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ON DISTANCES IN SIERPIŃSKI GRAPHS: ALMOST-EXTREME VERTICES AND METRIC DIMENSION

Sandi Klavžar, Sara Sabrina Zemljič

Sierpiński graphs S_p^n form an extensively studied family of graphs of fractal nature applicable in topology, mathematics of the Tower of Hanoi, computer science, and elsewhere. An almost-extreme vertex of S_p^n is introduced as a vertex that is either adjacent to an extreme vertex of S_p^n or is incident to an edge between two subgraphs of S_p^n isomorphic to S_p^{n-1}. Explicit formulas are given for the distance in S_p^n between an arbitrary vertex and an almost-extreme vertex. The formulas are applied to compute the total distance of almost-extreme vertices and to obtain the metric dimension of Sierpiński graphs.

1. INTRODUCTION

Sierpiński graphs S_p^n were introduced for at least three reasons. In [**18**], they were motivated by topological studies of universal spaces (cf. [**17**]) and the fact that the base-3 Sierpiński graphs S_3^n are isomorphic to the Tower of Hanoi graphs on 3 pegs. Independently, a class of graphs called WK-recursive networks was introduced in computer science in [**3**], see also [**5**]. WK-recursive networks are very similar to Sierpiński graphs, they can be obtained from Sierpiński graphs by adding a link (an open edge) to each of its extreme vertices.

Graphs S_p^n were studied by now from numerous points of view, the reader is invited to read the recent paper [**12**] about colorings of these graphs and references therein; see also [**6**] for more coloring results. Of the many other investigations, we only mention a few explicitly. An appealing application of Sierpiński graphs is

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due to ROMIK [**23**] who designed, based on Sierpiński labelings, a finite automaton particularly useful for the Tower of Hanoi problem. In [**19**] the structure of Sierpiński graphs was the key to determine for the first time the exact genus of infinite families of fractal graphs. Recently, the hub number of Sierpiński-like graphs was determined in [**15**].

Metric issues received a special attention on Sierpiński graphs. This is in particular due to the fact that shortest paths in the base-3 Sierpiński graphs correspond to optimal solutions in the Tower of Hanoi puzzle. In the seminal paper [**18**] a formula for the distance between vertices in S_p^n was proved, we state it as Theorem 2. Then, in [**11**], additional metric properties of these graphs were investigated, in particular establishing a connection with Stern's diatomic sequence. PARISSE [**20**] followed with a paper in which he studied, among other matters, the diameter, the eccentricity, the radius, and the center of these graphs. WIESENBERGER [**25**] obtained a formula for the average distance in S_p^n. The formula is far from being trivial, it extends over several lines! Very recently, HINZ and PARISSE [**13**] succeeded in determining the average eccentricity and its standard deviation for all Sierpiński graphs.

The metric dimension of a graph turned out to be a natural concept while studying several different problems and was consequently also reinvented in numerous disguises. (An impressive list of its applications can be found in [**10**]) It is thus clear that this dimension presents an intrinsic graph invariant. For the first time it was independently introduced in 1974 and 1975 by HARARY and MELTER [**9**] and SLATER [**24**], respectively. We refer to the recent semi-survey paper of BAILEY and CAMERON [**1**] for a great source on historical developments, connections to other invariants, non-standard terminology, and a long list of references. Another survey source for the dimension is [**7**]. Here we just recall that the metric dimension has been studied on Cartesian products of graphs [**2, 22**], distance-regular graphs [**8**], and circulant graphs [**14**].

Our paper is organized as follows. In the next section definitions, concepts, and results needed in this paper are given. Then, in Section 3, we obtain distances between almost-extreme vertices and other vertices. The advantage of the new formulas compared to Theorem 2 is that we do not need to compute the minima of related expressions. As a by-product the metric dimension of the Sierpiński graphs is determined. We point out here that in general it is very difficult to determine the exact metric dimension, see [**10**] and references therein for complexity issues on metric dimension. In the final section we use the derived formulas to compute the total distance of almost-extreme vertices.

