \tag{2.4}$$

where

$$\hat{\sigma}_\lambda(\xi) = \sum_{|m|^2=\lambda} e^{2\pi i m\cdot\xi} \tag{2.5}$$

is the Fourier transform of σ_λ, and Π^n denotes the n-dimensional torus.

Note that $|\hat{\sigma}_\lambda(\xi)| \le \hat{\sigma}_\lambda(0) \lesssim \lambda^{n/2-1}$ for $n \ge 5$, using the well-known fact in number theory, that $|\{m \in \mathbb{Z}^n :\ |m|^2 = \lambda\}| \lesssim \lambda^{n/2-1}$. Thus if $A \subseteq B_R$ then by (2.4):

$$N(A,\lambda) \lesssim |A| \lambda^{n/2-1} \le R^n \lambda^{n/2-1} \tag{2.6}$$

Here $A \lesssim B$ means that $A \le c_n B$ with a constant $c_n > 0$ depending only on the dimension n, and whose exact value may change from place to place.

The behavior of the exponential sum $\hat{\sigma}_\lambda(\xi)$ is described in [MSW] and summarized in the asymptotic formula (3.3). We'll use the fact that it is concentrated near rational points of small denominator. More precisely given $\varepsilon > 0$, there is a $Q_\varepsilon \in \mathbb{N}$ and a $\lambda_\varepsilon > 0$ depending only on ε, such that for $\lambda \ge \lambda_\varepsilon$

$$|\hat{\sigma}_\lambda(\xi)| \lesssim \varepsilon^3 \lambda^{\frac{n}{2}-1} \quad \text{if} \quad |\xi - l/Q_\varepsilon| \gtrsim \varepsilon^{-\frac{6}{n-1}}\, \lambda^{-\frac{1}{2}}$$

for every rational point l/Q_ε, $l \in \mathbb{Z}^n$. This implies

$$\int_{\Pi^n} |\hat{1}_A(\xi)^2\, \hat{\sigma}_\lambda(\xi)\ (1 - \sum_{l\in\mathbb{Z}^n} \hat{\psi}^1_\lambda(\xi - l/Q_\varepsilon))|\, d\xi \lesssim \varepsilon^3 R^n \lambda^{\frac{n}{2}-1} \tag{2.7}$$

where $0 \le \hat{\psi}^1_\lambda(\xi) \le 1$ is smooth cut-off function, such that $\hat{\psi}^1_\lambda(0) = 1$ and is supported on the ball $|\xi| \lesssim \varepsilon^{-\frac{6}{n-1}}\, \lambda^{-\frac{1}{2}}$. This will be proved in Section 3.

In Section 4, we prove our key estimate, namely that if $A \subseteq B_R$, $|A| \ge \varepsilon R^n$ and if A is uniformly distributed in the congruence classes of a certain modulus q_ε, then

$$\int_{\Pi^n} |\hat{1}_A(\xi)|^2 \hat{\sigma}_\lambda(\xi) \sum_{l\in\mathbb{Z}^n} \hat{\psi}^2_\lambda(\xi - l/Q_\varepsilon)\, d\xi \gtrsim \varepsilon^3 R^n \lambda^{\frac{n}{2}-1} \tag{2.8}$$

where ψ_λ^2 is a smooth function whose Fourier transform $\hat{\psi}_\lambda^2(\xi)$ is supported on $|\xi| \lesssim \lambda^{-\frac{1}{2}}$.

Now assume that, in the settings of Theorem 2, $N(A,\lambda) = 0$ for each $\lambda = \lambda_j Q_\varepsilon^2$ $(J_\varepsilon/2 \le j \le J_\varepsilon)$. Using the decomposition

$$1 = (1 - \sum_{l\in\mathbb{Z}^n} \hat{\psi}_\lambda^1(\xi - l/Q_\varepsilon)) + \sum_{l\in\mathbb{Z}^n} \hat{\psi}_\lambda^2(\xi - l/Q_\varepsilon) + \sum_{l\in\mathbb{Z}^n} \hat{\phi}_\lambda(\xi - l/Q_\varepsilon)$$

in (2.4), it follows from (2.7) and (2.8) and from the uniform bound: $|\hat{\sigma}_\lambda(\xi)| \lesssim \lambda^{\frac{n}{2}-1}$, that