2. PRELIMINARIES

The graphs considered are simple and connected. The *distance* $d_G(u,v)$ between vertices u and v in a graph G is the standard shortest path distance. For a vertex u of G the *total distance* $d_G(u)$ *of* u is $d_G(u) = \sum_{v\in V(G)} d_G(u,v)$. Whenever G

is clear from the context we write $d(u, v)$ and $d(u)$ instead of $d_G(u, v)$ and $d_G(u)$, respectively.

The set $\{1, 2, \ldots, n\}$ is shortly denoted by $[n]$ and the set $\{0, 1, \ldots, n-1\}$ by $[n]_0$.

Let G be a graph, then $R \subseteq V(G)$ is a *resolving set* if each vertex of G is uniquely determined by the distances to the vertices of R. More precisely, let $R = \{u_1, \ldots, u_k\}$, $k \geq 1$, then R is resolving if $(d(x, u_1), \ldots, d(x, u_k)) \neq (d(y, u_1), \ldots, d(y, u_k))$ holds for any two distinct vertices $x, y \in V(G)$. In other words, any two distinct vertices $x, y \in V(G)$ are resolved by some vertex of R, that is, there exists a vertex $u_i \in R$ such that $d(x, u_i) \neq d(y, u_i)$. The *metric dimension* of G, denoted $\mu(G)$, is the size of a minimum resolving set.

Let $p \in \mathbb{N}$, $p \geq 2$, throughout. For $n \in \mathbb{N}_0$ the *Sierpiński graph* S_p^n is defined on the vertex set $[p]^n$. Two vertices, written as $s = s_n \ldots s_1$ and $t = t_n \ldots t_1$, are adjacent if and only if they are of the form $s = \underline{s} s_\delta t_\delta^{\delta-1}$, $t = \underline{s} t_\delta s_\delta^{\delta-1}$ with $\delta \in [n]$, $\underline{s} \in [p]^{n-\delta}$, and $s_\delta \neq t_\delta$.

Note that $S_p^0 \cong K_1$, $S_p^1 \cong K_p$ for any p and that $S_2^n \cong P_{2^n}$ for every n. For S_5^3 see Figure 1. For $i \in [p]$, let iS_p^n be the subgraph of S_p^{n+1} induced by the vertices of the form $s = is_n \ldots s_1$; this subgraph is isomorphic to S_p^n.

Let $n \in \mathbb{N}$. Then S_p^n contains p *extreme vertices* of the form $i \ldots i = i^n$; they have degree $p - 1$, while all the other vertices are of degree p. We also introduce *almost-extreme vertices* of S_p^{n+1} as those vertices which are either of the form $i^n j$ or ij^n, where $i \neq j$. In Figure 1 the extreme vertices of S_5^3 are emphasized with filled circles and the almost-extreme vertices are emphasized as triangles (vertices of the form ij^2) and as diamonds (vertices of the form $i^2 j$).

Obviously, for $n \geq 2$ the graph S_p^{n+1} contains $p(p-1)$ vertices of the form $i^n j$ and also $p(p-1)$ vertices of the form ij^n. The almost-extreme vertex $i^n j$ is adjacent to the extreme vertex i^{n+1} and the almost-extreme vertex ij^n is incident with the edge between iS_p^n and jS_p^n. Thus, there are $2p(p-1)$ almost-extreme vertices. For $n = 1$ the vertices $i^n j$ and ij^n coincide, hence in S_p^2 there are exactly $p(p-1)$ almost-extreme vertices and any vertex is either extreme or almost-extreme.

The distance between a vertex of S_p^n and an extreme vertex can be computed as follows, where we use Iverson's convention that $(X) = 1$, if statement X is true, and $(X) = 0$, if X is false.

Lemma 1. **[18]** *For any $j \in [p]$ and any vertex $s = s_n \ldots s_1$ of S_p^n,*

$$d(s, j^n) = \sum_{d=1}^{n} (s_d \neq j) \cdot 2^{d-1}\,.$$

Moreover, there is exactly one shortest path between s and j^n.

An immediate consequence of Lemma 1 is that for any vertex s of S_p^n,

$$\sum_{i=1}^{p} d(s, i^n) = (p-1)(2^n - 1)\,. \tag{1}$$

(Cf. also [**20**, Proposition 2.5].) It follows that $\{i^n \mid i \in [p-1]\}$ is a resolving set for S_p^n (cf. [**21**, Lemme 3.5]): let s and t be vertices with $d(s, i^n) = d(t, i^n)$ for all $i \in [p-1]$, such that by (1) also $d(s, p^n) = d(t, p^n)$ holds; but then, by the formula in Lemma 1, $s = t$.

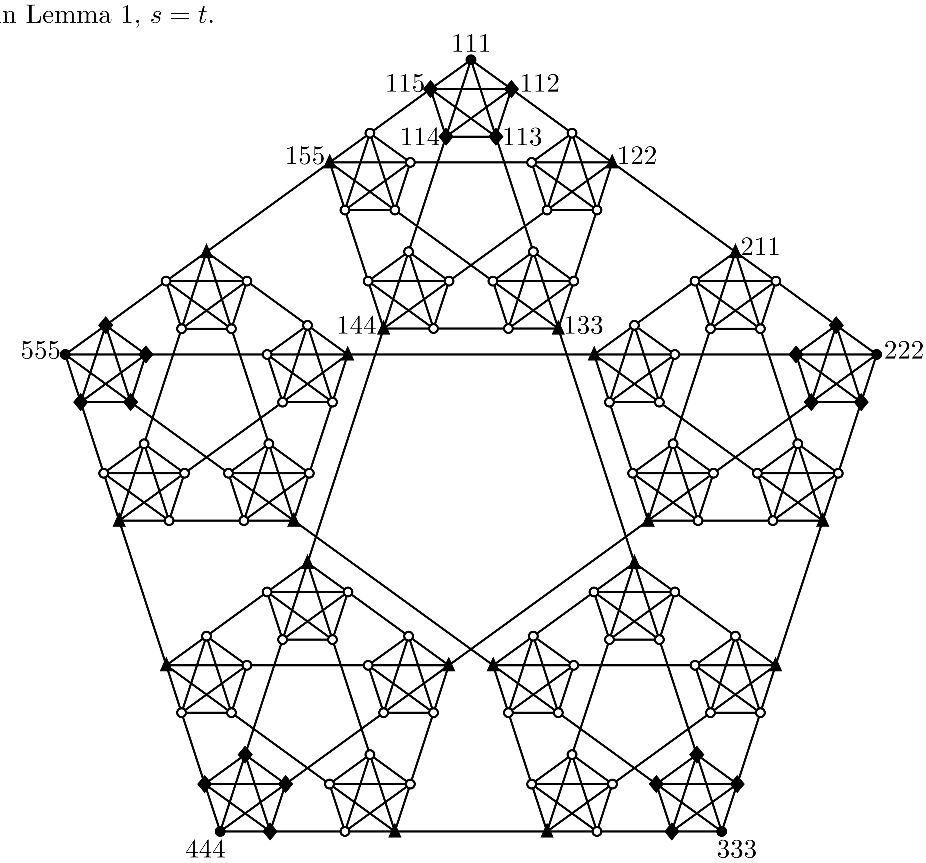

Figure 1. S_5^3 with its extreme and almost-extreme vertices emphasized

Note further that $d(i^n, j^n) = 2^n - 1$ for any $i \neq j$. More generally, the distance between arbitrary vertices of S_p^n can be determined in the following way:

Theorem 2. [**18**] *For* $i, j \in [p]$, $i \neq j$, $\delta \in [n]$, $\overline{s}$, $\overline{t} \in [p]^{\delta-1}$, *and* $\underline{s} \in [p]^{n-\delta}$, *let*

$$d_0(\underline{s}i\overline{s}, \underline{s}j\overline{t}) = d(\overline{s}, j^{\delta-1}) + 1 + d(\overline{t}, i^{\delta-1}),$$
$$\forall \ell \in [p] : \; d_\ell(\underline{s}i\overline{s}, \underline{s}j\overline{t}) = d(\overline{s}, \ell^{\delta-1}) + 1 + 2^{\delta-1} + d(\overline{t}, \ell^{\delta-1}).$$

Then,

$$d(\underline{s}i\overline{s}, \underline{s}j\overline{t}) = \min\left\{d_\ell(\underline{s}i\overline{s}, \underline{s}j\overline{t}) \mid \ell \in [p+1]_0\right\}.$$

Remark 3. The above minimum can be equivalently written as

$$\min\left\{d_\ell(\underline{s}i\overline{s}, \underline{s}j\overline{t}) \mid \ell \in [p+1]_0 \setminus \{i, j\}\right\}.$$

The respective paths realizing the values $d_\ell(\underline{s}i\overline{s}, \underline{s}j\overline{t})$ are unique. The minimum can be obtained by at most one $\ell \in [p]$. Therefore, there are at most two shortest paths between any two vertices.