$$\int_{\Pi^n} |\hat{1}_A(\xi)|^2 \left| \sum_{l\in\mathbb{Z}^n} \hat{\phi}_\lambda(\xi - l/Q_\varepsilon) \right| d\xi \gtrsim \varepsilon^3 R^n \tag{2.9}$$

where $\hat{\phi}_\lambda(\xi) = \hat{\psi}_\lambda^1(\xi) - \hat{\psi}_\lambda^2(\xi)$ is a smooth function essentially supported on the annulus: $\lambda^{-1/2} \le |\xi| \le \varepsilon^{-6/(n-1)}\,\lambda^{-1/2}$. Thus the supports of the integrands in (2.9) for well-separated values of $\lambda = \lambda_j Q_\varepsilon^2$ are (essentially) disjoint, and hence the sum of the left side (2.9) over such j's would be bounded by $\int_{\Pi^n} |\hat{1}_A(\xi)|^2\, d\xi = |A| \le R^n$ which contradicts (2.9) if J_ε is chosen large enough.

3. Upper bounds.

In this section we prove the upper bound (2.7), which is an easy corollary of the following asymptotic formula, proved in [MSW] (see Proposition 4.1):

$$\hat{\sigma}_\lambda(\xi) = \omega_n \lambda^{n/2-1} \sum_{r=1}^{\infty} m_{r,\lambda}(\xi) + E(\xi,\lambda) \quad \text{where} \tag{3.1}$$
$$\sup_\xi |E(\xi,\lambda)| \lesssim \lambda^{n/4}$$

and the main terms are of the form

$$m_{r,\lambda}(\xi) = \sum_{k\in\mathbb{Z}^n} S(k,r,\lambda)\, \phi(r\xi - k)\, d\tilde{\sigma}(\lambda^{1/2}(\xi - k/r)) \tag{3.2}$$

Moreover by the standard estimate for Gauss sums

$$|S(k,r,\lambda)| = |q^{-n} \sum_{(a,r)=1} \sum_{s\in\mathbb{Z}_r^n} e^{2\pi i \frac{a(|s|^2-\lambda)+s\cdot l}{r}}| \le r^{-n/2+1} \tag{3.3}$$

The cut-off function $\phi(\xi)$ is supported in a small neighborhood of the origin, and the Fourier transform of the surface area measure of the unit sphere satisfies

$$|d\tilde{\sigma}(\xi)| \lesssim (1+|\xi|)^{-\frac{n-1}{2}} \tag{3.4}$$

Proposition 1. *Let $n \geq 5,\ \varepsilon > 0$ and $c > 0$ be given. Then there is a constant $C_n > 0$ depending only on c and the dimension n, such that the following holds.*

If $Q \in \mathbb{N}$ is such that r divides Q for all $r \leq C_n\, \varepsilon^{-\frac{6}{n-4}}$, moreover if $\lambda \geq C_n\, \varepsilon^{-\frac{12}{n-4}}$ and $\xi \in \Pi^n$ satisfies

$$|\xi - l/Q| \geq C_n\, \varepsilon^{-\frac{6}{n-1}} \lambda^{-\frac{1}{2}} \tag{3.5}$$

for all $l \in \mathbb{Z}^n$, then one has

$$|\hat{\sigma}_\lambda(\xi)| \leq c\, \varepsilon^3 \lambda^{\frac{n}{2}-1} \tag{3.6}$$

Proof. By equation (3.1), one has for $n \geq 5$

$$|E(\xi,\lambda)| \leq c_1 \lambda^{\frac{n}{4}} \leq \frac{c}{3}\, \varepsilon^3 \lambda^{\frac{n}{2}-1} \tag{3.7}$$

if $\lambda \geq C_n\, \varepsilon^{-\frac{12}{n-4}}$ and C_n is large enough w.r.t. $c_1,\ c$ and n. There is at most one non-zero term in the expression for $m_{r,\lambda}(\xi)$, hence by (3.3)

$$\sum_{r \geq K} |m_{r,\lambda}(\xi)| \leq c_2 K^{-n/2+2} \leq \frac{c}{3}\, \varepsilon^3 \tag{3.8}$$

if one chooses $K \geq C_n\, \varepsilon^{-\frac{6}{n-4}}$.