It is clear from the theorem that the distance between two vertices does not depend on a common prefix; in particular, for $i \in [p]$, $n \in \mathbb{N}_0$, and $s, t \in [p]^n$,

$$d(is, it) = d(s, t). \tag{2}$$

3. DISTANCES TO ALMOST-EXTREME VERTICES

In this section we apply Theorem 2 to the case of almost-extreme vertices and begin with the almost-extreme vertices that are adjacent to extreme vertices.

Proposition 4. *Let $i, j, k \in [p]$, $i \neq j$, $n \in \mathbb{N}_0$, and $s \in [p]^n$. Then*

$$d_{S_p^{n+1}}(is, j^n k) = d(s, j^n) + 2^n - (i = k)\,.$$

Proof. We may assume that $n \in \mathbb{N}$. By the definition of the almost-extreme vertices, $j \neq k$. Then, for $\ell \in [p] \setminus \{j\}$ and using Lemma 1,

$$\begin{aligned} d_0(is, j^n k) &= d(s, j^n) + 1 + d(j^{n-1}k, i^n) = d(s, j^n) + 2^n - (i = k) \\ &\le 2^{n+1} - 1 \le 1 + 2^n + d(j^{n-1}k, \ell^n) \le d_\ell(is, j^n k)\,. \end{aligned}$$

(Here equality holds if and only if $i \neq k = \ell$, $d(s, j^n) = 2^n - 1$, and $d(s, \ell^n) = 0$, i.e. for $s = k^n$ and d_k. Only in this case there are two shortest paths between is and $j^n k$.)

Remark 5. It follows immediately from Proposition 4 that $d(is, j^n k) = d(is, j^{n+1})$ if $|\{i, j, k\}| = 3$.

This observation now allows us to approach the question of metric dimension.

Corollary 6. *For any $n \in \mathbb{N}_0$,*

$$\mu(S_p^{n+1}) = p - 1\,.$$

Moreover, if R is a minimum resolving set, then $|R \cap jS_p^n| \le 1$ holds for any $j \in [p]$.

Proof. Let $R \subset V(S_p^{n+1})$. Assume that $R \cap jS_p^n = \emptyset = R \cap kS_p^n$ for some $j \neq k$. It then follows from Remark 5 that for each $r \in R$ we have $d(r, j^n k) = d(r, j^{n+1})$, such that R cannot be a resolving set for S_p^{n+1}. Hence each resolving set must contain at least $p-1$ elements. Since we have seen earlier that (any) $p-1$ extreme vertices form a resolving set, we deduce that $\mu(S_p^{n+1}) = p - 1$ and, with recourse to the pigeonhole principle, that no jS_p^n can contain more than one element of a minimal resolving set. □

The first assertion of Corollary 6 has been found independently and at the same time by Aline Parreau [**21**, Théorème 3.6].

We now turn to the other class of almost-extreme vertices of S_p^{n+1}. To facilitate the formulation of a formula for $d(is, jk^n)$, we call $s \in [p]^n$ *special* (with respect to $i, j, k \in [p]$, $|\{i, j, k\}| = 3$, i.e. if $p \geq 3$), if there is a $\delta \in [n]$ such that $s = \underline{s}k\overline{s}$ with $\underline{s} \in ([p] \setminus \{j, k\})^{n-\delta}$ and $\overline{s} \in [p]^{\delta-1}$. Then the following holds.

Proposition 7. *Let $i, j, k \in [p]$, $i \neq j$, $j \neq k$, $n \in \mathbb{N}$, and $s \in [p]^n$. Then*

$$d_{S_p^{n+1}}(is, jk^n) = \begin{cases} d(s, k^n) + 2^n + 1, & \text{if } s \text{ is special,} \\ d(s, j^n) + 2^n - (i = k)(2^n - 1), & \text{otherwise.} \end{cases}$$

Proof. We have

$$d_0(is, jk^n) = d(s, j^n) + 1 + d(i^n, k^n) = d(s, j^n) + 2^n - (i = k)(2^n - 1)$$

and for $\ell \in [p] \setminus \{i, j\}$,

$$d_\ell(is, jk^n) = d(s, \ell^n) + 1 + 2^n + d(\ell^n, k^n)\,.$$

This is strictly larger than $d_0(is, jk^n)$, if $\ell \neq k$. So we may assume that $k \neq i$ and have to compare $d_0(is, jk^n)$ with

$$d_k(is, jk^n) = d(s, k^n) + 1 + 2^n$$

i.e. we look at the sign of

$$\begin{aligned} \rho(s) &:= d_0(is, jk^n) - d_k(is, jk^n) \\ &= d(s, j^n) - d(s, k^n) - 1 \\ &= \sum_{d=1}^{n}((s_d = k) - (s_d = j)) \cdot 2^{d-1} - 1\,. \end{aligned}$$

Now $\sum_{d=1}^{n} \tau_d \cdot 2^{d-1} \geq 1$, if and only if $\tau = \tau_n \ldots \tau_1 \in \{-1, 0, 1\}^n$ has the special form $0^{n-\delta}1\overline{\tau}$ with $\overline{\tau} \in \{-1, 0, 1\}^{\delta-1}$ for some $\delta \in [n]$ (with equality if and only if $\overline{\tau} = (-1)^{\delta-1}$). This is equivalent to s being special. (Note that there are two shortest paths if and only if $s = \underline{s}kj^{\delta-1}$.) □

Proposition 7 is illustrated in Figure 2 on S_6^4. The subgraph $1S_6^3$ is drawn explicitly and special vertices with respect to $i = 1$, $j = 2$, $k = 5$ are drawn with filled circles.