If Q is such that r divides Q for all $r \leq K$, then every rational point k/r can be written in the form l/Q, then (3.4) and (3.5) implies

$$\sum_{r \leq K} |m_{r,\lambda}(\xi)| \leq c_3 \max_{k,\, r \leq K} |\lambda^{1/2}(\xi - k/r)|^{-\frac{n-1}{2}} \leq \frac{c}{3}\, \varepsilon^3 \tag{3.9}$$

The Proposition follows by adding (3.7)-(3.9) $\square$

In what follows, C_n will denote a large enough constant of our choice, which guarantees the validity of certain inequality, and whose exact value may change from place to place.

The above estimate shows that $\hat{\sigma}_\lambda(\xi)$ is uniformly small on the complement of the neighborhoods $U_l = \{\xi :\ |\xi - l/Q_\varepsilon| \geq C_n\, \varepsilon^{-\frac{6}{n-1}} \lambda^{-\frac{1}{2}}\}$. Thus it can be used to bound the contribution of this set to the integral $N(A,\lambda)$ given in (2.4). To be more precise, let $A \subseteq B_R$ such that $|A| \geq \varepsilon R^n$, and let $\psi > 0$ be a smooth function satisfying

$$1 = \hat{\psi}(0) \geq \hat{\psi}(\xi) > 0 \ \ \forall\, \xi \quad \text{ and } \quad supp\ \hat{\psi} \subseteq [-1/2, 1/2]^n \tag{3.10}$$

For $Q \in \mathbb{N}$ and $L > 0$ define the following expressions

$$N_1(A,\lambda,Q,L) = \int_{\Pi^n} |\hat{1}_A(\xi)|^2\, \hat{\sigma}_\lambda(\xi)\, (1 - \sum_{l\in\mathbb{Z}^n} \hat{\psi}(L(\xi - l/Q))\,)\, d\xi \tag{3.11}$$

$$N_2(A,\lambda,Q,L) = \int_{\Pi^n} |\hat{1}_A(\xi)|^2\, \hat{\sigma}_\lambda(\xi) \sum_{l\in\mathbb{Z}^n} \hat{\psi}(L(\xi - l/Q))\, d\xi$$

Lemma 1. *Let $\varepsilon > 0$ and $c > 0$ be given. Then there is a constant $C_n > 0$, such that if $Q \in \mathbb{N}$ is a multiple of all $r \leq C_n\, \varepsilon^{-\frac{6}{n-4}}$ moreover if $L \geq C_n\, Q$ and $\lambda \geq C_n\, \varepsilon^{-\frac{12}{n-1}-3} L^2$ then*

$$|N_1(A,\lambda,Q,L)| \leq c\, \varepsilon^3 \lambda^{\frac{n}{2}-1} R^n \tag{3.12}$$

Proof. Since $\int_{\Pi^n} |\hat{1}_A(\xi)|^2\, d\xi \ = |A| \leq R^n$ it is enough to show that

$$\sup_{\xi\in\Pi^n} |\hat{\sigma}_\lambda(\xi)|\ |1 - \sum_{l\in\mathbb{Z}^n_Q} \hat{\psi}(L(\xi - l/Q)| \leq c\, \varepsilon^3 \lambda^{n/2-1} \tag{3.13}$$

Note that the supports of the functions $\hat{\psi}(L(\xi - l/Q))$ are disjoint for different values of l, thus if there is an l_0 such that: $|\xi - l_0/Q| \leq c_1\, \varepsilon^{\frac{3}{2}} L^{-1}$, where c_1 is small enough w.r.t. c, then

$$|1 - \sum_{l\in\mathbb{Z}^n} \hat{\psi}(L(\xi - l/Q))| = |1 - \hat{\psi}(L(\xi - l_0/Q))| \leq c\, \varepsilon^3$$

using $|1 - \hat{\psi}(\eta)| \lesssim |\eta|^2$ and (3.13) follows. In the opposite case:

$$|\xi - l/Q| > c_1\, \varepsilon^{\frac{3}{2}} L^{-1} \geq C_n \varepsilon^{-\frac{6}{n-1}} \lambda^{-\frac{1}{2}}$$

for all $l \in \mathbb{Z}^n$, by the assumptions on L and λ, and (3.13) follows from (3.6). $\square$

4. Lower bounds.

In this section we prove the lower bound (2.8). We start by proving an analogous estimate in the settings of the group of congruence classes of the modulus Q: $\mathbb{Z}^n_Q = (\mathbb{Z}/Q\mathbb{Z})^n$.

Let $\varepsilon > 0$ and let $q,\ Q \in \mathbb{N}$ such that q divides Q. Let $f : \mathbb{Z}^n_Q \to [0,1]$ be a function satisfying the following two conditions

$$\sum_{m\in\mathbb{Z}_Q^n} f(m) \geq \frac{4\varepsilon}{5}\, Q^n \tag{4.1}$$

and for each $s\in\mathbb{Z}_Q^n$

$$\sum_{m\in\mathbb{Z}_Q^n} f(qm+s) \leq \frac{10\varepsilon}{9}\, Q^n \tag{4.2}$$

Note that if f is the characteristic function of a set $A\subseteq\mathbb{Z}_Q^n$ then by equation (4.1) the density of A is at least $4\varepsilon/5$, while by (4.2) the density of A is at most $10\varepsilon/9$ in any of the congruence classes of the modulus q. We say that the set A is well-distributed in these congruence classes.

For $\lambda\in\mathbb{Z}$ we consider the following quantity:

$$N = N(f,Q,\lambda) = \sum_{m,l\in\mathbb{Z}_Q^n} f(m)f(m-l)\,\omega_{\lambda,Q}(l) \tag{4.3}$$

where the function: $\omega_{\lambda,Q}:\ \mathbb{Z}_Q^n\to\mathbb{N}$ is defined by

$$\omega_{\lambda,Q}(l) = |\{k\in\mathbb{Z}^n:\ |k|^2=\lambda,\ k\equiv l\ (mod\ Q)\} \tag{4.4}$$

that is the number of lattice points of length $\lambda^{1/2}$ which are congruent to l modulo Q. We will make use of the Fourier transform on $\mathbb{Z}_Q^n$:

$$\hat f(s) = \sum_{m\in\mathbb{Z}_Q^n} e^{-2\pi i\frac{m\cdot s}{Q}} f(m) \qquad \text{and note that}$$

$$\hat\omega_{\lambda,Q}\,(s) = \sum_{k\in\mathbb{Z}^n,\ |k|^2=\lambda} e^{-2\pi i\frac{k\cdot s}{Q}} = \hat\sigma_\lambda(s/Q) \tag{4.5}$$

Lemma 2. *Let $0<\varepsilon<1$ and let q,Q,λ be positive integers such that k divides q for all $k\leq C_n\,\varepsilon^{-\frac{6}{n-4}}$, q divides Q, and $\lambda > C_n\,Q^2\,\varepsilon^{-\frac{12}{n-1}}$.*

If $f\colon\mathbb{Z}_Q^n\to[0,1]$ is a function satisfying (4.1) and (4.2) then for C_n large enough, one has

$$N(f,Q,\lambda) \geq c\,\lambda^{\frac{n}{2}-1}Q^n\varepsilon^2 \tag{4.6}$$

where $c>0$ is a constant depending only on the dimension n.

Proof. Using the Fourier transform on $\mathbb{Z}_Q^n$, similarly as in (2.4), one has

$$N = N(f,Q,\lambda) = \frac{1}{Q^n}\sum_{s\in\mathbb{Z}_Q^n} |\hat f(s)|^2\,\hat\sigma_\lambda(s/Q)$$

Write $Q = Q_1 q$ and decompose the summation into two terms according to whether Q_1 divides s;

$$N = \frac{1}{Q^n} \sum_{s_1 \in \mathbb{Z}_q^n} |\hat{f}(Q_1 s_1)|^2 \, \hat{\sigma}_\lambda(s_1/q) + \frac{1}{Q^n} \sum_{Q_1 \nmid s} |\hat{f}(s)|^2 \, \hat{\sigma}_\lambda(s/Q) = M + E$$

Here the main term M is obtained by writing $s = Q_1 s_1$ where s_1 is running through $\mathbb{Z}_q^n$.