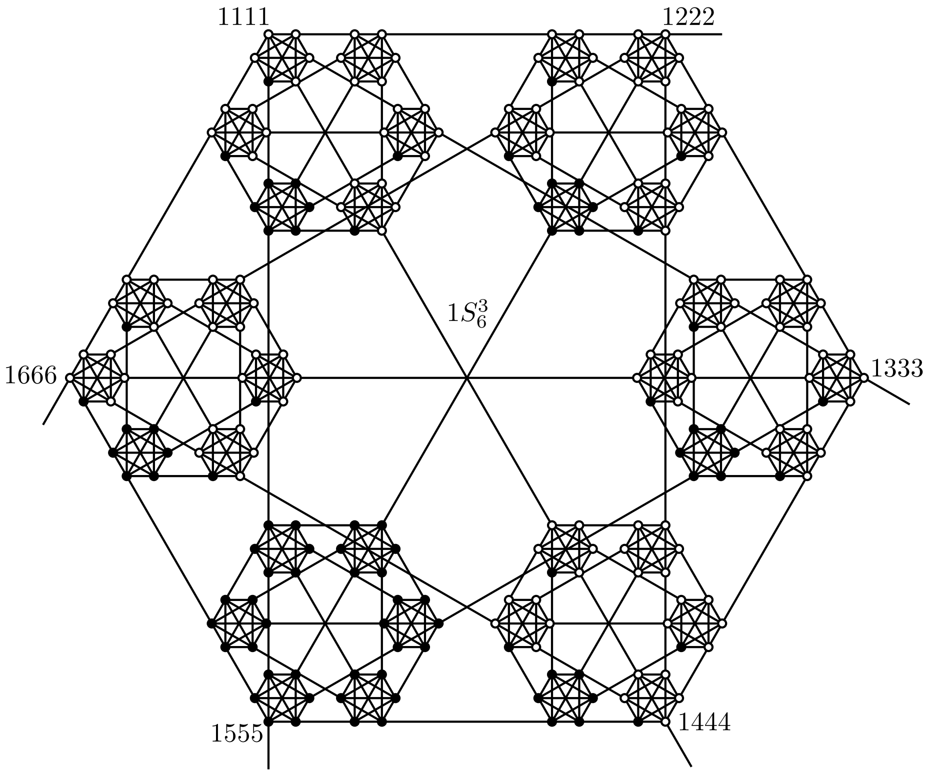

Figure 2. Illustration of Proposition 7 on S_6^4

4. TOTAL DISTANCE OF ALMOST-EXTREME VERTICES

The total distance of a vertex in particular plays an important role in mathematical chemistry, cf. [**16**], because it is a building block for the extensively investigated Wiener index of a graph. In this section we determine the total distance of almost-extreme vertices of Sierpiński graphs. To make the paper self-contained we first reprove the following result that can be found in [**25**] as well as in the proof of [**20**, Corollary 2.6].

Lemma 8. *For any $n \in \mathbb{N}$ and each $i \in [p]$,*

$$d_{S_p^n}(i^n) = p^{n-1}(p-1)(2^n-1).$$

Proof. Since for every $d \in [p]$ there are $p^{n-1}(p-1)$ vertices $s = s_n \dots s_1$ with $s_d \neq i$, $i \in [p]$, it follows by Lemma 1

$$\sum_{s\in[p]^n} d(s,i^n) = \sum_{s\in[p]^n}\sum_{d=1}^{n}(s_d \neq i)\cdot 2^{d-1} = \sum_{d=1}^{n}\left(\sum_{s\in[p]^n}(s_d\neq i)\right)\cdot 2^{d-1}$$

$$= p^{n-1}(p-1)\sum_{d=1}^{n} 2^{d-1} = p^{n-1}(p-1)\,(2^n-1)\ .$$

Theorem 9. *Let $j,k \in [p]$, $j \neq k$, and $n \in \mathbb{N}_0$. Then*

$$d_{S_p^{n+1}}(j^n k) = \frac{p-1}{p}(2p)^{n+1} - \left(1 + \frac{1}{p(p-1)}\right)p^{n+1} + \frac{p}{p-1}.$$

Proof. Set $x_0 = 1$ and $x_{n+1} = d_{S_p^{n+1}}(j^n k)$, $n \geq 0$. Then, using (2), Proposition 4, and Lemma 8,

$$\begin{aligned} x_{n+1} &= \sum_{i\in[p]}\sum_{s\in[p]^n} d(is, j^n k) \\ &= \sum_{s\in[p]^n} d(js, j^n k) + \sum_{s\in[p]^n} d(ks, j^n k) + \sum_{i\in[p]\setminus\{j,k\}}\sum_{s\in[p]^n} d(is, j^n k) \\ &= x_n + \frac{2p-1}{p}p^n(2^n-1) + (p-2)\left(\frac{2p-1}{p}(2p)^n - \frac{p-1}{p}p^n\right) \\ &= x_n + \frac{(2p-1)(p-1)}{p}(2p)^n - \left(1 + \frac{(p-1)^2}{p}\right)p^n. \end{aligned}$$

A straightforward calculation leads to the desired result.

Remark 10. The expression of Theorem 9 can be further transformed as follows:

$$\begin{aligned} d_{S_p^{n+1}}(j^n k) &= \frac{p-1}{p}(2p)^{n+1} - \left(1 + \frac{1}{p(p-1)}\right)p^{n+1} + \frac{p}{p-1} \\ &= p^n(p-1)2^{n+1} - p^n(p-1) + p^n(p-1) - p^{n+1} - \frac{p^n}{p-1} + \frac{p}{p-1} \\ &= p^n(p-1)(2^{n+1}-1) - p\cdot\frac{p^n-1}{p-1} \\ &= d_{S_p^{n+1}}(j^{n+1}) - \sum_{\ell=1}^{n} p^\ell\,. \end{aligned}$$

This alternative way to calculate $d_{S_p^{n+1}}(j^n k)$ can be interpreted as $d_{S_p^{n+1}}(j^{n+1})$ minus the additional step to all the vertices reachable directly from $j^n k$ and there are $p + p^2 + p^3 + \cdots + p^n$ such vertices.

Based on (2), Lemma 8, and Proposition 7, the corresponding result for the other almost-extreme vertices reads as follows.