Let $f_q\colon\ \mathbb{Z}_q^n \to [0,1]$ be defined by: $f_q(m) = Q_1^{-n} \sum_{k \in \mathbb{Z}_{Q_1}^n} f(m+qk)$, that is the average of f over the congruence class of m with respect to the modulus q. Then $\hat{f}_q(s) = Q_1^{-n} \hat{f}(Q_1 s)$ thus by equation (4.5) and Plancherel

$$M = \frac{Q_1^{2n}}{Q^n} \sum_{s_1 \in \mathbb{Z}_q^n} |\hat{f}_q(s_1)|^2 \, \hat{\sigma}_\lambda(s_1/q) \ = \ Q_1^n \sum_{m,l \in \mathbb{Z}_q^n} f_q(m) f_q(l) \, \omega_{\lambda,q}(m-l) \tag{4.7}$$

Note that (4.1) and (4.2) is equivalent to

$$\sum_{m \in \mathbb{Z}_q^n} f_q(m) \geq \frac{4\varepsilon}{5} \, q^n \quad \text{ and } \quad f_q(m) \leq \frac{10\varepsilon}{9}\ , \quad \forall\, m \in \mathbb{Z}_q^n \tag{4.8}$$

If $G = \{m \in \mathbb{Z}_q^n :\ f_q(m) \geq \varepsilon/10\,\}$ then the sum of $f_q(m)$ for $m \notin G$ is at most $q^n \varepsilon/10$ thus by (4.8)

$$|G| \geq \frac{9}{10\varepsilon} \sum_{m \in G} f_q(m) \geq \frac{63}{100} \, q^n \tag{4.9}$$

Hence for every $l \in \mathbb{Z}_q^n$

$$\sum_{m \in \mathbb{Z}_q^m} f_q(m) \, f_q(m-l) \geq \frac{\varepsilon^2}{100} |G \cap (G+l)| > \frac{\varepsilon^2}{500} \, q^n$$

Substituting back to (4.7) one has

$$M \geq \frac{\varepsilon^2 Q^n}{500} \sum_{l \in \mathbb{Z}_q^n} \omega_{\lambda,q}(l) = \frac{\varepsilon^2 Q^n}{500} \ \hat{\sigma}_\lambda(0) \geq c_n \, Q^n \varepsilon^2 \lambda^{\frac{n}{2}-1} \tag{4.10}$$

where the constant $c_n > 0$ depends only on the dimension n.

Now let C_n be chosen as in Proposition 1. w.r.t. $c = c_n/2$. If Q_1 does not divide s, then for every $l \in \mathbb{Z}^n$

$$\left|\frac{s}{Q}-\frac{l}{q}\right| \geq \frac{1}{Q} \geq \varepsilon^{-\frac{6}{n-1}}\,\lambda^{-\frac{1}{2}}$$

The conditions of Lemma 1. are satisfied, thus

$$|\hat{\sigma}_\lambda(s/Q)| \leq \frac{c_n}{2}\,\varepsilon^3\,\lambda^{\frac{n}{2}-1}$$

hence by Plancherel $|E| \leq \frac{c_n}{2}\,\varepsilon^3 Q^n \lambda^{\frac{n}{2}-1}$ and the Lemma follows with $c = \frac{c_n}{2}$. $\square$

Next, our aim is to reduce estimate (2.8) to that of (4.6). First, one has

Proposition 2. *Let $Q, \lambda \in \mathbb{N}$ and let $L \geq \lambda^{1/2}$. Then one has*

$$N_2(A,\lambda,Q,L) \geq c_n\,Q^n\,L^{-n} \sum_{m\in\mathbb{Z}^n} \sum_{\substack{l\,\in\,\mathbb{Z}^n\\ |l|\,\leq\,\sqrt{n}L}} 1_A(m)\,1_A(m-l)\,\omega_{\lambda,Q}(l) \tag{4.11}$$

Proof. Define the distribution δ_Q by

$$< \delta_Q, \phi >= \sum_{m\in\mathbb{Z}^n} Q^n\,\phi(Qm)$$

then by Poisson summation

$$< \hat{\delta}_Q, \phi >=< \delta_Q, \hat{\phi} >= \sum_{l\in\mathbb{Z}^n} \phi(l/Q) \tag{4.12}$$

Thus by Plancherel

$$N_2(A,\lambda,Q,L) =< \hat{1}_A, \hat{1}_A \hat{\sigma}_\lambda(\hat{\psi}_L * \hat{\delta}_Q) >=< 1_A, 1_A * \sigma_\lambda * (\psi_L \delta_Q) >$$

where $\psi_L(x) = L^{-n}\psi(x/L)$. If $l \in \mathbb{Z}^n$ such that $|l| \leq \sqrt{n}L$ then

$$\sigma_\lambda * (\psi_L\delta_Q)(l) = \sum_{k\in\mathbb{Z}^n} \sigma_\lambda(k)\,\psi_L(l-k)\,\delta_Q(l-k) \geq$$

$$\geq c_n\,Q^n L^{-n} \sum_{k:\ Q|l-k} \sigma_\lambda(k) = c_n Q^n L^{-n} \omega_{\lambda,Q}(l)$$

Indeed if $\sigma_\lambda(k) \neq 0$ then $|k| = \lambda^{1/2} \leq L$ hence $|l-k| \leq (\sqrt{n}+1)L$ and $\psi_L(l-k) \geq c_n L^{-n}$. This proves the Proposition. $\square$

Assume that $R > 10\lambda^{\frac{1}{2}}$, choose L such that $\lambda^{1/2} \leq L \leq 2\lambda^{1/2}$ and R/L is an integer. Divide the box B_R into R^n/L^n boxes $B_L(j)$ of equal size L, and let $\mathcal{F}$ denote the set of boxes in which the density of A remains large:

$$\mathcal{F} = \{j:\ |A \cap B_L(j)| \geq \frac{4\varepsilon}{5}\, L^n\} \tag{4.13}$$

It is easy to see that $|\mathcal{F}| \geq \frac{\varepsilon}{10}\frac{R^n}{L^n}$. If f_j denotes the characteristic function of the set $A \cap B_L(j)$, then by (4.12)

$$N_2(A,\lambda,Q,L) \geq c_n\, Q^n L^{-n} \sum_{j\in\mathcal{F}} \sum_{m,l\in\mathbb{Z}^n} f_j(m)\, f_j(m-l)\, \omega_{\lambda,Q}(l) \tag{4.14}$$

since the diameter of each box $B_L(j)$ is at most $\sqrt{n}\, L$.

The function $\omega_{\lambda,Q}$ is constant on the congruence classes mod Q, hence the inner sum in (4.15) can be written in the form

$$\frac{L^{2n}}{Q^{2n}} \sum_{m,l\in\mathbb{Z}^n_Q} f_{j,Q}(m)\, f_{j,Q}(m-l)\, \omega_{\lambda,Q}(l) \qquad \text{where}$$

$$f_{j,Q}(m) = \sum_{k\in\mathbb{Z}^n} f_j(Qk+m) = \frac{Q^n}{L^n}\, |\{m' \in A\cap B_L(j):\ m' \equiv m\ (mod\ Q)\}|$$

If one assumes that $A \cap B_L(j)$ is well-distributed in the congruence classes mod q, that is if for every $s \in \mathbb{Z}^n$ and $j \in \mathcal{F}$

$$\sum_{m\in\mathbb{Z}^n_Q} f_{j,Q}\,(qm+s) = |\{m \in A\cap B_L(j):\ m \equiv s\ (mod\ q)\}| \leq \frac{10\varepsilon}{9}\frac{L^n}{q^n} \tag{4.15}$$

then the functions $f_{j,Q}$ satisfy (4.1) and (4.2)

$$\sum_{m\in\mathbb{Z}^n_Q} f_{j,Q}\,(m) = \frac{Q^n}{L^n}\, |A\cap B_L(j)| \geq \frac{4\varepsilon}{5}\, Q^n$$

$$\sum_{m\in\mathbb{Z}^n_Q} f_{j,Q}\,(qm+s) = \frac{Q^n q^n}{L^n} \sum_{k\in\mathbb{Z}^n} f_j(qk+s) \leq \frac{10\varepsilon}{9}\, Q^n$$