Theorem 11. *Let $j,k \in [p]$, $j \neq k$, and $n \in \mathbb{N}_0$. Then*

$$d_{S_p^{n+1}}(jk^n) = \frac{p^2-2}{p(p+2)}(2p)^{n+1} - \frac{p-2}{2p}p^{n+1} - \frac{p}{2(p+2)}(p-2)^{n+1}\,.$$

Proof. Let us first calculate

$$d_0(jk^n) := \sum_{is\in[p]^{n+1}} d_0(is, jk^n) = d(k^n) + d(j^n) + p^n + (p-2)(d(j^n) + (2p)^n)$$
$$= (2p-3)(2p)^n - (p-2)p^n\,.$$

However, if $p \geq 3$, this value over-estimates $d(jk^n)$, because we did not take into account the smaller distance between is and jk^n if s is special with respect to i, j, k. We therefore have to calculate the sum $\mathrm{P} := \sum \rho(s)$ over all such special s and, for symmetry reasons, a fixed $i \in [p] \setminus \{j, k\}$ with ρ defined as in the proof of Proposition 7. We get

$$\mathrm{P} = \sum_{\delta=1}^{n}\left((p-2)^{n-\delta}p^{\delta-1}(2^{\delta-1}-1) + \sum_{\overline{s}\in[p]^{\delta-1}}\sum_{d=1}^{\delta-1}((\overline{s}_d = k) - (\overline{s}_d = j))\cdot 2^{d-1}\right).$$

The sum inside the large brackets is zero, because $\overline{s}_d$ is equal to k as often as it is equal to j. Therefore,

$$\mathrm{P} = \sum_{\delta=1}^{n}(p-2)^{n-\delta}(2p)^{\delta-1} - \sum_{\delta=1}^{n}(p-2)^{n-\delta}p^{\delta-1} = \frac{1}{p+2}(2p)^n - \frac{1}{2}p^n + \frac{p}{2(p+2)}(p-2)^n\,.$$

The statement of the theorem now follows from $d(jk^n) = d_0(jk^n) - (p-2)\mathrm{P}$. □

Note that for $n = 2$, both kinds of almost-extreme vertices coincide and their total distances must be equal. Indeed, for $n = 2$, Theorems 9 and 11 both give the value $d_{S_p^2}(jk) = p(3p-4)$. We also add that the expression of Theorem 11 can be rewritten as follows:

$$d_{S_p^{n+1}}(jk^n) = \frac{1}{2}p^n(p-2)(2^{n+1}-1) + \frac{p}{2}\sum_{\ell=0}^{n}(2p)^{n-\ell}(p-2)^{\ell}\,.$$

In this case, however, we have no interpretation for the formula such as in Remark 10.

For the classical case $p = 3$, where S_3^n is isomorphic to the Hanoi graph H_3^n with extreme vertices mapped onto perfect ones and almost-extreme vertices being transformed into vertices of the same form, we finally obtain from Lemma 8 and Theorems 9 and 11:

Corollary 12. *Let $i, j, k \in [p]$, $j \neq k$, and $n \in \mathbb{N}_0$. Then*

$$d_{S_3^n}(i^n) = \frac{2}{3}3^n(2^n-1) = d_{H_3^n}(i^n)\,.$$
$$d_{S_3^{n+1}}(j^nk) = \frac{2}{3}\cdot 6^{n+1} - \frac{7}{6}\cdot 3^{n+1} + \frac{3}{2} = d_{H_3^{n+1}}(j^nk)\,,$$
$$d_{S_3^{n+1}}(jk^n) = \frac{7}{15}\cdot 6^{n+1} - \frac{1}{6}\cdot 3^{n+1} - \frac{3}{10} = d_{H_3^{n+1}}(jk^n)\,.$$

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Faculty of Mathematics and Physics,
University of Ljubljana,
Jadranska 19, 1000 Ljubljana
Slovenia

and

Faculty of Natural Sciences and Mathematics,
University of Maribor,
Koroška 160, 2000 Maribor
Slovenia
E-mail: sandi.klavzar@fmf.uni-lj.si

Institute of Mathematics, Physics and Mechanics,
Jadranska 19, 1000 Ljubljana
Slovenia
E-mail: sara.zemljic@gmail.com