If the parameters q, Q, λ satisfy the conditions of Lemma 2, then from (4.6) and (4.15) one obtains the lower bound

$$N_2(A,\lambda,Q,L) \,\geq\, c_n\, Q^{-n} L^n\, |\mathcal{F}|\, \varepsilon^2 Q^n \lambda^{\frac{n}{2}-1} \,\geq\, c_n\, \varepsilon^3 R^n \lambda^{\frac{n}{2}-1} \tag{4.16}$$

5. Proof of Theorem 2.

We are in a position to apply an induction argument to prove our main result. Let $\varepsilon_0 = Q_0 = J_0 = 1$ and for $k = 1, 2, \ldots$, define

$$\varepsilon_k = (9/10)^k \ , \quad q_k = \left[C_n\, \varepsilon_k^{-\frac{6}{n-4}} \right]!! \quad \text{and} \quad Q_k = q_k Q_{k-1}. \tag{5.1}$$

Moreover we define J_k be the smallest integer satisfying

$$J_k \geq 2J_{k-1} + C_n \log\, Q_k + C_n\, \varepsilon_k^{-3} \log\,(\varepsilon_k^{-1}). \tag{5.2}$$

where [ ] stands for the integer part, and $M!!$ denotes the least common multiple of the natural numbers $1 \leq m \leq M$. Here $C_n > 0$ is a large constant to be chosen later. Note that, by the prime number theorem, it is easy to see that $\log\,(Q_k) \lesssim \varepsilon_k^{-6/\,n-4} \leq \varepsilon_k^{-3}$ if $n \geq 6$, hence in that case the second term of (5.2) may be omitted.

We prove by induction that $(\varepsilon_k, Q_k, J_k)$ is regular for all k in the sense described in the introduction. So assume on the contrary that
$(\varepsilon_{k-1}, Q_{k-1}, J_{k-1})$ is regular, but $(\varepsilon_k, Q_k, J_k)$ is not. Then there exists a sequence: $\lambda_1, \lambda_2, \ldots, \lambda_{J_k}$ with $10\lambda_j < \lambda_{j+1}$ a cube B_R of size $R \geq 10\, \lambda_{J_k}^{1/2}$, and a set $A \subset B_R$ with $|A| \geq \varepsilon_k R^n$, such that for all $1 \leq j \leq J_k$

$$\lambda_j\, Q_k^2 \notin d^2(A) \tag{5.3}$$

First we show that for every $L \geq (\lambda_{J_k/2})^{1/2}\, Q_k$ and for every cube $B_L \subset B_R$ of size L and $s \in \mathbb{Z}^n$, one has

$$|A'| = |\{m \in \mathbb{Z}^n :\ q_k m + s \in A \cap B_L\}| \geq \frac{10\varepsilon}{9}\, \frac{L^n}{q_k^n} \tag{5.4}$$

Indeed, otherwise the set A' is contained in a cube $B_{R'}$ of size $R' = L/q_k$ of density at least ε_{k-1}. Since $R' = L/q_k > \lambda_{J_k/2}^{1/2} \geq 10\lambda_{J_{k-1}}^{1/2}$, by induction, there is a $j \leq J_{k-1}$ such that: $\lambda_j Q_{k-1}^2 \in d^2(A')$. Then $\lambda_j Q_{k-1}^2 q_k^2 = \lambda_j Q_k^2 \in d^2(A)$ contradicting our indirect assumption.

Let $J_k/2 \leq j \leq J_k$. To get in agreement with the notations of the previous sections, let $\varepsilon = \varepsilon_k$, $q = q_k$, $Q = Q_k$. If the constant C_n is chosen large enough then, by (5.1) and (5.2), the conditions of inequality (3.12) are satisfied for $\lambda = \lambda_j Q^2$ and $L' = L'_j = \delta \lambda^{1/2}$ with $\delta = \varepsilon^{\frac{6}{n-1}+\frac{3}{2}}$. Thus

$$N_1(A,\lambda,Q,L') \leq \frac{c_n}{2}\,\varepsilon^3 R^n \lambda^{\frac{n}{2}-1} \tag{5.5}$$

Let $\lambda^{1/2} \leq L_j \leq 2\lambda^{1/2}$ be chosen such that R/L_j is an integer. Then inequality (4.17) applies with $L = L_j$, thus

$$N_2(A,\lambda,Q,L) \geq \; c_n\,\varepsilon^3 R^n \lambda^{\frac{n}{2}-1} \tag{5.6}$$

By our indirect assumption, $N(A,\lambda) = 0$ for each $\lambda = \lambda_j Q^2,\ J_k/2 \leq j \leq J_k$. We decompose the integral $N(A,\lambda)$ into three terms, as described in the introduction

$$N(A,\lambda) = N_1(A,\lambda,Q,L') + N_2(A,\lambda,Q,L) + N_3(A,\lambda,Q,L,L') \tag{5.7}$$

where $N_3(A,\lambda,Q,L,L')$ is defined by the above equation. Thus by (5.5) and (5.6) one has

$$|N_3(A,\lambda_j Q^2, L_j, L'_j)| = |\int_{\Pi^n} |\hat{1}_A(\xi)|^2 \hat{\sigma}_\lambda(\xi)\Phi_j(\xi)\,d\xi| \geq \frac{c_n}{2}\,\varepsilon^3 R^n \lambda^{\frac{n}{2}-1} \tag{5.8}$$

where

$$\Phi_j(\xi) = \sum_{l\in\mathbb{Z}^n} \hat{\psi}(L'_j(\xi - l/Q)) - \hat{\psi}(L_j(\xi - l/Q)) \tag{5.9}$$

Note that the integral $N_3(A,\lambda_j Q^2, L_j, L'_j)$ captures the contribution of the region: $\{\xi :\ L_j^{-1} \leq |\xi - l/Q| \leq \delta^{-1}L_j^{-1},\ l\in\mathbb{Z}^n\}$ to the integral $N(A,\lambda)$. Since $|\hat{\sigma}_\lambda(\xi)| \lesssim \lambda^{n/2-1}$ one has

$$N_3(j) := \int_{\Pi^n} |\hat{1}_A(\xi)|^2 |\Phi_j(\xi)| \geq \; c_n\,\varepsilon^3 R^n \tag{5.10}$$

On the other hand the integrands are essentially supported on disjoint sets, in fact one has

$$\sum_{J_k/2\leq j\leq J_k} |\Phi_j(\xi)| \lesssim \; \log(\varepsilon^{-1}) \tag{5.11}$$

which implies

$$\sum_{J_k/2\leq j\leq J_k} N_3(j) \lesssim \; \log(\varepsilon^{-1}) R^n \tag{5.12}$$

This clearly contradicts (5.10) as J_k has been chosen to satisfy:
$J_k > C_n\,\varepsilon_k^{-3}\log(\varepsilon_k^{-1})$ with a large enough constant C_n. Finally, to see (5.11) first note that the functions:

$$\sum_{J_k/2 \leq j \leq J_k} \hat{\psi}(L'_j(\xi - l/Q)) - \hat{\psi}(L_j(\xi - l/Q))$$

have disjoint supports for different values of l. Thus it is enough to show that for fixed l, for $\eta = \xi - l/Q$:

$$\sum_{J_k/2 \leq j \leq J_k} |\hat{\psi}(L'_j\eta) - \hat{\psi}(L_j\eta)| \lesssim \sum_{j:\ L'_j|\eta|<1} \min(L_j|\eta|, 1)$$
$$\lesssim \sum_{j:\ L_j|\eta|<1} L_j|\eta| \ + \sum_{j:\ 1 \leq L_j|\eta| < \delta^{-1}} 1 \lesssim \log(\varepsilon^{-1}).$$

This follows using the properties of $\hat{\psi}$ given in (3.10) and the fact that $L'_j \leq \delta L_j$. We have reached a contradiction to our indirect assumption and Theorem 2 is proved. $\square$

It is not hard to estimate the size of Q_ε for any given $\varepsilon > 0$. Indeed if $\varepsilon_k \leq \varepsilon < \varepsilon_{k-1}$ then we take $Q_\varepsilon = Q_k$. Now $Q_k = \prod_{l=1}^k q_l$ where $q_l = M_l\,!! \leq \exp(C_n\, \varepsilon_l^{-6/\,n-4})$. It follows $Q_\varepsilon \leq \exp(C_n\, \varepsilon^{-6/\,n-4})$ for some constant C_n depending only on the dimension n.

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